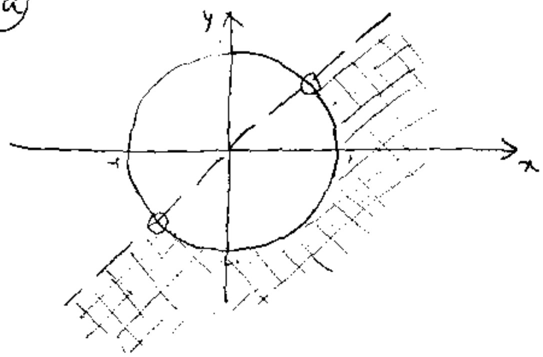
I ① $D_f = \{(x,y)\in\mathbb{R}^2 : x-y>0 \wedge x^2+y^2-1\geq 0\}$

ⓐ



ⓑ

$fr(D_f) = \{(x,y)\in\mathbb{R}^2 : x-y=0 \wedge x^2+y^2-1\geq 0\} \cup$

$\cup\{(x,y)\in\mathbb{R}^2 : x-y\geq 0 \wedge x^2+y^2=1\}$

② $\begin{cases}\frac{\partial f}{\partial x}=0\\ \frac{\partial f}{\partial y}=0\end{cases} \Leftrightarrow \begin{cases}24x-24y=0\\ 24y^2-24x=0\end{cases} \Leftrightarrow \begin{cases}x=y\\ 24(x^2-x)=0\end{cases} \Leftrightarrow \begin{cases}y=x\\ x=0 \vee x=1\end{cases}$ $\therefore$ $(0,0)$ e $(1,1)$ são os pts críticos

- $\frac{\partial^2 f}{\partial x^2}=24$; $\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x} = -24$; $\frac{\partial^2 f}{\partial y^2} = 48y$

- $H_f(0,0) = \begin{bmatrix}24 & -24\\ -24 & 0\end{bmatrix}$; $|H_f(0,0)-\lambda I|=0 \Leftrightarrow (24-\lambda)(-\lambda)-24^2=0 \Leftrightarrow$
  $\Leftrightarrow \lambda^2-24\lambda-24^2=0 \Leftrightarrow \lambda = \frac{24\pm 24\sqrt{5}}{2}$

  Como $H_f(0,0)$ tem 1 valor pp >0 e outro <0, $H_f(0,0)$ é indef e portanto (0,0) é pto de sela

- $H_f(1,1) = \begin{bmatrix}24 & -24\\ -24 & 48\end{bmatrix}$ $\Delta_1 = 24>0$, $\Delta_2 = 24\times 48-24\times 24>0$ $\Rightarrow H_f(1,1)$ DP $\Rightarrow (1,1)$ minimizante local

③ • Resol. da eq. homogénea $y_t+2y_{t-1}=0$:

$y_t=-2y_{t-1} \Leftrightarrow y_t^h = (-2)^t C$

- Determinação duma solução particular da eq. $y_t+2y_{t-1}=3t$

  1ª sol. ensaio $y_t^p = At+B$

  $y_t^p + 2y_{t-1}^p = 3t \Leftrightarrow At+B+2(A(t-1)+B)=3t \Leftrightarrow$

  $\Leftrightarrow 3At+3B-2A=3t \Rightarrow$

  $\Rightarrow \begin{cases}3A=3\\ 3B-2A=0\end{cases} \Leftrightarrow \begin{cases}A=1\\ B=2/3\end{cases} \Rightarrow y_t^p = t+\frac{2}{3}$

- A solução geral da eq. dada é $y_t = C(-2)^t + t+\frac{2}{3}$, $\forall c\in\mathbb{R}$

I 4 $Q(x,y,z) = [x\ y\ z]\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & \alpha/2 \\ 0 & \alpha/2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$\Delta_1 = 1 > 0$

$\Delta_2 = -4 < 0$

$\Delta_3 = -\frac{\alpha^2}{4} - 4 < 0 \implies$ F.Q. é indefinida, $\forall \alpha \in \mathbb{R}$

($\Delta_3 \neq 0 \implies$ F.Q. ñ é Semidef; $\Delta_1 > 0 \implies$ F.Q ñ é DN;
$\Delta_2 < 0 \implies$ F.Q. ñ é DP e portanto a F.Q. é IND)

