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MODELLING ELECTRICITY AND EMISSION MARKETS

JOÃO FRANCISCO MESQUITA D'ÁGUA

SUPERVISION:

JOÃO GUERRA

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Glossary

PDE - Partial Differential Equation
EUA - European Emission Allowances
FBSDE - Forward Backward Stochastic Differential Equation
EU - European Union
EU ETS - European Union Emission Trading System
CO_2 - Carbon Dioxide

ABSTRACT

The global energy landscape is undergoing a transformative shift driven by ambitious environmental policies aimed at mitigating climate change and promoting renewable energy sources. The Kyoto Protocol and the European Union Emissions Trading Scheme (EU ETS) have played central roles in reshaping energy markets and emission certificate pricing. This thesis delves into the mathematical modeling of energy markets and CO_2 emission certificates, using forward-backward stochastic differential equations (FBSDEs) to unravel intricate pricing mechanisms.

The research has two key objectives. Firstly, it aims to derive fundamental partial differential equations (PDEs) for pricing emission certificates within standard and joint models. These PDEs serve as the foundation for subsequent analysis. Secondly, the thesis seeks to develop efficient numerical methods, including finite difference schemes and alternating direction finite schemes, to solve these PDEs, enabling the interpretation of real-world pricing dynamics.

The findings reveal the interplay between emission certificate prices, electricity demand, and cumulative emissions, highlighting their influence on allowance prices. The analysis underscores the discrete nature of allowance prices at the end of the compliance period.

Moreover, the study emphasizes the significance of incorporating fossil fuel prices into the modeling framework, which drives variations in initial allowance certificate prices. In summary, this research contributes to understanding emission market pricing mechanisms in the evolving energy landscape.

KEYWORDS: Forward Backward Stochastic Differential Equation, Numerical Methods, Emission Markets, Bid-Stack, Allowance Certificates.

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1 Introduction

The structural transformation of energy markets is currently underway at a global level, primarily driven by ambitious environmental policy initiatives aimed at mitigating the effects of climate change and fostering a transition to renewable energy sources and low-carbon technologies. As a result, the composition of energy sources, energy demand, and the dynamics of the energy market have undergone significant changes in recent years. The adoption of environmental policies and the increasing focus on sustainable practices have necessitated a profound shift in the electricity market landscape. This thesis delves into the mathematical modelling of energy markets and carbon dioxide (CO_2) emission certificates, examining how these changes have influenced pricing mechanisms.

The Kyoto Protocol, an international treaty adopted in 1997, has played a pivotal role in shaping the response to climate change (see [6]). The agreement sets binding emission reduction targets for developed countries and established the framework for international cooperation on emissions reduction. As a direct consequence, the electricity market has experienced substantial transformations in line with the commitment to reduce greenhouse gas emissions. The integration of renewable energy sources and the development of cleaner technologies have become central to the strategies employed by energy market participants to align with the Kyoto Protocol's objectives.

Emission Trading, as implemented in the European Union Emissions Trading Scheme (EU ETS), is a market-based approach to control greenhouse gas emissions. Within the EU ETS, a cap is set on the total amount of certain greenhouse gases that can be emitted by covered installations. These installations are issued a limited number of emission allowances, which they can trade amongst themselves. This mechanism creates a market for emission certificates, with prices being determined by the supply and demand dynamics of the trading participants. The EU ETS has become one of the largest and most influential emissions trading systems globally, significantly impacting the pricing of emission certificates.

One of the key focal points of this thesis is the mathematical modelling of energy markets and CO_2 emission certificates. Specifically, the study revolves around a risk-neutral valuation approach, which involves considering a forward-backward stochastic differential equation (FBSDE) to determine the price of emission certificates. This approach considers structural models and links emission certificates to both the electricity demand process and cumulative emissions. Such mathematical models provide crucial insights into pricing mechanisms and risk assessment within energy and emission markets.

The scientific objectives of this research project are twofold. Firstly, the thesis aims to derive the partial differential equations (PDEs) necessary for pricing emission certificates within the standard model, as proposed in [17], and the joint model for electricity and emissions, as introduced in [15]. Understanding and establishing the fundamental PDEs for pricing emission certificates will lay the foundation for subsequent analysis.

Secondly, the thesis seeks to develop and implement efficient numerical methods for solving the pricing PDEs. Two numerical schemes will be used, such as a Crank-Nicolson scheme and a forward Euler scheme. These will be tested to identify the most effective and accurate methods for solving the pricing PDEs. These numerical methods will be instrumental in analyzing and interpreting real-world data and pricing dynamics within electricity and emission markets.

To achieve our proposed goal, our thesis is structured as following: we will start with a literature review that delves into prior research on energy markets, emission trading, and mathematical modelling techniques. The next chapters will present the derivations of the pricing PDE's, the development and testing of numerical methods, and the analysis of results. Finally, the thesis will conclude with a summary of findings and their implications, along with potential paths for future research.

2 Literature Review

Electricity markets play a crucial role in meeting the energy needs of societies around the world. The availability and pricing of electricity are influenced by numerous factors such as supply-demand dynamics, fuel prices, government policies, technological advancements, and environmental regulations [4]. Modelling these intricate aspects allows researchers to develop mathematical models that aid in predicting electricity prices [4].

Similarly, emission markets focus on the trading of greenhouse gas emissions permits or credits. These markets have emerged as a response to global efforts to mitigate climate change by reducing carbon emissions. Modelling emission markets involves understanding risk-neutral pricing techniques for financial instruments within this domain [17].

To provide an overview of modelling electricity and emission markets, it is essential first to define these terms. Electricity markets refer to the marketplace where buyers and sellers trade wholesale or retail power contracts [1]. On the other hand, emission markets involve buying or selling permits or credits related to greenhouse gas emissions [22]. The emission markets were first studied by Dales [12] and then by Montgomery [19].

The emission rates originating from electricity production are inherently linked to the technology and energy sources employed in the generation process. To gain insights into which electricity sources or generators hold preference in the market, a useful tool known as the "bid stack" comes into play. The bid stack can be thought as a map that organizes market supply based on the ascending order of electricity production costs. This idea was first introduced by Barlow [2], and then some of its main properties were studied in [16].

In the realm of emission markets, a no-arbitrage approach, typical in mathematical finance, offers a means to determine certificate prices. These prices can be conceptualized as the anticipated discounted future returns of the contract under a risk-neutral framework. A notable contribution to this field can be found in [17], which introduces a structural model and risk-neutral valuation technique. Within this framework, emissions certificates are treated as financial derivatives that depend on both demand patterns and cumulative emissions. Importantly, the emissions process aligns with the demand process through the bid stack map, allowing for the derivation of a forward-backward stochastic differential equation (FBSDE) governing the certificate price. In [17], the authors use Itô stochastic calculus to demonstrate that the FBSDE solution satisfies a semilinear partial differential equation (PDE).

Establishing the existence and uniqueness of solutions for the FBSDEs related to emission certificates presents a challenging problem. This complexity arises from the degeneracy in one of the forward components and the singularity in the terminal condition. Nevertheless, in the works of [7], [8], and [20], the authors demonstrate the existence and uniqueness of solutions. They achieve this by imposing regularity assumptions on the coefficients within the stochastic differential equation governing the demand process.

Expanding upon the model introduced in [17], [11] incorporates stochastic fuel prices as variable costs impacting firms' bidding strategies. This augmentation proves valuable in pricing contracts like clean spread options.

To numerically address the semilinear PDE governing certificate prices, various finite difference schemes come into play. One such scheme, detailed in [17], employs a backward finite difference approach for the time derivative and an explicit scheme. An alternative method, presented in [15], utilizes an alternating direction implicit finite difference scheme to solve a similar PDE numerically. Moreover, [15] puts forth a combined model encompassing both the electricity spot market and the emission market. This integrated model facilitates simulations of changing market parameters and policy impacts. The authors deduce a partial differential equation for pricing emission certificates, and propose an alternating direction implicit finite difference scheme in order to obtain a numerical solution for the PDE.

3 Stochastic Models for the Carbon Emission Markets

3.1 Overview of Carbon Emission Market

We will begin by exploring the intricate relationship between electricity and emissions markets. In a liberalized electricity market the consumers are free to choose their electricity suppliers. Importantly, an electricity supplier need not be an electricity producer; they can purchase electricity from producers and then distribute it to consumers. Our primary focus will be on electricity producers who predominantly generate electricity through the combustion of fossil fuels. This method, unfortunately, leads to the emission of carbon dioxide (CO_2) and other greenhouse gases. In response to the Kyoto Protocol's emissions reduction goals, the European Union (EU) introduced the EU Emissions Trading System (EU ETS).

The EU ETS functions as a market mechanism built upon a cap-and-trade framework. Within this system, the EU establishes an annual cap on total CO_2 emissions and allocates a finite number of Emission Allowance Certificates. Each certificate grants producers the privilege to emit one metric ton of CO_2 or its greenhouse gas equivalent. At the close of the compliance period (typically one year), electricity producers must possess enough certificates to offset their emissions throughout that period. Any surplus certificates can be traded with other producers or retained for subsequent years. In cases of non-compliance, producers are subject to fines.

The imposition of a cap on the number of allowance certificates ensures their monetary value, giving rise to a marketplace for emissions trading, often referred to as the carbon market. Notably, some argue, that under perfect information, imposing a carbon emissions tax is essentially equivalent to operating within a carbon market. In this section, we present a structural model for a simplified version of the electricity and emissions markets, aiming to determine a fair price for an allowance certificate.

Within this framework, the electricity market is characterized by the following (see [5]):

- The market is open to both suppliers and consumers, simplistically assuming that producers also act as suppliers.
- A market administrator is responsible for matching supply and demand, meaning consumers do not directly select their suppliers; instead, each consumer is assigned their optimal supplier.
- Electricity suppliers submit bids to the market administrator, consisting of both a quantity (of electricity) and an associated price.
- The market administrator organizes supply bids in merit order to match demand with the lowest available current supply bid, creating what is commonly known as a bid stack.
- Capacity limits exist for electricity production, and in cases of excess demand, the maximum capacity is supplied.
- For simplicity, we disregard the existence of alternative electricity production methods, including nuclear and environmentally friendly options.

