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MASTER

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MASTER'S FINAL WORK

DISSERTATION

TWO PLAYER GAME WITH TWO STRATEGIES: REGULATION AND INNOVATION IN FINANCE

BERNARDO AMARAL SARDINHA FERREIRA DA SILVA

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SUPERVISION:

TELMO JORGE LUCAS PEIXE

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Bernardo Silva

Abstract

In this work we analyze a paper that studies a concrete bimatrix game. Namely, a model that studies the relation between financial regulation and financial innovation using the bimatrix replicator equation.

It is possible to fully explore bimatrix games by using game theory and reaching the respective replicator equation for the game, allowing the use of previous knowledge on differential equations in order to visualize how the solutions for the game behave in each case. By the payoff matrix that defines each game, it is possible to study how the solution of the associated system behaves. Since this field has been studied in depth it is possible to apply it to a concrete model as in the original paper, hence, an analysis of a real life model is made.

The bimatrix replicator system is restrictive in the sense that it only considers interactions between "individuals" of different groups. An extension of this model can be made if we consider interactions between any "individuals" of the population, including those of the same group. That is the case of the polimatrix replicator that we also study in the end of this work. In particular, in this case where the considered population is divided in two groups, each one with two possible strategies, we can observe significant different possibilities for the model dynamics. For example, we can give conditions for the existence of limit cycles in these 2-dimensional models.

Keywords: Evolutionary game theory, Financial innovation, Financial regulation, Bimatrix games, Polimatrix replicators.

Resumo

No presente trabalho analisámos um artigo que estuda um jogo bimatricial concreto. Nomeadamente, um modelo que estuda a relação entre inovação financeira e regulamentação financeira usando o replicador bimatricial.

Através da mesma, pode-se analisar a teoria de jogos bimatriciais em toda a sua extensão ao alcançar as respetivas equações replicadoras, permitindo a mesma o uso de conhecimento inerente à teoria de equações diferenciais de modo a visualizar como se estabelece e processa a solução do jogo em cada caso analisado.

Através da matriz de payoff que define cada jogo, é possível estudar como a solução do sistema associado se irá comportar.

Constata-se que ao estar esta disciplina científica verificada e comprovada, é possível aplicá-la a um modelo concreto como no artigo original, sendo deste modo feita a análise a um modelo realista.

O sistema replicador bimatricial é restritivo no sentido que apenas considera interações entre "indivíduos" de grupos diferentes. Uma extensão deste modelo pode ser feita se forem consideradas interações entre qualquer "indivíduo" da população, incluindo do mesmo grupo. Este é o caso do replicador polimatricial que também é estudado neste trabalho.

Em particular, o caso em que a população considerada é dividida em dois grupos, cada um com duas estratégias possíveis, verifica-se diferentes possíveis dinâmicas de modelos. Por exemplo, é possível impôr condições para a existência de ciclos limite em modelos 2-dimensionais.

Palavras-chave: Teoria dos jogos evolutivos, Inovação financeira, Regulamentação financeira, Jogos bimatriciais, Replicadores polimatriciais.

Contents

1. Introduction

Evolutionary game theory stands as a powerful framework for understanding the dynamics of strategic interactions among individuals in evolving populations. It is grounded in the fundamental principle that the success of a strategy depends not only on its inherent properties but also on its relative frequency in a given population. In this thesis, we delve into the intricate world of evolutionary game theory, exploring not only its structure but also possible applications.

The origins of evolutionary game theory can be traced back to the groundbreaking work of John Maynard Smith and George R. Price[14], who applied concepts from classical game theory to biological contexts. They proved that the principles of strategic interactions and natural selection could be seamlessly integrated, allowing us to analyze and predict the strategies that evolve within populations of organisms, allowing the modelling of Darwinian competition.

The central tenet of evolutionary game theory lies in the notion of fitness, where the fitness of an individual is determined by its ability to obtain resources, reproduce, and pass on its traits to the next generation. Strategies, whether cooperative or competitive, are the vehicles through which individuals interact to maximize their fitness.

Starting by a mathematical breakdown of a general model with two players each with two strategies and implementing this mathematical theory to a financial model, where financial institutions compete with regulation institutions to try and find the optimal balance where there exists a healthy development of economy and finance, avoiding excessive innovation and unnecessary regulation. A broader mathematical model is considered and deconstructed at the end in order to allow an analysis with less restrictions.