II

1 a $\frac{\partial f}{\partial x}(3,0) = \lim_{t \to 0} \frac{f(3+t,0) - f(3,0)}{t} = \lim_{t \to 0} \frac{\frac{t^2 \cdot 0}{t^2} - 0}{t} = \lim_{t \to 0} 0 = 0$

$\frac{\partial f}{\partial y}(3,0) = \lim_{t \to 0} \frac{f(3,t) - f(3,0)}{t} = \lim_{t \to 0} \frac{\frac{0^2\sqrt{|t|}}{t^2} - 0}{t} = \lim_{t \to 0} \frac{0}{t} = \lim_{t \to 0} 0 = 0$

$\therefore \nabla f(3,0) = (0,0)$

b $$\varepsilon(h_1,h_2) = \frac{f[(3,0)+(h_1,h_2)] - \overbrace{f(3,0)}^{=0} - \overbrace{\frac{\partial f}{\partial x}(3,0)}^{=0}h_1 - \overbrace{\frac{\partial f}{\partial y}(3,0)}^{=0}h_2}{\sqrt{h_1^2+h_2^2}}$$

$$= \frac{\frac{h_1^2\sqrt{|h_2|}}{h_1^2+h_2^2}}{\sqrt{h_1^2+h_2^2}} = \frac{h_1^2\sqrt{|h_2|}}{(h_1^2+h_2^2)^{3/2}}$$

$$\lim_{\substack{(h_1,h_2)\to(0,0) \\ h_2 = mh_1}} \varepsilon(h_1,h_2) = \lim_{h_1 \to 0} \frac{h_1^2\sqrt{|mh_1|}}{(h_1^2(1+m^2))^{3/2}} = \lim_{h_1 \to 0} \frac{\sqrt{|mh_1|}}{|h_1|(1+m^2)^{3/2}} = +\infty$$

$\therefore f$ ñ é diferenciável em $(3,0)$

c $$\delta_{\bar{u}} f(3,0) = \lim_{t \to 0} \frac{f(3+tu_1, 0+tu_2) - f(3,0)}{t} = \lim_{t \to 0} \frac{(tu_1)^2\sqrt{|tu_2|}}{t(t^2u_1^2 + t^2u_2^2)} =$$

$$= \lim_{t \to 0} \frac{u_1^2\sqrt{|t||u_2|}}{t(u_1^2+u_2^2)} = \begin{cases} \infty, & \text{se } u_1 \times u_2 \neq 0 \\ \frac{u_1^2\sqrt{|u_2|}}{[illegible]}, & \text{se } u_1 \times u_2 = 0 \end{cases} \longrightarrow$$

$\therefore$ existe derivada apenas segundo os vectores $(u_1, 0)$ e $(0, u_2)$, $u_1, u_2 \neq$

② $\left.\begin{array}{l} \mathrm{int}(A) = \emptyset \\ \mathrm{fr}(A) = A \end{array}\right\} \Rightarrow A = \mathrm{ad}(A) \Leftrightarrow A$ é fechado

A é limitado porque $A \subset B_r(0,0)$, $r > \sqrt{8}$

$\Rightarrow$ A é compacto

- $f(x,y)$ é fç polinomial e portanto é contínua em A

$\therefore$ Pelo T. Weierstrass f tem máximo e mínimo absolutos no conjunto A.

Calculo do max e min:

$$\mathcal{L}(x,y;\lambda) = x^2 + 2xy + y^2 + \lambda(x^2+y^2-8)$$

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x} = 0 \\ \frac{\partial \mathcal{L}}{\partial y} = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = 0 \end{cases} \Leftrightarrow \begin{cases} 2x + 2y + 2\lambda x = 0 \\ 2x + 2y + 2\lambda y = 0 \\ x^2 + y^2 = 8 \end{cases} \Leftrightarrow \begin{cases} - \\ 2\lambda y - 2\lambda x = 0 \\ - \end{cases} \Leftrightarrow \begin{cases} - \\ 2\lambda(y-x) = 0 \\ - \end{cases}$$

- Se $\lambda = 0 \xRightarrow{1^a\ eq} x = -y \xRightarrow{3^a\ eq} 2y^2 = 8 \Leftrightarrow y = \pm 2$

