Mathematics II

Undergraduate Degrees in Economics and Management

Regular period, January 5th, 2016

Part II

1. Determine and classify all critical points of $f(x,y,z) = 2x^4 + y^2 + z^2 - xy^2 + x - 2z$.

Solution: The critical points of f are the solutions of the nonlinear system $\nabla f = 0$, that is

$$\nabla f = 0 \Leftrightarrow \begin{cases} \dfrac{\partial f}{\partial x} = 0 \\ \dfrac{\partial f}{\partial y} = 0 \\ \dfrac{\partial f}{\partial z} = 0 \end{cases} \Leftrightarrow \begin{cases} 8x^3 - y^2 + 1 = 0 \\ 2y - 2xy = 0 \\ 2z - 2 = 0 \end{cases} \Leftrightarrow \begin{cases} 8x^3 - y^2 + 1 = 0 \\ 2y(1-x) = 0 \\ z = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} 8x^3 + 1 = 0 \\ y = 0 \\ z = 1 \end{cases} \vee \begin{cases} y^2 = 9 \\ x = 1 \\ z = 1 \end{cases} \Leftrightarrow \begin{cases} x = -\frac{1}{2} \\ y = 0 \\ z = 1 \end{cases} \vee \begin{cases} y = \pm 3 \\ x = 1 \\ z = 1 \end{cases}$$

So, the critical points are: $(-\frac{1}{2}, 0, 1); (1, 3, 1); (1, -3, 1)$. They can be classified using the hessian matrix

$$H_f(x,y,z) = \begin{pmatrix} 24x^2 & -2y & 0 \\ -2y & (2-2x) & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

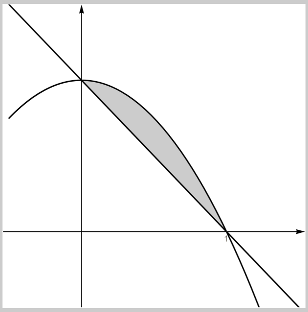
$$H_f(-\frac{1}{2}, 0, 1) = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad H_f(1, \pm 3, 1) = \begin{pmatrix} 24 & \mp 6 & 0 \\ \mp 6 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

For the point $(-\frac{1}{2}, 0, 1)$ we have $\Delta_1 = 6 > 0, \Delta_2 = 18 > 0, \Delta_3 = 36 > 0$ and so the point is a local minimum.

For the other points we have that $\Delta_1 = 24 > 0, \Delta_2 = -36 < 0, \Delta_3 = -72 < 0$, which means that the hessian matrix in indefinite and the points are saddle points.

2. Compute $\iint_\Omega x(y-1)dxdy$, where $\Omega = \{(x,y) \in \mathbb{R}^2 : x+y \geq 1, y \leq 1-x^2, x \geq 0\}$.

Solution:



$$\iint_\Omega x(y-1)dxdy = \int_0^1 \int_{1-x}^{1-x^2} x(y-1)dydx$$
$$= \int_0^1 x\left[\frac{(y-1)^2}{2}\right]_{y=1-x}^{y=1-x^2} dx$$
$$= \int_0^1 \frac{x}{2}(x^4-x^2)dx = \frac{1}{2}\left[\frac{x^6}{6}-\frac{x^4}{4}\right]_{x=0}^{x=1} = -\frac{1}{24}$$

3. After certain economic considerations, it was possible to establish the following differential equation, for the dynamic equilibrium price of a given commodity.

$$y''(x) + a\,y'(x) + b\,y(x) = c,$$

where $y(x)$ is the price at time x and a, b, c are known constants.

(a) Assuming that $a = 1$, $b = \frac{1}{2}$, $c = 1$, and $y(0) = y'(0) = 1$, determine the equilibrium price $y(x)$. Provide a long run estimate $(t \to +\infty)$ of the equilibrium price.

Solution: Substituting all the given parameters in the differential equation, we get the initial value problem $y''(x) + y'(x) + \frac{1}{2}y(x) = 1, y(0) = y'(0) = 1$. Using the superposition principle, the general solution to this linear non-homegeneous equation can be obtained as $y(x) = y_h(x) + y_p(x)$, where $y_h(x)$ is the general of the homogeneous equation and $y_p(x)$ is a particular solution of the equation.

i. **Solution of the homegeneous equation.** The characteristic polynomial is given by $P(D) = D^2 + D + \frac{1}{2}$ and has two complex roots, $-\frac{1}{2} \pm \frac{1}{2}i$. Hence we have
$$y_h(x) = e^{-\frac{x}{2}}\left(c_1 \cos\frac{x}{2} + c_2 \sin\frac{x}{2}\right).$$

ii. **Particular solution of the equation.** Since the right hand side is a constant, we shall try a particular solution of the form $y_p(x) = k$, which leads to $y_p(x) = 2$.