Turning to the emissions market, we make the following assumptions (see [5]):

- All CO_2 emissions subject to the emissions cap originate solely from electricity production.
- A regulatory body enforces the emissions limit using emission allowances, allocated at the beginning of each period (whether through auctions, sales, or distribution remains unspecified).
- Emission allowances are tradeable assets.
- For simplicity, we operate within a single compliance period, negating any transition value of allowances across multiple periods.

This comprehensive framework highlights the intricate interplay between electricity and emissions markets, underscoring their critical role in mitigating CO_2 emissions and addressing climate change concerns.

3.2 Construction of the Standard Model

For the construction of the standard model, we will follow [17]. This model's goal is to determine the price of the allowances certificates, given the total cumulative emissions of CO_2, and the demand levels for electricity.

3.2.1 Market Setup

The first step would be to define our probability space. Instead of initially modeling the market under an empirical probability measure, denoted as $\mathbb{P}$, we opt to operate directly under the risk-neutral measure $\mathbb{Q}$. This choice is driven by the realization that the absence of arbitrage is, in fact, equivalent to the existence of a specific measure $\mathbb{Q}$ that is equivalent to $\mathbb{P}$. Essentially, this means that if $\mathbb{P}(N) = 0$, then $\mathbb{Q}(N) = 0$, and vice versa. Furthermore, it implies that all the discounted prices of the allowance certificates behave as martingales under $\mathbb{Q}$ (see [17]). This motivates the following assumption:

Assumption 1: *There exists an equivalent martingale measure $\mathbb{Q}$ equivalent to $\mathbb{P}$, under which the discounted price of any tradeable asset follows a martingale process.*

With this assumption in mind, we can now define the equation governing the dynamics of the allowance certificate prices. We initiate this process by considering a probability space denoted as $(\Omega, \mathcal{F}, \mathbb{Q})$, where we focus on a single compliance period $[0, T]$. We consider also a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, generated by a standard $\mathbb{Q}$-Brownian motion or Wiener process $W = \{W_t\}_{t\in[0,T]}$.

In our market setup, we have two essential processes: the demand process $D = \{D_t\}_{t\in[0,T]}$ and the aggregate supply of electricity process $\xi = \{\xi_t\}_{t\in[0,T]}$. Both of these processes are measured in megawatts. The demand for electricity is considered to be perfectly inelastic, as is frequently justifiable in electricity markets (see [9]), meaning it does not depend on the price of electricity. This assumption is reasonable because there are no close substitute goods for electricity, and elasticity levels between demand and price are low. Additionally, we assume that the existing supply always meets the demand, since they are related by a Walrasian equilibrium assumption [21], which is expressed as follows:

$$0 \leq D_t = \xi_t \leq \xi_{max} \quad \text{for } 0 \leq t \leq T.$$

The cumulative emissions of CO_2 during the time interval $[0, t]$ are represented by the process $E = \{E_t\}_{t\in[0,T]}$, measured in tonnes of CO_2, and are bounded:

$$0 \leq E_t \leq E_{max} \quad \text{for } 0 \leq t \leq T, \text{where } E_{max} > 0.$$

Finally, we consider the process $A = \{A_t\}_{t\in[0,T]}$ representing the price of an allowance certificate. We will also consider a riskless money market account with constant risk-free interest rate $r \geq 0$.

3.2.2 The Bid and Emissions Stacks

In order to model the process $E = \{E_t\}_{t\in[0,T]}$ effectively, it is essential to address both the bid and emissions stacks. We introduce the notation BAU (representing "business-as-usual") to denote a function unaffected by CO_2 emission constraints. In other words, it represents the function in its pre-emission limitation state.

As outlined in Section 3.1, the market administrator organizes production bids in ascending order, known as the merit order. This approach ensures that energy is supplied at the most economical price for each level of demand. Therefore, the BAU bid stack can be characterized by the continuous function

$$b^{BAU}(\xi) : [0, \xi_{max}] \to \mathbb{R}_0^+,$$

where $b^{BAU}(\xi) \in C^1(0, \xi_{max})$ and $\frac{db^{BAU}}{d\xi} > 0$. Considering the discrete nature of this problem, our bid stack should ideally take the form of an increasing step function on its domain of definition, serving as an approximation to the actual continuous function. Given this, we take the business-as-usual bid stack to have the following shape

$$b^{BAU}(\xi) := \underline{b} + \left(\frac{\bar{b} - \underline{b}}{\xi_{max}^{\theta_1}}\right) \xi^{\theta_1} \quad \text{for } 0 \leq \xi \leq \xi_{max}, \tag{1}$$

where $\underline{b}, \bar{b} \geq 0$ and $2 < \theta_1 < \infty$. The parameters $\underline{b}, \bar{b}$, represent, respectively, the minimum and maximum values of electricity that the model can produce. The parameter θ_1 governs the steepness of the stack and, more specifically, the rate at which the marginal costs of generators rise.

For the marginal emissions stack functions, we will use the same idea.

$$e(\xi) : [0, \xi_{max}] \rightarrow \mathbb{R}_0^+,$$

where $e(\xi) \in C^1(0, \xi_{max})$. Using the provided definition, the function $e(\xi)$ links a particular electricity supply, ξ, with the emissions rate of the marginal unit, measured in metric tons of CO_2 per MWh. Given this, we have

$$e(\xi) := \bar{e} - \left(\frac{\bar{e} - \underline{e}}{\xi_{max}^{\theta_2}}\right) \xi^{\theta_2} \quad \text{for } 0 < \xi \leq \xi_{max}, \tag{2}$$

where $\underline{e}, \bar{e} \geq 0$ and $0 \leq \theta_2 < 1$. The parameters $\underline{e}, \bar{e}$, represent, respectively, the minimum and maximum values of marginal emissions rates in the market. The parameter θ_2 governs the fuel mix in the market. A reduced θ_2 value corresponds to a decreased portion of the market capacity being fulfilled by the pollution-intensive technology.

With these definitions, we can derive the market emissions rate function

$$\mu_E^{BAU}(D_t) := \kappa \int_0^{D_t} e(\xi) d\xi \quad \text{for } 0 \leq D_t \leq \xi_{max}, \tag{3}$$

where the scaling constant κ represents the relationship between the emissions period T and the time unit associated with the marginal emissions stack e. Typically, T is measured in years, and κ corresponds to the number of hours per year.

3.2.3 Load Shifting: A Short-Term Abatement Measure

Now, let us consider the impact of emissions trading on the previously introduced business-as-usual economy. Emissions trading effectively assigns a value to carbon emissions, which, in turn, raises the operational costs for firms. Particularly, this makes production more expensive for those firms heavily reliant on emission-intensive technologies. For every unit of CO_2 they emit beyond their initial allocation, they must purchase an allowance contract to avoid penalties, turning the cost of carbon into a tangible expense. Conversely, if a firm possesses excess allowances, they can sell them in the market, where the cost of carbon becomes an opportunity cost.

We won't account for firms investing in long-term abatement projects, focusing solely on the direct impact on the bid stack. Our assumption is that firms pass on the increased production costs linked to emissions to consumers to preserve their profit margins. Since the price of an allowance certificate represents the cost of carbon, each firm's business-as-usual bids rise by an amount equivalent to the allowance price multiplied by the firm's marginal emissions rate.

Additionally, we consider the phenomenon of load shifting, which involves reallocating energy production from emission-intensive to environmentally friendly resources. This action leads to a cap on total emissions. The introduction of a carbon market effectively acts as an invisible carbon tax, causing emission allowance certificates to incur additional production costs. To avoid fines, companies must purchase an allowance contract for each additional unit of CO_2 they release beyond their initial allocation. As in the previous scenario, businesses are assumed to pass on the rise in production costs to consumers to safeguard their profit margins. Consequently, the price of an allowance certificate, represented as A, leads to an increase in each company's business-as-usual bids, calculated as A multiplied by its marginal emissions rate. This transforms into the function g, where

$$g(A,\xi) := b^{BAU}(\xi) + Ae(\xi) \quad \text{for } 0 \leq A \leq \infty, 0 \leq \xi \leq \xi_{max}. \tag{4}$$

As the cost of carbon rises, the production expenses for businesses using polluting fuels increase, leading to higher bids associated with elevated marginal emission rates. Consequently, adhering to our merit order assumption, the market administrator will activate generators in ascending order based on their bid levels. Thus, for a given allowance A and energy price p, we can define the set of potential operational generation units as follows

$$S(A,p) = \{\xi \in [0,\xi_{max}] : g(A,\xi) \leq p\} \quad for\ 0 \leq A, p < \infty. \tag{5}$$

To put it differently, S represents the range of production levels that can be reached at the specified electricity and allowance certificate prices without incurring financial losses.

According to the definition of a sublevel set, the mapping from $P \to \lambda(S(\cdot,P))$, where λ represents the Lebesgue measure, is not only strictly increasing but also continuous and thus invertible under the following assumption:

$$\lambda\left(\{\xi \in (0,\xi_{max}) : \frac{\partial b^{BAU}}{\partial \xi}(\xi) + A\frac{\partial e}{\partial \xi}(\xi) = 0\}\right) = 0 \ \text{ for } 0 \leq A < \infty. \tag{6}$$

With equation (5), we can now define the market bid stack b, for certain values of the allowance price, as

$$b(A,\xi) := \lambda(S(A,\cdot))^{-1}(\xi) \text{ for } 0 \le A < \infty,\ 0 \le \xi \le \xi_{max}. \tag{7}$$

Given this, the price of electricity P is then given by

$$P := b(A,D) \ \ for\ 0 \le A < \infty,\ 0 \le D \le \xi_{max}. \tag{8}$$

With the market regulator now imposing an emissions cap through emission permits, we can formally define the capped market emissions rate as

$$\mu_E(A,D) = \kappa \int_{S_p(A,D)} e(\xi) \quad \text{for } 0 \le A < \infty, 0 \le D \le \xi_{max}, \tag{9}$$

where $S_p(A,D) := S(A,b(A,D))$. The set $S_p(A,D)$ can be calculated as explained in detail in Appendix A. In the subsequent lemma, we establish various technical properties of μ_E. These properties demonstrate that the model presented for the market emissions rate aligns with common-sense expectations and results in a function that possesses the necessary regularity.