$\therefore$ Para $\lambda = 0$ os possíveis ext. são $(2,-2)$ e $(-2,2)$

- Se $y = x \xRightarrow{3^a\ eq} x = \pm 2$

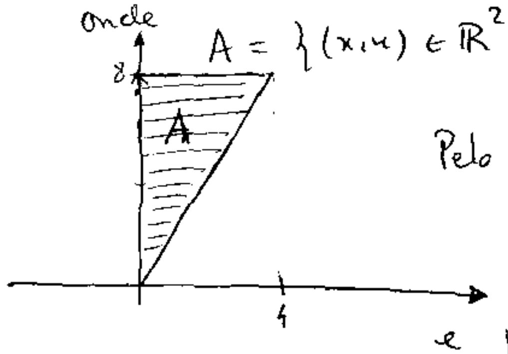
$\xRightarrow{1^a\ eq} 4x + 2\lambda x = 0 \Leftrightarrow 2x(2+\lambda) = 0 \xRightarrow{x \neq 0} \lambda = -2$

$\therefore$ Para $\lambda = -2$, os possíveis ext. são $(2,2)$ e $(-2,-2)$

$$\left.\begin{array}{l} f(2,2) = 16 \\ f(-2,-2) = 16 \\ f(-2,2) = 0 \\ f(2,-2) = 0 \end{array}\right\} \xRightarrow{\text{T. Weierstrass}} \begin{array}{l} 16 \text{ é o valor máximo de } f \text{ em } A \\ 0 \text{ é o valor mínimo de } f \text{ em } A \end{array}$$

II. (3) $\int_0^4 \int_{2x}^8 \sin y^2 \, dy \, dx = \iint_A \sin y^2 \, dy \, dx$,

onde $A = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 4 \, , \, 2x \leq y \leq 8\}$



Pelo esboço gráfico, podemos reescrever A

$$A = \{(x,y) : 0 \leq y \leq 8 \quad , \quad 0 \leq x \leq y/2\}$$

e portanto,

$$\int_0^4 \int_{2x}^8 \sin(y^2) \, dy \, dx = \int_0^8 \int_0^{y/2} \sin(y^2) \, dx \, dy =$$

$$= \int_0^8 \sin(y^2) \cdot x \Big]_0^{y/2} dy = \int_0^8 \sin(y^2) \times y/2 \, dy =$$

$$= \frac{1}{4} \int_0^8 2y \cdot \sin(y^2) \, dy = \frac{1}{4} \; -\cos y^2 \Big]_0^8 =$$

$$= \frac{1}{4}\left(-\cos 64 + 1\right)$$

(4) $\frac{\partial g}{\partial x} = \frac{df}{du} \cdot \frac{\partial u}{\partial x}$, onde $u(x,y) = \frac{y}{x}$

$$= \frac{df}{du} \times \left(-\frac{y}{x^2}\right)$$

$$\frac{\partial^2 g}{\partial y \, \partial x} = \left(\frac{d^2 f}{du^2} \cdot \frac{\partial u}{\partial y}\right)\left(-\frac{y}{x^2}\right) + \frac{df}{du} \times \left(-\frac{1}{x^2}\right)$$

$$= \frac{d^2 f}{du^2} \cdot \frac{1}{x}\left(-\frac{y}{x^2}\right) + \frac{df}{du}\left(-\frac{1}{x^2}\right)$$

$$\frac{\partial^2 g}{\partial x^2} = \left(\frac{d^2 f}{du^2}\,\frac{\partial u}{\partial x}\right)\left(-\frac{y}{x^2}\right) + \frac{df}{du}\,\frac{2y}{x^3}$$

$$= \frac{d^2 f}{du^2}\left(\frac{y^2}{x^4}\right) + \frac{df}{du}\,\frac{2y}{x^3}$$

Assim, $x\,\dfrac{\partial^2 g}{\partial x^2} + y\,\dfrac{\partial^2 g}{\partial y \partial x} =$

$$= \left(\cancel{\frac{y^2}{x^3}\,\frac{d^2 f}{du^2}} + \frac{2y}{x^2}\,\frac{df}{du}\right) + \left(\cancel{-\frac{y^2}{x^3}\,\frac{d^2 f}{du^2}} - \frac{df}{du}\,\frac{y}{x^2}\right)$$

$$= \frac{y}{x^2}\,\frac{df}{du} = -\frac{\partial g}{\partial x} \qquad \text{c.q.p.}$$

Interpretação: A função derivada parcial de g em ordem a x, $\dfrac{\partial g}{\partial x}(x,y)$, verifica a igualdade de Euler, com $\alpha = -1$.