I will now present a brief summary of the structure of this thesis. In chapter 2 we introduce some theory notions that are essential for the understanding and development of the thesis. In chapter 3 we apply and analyse mathematically a bimatrix replicator to a concrete financial example . Finally in chapter 4 we extend the mathematical theory by considering polimatrix replicators, expanding the model possibilities. In the end we present a discussion about some conclusions we can obtain from this work. Moreover, we establish some possible lines of future research that can be done based on this work.

2. Theory - preliminaries

To gain a better understanding of the mathematical theory behind this work, some basic topics of several areas, such as differential equations and evolutionary game theory, will be presented.

2.1 Differential equations

The theory of differential equations which we briefly present in this section is general. However, for this presentation, we followed [7].

One way to represent a system of differential equations in dimension 2 with 2 variables is,

$$\begin{cases} \dot{x} = f_1(x,y) \\ \dot{y} = f_2(x,y) \end{cases}. \tag{2.1}$$

The respective Jacobian matrix is calculated using the first-order partial derivatives. Allowing to understand the behaviour of the system in a neighbourhood of the equilibria point.

$$J = \begin{bmatrix} \frac{\partial f_1(x,y)}{\partial x} & \frac{\partial f_1(x,y)}{\partial y} \\ \frac{\partial f_2(x,y)}{\partial x} & \frac{\partial f_2(x,y)}{\partial y} \end{bmatrix}.$$

A simpler representation of the Jacobian matrix can be used, such as $J = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with each entry equal to the partial derivative as it is written in the original representation.

Since it is a 2x2 matrix it is possible to extract the eigenvalues, which are the roots of the characteristic polynomial. These allow to have some idea regarding the geometry of solutions of the differential equations.

$$\lambda^2 \text{ - (a+d) } \lambda + \text{(ad - bc) = 0.}$$

The constant term in this equation is the determinant of J, generally known by det J or D. The quantity a+d is known as the trace of J, known by tr J or T, so the equation can be rewritten as,

$$\lambda^2 \text{ - (tr J) } \lambda + \text{det J = 0.}$$

Hence an analysis of a matrix can be reduced to its trace and determinant, corresponding to the point (T,D) on the trace-determinant plane (see Figure 2.1). The location in this plane gives an idea of the geometry of the phase portrait. A visual classification of the solutions is possible, that is, whether it spirals into or away from a given point, whether it is a center, and so forth, since these two values give information regarding the eigenvalues of the matrix,

1. Complex with nonzero imaginary part if T^2 - 4D < 0
2. Real and distinct if if T^2 - 4D > 0
3. Real and repeated if T^2 - 4D = 0.

Regarding the phase portraits, more can be said. If T^2 - 4D < 0, eigenvalues are complex with nonzero imaginary part, and the real part is $\frac{T}{2}$, so the solution is a

1. Spiral sink if T < 0
2. Spiral source if T > 0
3. Center if T = 0.

In the case that T^2 - 4D > 0, both eigenvalues are real. If D < 0, it is a saddle, since D is the product of eigenvalues, one of which positive whereas the other is negative. If D > 0 and T < 0, then it is a (real) sink, whereas if D > 0 and T > 0 leads to a (real) source.

It is important to note that the trace-determinant plane is a two-dimensional representation of a four-dimensional space, since 2x2 matrices are determined by four parameters, the entries of the matrix.Therefore there are infinitely many different matrices corresponding to each point in the TD-plane.