Lemma 1: We have then the following properties about the market emissions rate μ_E (for proofs of these properties, see [17]).

1. *The map* $D \to \mu_E(.,D)$ *is*
   (a) *strictly increasing and*
   (b) *Lipschitz continuous.*
2. *The map* $A \to \mu_E(A,.)$ *is*
   (a) *nonincreasing and*
   (b) *Lipschitz continuous.*
3. μ_E is bounded.

3.3 Risk-Neutral Pricing of Allowance Certificates

When we defined the market setup, we stated the fact that the absence of arbitrage was, in fact, equivalent to the existence of a risk neutral measure $\mathbb{Q}$ that is equivalent to $\mathbb{P}$. With this in mind, we can make some more assumptions regarding the demand and cumulative emissions processes denoted, respectively, as $D = \{D_t\}_{t\in[0,T]}$ and $E = \{E_t\}_{t\in[0,T]}$. Additionally, we assume that at the initial time, $t = 0$, the demand for electricity is known and evolves according to an Itô diffusion process. Specifically, for $0 \leq t \leq T$, under the measure $\mathbb{Q}$, the demand for electricity follows the general stochastic differential equation

$$dD_t = \mu_D(D_t)dt + \sigma_D(D_t)d\tilde{W}_t, \tag{10}$$

where $\tilde{W} = \{\tilde{W}_t\}_{t\in[0,T]}$ is $\{\mathcal{F}_t\}_{t\in[0,T]}$-adapted and is a $\mathbb{Q}$-standard Brownian motion.

In order to consider a more specific model, let us consider that the demand process satisfies the stochastic differential equation

$$dD_t = -\eta(D_t - \bar{D})dt + \sqrt{2\eta\bar{\sigma}_D D_t(\xi_{max} - D_t)}d\tilde{W}_t, \tag{11}$$

where $\bar{\sigma}_D, \eta > 0$, are constants. The strong solution of this stochastic differential equation is the so-called Jacobi diffusion process. It degenerates on the boundary, and it has a mean-reverting, linear drift component. Furthermore, subject to $\bar{D} \in (0, \xi_{max})$ and $\min(\bar{D}, \xi_{max} - \bar{D}) \geq \xi_{max}\bar{\sigma}_D$, the process remains within the interval $(0, \xi_{max})$. Its stationary distribution is a beta distribution, and its mean is given by $\bar{D}$ (see [14] for more details).

It is worth noting that our assumption of perfectly inelastic demand is reflected in the fact that both coefficients are functions solely dependent on demand. It is also important to mention that if there were feedback from price into demand in the model, it would introduce additional nonlinearities not present in the current formulation. Additionally, in practical scenarios, demand for electricity often exhibits seasonal periodicity, which would cause μ_D to explicitly depend on time. However, for simplicity, we opt to disregard this feature in our model.

Cumulative emissions are measured starting from the commencement of the compliance period at $t = 0$, setting $E_0 = 0$. Subsequently, they are determined through integration based on the market emissions rate μ_E derived in the previous chapter. As a result, the cumulative emissions process is represented by an absolutely continuous process. For $0 \leq t \leq T$,

$$dE_t = \mu_E(A_t, D_t)dt, \quad E_0 = 0. \tag{12}$$

It is noteworthy that this definition ensures that the process E is nondecreasing, which aligns with its interpretation as a cumulative quantity.

3.3.1 One Compliance Period

Now, in order to finish the presentation of our standard model, we need to characterize the allowance certificate price process $A = \{A_t\}_{t\in[0,T]}$. Contrary to what happens when we define the processes E and D, we do not know its value at $t = 0$. However, we know that the noncompliance event at the end of the compliance period can be represented by $\{E_T \geq E_{cap}\}$. Given this, we have that the terminal value of the allowance certificates is given by the following terminal condition:

$$A_T = \begin{cases} 0 & \text{if } 0 \leq E_T \leq E_{cap} \\ \Pi & \text{if } E_{cap} \leq E_T \leq E_{max}. \end{cases} \tag{13}$$

Given **Assumption 1**, we are aware that the process $e^{-rt}A_t$ is a martingale under the martingale measure $\mathbb{Q}$, where r represents the risk-free rate. Consequently, following the Martingale representation theorem [18], we can express a martingale as an Itô integral with respect to the $\mathbb{Q}$-Brownian motion $\tilde{W}$, involving an adapted process $Z = \{Z_t\}_{t\in[0,T]}$, such that

$$d(e^{-rt}A_t) = Z_t d\tilde{W}_t \quad \text{for} \;\; 0 \leq t \leq T. \tag{14}$$

Now, applying the Itô formula, we have

$$dA_t = rA_t dt + e^{rt} Z_t d\tilde{W}_t. \tag{15}$$

Arriving here, it is possible to combine the processes (12) and (10) for cumulative emissions and demand with the equation (15) and the terminal condition (13), such that the pricing problem is presented as follows

$$\begin{cases} dD_t = \mu_D(D_t)dt + \sigma_D(D_t)d\tilde{W}_t, \quad D_0 = d \in (0, \xi_{max}), \\ dE_t = \mu_E(A_t, D_t)d_t, \quad E_0 = 0, \\ dA_t = rA_t dt + e^{rt} Z_t d\tilde{W}t, \quad A_T = \Pi \, \mathbb{I}_{[E_{cap},\infty)}(E_T). \end{cases} \tag{16}$$

The question of the existence and uniqueness of a solution to the FBSDE (16) is intricate (see [5]). The uniqueness of this type of equation is challenged by its nonstandard nature, stemming from the degeneracy of one of its forward components (in our case, the emissions process E combined with the singularity of the terminal condition. Results have confirmed the existence and uniqueness of a solution to the FBSDE (16) under weaker conditions regarding the regularity of the coefficients μ_D and σ_D than required in [7] and [8]. In fact, it is enough for μ_D and σ_D to exhibit sufficient regularity to ensure that the stochastic differential equation for D possesses a strong solution. Since this proof is out of the scope of this thesis, we direct interested readers to the thesis [20].

Building on the earlier observation, we posit that within our Markovian framework, there is a function $\alpha(t, D, E) : [0, T] \times [0, \xi_{max}] \times [0, E_{max}] \to [0, \Pi]$ such that $A_t = \alpha(t, D_t, E_t)$ for $0 \leq t \leq T$. This function exhibits sufficient regularity on $[0, T)$ to be considered a classical solution to the PDE

$$\frac{\partial \alpha}{\partial t} + \frac{1}{2}\sigma_D^2(D)\frac{\partial^2 \alpha}{\partial D^2} + \mu_D(D)\frac{\partial \alpha}{\partial D} + \mu_E(\alpha(t,D,E),D)\frac{\partial \alpha}{\partial E} - r\alpha = 0, \quad \text{on } U \text{ and } 0 \leq t < T, \tag{17}$$

$$\alpha = \Pi \mathbb{I}_{[E_{cap},\infty)}(E) \qquad \text{on } U := (0,\xi_{\max}) \times (0, E_{\max}) \text{ and } t = T.$$

The procedure to obtain equation (17) is straightforward. Under the risk-neutral measure $\mathbb{Q}$, the asset with price process A is a traded asset with drift equal to the risk-free interest rate (last equation of 16). Moreover, we apply Itô's formula to $A_t = \alpha(t, D_t, E_t)$, we use equations (16) and take the expected value, all the while assuming the existence of a classical solution to the PDE. Important to note that, despite having the terminal condition, we will also need suitable boundary conditions, which will be introduced in a later section (see Section 4.1).

3.4 Construction of the Joint Model for Electricity/Emissions

In this section, we present the joint model for electricity/emissions, which is very close to the model introduced in the previous chapter. However, we expand upon the findings of the previous chapter by accommodating the stochastic equilibrium bids of generators influenced by fuel prices.

Given this, and keeping in mind the framework of the electricity and emission market discusses in Section 3.1, we start by presenting the main differences in this model, and then show a reformulated solution of the FBSDE of the pricing problem.

3.4.1 Stochastic Bid Curves and the Price for Electricity

In our analysis, we delve into a market existing at time t with a demand D_t for energy and a capacity C_t measured in MWh. In day-ahead auctions, energy companies submit m bids, each comprising a quantity q_j and a price p_j for $j = 1, ..., m$. These bids are organized in merit order, as in the previous model, and the market operator engages generators until the current demand is satisfied. Let us denote the distribution function $F_i(S_t)$ as the fraction of bids priced below S_t €/MWh for generators of fuel type $i = 1, ..., n$. Consequently, the spot price S_t, which represents the current price in the marketplace at which electricity can be bought, solves the equation

$$F(S_t) = \sum_{i=1}^{n} w_i F_i(S_t) = \frac{D_t}{C_t}, \quad \text{where} \sum_{i=1}^{n} w_i = 1. \tag{18}$$

Thus, the electricity spot price can be expressed as

$$S_t = F(.)^{-1}\left(\frac{D_t}{C_t}\right). \tag{19}$$

Let us introduce a truncated domain with fixed lower and upper bounds, denoted b_L and b_U, where $b_L < \frac{D_t}{C_t} < b_U$ must hold. Subsequently, the demand and capacity can be rescaled as follows

$$\hat{D}_t := D_t - b_L C_t, \quad \hat{C}_t := (b_U - b_L)C_t$$

. This rescaling leads to the spot price satisfying the equation

$$F(S_t) = \sum_{i=1}^{n} w_i F_i(S_t) = \frac{\hat{D}_t}{\hat{C}_t}. \tag{20}$$

3.4.2 An FBSDE for the Allowance Price

After discussing stochastic bid curves and the price for electricity, we should move onto the equations that form the pricing problem. It is important to notice, to avoid the difficulties of estimating market prices of risk, we are operating under a risk-neutral measure, so **assumption 1** is still valid for this model. Moreover, we choose to disregard aspects of market incompleteness (such as the uniqueness of $\mathbb{Q}$), transaction costs, illiquidity, inelastic demand (and the unlikely scenario of demand surpassing supply, leading to issues like blackouts), and non-power sector emissions. Although these details are noteworthy in market dynamics, we contend that considering these factors should not significantly alter the crucial qualitative insights derived from the model.

