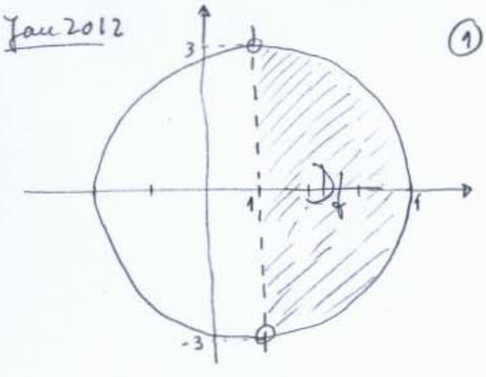
MAT2 - Época Normal - Esboço de resolução - Jan 2012



(1) $D_f = \{(x,y) \in \mathbb{R}^2 : x-1>0 \ \wedge \ 9-(x-1)^2-y^2 \geq 0\}$

a) $= \{(x,y) \in \mathbb{R}^2 : x>1 \ \wedge \ (x-1)^2+y^2 \leq 9\}$

b) $fr(D_f) = \{(x,y) \in \mathbb{R}^2 : x=1 \wedge (x-1)^2+y^2 \leq 9\} \cup$

$\cup \{(x,y) \in \mathbb{R}^2 : (x-1)^2+y^2 = 9 \ \wedge \ x \geq 1\}$

D_f ñ é fechado porque $fr(D_f) \not\subset D_f$ e, portanto, ñ é compacto

(2) · resol. da eq. homog associada:

$D^2-4=0 \Leftrightarrow D=\pm 2 \Rightarrow y_h(x) = Ae^{2x}+Be^{-2x}$, $\forall A,B \in \mathbb{R}$

· determinação duma sol. particular:

1ª sol. ensaio: $y_p(x) = ke^{3x}$

$y_p'' - 4y_p = e^{3x} \Leftrightarrow 9ke^{3x} - 4ke^{3x} = e^{3x} \Leftrightarrow 5ke^{3x} = e^{3x} \Leftrightarrow 5k=1 \Leftrightarrow k=\frac{1}{5}$

$\therefore y_p(x) = \frac{1}{5}e^{3x} \Rightarrow y_g(x) = Ae^{2x}+Be^{-2x}+\frac{1}{5}e^{3x}$, $\forall A,B \in \mathbb{R}$

(3) a) $0 \leq |f(x,y)| = \frac{y^2|\sin(x-1)|}{(x-1)^2+y^2} \leq \frac{[(x-1)^2+y^2]\cdot|\sin(x-1)|}{(x-1)^2+y^2} = |\sin(x-1)|$

$\downarrow (x,y)\to(1,0)$ $\quad 0$ $\qquad\qquad \swarrow (x,y)\to(1,0)$ $\quad 0$

$\therefore$, pelo teo. do enquadramento, $\lim\limits_{(x,y)\to(1,0)} |f(x,y)| = 0$ e portanto,

$\lim\limits_{(x,y)\to(1,0)} f(x,y) = 0$. Como $f(1,0)=0$, concluímos que f é contínua em $(1,0)$.

b) $\delta_{(v_1,v_2)} f(1,0) = \lim\limits_{t\to 0} \frac{f(1+tv_1, tv_2) - f(1,0)}{t} = \lim\limits_{t\to 0} \frac{\frac{t^2v_2^2\sin(tv_1)}{(tv_1)^2+(tv_2)^2} - 0}{t} =$

$= \lim\limits_{t\to 0} \frac{\frac{\cancel{t^2}\cdot v_2^2\sin(tv_1)}{\cancel{t^2}(v_1^2+v_2^2)}}{t} = \lim\limits_{t\to 0} \frac{v_2^2\sin(tv_1)}{t} = \lim\limits_{t\to 0} \frac{1}{2}\frac{\sin\left(\frac{\sqrt{2}}{2}t\right)}{t\times\frac{\sqrt{2}}{2}} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4}$

$\|v\|=1$ $\qquad (v_1,v_2) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ $\qquad \lim\limits_{x\to 0}\frac{\sin x}{x} = 1$

c) $\frac{\partial f}{\partial x}(1,0) = \lim_{t\to 0} \frac{f(1+t,0)-f(1,0)}{t} = \lim_{t\to 0} \frac{\frac{0^2 \sin t}{t^2} - 0}{t} = \lim_{t\to 0} \frac{0}{t} = \lim_{t\to 0} 0 = 0$ //

$\frac{\partial f}{\partial y}(1,0) = \lim_{t\to 0} \frac{f(1,t)-f(1,0)}{t} = \lim_{t\to 0} \frac{\frac{t^2 \sin 0}{t^2} - 0}{t} = \lim_{t\to 0} \frac{0}{t} = 0$ //

$$\lim_{(h_1,h_2)\to(0,0)} \frac{\varepsilon(h_1,h_2)}{\sqrt{h_1^2+h_2^2}} = \lim_{(h_1,h_2)\to(0,0)} \frac{f(1+h_1,h_2)-f(1,0)-\frac{\partial f}{\partial x}(1,0)h_1 - \frac{\partial f}{\partial y}(1,0)h_2}{\sqrt{h_1^2+h_2^2}} =$$

$$= \lim_{(h_1,h_2)\to(0,0)} \frac{f(1+h_1,h_2)}{\sqrt{h_1^2+h_2^2}} = \lim_{(h_1,h_2)\to(0,0)} \frac{h_2^2 \sin h_1}{(h_1^2+h_2^2)\sqrt{h_1^2+h_2^2}}$$