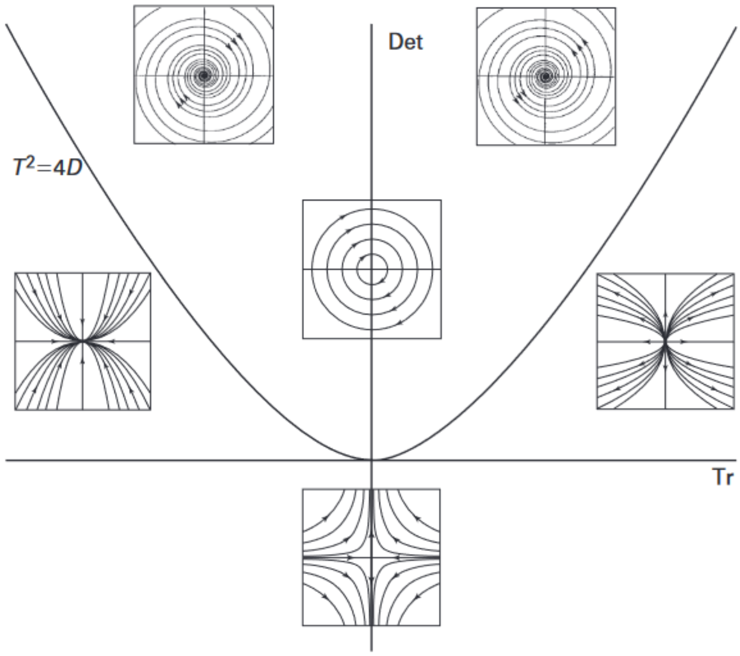

Figure 2.1: Trace Determinant plane [7]

For planar systems the Poincaré-Bendixson theorem is a crucial and very useful result that allows under certain conditions to characterize the long term behaviours of its orbits.

Theorem 2.1.1 (Poincaré-Bendixson, [7])**.** *Suppose that Ω is a nonempty, closed and bounded limit set of a planar system of differential equations that contains no equilibrium point. Then Ω is a closed orbit.*

2.2 EVOLUTIONARY GAME THEORY

The initial aim of game theory was to find principles of rational behaviour, by means of thought experiments involving fictitious player who were assumed to be acquainted with such theory, and to be aware that their fictitious co-players would use it. At the same time, it was expected that rational behaviour would prove to be optimal against irrational behaviour, which turned out to be too much to ask for. Hence, players shouldn't be constrained to rationality, but be able to learn, adapt and evolve.

In the 1970s, J. Maynard Smith and G. Price[14], drawing from the work of J. von Neumann and O. Morgenstern in the 1940s [6], harnessed the principles of strategic game theory to explore the dynamic mechanisms governing biological populations. This groundbreaking work paved the way for the emergence of Evolutionary Game Theory (EGT) as a distinct field.

2.2.1 Replicator dynamics

In 1978, Taylor and Jonker [16] introduced a set of differential equations, later labeled as the replicator equation by Schuster and Sigmund in 1983 [12]. These equations have primarily been examined within the framework of Evolutionary Game Theory. Replicator dynamics describe the evolution of the frequencies of strategies in a population [8].

Assuming that the population is divided into n types E_1 to E_n with respective frequencies x_1 to x_n. Let f_i of E_i be a function of the composition of the population, that is of a state x in time. It is assumed that the state x(t) evolves on the simplex

$$\Delta^{n-1} = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_i \geq 0, \sum_{i=1}^{n} x_i = 1, \forall i \in \{1, ..., n\}\}$$

as a differentiable function of t. The rate of increase $\frac{\dot{x_i}}{x_i}$ of type E_i, hence by the basic principle of darwinism it can be represented as,

$$\frac{\dot{x_i}}{x_i} = \text{fitness of } E_i \text{ - average fitness}$$

which leads to the replicator equation,

$$\dot{x_i} = x_i(f_i(\text{x}) - \bar{f}(\text{x}))$$

The main interest of our analysis is the case of linear f_i. Considering an n×n matrix A = (a_{ij}) such that f_i(x) = $(A\text{x})_i$, hence the equation takes the form,

$$\dot{x_i} = x_i((A\text{x})_i - \text{x} \cdot A\text{x})$$

Where $x_i \geq 0$, denotes the population density of species i at time t, and each coefficient $a_{i,j}$ means the influence of species j on the population of species i. In other words, it represents the payoff of a player employing strategy i when interacting with a player utilizing strategy j.

From a game-theoretical interpretation of the replicator dynamics, there is an underlying normal form game with N pure strategies R_1 to R_N and a payoff function given by an N×N matrix U, where a strategy is defined by a point in Δ^{N-1} : the types E_1 to E_n correspond to n points p^1, ..., p^n. Defining the matrix A with a_{ij} = $p_i \cdot \text{U} p_j$ and obtain the fitness f_i(x) of the type E_i,

$$f_i(\text{x}) = \sum_j a_{ij}\, x_j = (A\text{x})_i$$

A point $\hat{\text{x}} \in \Delta^n$ is a Nash equilibrium if

$$\text{x} \cdot \text{A}\hat{\text{x}} \leq \hat{\text{x}} \cdot \text{A}\hat{\text{x}} \text{ for all x} \in \Delta^{n-1}$$

and evolutionary stable state if,

$$\hat{\text{x}} \cdot \text{Ax} > \text{x} \cdot \text{Ax for all x} \neq \hat{\text{x}} \text{ in a neighbourhood of } \hat{\text{x}}.$$