Before presenting these factors, we should make a remark regarding the notation used in the following sections. Consider a time horizon represented by $T \in \mathbb{R}_+$. In this context, we have a $(n+1)$-dimensional standard Brownian motion or Wiener process denoted as $\{\tilde{W}_t^0, \tilde{W}_t\}_{t\in[0,T]}$. This process exists within a probability space represented as $(\Omega, \mathcal{F}, \mathbb{Q})$, where Ω is the sample space, $\{\mathcal{F}_t^0\}_{t\in[0,T]}$ denotes the filtration generated by $\tilde{W}^0$, $\{\mathcal{F}_t^W\}_{t\in[0,T]}$ represents the filtration generated by $\tilde{W}$, and $\mathcal{F}_t := \mathcal{F}_t^0 \cup \mathcal{F}_t^W$. We make some assumptions for our model. We start by defining that at time $t = 0$, we have knowledge of electricity demand, which then evolves based on an Itô diffusion process. We assume that, for $t \in [0, T]$, the demand for electricity $D = \{D_t\}_{t\in[0,T]}$ follows the general stochastic differential equation

$$dD_t = \mu_d(t, D_t)dt + \sigma_d(D_t)d\tilde{W}_t^0, \quad D_0 = d_0 \in (0, \xi_{max}). \tag{21}$$

In general, the drift can be time dependent to account for seasonal variations in electricity demand. However, in order to keep the model simple, we do not consider these seasonal effects and assume the specific model

$$dD_t = -\eta(D_t - \bar{D})dt + \sqrt{2\eta\bar{\sigma}_D D_t(\xi_{max} - D_t)}d\tilde{W}_t. \tag{22}$$

As discussed before, the solution of this SDE is a Jacobi diffusion process. It degenerates on the boundary, and it has a mean-reverting and linear drift component. Furthermore, subject to $\bar{D} \in (0, \xi_{max})$ and $\min(\bar{D}, \xi_{max} - \bar{D}) \geq \xi_{max}\bar{\sigma}_D$, the process remains within the interval $(0, \xi_{max})$.

The prices of fuels used in the electricity production also follow a $\mathcal{F}_t^W$-adapted stochastic process $S = \{S_t\}_{t\in[0,T]}$ taking values in $\mathbb{R}^n$ and where $S_t := (S_t^1, ..., S_t^n)$. In vector form:

$$dS_t = \mu_s(S_t)dt + \sigma_s(S_t)d\tilde{W}_t, \quad S_0 = s_0 \in \mathbb{R}^n, t \in [0, T] \tag{23}$$

These equations are driven by the $\mathbb{Q}$-standard Brownian motion $\tilde{W}$. In this thesis, we will only consider the prices of coal S^c and gas S^g. We assume that these processes follow the SDE

$$dS_t^i = -\eta_i(logS_t^i - \bar{s}_i - \frac{\hat{\sigma}_i^2}{2\eta_i})S_t^i\mathrm{d}t + \hat{\sigma}_i S_t^i \mathrm{d}\tilde{W}_t^i, \quad S_0^i = s_0^i \in \mathbb{R}_+, \tag{24}$$

where $d\tilde{W}_t^c$ and $d\tilde{W}_t^g$ have a correlation factor ρdt, $i \in \{c, g\}$ and $t \in [0, T]$. Applying Itô's Lemma, it is easy to show that the processes $K_t^i := log(S_t^i)$ follow correlated Onrstein-Uhlenbeck processes under the measure $\mathbb{Q}$.

The market emission rate, denoted as μ_e, is a crucial factor in this model. It is a positive and bounded function that is influenced by various factors, including the current energy demand, available capacity, and the prices of coal and gas. Additionally, μ_e is sensitive to changes in CO_2 allowance prices, which aligns with the goals of the EU ETS. When carbon prices fluctuate, we anticipate shifts in the merit order, leading to variations in greenhouse gas emissions.

To incorporate the feedback loop from European Emission Allowances (EUAs) prices, we assume that the electricity spot price responds to changes in allowance prices. This assumption is grounded in the requirement for generators to purchase certificates to offset their emissions. Given that these additional costs are typically passed on, at least partially, to the market, this assumption appears to be quite relevant. To determine the current emission rate we need to calculate the amount of energy supplied by environmentally unfriendly generators. The proportion of clean energy is denoted as b_L. Once these bids have been excluded from the dataset, $\hat{D}_t$ represents the remaining demand that must be met by conventional generators.

Using model (20), we can compute the percentage of the remaining demand satisfied by generators using fuel $i = 1, 2$ through $w_i F_i(S_t)$. Their contribution to the total demand

$\hat{D}_t$ is then given by $w_i F_i(S_t)\hat{C}_t$ for $i = 1, 2$. If we make the assumption that each type of generator has a specific emission rate $\hat{e}_i$ (tCO_2/MWh), the overall emission rate is then

$$\mu_e(\hat{D}_t, A_t, S_t^c, S_t^g) = \widehat{e}_1 w_1 F_1(S_t)\widehat{C}_t + \widehat{e}_2 w_2 F_2(S_t)\widehat{C}_t \tag{25}$$

Cumulative emissions begin at zero at the start of the compliance period $(t = 0)$ and are calculated by integrating over the market emission rate μ_e. Assuming we know the price A_t of an allowance certificate, the cumulative emissions process is continuous and can be described as follows:

$$dE_t = \mu_e(D_t, A_t, S_t^c, S_t^g)dt, \quad E_0 = 0. \tag{26}$$

This definition ensures that the cumulative emissions process E remains non-decreasing, aligning with its cumulative nature.

In order to complete our pricing model, we need to define the allowance certificate price process $A = \{A\}_{t\in[0,T]}$. In a competitive equilibrium for a single compliance period, the price of the allowance certificate at the end of the period depends on the cumulative emissions and is represented by a deterministic function:

$$A_T = \phi(E_T). \tag{27}$$

Here, $\phi : \mathbb{R} \to \mathbb{R}$ is bounded, measurable, and non-decreasing. Typically, $\phi(\cdot)$ takes the form $\phi(\cdot) := \pi\mathbb{1}[\Gamma, \infty)(\cdot)$, where $\pi \in \mathbb{R}_+$ represents the penalty for non-compliance, and is the cap set by the regulator for the total allocation of certificates. As the discounted allowance price is a martingale under $\mathbb{Q}$, it equals the conditional expectation of its final value, expressed as:

$$A_t = \exp(-r(T - t))\mathbb{E}_Q[\phi(E_T)|\mathcal{F}_t], \quad \text{for } t \in [0, T]. \tag{28}$$

This equation also implies that the allowance price process remains bounded. Given that the filtration $\{\mathcal{F}\}_{t\in[0,T]}$ is generated by the Brownian motion, we can apply the Martingale Representation Theorem to represent the allowance price as an Itô integral with respect to the Brownian motion $\{\tilde{W}_t^0, \tilde{W}_t\}$. This results in the stochastic differential equation

$$dA_t = rA_t dt + Z_t^0 d\tilde{W}_t^0 + Z_t \cdot d\tilde{W}_t, \quad \text{for } t \in [0, T]. \tag{29}$$

Here, $\{Z_t^0, Z_t\}$ is an $\{\mathcal{F}_t\}$-adapted, square integrable process.

Combining the previous equations, we can put the pricing problem as the solution of the FBSDE

$$\begin{cases} dD_t = \mu_D(D_t)dt + \sigma_D(D_t)d\tilde{W}_t, \quad D_0 = d \in (0, \xi_{max}), \\ dS_t = \mu_s(S_t)dt + \sigma_s(S_t)d\tilde{W}_t, \quad S_0 = s_0 \in \mathbb{R}^n, \\ dE_t = \mu_E(A_t, D_t, S_t^c, S_t^g)dt, \quad E_0 = 0, \\ dA_t = rA_t dt + Z_t^0 d\tilde{W}_t^0 + Z_t d\tilde{W}_t, \quad A_T = \phi(E_T). \end{cases} \tag{30}$$

Regarding the existence of solution, we have that if the function μ_e, which defines the emission rate, exhibits Lipschitz continuity and $\mu_e(x, 0, s)$ remains uniformly bounded in both *x* and *s*, and if the function ϕ, representing the terminal condition, is bounded, non-decreasing, and Lipschitz, then the FBSDE (30) possesses a unique square integrable solution (for a proof and more details, see [9]).