iii. The **general solution** is then given by

$$y(x) = 2 + e^{-\frac{x}{2}} \left(c_1 \cos \frac{x}{2} + c_2 \sin \frac{x}{2} \right).$$

iv. Finally, the initial conditions must be imposed

$$\begin{cases} y(0) = 1 \\ y'(0) = 1 \end{cases} \Leftrightarrow \begin{cases} 2 + c_1 = 1 \\ -\frac{1}{2}c_1 + \frac{1}{2}c_2 = 1 \end{cases} \Leftrightarrow \begin{cases} c_1 = -1 \\ c_2 = 1 \end{cases}$$

yielding the particular solution

$$y(x) = 2 + e^{-\frac{x}{2}} \left(-\cos \frac{x}{2} + \sin \frac{x}{2} \right).$$

The long run behavior of the equilibrium price can be determined computing the limit

$$\lim_{x \to +\infty} y(x) = \lim_{x \to +\infty} \left[2 + e^{-\frac{x}{2}} \left(-\cos \frac{x}{2} + \sin \frac{x}{2} \right) \right] = 2.$$

(b) Show that if $a = 0$ and $b > 0$, the equilibrium price is a periodic function of x.

Solution: If $a = 0$ and $b > 0$ the characteristic polynomial, $P(D) = D^2 + b$ has two complex roots given by $\pm\sqrt{b}i$. The general solution is in this case given by (see previous calculations)

$$y(x) = \frac{c}{b} + c_1 \cos(\sqrt{b}x) + c_2 \sin(\sqrt{b}x),$$

which is a periodic function (with period $\frac{2\pi}{\sqrt{b}}$).

4. Suppose that the total income in a closed economy, denoted by Y_t, follows the difference equation $Y_t = 2Y_{t-1} - Y_{t-2} + G$, where G is a known constant.

(a) Set $Y_0 = Y_1 = 10$, $G = 2$ and determine the total income Y_t, for any $t \geq 2$.

Solution: This is a linear second order difference equation with constant coefficients, whose characteristic polynomial is $p(\lambda) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$. This polynomial a a real root ($\lambda = 1$) with multiplicity 2, so the general solution to the homogeneous equation is given by $Y_t^h = (c_1 t + c_2)1^t = c_1 t + c_2$. On the other hand, we must look for a particular solution of the form $y_t^* = kt^2$ (polynomials of degree 0 or 1 will not work as they are solutions to the homogeneous equation), which leads to the general solution $Y_t = c_1 t + c_2 + t^2$. Finally, using the initial conditions, we can compute the constants c_1, c_2

$$\begin{cases} Y_0 = 10 \\ Y_1 = 10 \end{cases} \Leftrightarrow \begin{cases} c_2 = 10 \\ c_1 + c_2 + 1 = 10 \end{cases} \Leftrightarrow \begin{cases} c_2 = 10 \\ c_1 = -1 \end{cases}$$

obtaining the particular solution $Y_t = t^2 - t + 10$.

(b) What should be the value of G for the total income to be linear in t?

Solution: For general G, using the same procedure as above, the particular solution is given by $Y_t^* = \frac{G}{2}t^2$. Since the particular solution is the only non-linear component of the solution, setting $G = 0$ makes the solution linear in t.

Point values: 1. 3,0 **2**. 2,0 **3**. (a) 2,5 (b) 0,5 **4**. (a) 1.5 (b) 0.5

LISBON SCHOOL OF ECONOMICS & MANAGEMENT
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Part I

1. Classify the quadratic form $Q(x, y, z) = 4x^2 + 4xy + 2y^2 - 2yz + 2z^2$. Is there any vector $(a, b, c) \in \mathbb{R}^3$ such that $Q(a, b, c) = 0$? And such that $Q(a, b, c) < 0$? Justify.

Solution: The quadratic form is given by $Q(x, y, z) = (x, y, z)^T A(x, y, z)$, where

$$A = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

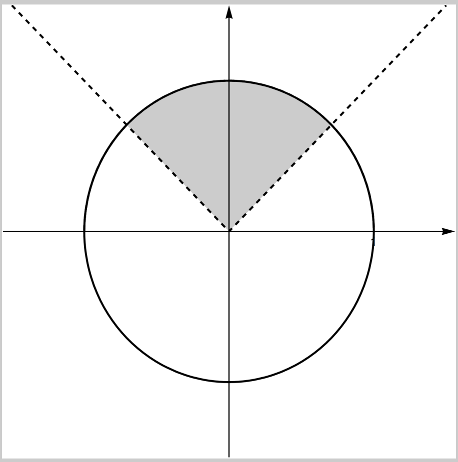
The determinants of the principal minors are $\Delta_1 = 4 > 0, \Delta_2 = 4 > 0, \Delta_3 = 4 > 0$. Since all these numbers are positive, the matrix is positive definite and so is the quadratic form. Taking $(a, b, c) = (0, 0, 0)$, we do have $Q(a, b, c) = 0$. Q being positive definite means that $Q(a, b, c) > 0, \forall (a, b, c) \neq (0, 0, 0)$ and so there is no vector (a, b, c) such that $Q(a, b, c) < 0$.

2. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = \ln(y - |x|) + \sqrt{1 - x^2 - y^2}$

(a) Determine the domain of f, D_f, analytically and represent it graphically.