2.3 BIMATRIX REPLICATOR

Traditional game theory aimed to find principles of rational behaviour, by means of thought experiment involving fictitious players, who were assumed to know such theory. Hence requiring players to make rational choices. Whereas evolutionary game theory does not require players to act rationally - only that they have a strategy. The results of a game show how good the strategy was.

To be more precise, classical game theory describes socially and temporally isolated encounters, while evolutionary game theory describes macro-social behavioural regularities.

Since an evolutionary game model has no requirement for complete rationality or information, the stakeholder has to conduct repeated game by pairing in the group randomly, and eventually achieving a dynamic and stable state.

When modeling interactions between two populations or groups, each with its distinct set of strategies (asymmetric games), and all interactions involve individuals from different groups, the commonly used model is the bimatrix replicator equation. This model first appeared in [11] and [13].

Considering distinct players, in position I and another in position II. In position I, a player has n strategies, whereas in position II, a player has m strategies, with payoff matrices A and B, respectively. Thus a player in position I using strategy i against a player player in position II using strategy j, obtains the payoff a_{ij}, while the opponent obtains b_{ji}. The mixed strategies for player I are denoted by $p \in \Delta^{n-1}$ and those for player II are denoted by $q \in \Delta^{m-1}$, hence the respective payoffs are given by $p \cdot \text{A}q$ and $q \cdot \text{B}p$.

The pair $(\hat{p}, \hat{q}) \in \Delta^{n-1} \times \Delta^{m-1}$ is said to be a Nash equilibrium if $\hat{p}$ is a best reply to $\hat{q}$ and $\hat{q}$ a best reply to $\hat{p}$, that is if,

$$p \cdot \text{A}\hat{q} \leq \hat{p} \cdot \text{A}\hat{q}$$

for all $p \in \Delta^{n-1}$, and,

$$q \cdot \text{A}\hat{p} \leq \hat{q} \cdot \text{A}\hat{p}$$

for all $q \in \Delta^{m-1}$. The set of Nash equilibria for bimatrix games is always nonempty, [8].

Let $x \in \Delta^{n-1}$ and $y \in \Delta^{m-1}$ denote the frequencies of the strategies for the players in position I and II respectively. Associating that the rate of increase $\frac{\dot{x_i}}{x_i}$ of strategy i is equal to the difference between its payoff $(Ay)_i$ and the average payoff x · Ay in the population X, which leads to the system of differential equations,

$$\begin{cases} \dot{x_i} = x_i\left((A\text{y})_i - x \cdot Ay\right) \\ \dot{y_j} = y_j\left((B\text{x})_j - y \cdot Bx\right) \end{cases}$$

usually designated as the bimatrix replicator. The phase space of this system is $\Delta^{n-1} \times \Delta^{m-1}$, which is invariant for the bimatrix replicator.

2.3.1 Evolutionary model

A bimatrix replicator model with two players, each one with two strategies, is determined by the 2x2 matrices,

$$\text{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \text{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

whose entries describe the payoffs of each different interaction.

As proved in [11], it is possible to add a constant to each column and obtain an equivalent system, therefore obtaining the bimatrix replicator with respective payoff matrices,

$$\text{A} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \qquad \text{B} = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}$$

such that $a_1 = a_{11} - a_{21}$, $a_2 = a_{22} - a_{12}$, $b_1 = b_{11} - b_{21}$, $b_2 = b_{22} - b_{12}$.