Tomemos o limite segundo a direção $h_2=h_1$

$$\lim_{\substack{(h_1,h_2)\to(0,0)\\ h_2=h_1\\ h_1>0}} \frac{\varepsilon(h_1,h_2)}{\sqrt{h_1^2+h_2^2}} = \lim_{h_1\to 0^+} \frac{\cancel{h_1^2} \sin h_1}{2\cancel{h_1^2}\sqrt{2}\,h_1} = \frac{1}{2\sqrt{2}} \neq 0 \Rightarrow$$

$\Rightarrow \lim_{(h_1,h_2)\to(0,0)} \frac{\varepsilon(h_1,h_2)}{\sqrt{h_1^2+h_2^2}}$ se existir, é $\neq 0 \Rightarrow f$ ñ é dif. em $(1,0)$

4 a) $\cdot\, y' = (3x^2+k)y \Leftrightarrow y' \underbrace{-(3x^2+k)}_{a(x)} y = 0 \Rightarrow y(x) = e^{-P-(3x^2+k)}\left[P0 + C\right]$

eq. dif. linear de 1ª ordem

$\Rightarrow y(x) = Ce^{x^3+kx}, \ \forall c\in\mathbb{R}$

$\cdot\, y(0) = 3 \Leftrightarrow Ce^0 = 3 \Leftrightarrow C = 3 \Longrightarrow y(x) = 3e^{x^3+kx}$ //

b) Sendo $y(x) = Ce^{x^3+kx}, \ \forall c\in\mathbb{R}$ a solução geral da eq. dada, então $y(x) = e^{x^3+x}$ ser solução da equação implica $\boxed{k=1}$

5 $\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases} \Leftrightarrow \begin{cases} y(x-1) + xy = 0 \\ x(x-1) = 0 \end{cases} \Leftrightarrow \begin{cases} y(2x-1) = 0 \\ x(x-1) = 0 \end{cases} \Leftrightarrow \begin{cases} — \\ x=0 \vee x=1 \end{cases}$

Se $x=0$ então (1ª eq) $y=0$ $\longrightarrow$ $(0,0)$ é pto crítico

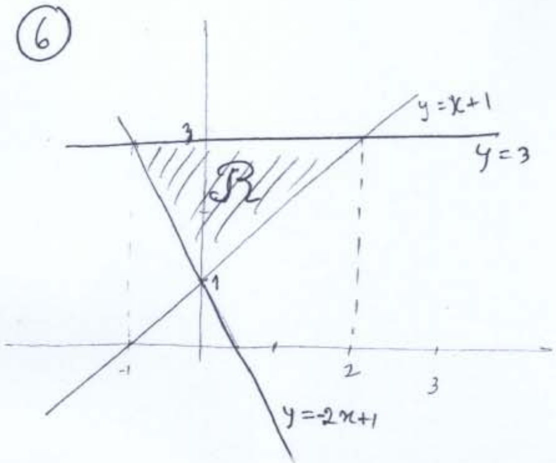
Se $x=1$ então (1ª eq) $y=0$ $\Longrightarrow$ $(1,0)$ tb é pto crítico

$Hf = \begin{bmatrix} 2y & 2x-1 \\ 2x-1 & 0 \end{bmatrix} \Rightarrow Hf(0,0) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, $\Delta_1 = 0$, $\Delta_2 = -1$ $\Rightarrow H_f(0,0)$ indef $\Rightarrow$ $(0,0)$ pto sela

$\Downarrow$

$Hf(1,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\Delta_1 = 0$, $\Delta_2 = -1$ $\Rightarrow Hf(1,0)$ IND $\Rightarrow$ $(1,0)$ tb é pto de sela.

6



$$\iint_{\mathcal{R}} 1 \, dx\,dy = \int_1^3 \left(\int_{-\frac{y}{2}+\frac{1}{2}}^{y-1} dx \right) dy =$$

$$= \int_1^3 \left(y - 1 + \frac{y}{2} - \frac{1}{2} \right) dy = \int_1^3 \frac{3y}{2} - \frac{3}{2} \, dy$$

$$= \left. \frac{3y^2}{4} - \frac{3}{2}y \right]_1^3 = \frac{27}{4} - \frac{9}{2} - \left(\frac{3}{4} - \frac{3}{2} \right) = 3$$

$\boxed{\text{ou}}$ $\iint_{\mathcal{R}} 1 \, dx\,dy = \int_{-1}^{0} \left(\int_{-2x+1}^{3} dy \right) dx + \int_0^2 \left(\int_{x+1}^{3} 1 \, dy \right) dx = \ldots = 3$

7 $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = x \left(g(x,y) + x \frac{\partial g}{\partial x} \right) + yx \frac{\partial g}{\partial y} = x\, g(x,y) + x \left[x \frac{\partial g}{\partial x} + y \frac{\partial g}{\partial y} \right] =$

$= x\, g(x,y) + x(-g(x,y)) = 0 \Rightarrow x \frac{\partial f}{\partial x} = -y \frac{\partial f}{\partial y}$, c.q.d.

$\downarrow$

g homog grau -1 e dif.
$\Downarrow$ T. Euler
$x \frac{\partial g}{\partial x} + y \frac{\partial g}{\partial y} = -g(x,y)$