Given the previous FBSDE, we can now derive the following PDE

$$\begin{aligned} \frac{\partial A}{\partial t} + \mu_d(t, D)\frac{\partial A}{\partial D} + \frac{1}{2}\sigma_d(D)^2\frac{\partial^2 A}{\partial D^2} + \mu_c(S_c)\frac{\partial A}{\partial S_c} + \frac{1}{2}\sigma_c S_c^2\frac{\partial^2 A}{\partial S_c^2} + \mu_g(S^g)\frac{\partial A}{\partial S_g} \\ + \frac{1}{2}\sigma_g S_g^2\frac{\partial^2 A}{\partial S_g^2} + \rho\sigma_k(K)\sigma_g(S_g)\frac{\partial^2 A}{\partial S_c \partial S_g} + \mu_e(D, A, S_c, S_g)\frac{\partial A}{\partial E} - rA = 0 \end{aligned} \tag{31}$$

with terminal condition at maturity $t = T$:

$$A(T, D, E, S_c, S_g) = \pi \mathbb{1}_{[E_{cap}, \infty]}(E) \tag{32}$$

We will specify the boundary condition, needed to find the solution, in the following chapter.

4 Numerical Results

In this chapter we will start by characterising the space and time discretization for each model, as well as the algorithms used. First, with the standard model and then move onto the joint model for electricity/emissions.

4.1 Discretization of the PDE

Both PDE's describe the evolution of an option price over time and space, and it includes several factors, such as volatility and interest rates. In order to solve them, we need to start by discretizing the time and space. We will use an explicit forward Euler method for the time variable for the standard model, and the Crank-Nicolson method for the joint model of electricity/emissions.

4.1.1 Standard Model discretization

The PDE we aim to solve is as follows:

$$\frac{\partial \alpha}{\partial t} + \frac{1}{2}\sigma_D^2(D)\frac{\partial^2 \alpha}{\partial D^2} + \mu_D(D)\frac{\partial \alpha}{\partial D} + \mu_E(\alpha(t,D,E),D)\frac{\partial \alpha}{\partial E} - r\alpha = 0, \quad \text{on } U \text{ and } 0 \leq t < T. \tag{33}$$

with the terminal condition:

$$\alpha = \Pi\mathbb{I}_{[E_{cap},\infty)}(E) \qquad \text{on } U := (0,\xi_{\text{max}}) \times (0, E_{\text{max}}) \text{ and } t = T \tag{34}$$

. Here, α represents the option price, D and E are the spatial variables, t is the time, $\sigma_D(D)$ is the volatility, $\mu_E(\alpha(t,D,E),D)$ and $\mu_E(D)$ are drift terms, r is the risk-free interest rate, Π is a constant, and $\mathbb{I}_{[E_{cap},\infty)}(E)$ is the indicator function.

As previously said, in order to solve this PDE numerically, we have to discretize both time and space. We will use an explicit forward Euler method to discretize the time variable. Let us discuss the solution to the allowance price valuation equation and establish the necessary boundary conditions. To determine the points on the boundary where conditions must be specified, we utilize the Fichera function. This function helps us identify the locations on the boundary where information either flows outward or inward for PDEs that degenerate at the boundary (see [17]).

Let us define n as the inward normal vector to the boundary, denoted as $n := (n_D, n_E)$. Fichera's function for equation (17) is

$$f(t,D,E) := (\mu_D(D) - \frac{1}{2}\frac{\partial}{\partial D}\sigma_D^2(D)n_D + \mu_E(\alpha(t,D,E),D)n_E \quad on\ \partial U_T. \tag{35}$$

Information flows outward over the boundary at sites where f is greater than or equal to zero, while it flows inward across points where f is less than zero. This distinction necessitates the specification of boundary conditions.

When $D = 0$ and $D = \xi_{max}$, $f \geq 0$ if and only if $min(\bar{D}, \xi_{max} - \bar{D}) \geq \xi_{max}\bar{\sigma}_D$. This condition is satisfied by the boundary condition derived from the Jacobi diffusion process. For $E = 0$, f is always greater than or equal to zero. The crucial boundary condition arises when $E = E_{max}$, and it takes the form:

$$\alpha(t,D,E) = e^{-r(T-t)}\Pi, \quad [0,T) \times (0,\xi_{max}) \times \{E = E_{max}\}. \tag{36}$$

This condition is based on the principle that as soon as cumulative emissions exceed the established cap, each additional metric ton of carbon dioxide incurs a penalty at a rate of Π at time T.

The time domain $[0, T]$ is divided into N_t time steps, where N_t is the total number of time steps. The time step size, represented by Δt, is given by

$$\Delta t = \frac{T}{N_t}. \tag{37}$$

The discrete time levels are $t_k = k \cdot \Delta t$ for $k = 0, 1, 2, \ldots, N_t$.

The spatial domain U is a rectangular region defined by $(0, \xi_{\text{max}}) \times (0, E_{\text{max}})$. To discretize this domain, we introduce a grid in both D and E directions. The spatial variable D is discretized by dividing the interval $(0, \xi_{\text{max}})$ into N_x equidistant grid points. The spatial step size, denoted as ΔD, is given by

$$\Delta D = \frac{\xi_{\text{max}}}{N_x}. \tag{38}$$

The discrete spatial points for D are $D_i = i \cdot \Delta D$ for $i = 0, 1, 2, \ldots, N_x$.

Similarly, the spatial variable E is discretized by dividing the interval $(0, E_{\text{max}})$ into N_y equidistant grid points. The spatial step size, denoted as ΔE, is given by

$$\Delta E = \frac{E_{\text{max}}}{N_y}. \tag{39}$$

The discrete spatial points for E are $E_j = j \cdot \Delta E$ for $j = 0, 1, 2, \ldots, N_y$.

With the discretization of time and space in place, we can rewrite the original PDE in discrete form. The discrete solution $\alpha_{i,j,k}$ represents the option price at the grid point (D_i, E_j) and time level t_k.

The discretized PDE considering the explicit forward Euler method, can be expressed as

$$\begin{aligned}&\frac{\alpha_{i,j,k+1} - \alpha_{i,j,k}}{\Delta t} + \frac{1}{2}\sigma_D^2(D_i)\frac{\alpha_{i+1,j,k+1} - 2\alpha_{i,j,k+1} + \alpha_{i-1,j,k+1}}{\Delta D^2} + \\ &\mu_D(D_i)\frac{\alpha_{i+1,j,k+1} - \alpha_{i-1,j,k+1}}{2\Delta D} + \mu_E(D_i)\frac{\alpha_{i,j+1,k+1} - \alpha_{i,j-1,k+1}}{2\Delta E} - r\alpha_{i,j,k+1} = 0,\end{aligned} \tag{40}$$

where ΔD and ΔE are discrete steps for D and E, respectively. Solving for $\alpha_{i,j,k}$ (since we will be moving backward in time), we get

$$\begin{aligned}&\alpha_{i,j,k} = \alpha_{i,j,k+1} + \Delta t[\frac{1}{2}\sigma_D^2(D_i)\frac{\alpha_{i+1,j,k+1} - 2\alpha_{i,j,k+1} + \alpha_{i-1,j,k+1}}{\Delta D^2} + \\ &\mu_D(D_i)\frac{\alpha_{i+1,j,k+1} - \alpha_{i-1,j,k+1}}{2\Delta D} + \mu_E(D_i)\frac{\alpha_{i,j+1,k+1} - \alpha_{i,j-1,k+1}}{2\Delta E} - r\alpha_{i,j,k}]\end{aligned} \tag{41}$$

With this procedure, there exists a numerical approximation update at each time step, based on the values of the previous time step. It is possible to use suitable boundary conditions to determine the values of α at the boundaries of the discretized domain U, as seen in equation (36).

4.1.2 Joint Model for Electricity/Emissions discretization

We need to solve numerically the following PDE

$$\begin{aligned}\frac{\partial A}{\partial t} + \mu_d(t,D)\frac{\partial A}{\partial D} + \frac{1}{2}\sigma_d(D)^2\frac{\partial^2 A}{\partial D^2} + \mu_c(S_c)\frac{\partial A}{\partial S_c} + \frac{1}{2}\sigma_c S_c^2\frac{\partial^2 A}{\partial S_c^2} + \mu_g(S^g)\frac{\partial A}{\partial S_g} \\ + \frac{1}{2}\sigma_g S_g^2\frac{\partial^2 A}{\partial S_g^2} + \rho\sigma_k(K)\sigma_g(S_g)\frac{\partial^2 A}{\partial S_c \partial S_g} + \mu_e(D,A,S_c,S_g)\frac{\partial A}{\partial E} - rA = 0,\end{aligned} \tag{42}$$

with terminal condition at maturity $t = T$:

$$A(T, D, E, S_c, S_g) = \pi \mathbb{1}_{[E_{cap},\infty]}(E). \tag{43}$$

Due to the high computational complexity of the PDE, we decided to remove the coal variable. While modeling coal with other variables would provide a more realistic representation of the energy and emission markets, it becomes important to reach a balance between realism and computational feasibility. Practical constraints, such as available computational power and simulation time, often require simplifications in modeling. Then, the new PDE is

$$\begin{aligned}&\frac{\partial A}{\partial t} + \mu_d(t,D)\frac{\partial A}{\partial D} + \frac{1}{2}\sigma_d(D)^2\frac{\partial^2 A}{\partial D^2} + \mu_g(S^g)\frac{\partial A}{\partial S_g} + \frac{1}{2}\sigma_g S_g^2\frac{\partial^2 A}{\partial S_g^2} + \\ &\mu_e(D,A,S_g)\frac{\partial A}{\partial E} - rA = 0,\end{aligned} \tag{44}$$

with terminal condition at maturity $t = T$:

$$A(T, D, E, S_g) = \pi \mathbb{1}_{[E_{cap},\infty]}(E) \tag{45}$$

Here, A represents the option price, D, E and S^g are the spatial variables, t is time, $\sigma_D(D)$ and $\sigma_G(G)$ are the volatilities, $\mu_D(D)$, $\mu_E(D, A, S^g)$ and $\mu_G(G)$ are drift terms, r is the risk-free interest rate, Π is a constant, and $\mathbb{I}_{[E_{cap},\infty)}(E)$ is the indicator function.