We have that this representation is essentially,

$$\Delta^1 \times \Delta^1 = \{\ (x_1,x_2,y_1,y_2) \in \mathbb{R}^4\text{: } x_i,y_i \geq 0,\ \sum x_i = 1,\ \sum y_i = 1\ \}$$

Proposition 1. *For these new two matrices A and B the bimatrix replicator becomes,*

$$\begin{cases} \dot{x} = x(1-x)[a_1 y - a_2(1-y)] \\ \dot{y} = y(1-y)[b_1 x - b_2(1-x)] \end{cases}$$

on the square $Q = \{(x,y) : 0 \leq x,y \leq 1\} \cong \Delta^1 \times \Delta^1$

Proof. This form is obtained intuitively, as the original form of any bi-matrix replicated dynamics equations is,

$$\begin{cases} \dot{x_i} = x_i((Ay)_i - x \cdot Ay) \\ \dot{y_i} = y_i((Bx)_i - y \cdot Ax) \end{cases}$$

Given that X = { $x, 1-x$ }, considering $x = x_1$ and $1 - x = x_2$, the same goes for the y variable, $y = y_1$ and $1 - y = y_2$, hence,

$$\begin{aligned}
\dot{x} &= x\left(\left(\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}\begin{bmatrix} y \\ 1-y \end{bmatrix}\right)_1 - \begin{bmatrix} x & 1-x \end{bmatrix}\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}\begin{bmatrix} y \\ 1-y \end{bmatrix}\right) \\
&= x\Big(a_1 y - \begin{bmatrix} xa_1 & (1-x)a_2 \end{bmatrix}\begin{bmatrix} y \\ 1-y \end{bmatrix}\Big) \\
&= x\Big(a_1 y - xya_1 - (1-x)(1-y)a_2\Big) \\
&= x\Big(a_1 y(1-x) - (1-y)a_2(1-x)\Big) \\
&= x(1-x)[a_1 y - a_2(1-y)]
\end{aligned}$$

This result can also be obtained for $\dot{x}$ with $\dot{x_2}$, since,

$$\dot{x} = (1-x)((Ay)_2 - x \cdot Ay)$$

The deduction for $\dot{y}$ is analogous considering $y = y_1$ and $1 - y = y_2$.

□

This phase space is a 2-dimensional projection of a 4-dimensional space, (x_1, x_2, y_1, y_2) which is equal to (x, $1-x$, y, $1-y$), essentially being reduced to (x, y), with e_1 = (1 , 0) and e_2 = (0 , 1), as represented in [9].

Vertex	$\mathbb{R}^2$	$\mathbb{R}^4$
v_1	$(0,0)$	$(0,1,0,1)$
v_2	$(0,1)$	$(0,1,1,0)$
v_3	$(1,1)$	$(1,0,1,0)$
v_4	$(1,0)$	$(1,0,0,1)$

Table 2.1: Representation of the four vertices of $[0,1]^2$ in $\mathbb{R}^2$ and in $\mathbb{R}^4$.

2.3.2 Analysis of the model

To analyse the model, the equilibria of the replicator equation must be known, by applying the notation used in (2.1),

$$\begin{cases} f_1(x,y)=0 \\ f_2(x,y)=0 \end{cases} \Leftrightarrow \begin{cases} x(1-x)[a_1 y - a_2(1-y)]=0 \\ y(1-y)[b_1 x - b_2(1-x)]=0 \end{cases}$$

Most of the solutions are straightforward, and represent the vertices of the $[0,1]\times[0,1]$ phase space, namely,

$$\begin{cases} x=0 \\ y=0 \end{cases} \vee \begin{cases} x=0 \\ y=1 \end{cases} \vee \begin{cases} x=1 \\ y=0 \end{cases} \vee \begin{cases} x=1 \\ y=1 \end{cases}$$

So the interior equilibria occurs when $0 < x < 1$ and $0 < y < 1$,

$$\begin{cases} x(1-x)[a_1 y - a_2(1-y)]=0 \\ y(1-y)[b_1 x - b_2(1-x)]=0 \end{cases} \Rightarrow \begin{cases} a_1 y - a_2(1-y)=0 \\ b_1 x - b_2(1-x)=0 \end{cases} \Rightarrow \begin{cases} (a_1+a_2)y - a_2=0 \\ (b_1+b_2)x - b_2=0 \end{cases}$$

Which leads to the only other equilibrium point,

$$\begin{cases} y = \dfrac{a_2}{(a_1+a_2)} \\ x = \dfrac{b_2}{(b_1+b_2)} \end{cases}$$

Proposition 2. *There is either no equilibria on the edges besides the vertex or it is a continuumn of equilibria.*

