LISBON SCHOOL OF ECONOMICS & MANAGEMENT
UNIVERSIDADE DE LISBOA

Mathematics II

Undergraduate Degrees in Economics and Management
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Part II

1. Consider the function $f(x,y) = e^x(kx + y^2)$.

(a) Determine and classify all critical points of f, when $k \neq 0$.

Solution: The critical points are the solutions of the system

$$\nabla f(x,y) = (0,0) \Leftrightarrow \begin{cases} e^x(kx+y^2) + ke^x = 0 \\ 2ye^x = 0 \end{cases} \Leftrightarrow \begin{cases} k(x+1) = 0 \\ y = 0 \end{cases}.$$

Since $k \neq 0$ the first equation yields $x = -1$ and we conclude that the only critical point is $(-1,0)$. The Hessian matrix is given by

$$H_f(-1,0) = \begin{bmatrix} k(x+2)e^x & 2ye^x \\ 2ye^x & 2e^x \end{bmatrix}_{|(-1,0)} = \begin{bmatrix} ke^{-1} & 0 \\ 0 & 2e^{-1} \end{bmatrix}$$

and its principal minors are $\Delta_1 = ke^{-1}, \Delta_2 = 2ke^{-2}$. Therefore, if $k > 0$ all minors are positive and (-1,0) is a local minimum point; if $k < 0$ all minors are negative and (-1,0) is a saddle point.

(b) Show that if $k = 0$, then f has a global minimum.

Solution: When $k = 0$ we have $f(x,y) = y^2e^x$. We can easily observe that $f(x,y) \geq 0$ and that $f(x,y) = 0 \Leftrightarrow y = 0$. Therefore we conclude that any point of the form $(x_0, 0)$ will be a global minimum point. The global minimum is $f(x_0, 0) = 0$.

(c) Justify that f attains a global maximum over the set $M = \{(x,y) \in \mathbb{R}^2 : x^2 - 1 \leq y \leq 1 - x^2\}$.(**note**: you are not required to compute the maximum).

Solution: Function f is the product of continuous functions (an exponential with polynomial exponent and a polynomial) and so it is continuous. The set M is compact because it is bounded, since $M \subset B_2((0,0))$ and closed since all boundary points belong to the set. Since f is a continuous function defined over a compact set, Weierstrass's theorem guarantees that it will have a global minimum and a global maximum over M. In particular, it will attain a global minimum, as we wanted to prove.

2. Compute $\int_\Omega xe^y dx\, dy$, where $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2+1 \leq y \leq 2\}$.

Solution:

$$\int_\Omega xe^y dxdy = \int_{-1}^{1}\int_{x^2+1}^{2} xe^y dydx = \int_{-1}^{1} x[e^y]_{y=x^2+1}^{y=2} dx = \int_{-1}^{1} x(e^2 - e^{x^2+1})dx$$
$$= \left[\frac{x^2e^2}{2} - \frac{e^{x^2+1}}{2}\right]_{x=-1}^{x=1} = 0$$

3. Consider a macroeconomic model where Q^S is the aggregate supply, P is the price level, and π is the expected rate of inflation. Assuming that the aggregate demand is given by $Q^D(t) = 2 - bP(t) + \pi(t)$, that the prices adjust according to $P'(t) = \frac{1}{2}(Q^D(t)-1)+\pi(t)$ and that expectations are adaptive ($\pi'(t) = k[P'(t)-\pi(t)]$), the price level follows the differential equation:

$$P''(t) - \frac{1}{2}(k-b)P'(t) + \frac{1}{2}kbP(t) = \frac{1}{2}k, \quad b,k>0. \tag{1}$$

(a) If the expected rate of inflation is constant ($k=0$), equation (1) reduces to $P''(t) + \frac{1}{2}bP'(t) = 0$. Knowing that $P(0) = P_0$ and that $\lim_{t\to+\infty} P(t) = P_\infty$, determine the price trajectory $P(t)$.

Solution: The characteristic polynomial of the differential equation $P''(t) + \frac{1}{2}bP'(t) = 0$ is $D^2 + \frac{1}{2}bD$, and its roots are $D=0$ and $D = -\frac{1}{2}b$. The general solution of this homegeneous equations is given by

$$P(t) = C_1e^{0\cdot t} + C_2e^{-\frac{1}{2}bt} = C_1 + C_2e^{-\frac{1}{2}bt}.$$

The constants C_1, C_2 can now be computed from the additional information that was provided:

$$\begin{cases} P(0) &= P_0 \\ \lim\limits_{t\to+\infty} P(t) &= P_\infty \end{cases} \Leftrightarrow \begin{cases} C_1+C_2 &= P_0 \\ C_1 &= P_\infty \end{cases} \Leftrightarrow \begin{cases} C_2 &= P_0 - P_\infty \\ C_1 &= P_\infty \end{cases}$$

and we finally conclude that

$$P(t) = P_\infty + (P_0 - P_\infty)e^{-\frac{1}{2}bt}.$$

note: $\lim\limits_{t\to+\infty} e^{-\frac{1}{2}bt} = 0$ because $b>0$.