We simplify the notation and we write

$$
\begin{aligned}
&A^n_{i,j,g} = A(t_n, D_i, E_j, G_g),\\
&\mu_{d;n,i} = \mu_d(t_n, D_i),\\
&\sigma_{d;i} = \sigma_d(D_i),\\
&\mu_{g;g} = \mu_g(G_g),\\
&\sigma_{g;g} = \sigma_g(G_g),\\
&\mu_{e;n,i,j,g} = \mu_e(D_i, A^n_{i,j,g}, G_g)-
\end{aligned}
\tag{46}
$$

In order to solve this PDE numerically, we will have to discretize both time and space. We will use the Crank-Nicolson method to discretize the time variable (see [10]). We will consider the set $\Omega = (0, D_{max}] \times (0, E_{cap}] \times (0, G_{max}]$ for all $t \in [0, T]$. As in the previous model we use the Fichera function, so we can find which points of the boundary are necessary to specify. let us consider the inward normal vector to the boundary $\partial\Omega$, denoted as $n = (n_D, n_E, n_G)$. The Fichera function is defined on the part of the boundary where the characteristic form equals zero (see [13]). This function is expressed as

$$
\begin{aligned}
&b = (\mu_d(D) - \frac{1}{2}\frac{\partial \sigma_d(D)^2}{\partial D})n_D + (\mu_g(G) - \frac{1}{2}\frac{\partial \sigma_g(G)^2}{\partial G})n_G +\\
&\mu_e(D, A, G)n_E.
\end{aligned}
\tag{47}
$$

If $b < 0$, it indicates that the flow of information is inward, requiring the specification of boundary conditions. Conversely, $b \geq 0$, it implies an outward flow, and no boundary conditions are needed.

At the boundary $\partial\Omega$, which corresponds to $D = 0$, we have $b \geq 0$, meeting the condition for an outward flow. Similarly, at the lower boundary the fuel process (G = 0), the Fichera function is also positive, indicating an outward flow. However, since the emission rate μ_e is always positive, and $n_E = -1$ at the boundary $E = E_{cap}$, the function becomes negative. Therefore, a boundary condition must be specified in this case. This condition becomes relevant when the emission cap has already been reached. Due to this, we reach the following boundary condition:

$$
A(t, D, E, G) = \pi \exp^{-r(T-t)} \text{ if } E \geq E_{cap}. \tag{48}
$$

The five dimension grid is discretized as

$$
\begin{aligned}
&0 < D_0 < D_1 < ... < D_{N_D-2} < D_{N_D-1} = D_{max},\\
&0 < G_0 < G_1 < ... < G_{N_G-2} < G_{N_G-1} = G_{max},\\
&0 = E_0 < E_1 < ... < E_{N_E-2} < E_{N_E-1} = E_{cap} - \Delta E,\\
&0 = t_0 < t_1 < ... < t_{N_T-1} < t_{N_T} = T,
\end{aligned}
\tag{49}
$$

where $\Delta D = D_{i+1} - D_i$, $\Delta E = E_{j+1} - E_j$, $\Delta G = G_{g+1} - G_g$, $\Delta t = t_{n+1} - t_n$ An Alternating Direction Implicit scheme will be used in order to solve the PDE. The following operator notation will be used in the scheme:

$$\begin{aligned}
&\delta_x^+ u(x) = u(x+\Delta x) - u(x)\\
&\delta_x^- = u(x) - u(x-\Delta x)\\
&\delta_x^0 u(x) = u(x+\Delta x) - u(x-\Delta x)\\
&\delta_x^2 u(x) = u(x+\Delta x) - 2u(x) + u(x-\Delta x)\\
&\delta_x^0\delta_y^0 u(x,y) = u(x+\Delta x, y-\Delta y) - u(x-\Delta x, y+\Delta y) -\\
&u(x+\Delta x, y-\Delta y) + u(x-\Delta x, y-\Delta y).
\end{aligned} \tag{50}$$

The scheme we use is then given by

$$\begin{aligned}
&(1 + \frac{1}{2}r\Delta t - \frac{1}{2}\mu_{d;n+\frac{1}{2},i}\frac{\Delta t}{\Delta D}\delta_D^{+/-} - \frac{1}{4}\sigma_{d,i}^2\frac{\Delta t}{\Delta D^2}\delta_D^2)\Delta A^* =\\
&(-r\Delta t + \mu_{d;n+\frac{1}{2},i}\frac{\Delta t}{\Delta D}\delta_D^{+/-} + \frac{1}{2}\sigma_{d;i}^2\frac{\Delta t}{\Delta D^2}\delta_D^2\\
&+ \mu_{e;n+1,i,j,g}\frac{\Delta t}{\Delta E}\delta_E^+)A^{n+1}\\
&(1 - \frac{1}{2}\mu_{g;g}\frac{\Delta t}{2\Delta G}\delta_G^0 - \frac{1}{4}\sigma_{g;g}^2\frac{\Delta t}{\Delta G^2}\delta_G^2)\Delta A^{**} = \Delta A^*\\
&(1 - \frac{1}{2}\mu_{e;n+1,i,j,g}\frac{\Delta t}{\Delta E}\delta_E^+)\Delta A = \Delta A^{**},
\end{aligned} \tag{51}$$

where $\Delta A = A^n - A^{n+1}$, and ΔA^* and ΔA^{**} are auxiliary variables. This scheme can be derived by factorizing a Crank-Nicolson scheme and rewriting the system in the so-called delta formulation. In the last leg we have to determine the boundary condition in the artificial variable ΔA. Since ΔA is the difference in an allowance value of two consecutive time steps, we choose the boundary condition as

$$\Delta A = \Pi(e^{-r(T-t_n)} - e^{-r(T-t_{n+1})})\ if E_t \geq E_{cap}. \tag{52}$$

Having defined both our models and discretization schemes, we will now show how both algorithms are used.

4.2 Numerical Algorithms

We now provide the algorithms that will be used to calculate the allowance price function, for both models.

4.2.1 Standard Model Algorithm

In this algorithm, we outline the steps to numerically solve the PDE. The PDE describes the evolution of a variable represented by α over a spatial grid and time intervals. The algorithm employs a forward Euler scheme, a popular method for solving PDEs numerically, to compute the values of α efficiently.

1. Define and initialize the parameters:
   - Set values for constants such as η, $\bar{D}$, κ, $\bar{e}$, θ_2, $\underline{e}$, ξ_{max}, r, E_{max}, E_{cap}, T, N_t, N_D, N_E, D_{min}.
   - Create arrays D and E with evenly spaced values within specified ranges.
   - Define the time step Δt and initialize the variable α as a three-dimensional array of zeros.

2. Calculate the terminal condition for α:
   - Set the values of α at the last time step N_t based on the Heaviside function and the condition $E \geq E_{\text{cap}}$.

3. Calculate the boundary condition for α:
   - Set the values of α along the boundary where E reaches its maximum value.

4. Calculate the μ_E variable:
   - Create an empty matrix and calculate its values using numerical integration.

5. Perform time-stepping for N_t time steps, starting from $N_t - 1$ and moving backward.

   (a) For each time step, iterate through the spatial grid.

   (b) Update the values of α at each grid point using a forward Euler scheme.

   (c) Calculate α at the current grid point based on neighboring grid points and time step size.

   (d) Update $\alpha[0, :, n]$, $\alpha[-1, :, n]$, $\alpha[:, 0, n]$, and $\alpha[:, -1, n]$ to account for boundary conditions and extrapolation.

   (e) Repeat the time-stepping loop for the previous time step $(n - 1)$ until all time steps are computed.

4.2.2 Joint Model for Electricity/Emissions Algorithm

As before, the PDE describes the evolution of a function represented by A over a spatial grid and time intervals. The algorithm employs a Crank-Nicolson scheme to compute the values of A efficiently.The given scheme can be solved by following these steps:

1. Initialize the parameters and variables: Set the values of constants such as r, μ, σ and the grid sizes Δt, ΔD, ΔE and ΔG.

2. Calculate the terminal condition for A:
   - Set the values of A at the last time step N_t based on the Heaviside function and the condition $E \geq E_{\text{cap}}$.

3. Calculate the boundary condition for α:
   - Set the values of A along the boundary where E reaches its maximum value.

4. Iterate over the time steps:

   (a) Use the given equation to solve for ΔA^* by rearranging the terms.

   (b) Use the equation $(1 - \frac{1}{2}\mu_{g;g}\frac{\Delta t}{2\Delta G}\delta^0 G - \frac{1}{4}\sigma^2 g; g\frac{\Delta t}{\Delta G^2}\delta_G^2)\Delta A^{**} = \Delta A^*$ to solve for ΔA^{**}.

   (c) Use the equation $(1 - \frac{1}{2}\mu_{e;n+1,i,j,k,g}\frac{\Delta t}{\Delta E}\delta_E^+)\Delta A = \Delta A^{**}$ to solve for ΔA.

   (d) Update the values of A^{n+1}: Update A^{n+1} by adding ΔA from A^n.

   (e) Repeat the steps for the next time step until the desired number of iterations is reached.

4.3 Numerical Experiments

In this final subsection we present the numerical experiments made in both models. Firstly, we use similar values to compare how the models behave. We also demonstrate how allowance prices are influenced by both demand and cumulative emissions, and how the inclusion of fuel prices would change this influence. Lastly, we do sensitivity tests, in order to check how the models react to different inputs.

The following tables show the values used for the numerical experiments to the standard model, in accordance to [17]. They do not represent any market in particular, however, these findings can be regarded as indicative of a moderately-sized market where the primary sources of energy generation are coal and natural gas (see [17]).

