Mathematics II

Undergraduate Degrees in Economics and Management

Regular period, January 4, 2017

Part II

1. Consider the function $f(x,y) = x^6 - 6xy + y^6$.

(a) Determine and classify its critical points.

Solution: The critical points of f are the solutions of the nonlinear system

$$\begin{cases} \dfrac{\partial f}{\partial x} = 0 \\ \dfrac{\partial f}{\partial y} = 0 \end{cases} \Leftrightarrow \begin{cases} 6x^5 - 6y = 0 \\ -6x + 6y^5 = 0 \end{cases} \Leftrightarrow \begin{cases} y = x^5 \\ -x + x^{25} = 0 \end{cases} \Leftrightarrow$$

$$\begin{cases} y = x^5 \\ x(-1 + x^{24}) = 0 \end{cases} \Leftrightarrow \begin{cases} y = 0 \\ x = 0 \end{cases} \vee \begin{cases} y = \pm 1 \\ x = \pm 1 \end{cases}$$

The critical points are $(0,0)$, $(1,1)$ and $(-1,-1)$. In order to classify them we need to compute the Hessian matrix, given by

$$H(x,y) = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 30x^4 & -6 \\ -6 & 30y^4 \end{bmatrix}$$

Computing the determinants of the principal minors for each critical point, we get:

$(0,0)$: $\Delta_1 = 0, \Delta_2 = -36 \neq 0$, **saddle point**.

$(\pm 1, \pm 1) : \Delta_1 = 30 > 0, \Delta_2 = 30^2 - 6^2 = 864 > 0$, **local minimum points**.

(b) Show that f does not have a global maximum.

Solution: Since f is differentiable in an open set ($\mathbb{R}^2$), the global maximum would occur at a critical point, that would also be a local maximum. However, according to (a), there are no local maximum points, and so there are also no global maximum points. Alternatively, we observe that $f(x,0) = x^6$ is unbounded and so f can take arbitrarily large values. This excludes the possibility of existing a global maximum.

2. Compute $\iint_{\Omega} xy^2 dx\, dy$, where $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \wedge x \geq 0\}$.

Solution:

$$\iint_{\Omega} xy^2 dxdy = \int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} xe^y dydx = \int_0^1 x \left[\frac{y^3}{3}\right]_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} dx = \int_0^1 \frac{2}{3}x(1-x^2)^{3/2}dx$$
$$= \left[-\frac{1}{3}\frac{(1-x^2)^{5/2}}{5/2}\right]_{x=0}^{x=1} = \frac{2}{15}$$

3. Consider an economic model where Q_S and Q_D are the quantity supplied and demanded, respectively, and these quantities relate to the market price of a given good, $P(t)$, according to the formulas

$$\begin{aligned} Q_S = &\ a_0 + a_1 P(t) + a_2 P'(t) + a_3 P''(t) \\ Q_D = &\ b_0 + b_1 P(t) + b_2 P'(t) + b_3 P''(t), \end{aligned}$$

where $a_0, a_1, a_2, a_3, b_0, b_2, b_3 \in \mathbb{R}$ with $a_3 \neq b_3$ and $a_1 \neq b_1$.

(a) Show that if we assume that $Q_D = Q_S$ for all $t > 0$, the price level follows the differential equation

$$P''(t) + \alpha P'(t) + \beta P(t) = \gamma, \quad t > 0 \tag{1}$$

where $\alpha = (a_2 - b_2)/(a_3 - b_3)$, $\beta = (a_1 - b_1)/(a_3 - b_3)$ and $\gamma = (b_0 - a_0)/(a_3 - b_3)$.