(b) Consider now the set of parameters $k = 1, b = 3$ and determine the general solution of the differential equation (1).

Solution: For this choice of parameters the differential equation becomes $P''(t) + P'(t) + \frac{3}{2}P(t) = \frac{1}{2}$, which is a non-homogeneous, second order, linear differential equation with constant coefficients. The general solution is given by $P(t) = P_*(t) + P_h(t)$, where $P_h(t)$ is the general solution of the associated homogeneous equations and $P_*(t)$ is a particular solution on the complete equation. Regarding the homogeneous equation,

$$P_h''(t) + P_h'(t) + \frac{3}{2}P(t) = 0 \Leftrightarrow (D^2 + D + \frac{3}{2})P_h = 0$$
$$\Leftrightarrow P_h(t) = e^{-\frac{t}{2}}(C_1 \cos(\sqrt{5}t/2) + C_2 \sin(\sqrt{5}t/2))$$

Regarding the particular solution, since the right hand side is a constant, we will search for a constant particular solution $P_*(t) = \alpha$, which leads to $P_*(t) = \frac{1}{3}$.

The general solution of the equation is thus given by

$$P(t) = \frac{1}{3} + e^{-\frac{t}{2}}(C_1 \cos(\sqrt{5}t/2) + C_2 \sin(\sqrt{5}t/2)).$$

Note 1: $D^2 + D + \frac{3}{2} = 0 \Leftrightarrow D = \dfrac{-1 \pm \sqrt{1-6}}{2} = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}i$

Note 2: $P_*(t) = \alpha \Rightarrow \alpha'' + \alpha' + \frac{3}{2}\alpha = \frac{1}{2} \Leftrightarrow \alpha = \frac{1}{3}$.

Point values: 1. (a) 2,0 (b) 1,5 (c) 1,0 **2**. 2,5 **3**. (a) 1,5 (b) 1,5

Part I

1. Consider matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix}$.

(a) Compute the eigenvalues of A, as well as their algebraic multiplicities.

Solution:

$$|A - \lambda I| = 0 \Leftrightarrow \begin{vmatrix} (1-\lambda) & 2 & 0 \\ 0 & (2-\lambda) & 2 \\ 0 & 2 & (2-\lambda) \end{vmatrix} \Leftrightarrow (1-\lambda)[(2-\lambda)^2 - 2^2] - 2(0-0) = 0$$

$$\Leftrightarrow (1-\lambda)(2-\lambda-2)(2-\lambda+2) = 0 \Leftrightarrow \lambda = 0 \vee \lambda = 1 \vee \lambda = 4$$

The eigenvalues of A are $\lambda = 0, 1, 4$, all with algebraic multiplicity 1.

(b) Let $Q : \mathbb{R}^3 \to \mathbb{R}$ be defined by $Q(\boldsymbol{x}) = \boldsymbol{x}^T A \boldsymbol{x}$. Write down the expression of the quadratic form Q and classify it.

Solution: Computing the quadratic form we get

$$Q(x_1, x_2, x_3) = x_1^2 + 2x_1x_2 + 2x_2^2 + 4x_2x_3 + x_3^2.$$

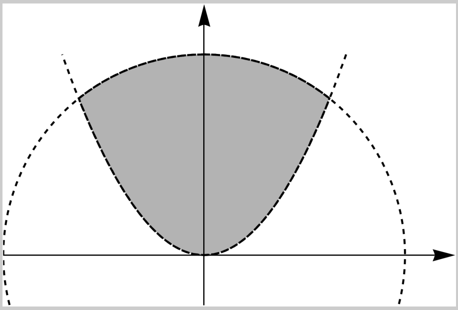
The symmetric matrix that represents this quadratic form is $B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix}$ and, for this matrix B, the principal minors are given by $\Delta_1 = 1 > 0, \Delta_2 = 1 > 0, \Delta_3 = -2 < 0$. This way we see that B is indefinite and so is Q.

2. Let $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ be defined by the expression $f(x, y) = \dfrac{\ln(4 - x^2 - y^2)}{\sqrt{y - x^2}}$.

(a) Determine the domain of f, Ω, analitically and geometrically.

Solution:

$$\begin{aligned}\Omega =& \{(x,y) \in \mathbb{R}^2 : 4 - x^2 - y^2 > 0 \wedge y - x^2 \geq 0 \wedge \sqrt{y - x^2} \neq 0\} \\ =& \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 4 \wedge y > x^2\}\end{aligned}$$



(b) Determine the boundary of Ω and decide if the set is open.