$\bar{b}$	$\underline{b}$	θ_1	$\bar{e}$	$\underline{e}$	θ_2	κ	ξ_{max}
200	0	10	1.2	0.4	0.4	8760	30000

TABLE I: Parameters for the bid and emissions stacks

η	$\bar{D}$	$\bar{\sigma}_D$	r
10	21000	0.05	0.05

TABLE II: Parameters for the demand process and the risk-free rate

E_{max}	E_{cap}	Π	T
1.6519×10^8	1.17×10^8	100	1

TABLE III: Parameters characterizing the emissions trading scheme

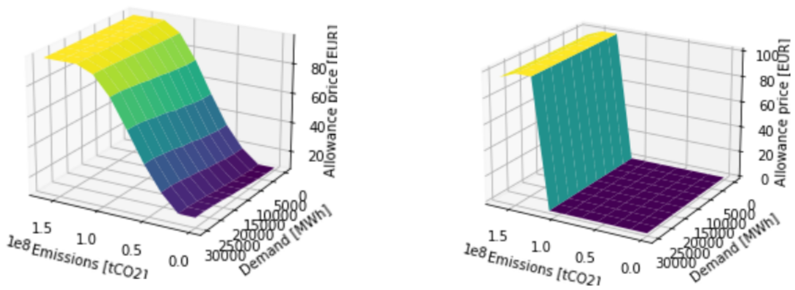
N_x	N_y	N_y	ΔD	ΔE	Δt
25	25	365	$\frac{\xi_m ax}{N_x}$	$\frac{E_m ax}{N_y}$	$\frac{T}{N_t}$

TABLE IV: Parameters for the grid discretisation

In Figure 1, we present the numerical results. Specifically, at the time $t = T/2$, the allowance price is influenced by both the cumulative emissions up to that point and the current demand level, as depicted in Figure 1(a). When we hold the emissions fixed at a constant value $E = E_{T/2}$, the function $\alpha(T/2, D, E_{T/2})$ increases as a function of the demand D, as expected, since a higher demand implies a higher probability that the emissions cap is surpassed.

We also have that, when we fix D, $\alpha(T/2, D_{T/2}, E)$ becomes an increasing function of E. Essentially, the current cumulative emissions establish a price range for allowances, and the electricity demand determines the precise price within that range. Additionally, it's worth noting that if cumulative emissions surpass the cap, the allowance price equals the discounted penalty.

As we approach the end of the compliance period, α is determined by the terminal condition (36). Figure (1(b)) illustrates the discrete nature of the price at this time and its lack of dependence on D.



((a)) $t = T/2$ ((b)) $t = T$

FIGURE 1: Plots of the allowance certificate price, in an emission market with one compliance period, at different times up to expiry, for the standard model.

The tables V, VI, VII, VIII and IX summarise the numerical parameters used to study the joint model for electricity/emissions. For the common parameters, we will use the same presented in the previous model. As for the parameters used related to the fuel price processes, they are based in the studies of [3] and [9].

b_L	b_U	w_1	$\hat{e}_1$	ξ_{max}
0.2	0.95	0.6984	8.2211	30000

TABLE V: Parameters for the bid and emissions stacks

η	$\bar{D}$	$\bar{\sigma}_D$	r
10	21000	0.05	0.05

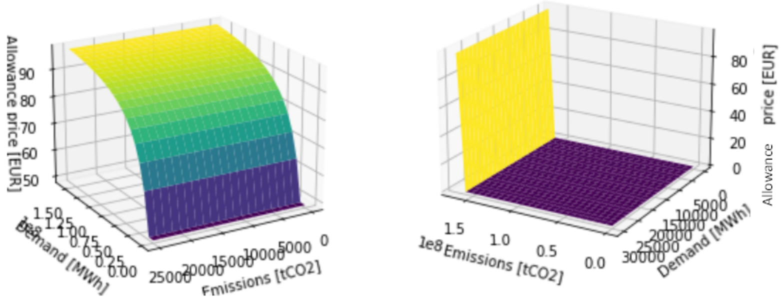
TABLE VI: Parameters for the demand process and the risk-free rate

E_{cap}	Π	T
1.6519×10^8	100	1

TABLE VII: Parameters characterizing the emissions trading scheme

η_g	$\bar{s}_g$	$\hat{\sigma}_g$	S_0^g
1.5	2	0.5	e^2

TABLE VIII: Parameters relating to the fuel price processes



((a)) $t = T/2$ ((b)) $t = T$

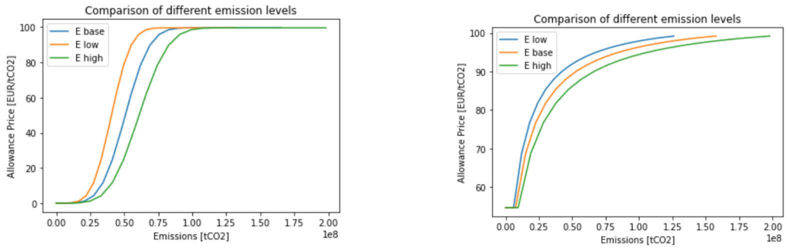
FIGURE 2: Plots of the allowance certificate price, in an emission market with one compliance period, at different times up to expiry, for the joint model for electricty/emissions.

N_x	N_y	N_g	N_t	ΔD	ΔE	ΔG	Δt
25	25	25	365	$\frac{\xi_{max}}{N_x}$	$\frac{E_{max}}{N_y}$	$\frac{S_g}{N_g}$	$\frac{T}{N_t}$

TABLE IX: Parameters for the grid discretisation

As before, in Figure (2), we present the numerical results. It is important to notice that we are depicting the impact of demand and emission of CO_2 in the allowance prices, with a fixed fuel price. Looking at the Figure (2(b)), we notice that there is no influence from D. The main difference would be that, since in this model $E_{cap} = E_{max}$, the allowance price only reaches $\Pi = 100$, when $E = E_{cap}$.

At the time $t = T/2$, we can see the effect of the demand and the cumulative emissions on the allowance price. In this model, when we fix the emissions at $E = E_{T/2}$, the function $\alpha(T/2, D, E_{T/2})$ exhibits a constant trend as D rises. When we fix D at $D = D_{T/2}$, the allowance price $\alpha(T/2, D_{T/2}, E)$ becomes an increasing function of E. The main difference between the two models would be the allowance price when $E = E_0$ and the constant trend of D in the second model. In the standard model, when $E = E_0$ the value of the allowance price is close to 0, for every value of demand. As for this model, we see that the value is around 60. So, as we can see, the introduction of the price of a fossil fuel in the model, induces a rise in the starting price of the allowance certificates. On the other hand, the lack of influence of the demand on the second model, might be explained by a couple of factors. The first one could be the disparity of value between some of the parameters. The second factor could be related to the changes made in the original equation, due to the lack of computational power and its long simulation time.



((a)) $t = T/2$ ((b)) $t = T$

FIGURE 3: Comparison of the allowance certificate price, for different emission levels.

4.3.1 Sensitivity tests

The price of the allowance certificates is influenced by multiple factors. The characteristics of power plants, including their efficiency, and fuel expenses are examples of it. It is anticipated that environmentally friendly power plants will experience advantages with rising EUA prices, while inefficient and environmentally harmful plants will have increased economic value when fuel costs are low, and the cost of emitting greenhouse gases is affordable.

In the following subsections we will simulate various scenarios and try to reach some conclusions.

4.3.1.1 Influence of Emission Cap

Our focus has now shifted to examining how the emission cap affects the allowance certificates price. Looking at the Figure 3(a), we can conclude that the lower the emission cap, the quicker will the allowance price rise. This was expected, since the lower the level of the emission cap, the higher the probability of it being reached.

Moving on to the joint model for electricity/emissions, we can see that, not only the allowance price rises faster when the level of the emission cap is lower, but it also starts at a higher price.

4.3.1.2 Influence of Demand

In this section we will study the influence of demand, in particular, we aim to simulate the impacts of both an economic boom and a recession. To model the economic boom, we will introduce a demand process that increases by 5%, representing industrial prosperity and a high demand for energy. Conversely, to simulate the recession, we will decrease the demand by 5%.

The starting price for the allowances, in the standard model, are $A_0^{base} = 5.02198$, $A_0^{+5\%} = 6.68101$ and $A_0^{-5\%} = 3.6543$. As for the second model we have $A_0^{base} = 22.5746$,

$A_0^{+5\%} = 24.4287$ and $A_0^{-5\%} = 20.6737$. These results highlight the sensitivity of emission allowance prices to changes in electricity demand. During an economic boom with an increase in demand, allowance prices tend to rise, reflecting the greater pressure on emission reduction targets. Conversely, during a recession with reduced demand, allowance prices may decrease as the need for emission reductions diminishes. It is essential for market participants and policymakers to consider these dynamics when analyzing and forecasting emission allowance prices in response to varying economic conditions.

5 CONCLUSION

This thesis has explored the intricate interplay between electricity and emission markets, shedding light on the profound transformations brought about by environmental policies and market dynamics. The transition toward cleaner energy sources, driven by international agreements like the Kyoto Protocol, has reshaped the energy landscape. The EU ETS has emerged as a key mechanism for controlling greenhouse gas emissions, substantially impacting emission certificate pricing.

Through rigorous mathematical modeling, this research has sought to unravel the pricing mechanisms governing emission certificates in the context of evolving energy markets. A risk-neutral valuation approach, based on FBSDEs, has been employed to determine certificate prices. This approach has considered structural models, establishing crucial connections between emission certificates, electricity demand, and cumulative emissions.

The objectives of this study were two-fold. Firstly, we aimed to derive the fundamental PDEs required for pricing emission certificates within both the standard model and the joint model for electricity/emissions. These PDEs serve as the cornerstone for subsequent analyses and pricing evaluations.

Secondly, we endeavored to develop and implement efficient numerical methods for solving these pricing PDEs. Numerical techniques such as finite difference schemes and methods based on alternating direction finite schemes, were examined and tested. These numerical methods are pivotal for interpreting real-world data and pricing dynamics within electricity and emission markets.

The findings presented in this thesis offer valuable insights into the intricate relationship between emission certificate prices, electricity demand, and cumulative emissions. Notably, our analysis revealed the influence of these factors on allowance prices at different points in time. As we approach the end of the compliance period, the discrete nature of the allowance price becomes evident, irrespective of demand levels.