Proof. The edges of the phase space, these are described by only one of the the following conditions, $x = 0 \vee x = 1 \vee y = 0 \vee y = 1$,

$$\begin{cases} x=0 \Rightarrow \dot{y} = y(1-y)(-b_2) \\ x=1 \Rightarrow \dot{y} = y(1-y)b_1 \\ y=0 \Rightarrow \dot{x} = x(1-x)(-a_2) \\ y=1 \Rightarrow \dot{x} = x(1-x)a_1 \end{cases} \xLeftrightarrow[\text{Equilibrium if } \dot{x}=0 \vee \dot{y}=0]{} \begin{cases} y=0 \Rightarrow y=0 \vee y=1 \vee b_2=0 \\ y=1 \Rightarrow y=0 \vee y=1 \vee b_1=0 \\ x=0 \Rightarrow x=0 \vee x=1 \vee a_2=0 \\ x=1 \Rightarrow x=0 \vee x=1 \vee a_1=0 \end{cases}$$

Which concludes that the equilibria on the border of the phase space are either the vertices, as we concluded $\big((0,0);(0,1);(1,0);(1,1)\big)$, or it means that a_i or b_i are equal to zero.

If any of these payoffs are equal to zero, it means that in the edge x (resp. y), either $x = 0$ or $x = 1$ (resp. $y = 0$ or $y = 1$), $y \in$]0,1[and $\dot{y} = 0$ (resp. $x \in$]0,1[and $\dot{x} = 0$), therefore if there is an equilibrium point in a edge, other than a vertex, the edge is a continuous of equilibrium. □

Considering the only possible interior equilibrium point is $\left(x = \frac{b_2}{b_1+b_2}, y = \frac{a_2}{a_1+a_2}\right)$, and that any equilibrium on the border besides the vertex has been ruled out. It must be checked for what values of a_1, a_2, b_1, b_2 it is indeed an interior equilibrium point, since it is only inside the phase space, Δ^1 x Δ^1, if,

$$\begin{cases} 0 < \dfrac{a_2}{(a_1+a_2)} < 1 \\ 0 < \dfrac{b_2}{(b_1+b_2)} < 1 \end{cases}$$

There are four possible cases,

(1) $a_1 + a_2 > 0$

(2) $a_1 + a_2 < 0$

(3) $b_1 + b_2 > 0$

(4) $b_1 + b_2 < 0$

that ought to be analysed one by one,

(1) $a_1 + a_2 > 0$

$$\begin{cases} \dfrac{a_2}{(a_1+a_2)} > 0 \\ \dfrac{a_2}{(a_1+a_2)} < 1 \end{cases} \Leftrightarrow \begin{cases} a_2 > 0 \\ a_2 < a_1 + a_2 \end{cases} \Leftrightarrow \begin{cases} a_2 > 0 \\ a_1 > 0 \end{cases}$$

(2) $a_1 + a_2 < 0$

$$\begin{cases} \dfrac{a_2}{(a_1+a_2)} > 0 \\ \dfrac{a_2}{(a_1+a_2)} < 1 \end{cases} \Leftrightarrow \begin{cases} a_2 < 0 \\ a_2 > a_1 + a_2 \end{cases} \Leftrightarrow \begin{cases} a_2 < 0 \\ a_1 < 0 \end{cases}$$

(3) $b_1 + b_2 > 0$

$$\begin{cases} \dfrac{b_2}{(b_1+b_2)} > 0 \\ \dfrac{b_2}{(b_1+b_2)} < 1 \end{cases} \Leftrightarrow \begin{cases} b_2 > 0 \\ b_2 < b_1 + b_2 \end{cases} \Leftrightarrow \begin{cases} b_2 > 0 \\ b_1 > 0 \end{cases}$$

(4) $b_1 + b_2 < 0$

$$\begin{cases} \dfrac{b_2}{(b_1+b_2)} > 0 \\ \dfrac{b_2}{(b_1+b_2)} < 1 \end{cases} \Leftrightarrow \begin{cases} b_2 < 0 \\ b_2 > b_1 + b_2 \end{cases} \Leftrightarrow \begin{cases} b_2 < 0 \\ b_1 < 0 \end{cases}$$

therefore, there is only an interior equilibrium in one of the following cases,

a_1,a_2 > 0 and b_1,b_2 > 0
a_1,a_2 > 0 and b_1,b_2 < 0
a_1,a_2 < 0 and b_1,b_2 > 0
a_1,a_2 < 0 and b_1,b_2 < 0

this exhaustive analysis describes all the cases where there exists an interior equilibrium point.