Solution: The condition $Q_S = Q_D$ is equivalent to

$$a_0 + a_1 P(t) + a_2 P'(t) + a_3 P''(t) = b_0 + b_1 P(t) + b_2 P'(t) + b_3 P''(t) \Leftrightarrow$$

$$(a_3 - b_3)P''(t) + (a_2 - b_2)P'(t) + (a_1 - b_1)P(t) + (a_0 - b_0) = 0 \Leftrightarrow$$

$$P''(t) + \underbrace{\left(\frac{a_2 - b_2}{a_3 - b_3}\right)}_{=\alpha} P'(t) + \underbrace{\left(\frac{a_1 - b_1}{a_3 - b_3}\right)}_{=\beta} P(t) = \underbrace{\left(\frac{b_0 - a_0}{a_3 - b_3}\right)}_{=\gamma} \Leftrightarrow$$

$$P''(t) + \alpha P'(t) + \beta P(t) = \gamma.$$

(b) Determine the solution of equation (1), with $\alpha = 1, \beta = \frac{1}{2}$ and $\gamma = 1$, given the initial conditions $P(0) = 10, P'(0) = 0$.

Solution: This is a second order linear differential equation with constant coefficients. The general solution can be written as $P = P_h + P_*$, where P_h is the general solution of the homogeneous equation and P_* is a particular solution of the full equation.

i. Calculation of P_h.

We must solve the homogeneous equation $(D^2 + D + \frac{1}{2})P_h = 0$. Since the roots of the characteristic polynomial are $-\frac{1}{2} \pm \frac{1}{2}i$, we know that

$$P_h(t) = e^{-\frac{1}{2}t}\left(C_1 \cos\left(\frac{t}{2}\right) + C_2 \sin\left(\frac{t}{2}\right)\right).$$

ii. Calculation of P_*.

Since the second member of our differential equation is a constant, we will try to find a constant particular solution, lets say $P_*(t) = K$. Substituting this in the differential equation we get

$$(K)'' + \alpha(K)' + \beta(K) = \gamma \Leftrightarrow 0 + 0 + \beta K = \gamma \Leftrightarrow K = \frac{\gamma}{\beta} = 2.$$

Since we are assuming that $a_1 \neq b_1$, we have that $\beta \neq 0$ and so $P_*(t) = 2$ is always a particular solution.

iii. Using the results from i. and ii. we can obtain the general solution

$$P(t) = P_h(t) + P_*(t) = e^{-\frac{1}{2}t}\left(C_1 \cos\left(\frac{t}{2}\right) + C_2 \sin\left(\frac{t}{2}\right)\right) + 2$$

and we cal also compute

$$\begin{aligned} P'(t) = &-\frac{1}{2}e^{-\frac{1}{2}t}\left(C_1 \cos\left(\frac{t}{2}\right) + C_2 \sin\left(\frac{t}{2}\right)\right) + \\ &e^{-\frac{1}{2}t}\left(-\frac{1}{2}C_1 \sin\left(\frac{t}{2}\right) + \frac{1}{2}C_2 \cos\left(\frac{t}{2}\right)\right) \end{aligned}$$

iv. Finally, we must compute the values of C_1, C_2 that yield the proposed initial conditions.

$$\left\{\begin{array}{c} P(0) = 10 \\ P'(0) = 0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{r} C_1 + 2 = 10 \\ -\frac{1}{2}C_1 + \frac{1}{2}C_2 = 0 \end{array}\right. \Leftrightarrow C_1 = C_2 = 8$$

The solution to our problem is then given by

$$P(t) = 8e^{-\frac{t}{2}}\left(\cos\frac{t}{2} + \sin\frac{t}{2}\right) + 2.$$

(c) Propose values of α, β for which the price level $P(t)$ is periodic in time (seasonal).

Solution: As we have seen before, $P_*(t) = \frac{\gamma}{\beta}$ is a particular solution of the equation. Therefore, $P(t)$ is periodic if and only if $P_h(t)$ is periodic. Now, $P_h(t)$ is periodic if the roots of the characteristic polynomial $D^2 + \alpha D + \beta$ are pure imaginary numbers (with no real part), which occurs when $\alpha = 0$ and $\beta > 0$, yielding the general solution

$$P(t) = C_1 \cos(\sqrt{\beta} t) + C_2 \sin(\sqrt{\beta} t) + 2,$$

a periodic function with period $\frac{2\pi}{\beta}$.

Point values: **1**. (a) 2,5 (b) 1,0 **2**. 2,0 **3**. (a) 1,0 (b) 2,5 (c) 1,0

Part I

1. Consider the matrix $A = \begin{bmatrix} a & 1 & b \\ 1 & -1 & 0 \\ b & 0 & -2 \end{bmatrix}$.

(a) Set $a = 1, b = 0$ and compute the eigenvalues of A, as well as the eigenvectors associated with one of the eigenvalues.