Solution:

$$Bdy(\Omega) = \{(x,y) \in \mathbb{R}^2 : (x^2 + y^2 = 4 \wedge y \geq x^2) \vee (y = x^2 \wedge x^2 + y^2 \leq 4)\}$$

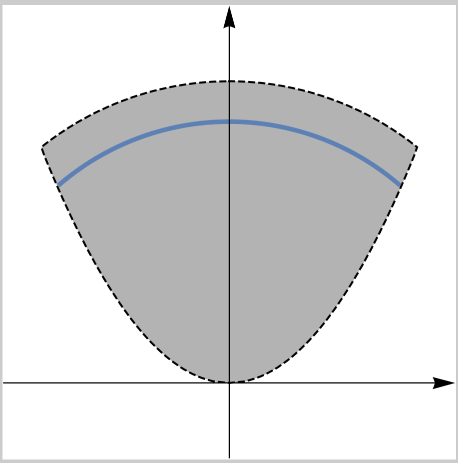
The points in the boundary do not belong to the set, which coincides with its interior and is therefore an open set.

(c) Sketch the zero levelset $C_0 = \{(x,y) \in \Omega : f(x,y) = 0\}$.

Solution:

$$f(x,y) = 0 \Leftrightarrow \frac{\ln(4 - x^2 - y^2)}{\sqrt{y - x^2}} = 0 \Leftrightarrow \ln(4 - x^2 - y^2) = 0 \Leftrightarrow x^2 + y^2 = 3.$$

The zero levelset will be the part of the circunference $x^2 + y^2 = 3$ that lies inside the domain of f, i.e. the thick line in the following image.



3. Consider $f(x,y) = \begin{cases} x+y & , y>0 \\ x+ye^y & , y \leq 0 \end{cases}$

(a) Compute $\dfrac{\partial f}{\partial x}(0,0)$, $\dfrac{\partial f}{\partial y}(0,0)$.

Solution:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0}\frac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0}\frac{t+0\cdot e^t-0}{t} = 1$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0}\frac{f(0,t)-f(0,0)}{t} = \lim_{t\to 0}\frac{f(0,t)}{t} \overset{(*)}{=} 1$$

Both partial derivatives are equal to one.

(*)

$$\lim_{t\to 0^+}\frac{f(0,t)}{t} = \lim_{t\to 0^+}\frac{0+t}{t} = 1$$
$$\lim_{t\to 0^-}\frac{f(0,t)}{t} = \lim_{t\to 0^+}\frac{0+te^t}{t} = 1$$

(b) Show that f is differentiable at $(0,0)$.

Solution: We must show that

$$\lim_{(u,v)\to(0,0)}\frac{f(u,v)-f(0,0)-u\frac{\partial f}{\partial x}(0,0)-v\frac{\partial f}{\partial y}(0,0)}{\sqrt{u^2+v^2}} = 0.$$

In order to do so, we only need to note that

$$\left|\frac{f(u,v)-f(0,0)-u\frac{\partial f}{\partial x}(0,0)-v\frac{\partial f}{\partial y}(0,0)}{\sqrt{u^2+v^2}} - 0\right| = \left|\frac{f(u,v)-u-v}{\sqrt{u^2+v^2}}\right|$$

$$= \begin{cases} \left|\dfrac{u+v-u-v}{\sqrt{u^2+v^2}}\right| & , v>0 \\ \left|\dfrac{u+ve^v-u-v}{\sqrt{u^2+v^2}}\right| & , v\leq 0 \end{cases} \leq \frac{|v||e^v-1|}{\sqrt{u^2+v^2}} \leq |e^v-1| \overset{u,v\to 0}{\longrightarrow} = 0.$$

4. Let $f:\mathbb{R}\to\mathbb{R}$ be a function of class C^1 and define $h(x,y) = xf(x/y)$. Show that for every $y \neq 0$ the following equation holds:

$$x\frac{\partial h}{\partial x} + y\frac{\partial h}{\partial y} - h = 0.$$

Solution:

$$\begin{aligned}\frac{\partial h}{\partial x} =& f(x/y) + \frac{x}{y} \cdot f'(x/y) \\ \frac{\partial h}{\partial y} =& x \cdot \frac{-x}{y^2} \cdot f'(x/y) = -\frac{x^2}{y^2} \cdot f'(x/y)\end{aligned}$$

$$\begin{aligned}x\frac{\partial h}{\partial x} + y\frac{\partial h}{\partial y} - h =& xf(x/y) + \frac{x^2}{y}f'(x/y) - \frac{x^2}{y}f'(x/y) - xf(x/y) \\ =& 0\end{aligned}$$

Point values: 1. (a) 1,5 (b) 1,0 **2**. (a) 1,5 (b) 1,0 (c) 1,0 **3**. (a) 1,0 (b) 2,0 **4**. 1,0