Solution:

$$D_f = \{(xy) \in \mathbb{R}^2 : y - |x| > 0 \wedge 1 - x^2 - y^2 \geq 0\} = \{(x, y) \in \mathbb{R}^2 : y > |x| \wedge x^2 + y^2 \leq 1\}$$



(b) Determine the interior and the boundary of D_f. Decide if D_f is open and if D_f is compact. Is Weierstrass's theorem applicable to f in D_f? Is f bounded from below in D_f? Justify.

Solution:

$$\begin{aligned}\text{Int}(D_f) =&\{(x,y) \in \mathbb{R}^2 : y > |x| \wedge x^2+y^2 < 1\}\\ \text{Bdy}(D_f) =&\{(x,y) \in \mathbb{R}^2 : y = |x| \wedge x^2+y^2 = 1\} \cup \{(x,y) \in \mathbb{R}^2 : y \geq |x| \wedge x^2+y^2 = 1\}\\ \overline{D_f} =&\text{Int}(D_f) \cup \text{Bdy}(D_f) = \{(x,y) \in \mathbb{R}^2 : y \geq |x| \wedge x^2+y^2 \leq 1\}\end{aligned}$$

Since $D_f \neq \text{Int}(D_f)$, the set is not open and since $D_f \neq \overline{D_f}$ the set is not closed and therefore not compact. Weierstrass's theorem is not applicable to f on D_f because D_f is not compact. Function f is not bounded from below on D_f because for example

$$\lim_{y \to 0^+} f(0,y) = \lim_{y \to 0^+} \left(\ln y + \sqrt{1-y^2}\right) = -\infty,$$

which shows that f can take arbitrarily large negative values on D_f.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} \dfrac{y^2\sqrt{|x|}}{x^2+y^2} & ,(x,y) \neq (0,0) \\ 0 & ,x=y=0 \end{cases}.$$

(a) Compute the directional limits at $(0,0)$.

Solution:

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} f(x,y) = \lim_{x\to 0} \frac{(mx)^2\sqrt{|x|}}{x^2+(mx)^2} = \lim_{x \to 0} \frac{m^2\sqrt{|x|}}{1+m^2} = 0.$$

(b) Show that f is continuous in $\mathbb{R}^2$.

Solution: If $(x,y) \neq (0,0)$ the function is obviously continuous (quotient of continuous functions with nonzero denominator). If f is also continuous in $(0,0)$, it will be continuous over $\mathbb{R}^2$. From (a) we know that all directional limits at $(0,0)$ are zero as so the only possible candidate to limit is zero. Also, since

$$0 \leq \left| \frac{y^2\sqrt{|x|}}{x^2+y^2} - 0 \right| \leq \frac{(x^2+y^2)\sqrt{|x|}}{x^2+y^2} = \sqrt{|x|} \to 0 \quad (\text{as}(x,y) \to (0,0)),$$

we can establish that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$, which allows us to conclude that f is continuous at $(0,0)$ and therefore (as discussed above) over $\mathbb{R}^2$.

(c) Compute the directional derivatives at $(0,0)$.

Solution:

$$\frac{\partial f}{\partial(u,v)}(0,0) = \lim_{t\to 0}\frac{f(0+ut,0+vt)-f(0,0)}{t} = \lim_{t\to 0}\frac{v^2t^2\sqrt{|ut|}}{t((ut)^2+(vt)^2)}$$
$$= \lim_{t\to 0}\frac{v^2\sqrt{|u|}}{u^2+v^2}\frac{\sqrt{|t|}}{t} = \begin{cases}\infty, & v\neq 0\\ 0, & v=0\end{cases}$$

4. Suppose that the income (Y) is a function of capital and labor, $Y = K^{1/2}L^{1/2}$, and that both capital and labor are functions of time (t). Use the chain rule to compute $Y'(t)$, considering $K = t, L = t^2$.

Solution:

$$Y'(t) = \frac{\partial Y}{\partial K}\cdot\frac{dK}{dt} + \frac{\partial Y}{\partial L}\cdot\frac{dL}{dt} = \frac{L^{1/2}}{2K^{1/2}}\cdot 1 + \frac{K^{1/2}}{2L^{1/2}}\cdot 2t = \frac{t}{2\sqrt{t}} + \frac{2t\sqrt{t}}{2t} = \frac{3}{2}\sqrt{t}$$

5. Compute the second order Taylor polynomial of $f(x,y) = y^2e^x$ around $(0,0)$ and use it to estimate $f(0.1, 0.1)$

Solution: The second order Taylor polynomial is given by

$$p_2(x,y) = f(0,0) + \frac{\partial f}{\partial x}(0,0)x + \frac{\partial f}{\partial y}(0,0)y + \frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}(0,0)x^2 + 2\frac{\partial^2 f}{\partial x\partial y}(0,0)xy + \frac{\partial^2 f}{\partial y^2}(0,0)y^2\right)$$
$$= \frac{1}{2!}2y^2 = y^2.$$

We can therefore consider the estimate $f(0.1,0.1) \approx p_2(0.1,0.1) = 0.1^2 = 0.01$.

Point values: 1. 1,5 **2**. (a) 1,5 (b) 1,5 **3**. (a) 1,0 (b) 1,5 (c) 1,0 **4**. 1,0 **5**. 1,0