Solution: The eigenvalues of A are the solutions of the equation $|A - \lambda I| = 0$,

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & -1-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0 \Leftrightarrow (-2-\lambda)\left[(1-\lambda)(-1-\lambda) - 1\right] = 0 \Leftrightarrow$$

$$\lambda = -2 \vee -1 - \lambda + \lambda + \lambda^2 - 1 = 0 \Leftrightarrow \lambda = -2 \vee \lambda = \pm\sqrt{2}$$

The eigenvectors corresponding to the eigenvalue $\lambda = -2$ are the solutions of the undetermined linear system $(A + 2I)u = 0$,

$$\left\{ \begin{array}{r} 3u_1 + u_2 = 0 \\ u_1 + u_2 = 0 \\ 0 = 0 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{r} u_2 = -3u_1 \\ u_1 - 3u_1 \ = 0 \\ 0 = 0 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{r} u_2 = 0 \\ u_1 = 0 \\ 0 = 0 \end{array} \right. \Leftrightarrow$$

The eigenvectors that we are searching are of the form $(0, 0, t), \quad t \neq 0$.

(b) Let $Q : \mathbb{R}^3 \to \mathbb{R}$ be defined by $Q(\boldsymbol{x}) = \boldsymbol{x}^T A \boldsymbol{x}$. Show that if $a > 0$ the quadratic form Q is indefinite.

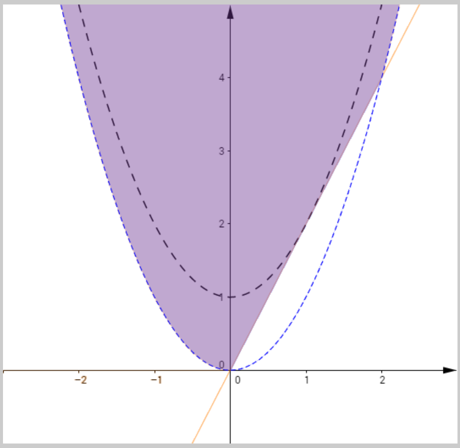
Solution: The determinants of the principal minors of A are $\Delta_1 = a > 0$, $\Delta_2 = -a - 1 < 0$ and $\Delta_3 = b^2 + 2a + 2 > 0$, which means that when $a > 0$ the matrix A is indefinite and so is the quadratic form $Q(x) = x^T A x$.

2. Let $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ be defined by the expression $f(x, y) = \dfrac{\sqrt{y - 2x}}{\ln(y - x^2)}$.

(a) Determine the domain of f, Ω, analitically and geometrically.

Solution:

$$\begin{aligned}\Omega =&\{(x,y)\in\mathbb{R}^2 : y-2x\geq 0 \wedge y-x^2>0\wedge \ln(y-x^2)\neq 0\}\\ =&\{(x,y)\in\mathbb{R}^2 : y\geq 2x\wedge y>x^2\wedge y\neq x^2+1\}\end{aligned}$$



(b) Determine the boundary of Ω and decide if the set is open.

Solution:

$$Bdy(\Omega) =\{(x,y)\in\mathbb{R}^2 : (y=x^2\wedge y\geq 2x)\vee(y=2x\wedge y\geq x^2)\vee y=x^2+1\}$$

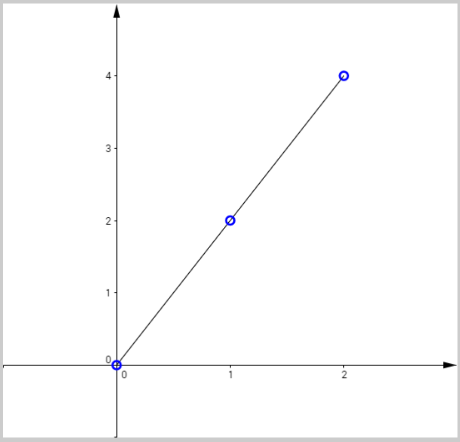
The set is not open because it includes points that are not interior, namely every point on the segment connecting $(0,0)$ and $(2,4)$ (except for $(0,0)$, $(1,2)$ and $(2,4)$).