Furthermore, our research highlighted the significance of introducing the price of fossil fuels into the modeling framework. This addition led to an increase in the starting price of allowance certificates, emphasizing the critical role of fuel prices in emission market dynamics.

In summary, this thesis has contributed to the understanding of emission market pricing mechanisms within the evolving energy landscape. By combining mathematical modeling and numerical results, we have unraveled complex interactions, providing a foundation for further research and policy considerations in the pursuit of sustainable energy and emissions management. Future research endeavors may explore additional factors, such as the impact of regulatory changes and technological advancements, to enhance our comprehension of these intricate markets.

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A Computation of the capped market emissions rate

A.1 Finding S_p interval

Let us recall the equations (1), (2), (4) and (5).

Under the hypothesis (6), the mapping $P \to \lambda(S(\cdot, P))$ is strictly increasing and also continuous. This implies the existence of an inverse function. Let us then define that inverse function as

$$(\lambda\{S(A,\cdot)\})^{-1} \quad \text{for } 0 \leq A < \infty,\ 0 \leq \xi \leq \xi_{max}. \tag{53}$$

With the previous definitions and hypothesis, it was possible for us to define the capped market emissions rate as (9). Considering the functions (1) and (2) in the mapping, the set $S_p(A, D)$ will always be an interval of the following form

$$S_p(A, D) = [\xi_1, \xi_2] \quad \text{for } 0 \leq \xi_1 < \xi_2 \leq \xi_{max} \tag{54}$$

In order to determine ξ_1 and ξ_2, we need to find ρ^* such that $\lambda(S(A, \rho^*)) = D$, which means that

$$S(A, \rho^*) = \{\xi \in [0, \xi_{max}] : g(A, \xi) \leq \rho^*\} = [\xi_1, \xi_2]. \tag{55}$$

We also have that

$$\lambda([\xi_1, \xi_2]) = D \Rightarrow [\xi_1, \xi_2] = [\xi_1, \xi_1 + D]. \tag{56}$$

Let us look at some of the values the interval S_p can take, depending on the value of A. When A $= 0$, we have the following

$S_p(0, D) = S(0, b(0, D)) = S(0, (\lambda\{S(0,\cdot)\})^{-1}(D) =$

$= \{\xi \in [0, \xi_{max}]: g(0, \xi) = b^{BAU}(\xi) \leq (\lambda\{S(0,\cdot)\})^{-1}(D)\}$.

Since b^{BAU} is strictly increasing, we have that ρ^* is such that $\lambda\{S(0, \rho^*)\} = D$ and we have the following expressions

$$b^{BAU}(\xi) \leq \rho^*, \forall \xi \leq D$$

and

$$b^{BAU}(\xi) > \rho^*, \forall \xi > D.$$

Then, we can conclude that $S_p(0, D) = [0, D]$.

If, on the other hand, A assumes values bigger than 0, in order to find the $S_p(A, D)$ interval, we have three possible cases. In the first case, we have to check if $g(A, 0) < g(A, D)$. In that case, we have that $\rho^* = g(A, D)$ and $S_p(A, D) = [0, D]$. In figure (4), we have a graphic representation of this case.

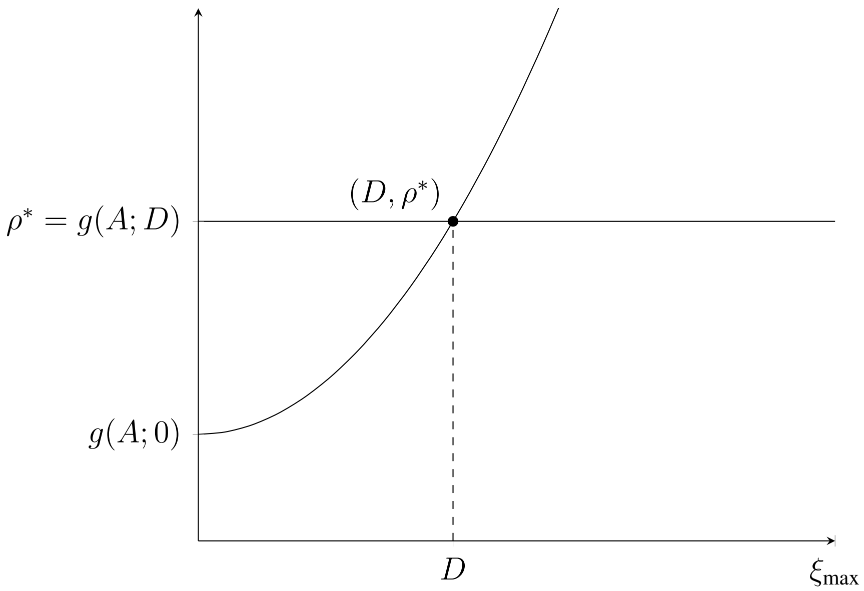

FIGURE 4: Case 1: $S_p(A, D) = [0, D]$

In the second case, we need to verify if

$$\begin{cases} g(A, \xi_{max}) < g(A, \xi_{max} - D), \\ g(A, 0) > g(A, D). \end{cases}$$

If the previous expression does indeed hold, we have $S_p(A, D) = [\xi_{max} - D, \xi_{max}]$ and $\rho^* = g(A, \xi_{max} - D)$.

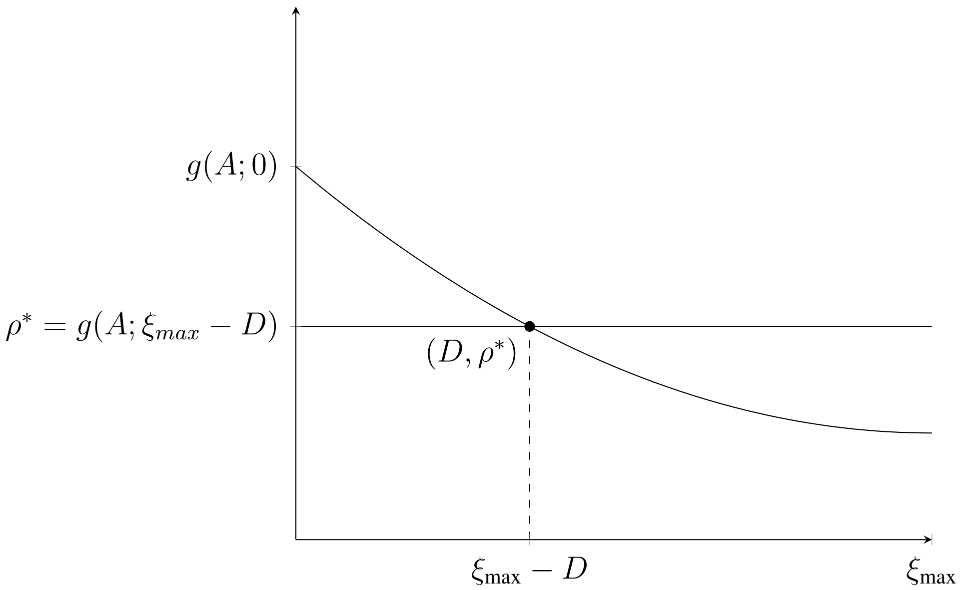

FIGURE 5: Case 2: $S_p(A, D) = [\xi_{max} - D, \xi_{max}]$

For the last case, we should verify the following expression

$$\begin{cases} g(A, \xi_{max}) > g(A, \xi_{max} - D), \\ g(A, 0) > g(A, D). \end{cases}$$

We then need to determine a ξ_1 such that it holds the subsequent equations

$$\begin{cases} g(A,\xi_1) = g(A,\xi_1 + D), \\ \xi_1 + D \leq \xi_{max}, \\ \xi_1 \geq 0. \end{cases}$$

In order to find ξ_1, we can use Newton's method

$$\begin{cases} g(A,\xi_1) - g(A,\xi_1 + D) = 0, \\ \xi_1 + D \leq \xi_{max}, \\ \xi_1 \geq 0. \end{cases}$$

The interval in that case will be $S_p(A,D) = [\xi_1, \xi_1 + D]$ and $\rho^* = g(A,\xi_1)$.

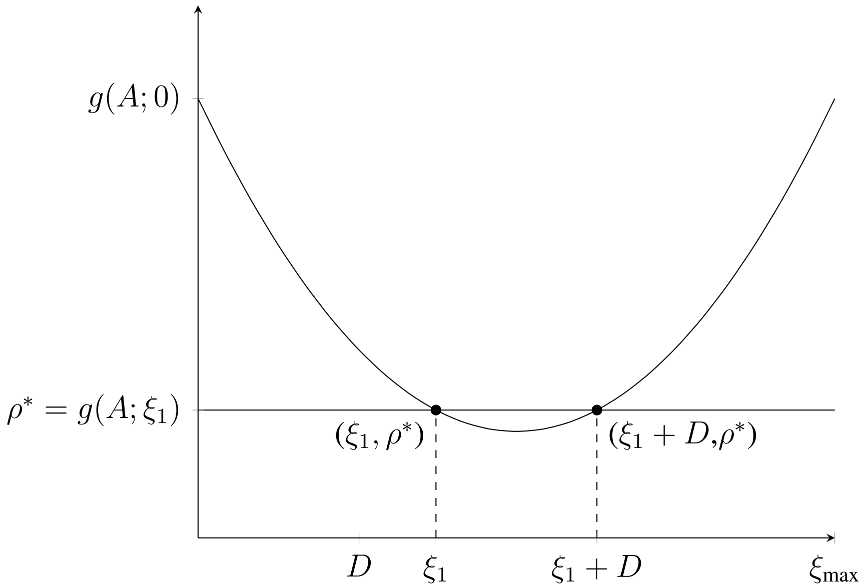

FIGURE 6: Case 3: $S_p(A,D) = [\xi_1, \xi_1 + D]$