(c) Sketch the zero levelset $C_0=\{(x,y)\in\Omega : f(x,y)=0\}$ and show that C_0 is bounded but not compact.

Solution: The zero levelset is given by

$$C_0=\{(x,y)\in\Omega : f(x,y)=0\}=\{(x,y)\in\Omega : y=2x\},$$

which is the line segment connecting $(0,0)$ and $(2,4)$, except for the points $(0,0)$, $(1,2)$ and $(2,4)$.



This set is bounded because it can be fit inside a ball, for example $B_5((0,0))$, but it is not closed (for example point (0,0) belongs to the adherence but not to the set) and it is thefore not compact.

3. Consider $f(x,y) = \begin{cases} \dfrac{2y^3 - xy}{\sqrt{x^2+y^2}} & , y > 0 \\ 0 & , y \leq 0 \end{cases}$

(a) Compute $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$.

Solution:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$
$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{f(0,h)}{h} \stackrel{*}{=} 0$$

(*)

$$\lim_{h\to 0^-} \frac{f(0,h)}{h} = \lim_{h\to 0^-} \frac{0}{h} = 0$$
$$\lim_{h\to 0^+} \frac{f(0,h)}{h} = \lim_{h\to 0} \frac{1}{h} \frac{2h^3}{\sqrt{h^2}} = \lim_{h\to 0} \frac{2h^2}{|h|} = 0$$

(b) Check if f is differentiable at $(0,0)$.

Solution: The function is differentiable at $(0,0)$ if

$$\lim_{(u,v)\to(0,0)} \frac{f(u,v) - f(0,0) - uf'_x(0,0) - vf'_y)(0,0)}{\sqrt{u^2+v^2}} = \lim_{(u,v)\to(0,0)} \frac{f(u,v)}{\sqrt{u^2+v^2}} = 0$$

The previous limit is zero if and only if

$$\lim_{\substack{(u,v)\to(0,0)\\ v\leq 0}} \frac{f(u,v)}{\sqrt{u^2+v^2}} = \lim_{\substack{(u,v)\to(0,0)\\ v>0}} \frac{f(u,v)}{\sqrt{u^2+v^2}} = 0.$$

Now,

$$\lim_{\substack{(u,v)\to(0,0)\\ v\leq 0}} \frac{f(u,v)}{\sqrt{u^2+v^2}} = \lim_{(u,v)\to(0,0)} \frac{0}{\sqrt{u^2+v^2}} = 0$$

and

$$\lim_{\substack{(u,v)\to(0,0)\\ v>0}} \frac{f(u,v)}{\sqrt{u^2+v^2}} = \lim_{(u,v)\to(0,0)} \frac{2v^3-uv}{u^2+v^2}$$

Computing directional limits we easily verify that this last limit does not exist, and we conclude that f is not differentiable at $(0,0)$.

4. Let $f:\mathbb{R}\to\mathbb{R}$ be a function of class C^1 and define $h(x,y)=\frac{1}{x}f(xy)$. Show that for every $x\neq 0$ the following equation holds:

$$x\frac{\partial h}{\partial x} - y\frac{\partial h}{\partial y} + h = 0.$$

Solution: The partial derivatives of $h(x,y)$ are given by

$$\frac{\partial h}{\partial x} = -\frac{1}{x^2}f(xy) + \frac{1}{x}\cdot y\cdot f'(xy)$$

$$\frac{\partial h}{\partial y} = \frac{1}{x}\cdot x\cdot f'(xy) = f'(xy)$$

$$x\frac{\partial h}{\partial x} - y\frac{\partial h}{\partial y} + h = -\frac{1}{x}f(xy) + yf'(xy) - yf'(xy) + \frac{1}{x}f(xy) = 0,$$

as we wanted to show.

Point values: 1. (a) 1,5 (b) 1,5 **2**. (a) 1,5 (b) 1,0 (c) 1,0 **3**. (a) 1,0 (b) 1,5 **4**. 1,0