It is important to remark that when there is no interior equilibrium point the system dynamic is fully determined by the border dynamics.

2.3.3 Equilibrium stable solutions under different parameters

The equilibria are used to observe the behaviour of each strategy, varying depending on whether the equilibrium point is a sink, a source or a center.

In order to analyse each strategy and the respective evolution process, the dynamics on the border is studied. The most interesting cases are displayed in order to clarify how the equilibrium stable solutions are obtained. It is also shown that the original article had typos in strategies 11, 12, 15 and 16, in the sense that these were exchanged in the table.

Strategy 1: a_1, b_1 > 0 $\wedge$ a_2, b_2 < 0 One of the basic strategies to showcase a simple example, starting by calculating the border dynamics,

y = 0 $\Rightarrow \dot{x} = x(1-x)(-a_2)$ > 0, since x $\in$]0,1[(as we are excluding the vertices), and the strategy defines a_2 < 0 , hence -a_2 > 0, therefore $\lim_{t\to\infty} x(t) = 1$

y = 1 $\Rightarrow \dot{x} = x(1-x)(a_1)$ > 0, using a similar logic to the previous case, $\lim_{t\to\infty} x(t) = 1$

x = 0 $\Rightarrow \dot{y} = y(1-y)(-b_2)$ > 0 , $\lim_{t\to\infty} y(t) = 1$

x = 1 $\Rightarrow \dot{y} = y(1-y)b_1$ > 0 , $\lim_{t\to\infty} y(t) = 1$

According to the previous calculations there is no interior equilibrium in this case, which is easily check-able, as with a_1> 0 $\wedge$ a_2 < 0, it is impossible for

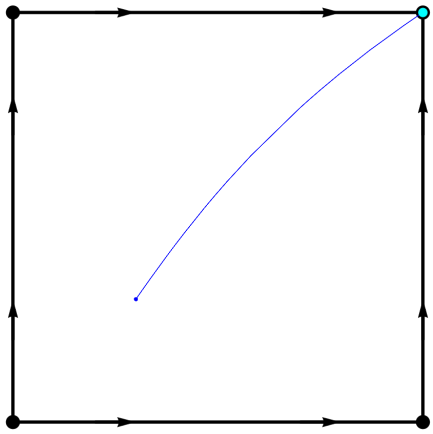

Figure 2.2: Phase portrait of strategy 1

$0 < \frac{a_2}{(a_1+a_2)} < 1$, placing this equilibrium point outside of the phase space, hence the dynamic is fully determined by the border dynamics.

The blue dot represents a random initial condition and the blue line the way it iterates through time, in the following images, a red dot will represent the interior equilibrium, that is not present here as expected.

Strategy 11: a_1, b_1, a_2, $b_2 > 0$

$y = 0 \Rightarrow \dot{x} = x(1-x)(-a_2) < 0$, therefore $\lim_{t\to\infty} x(t) = 0$

$y = 1 \Rightarrow \dot{x} = x(1-x)(a_1) > 0$, $\lim_{t\to\infty} x(t) = 1$

$x = 0 \Rightarrow \dot{y} = y(1-y)(-b_2) < 0$, $\lim_{t\to\infty} y(t) = 0$

$x = 1 \Rightarrow \dot{y} = y(1-y)b_1 > 0$, $\lim_{t\to\infty} y(t) = 1$

This is one of the cases where it was proven the existence of an interior equilibrium, which must be checked to whether it is a source, a sink or a center. In order to solve this issue, the Jacobian matrix must be considered.

Hence utilizing theory explained at the beginning of this chapter, and having into account that the partial derivatives are as follows,

$$\begin{cases} \dfrac{\partial f_1(x,y)}{\partial x} = (1-2x)(a_1 y + a_2(y-1)) \\ \dfrac{\partial f_1(x,y)}{\partial y} = (a_1+a_2)(1-x)x \\ \dfrac{\partial f_2(x,y)}{\partial x} = (b_1+b_2)(1-y)y \\ \dfrac{\partial f_2(x,y)}{\partial y} = (1-2y)(b_1 x + b_2(x-1)) \end{cases}$$

the Jacobian matrix of this dynamical system at the interior equilibrium point is,

$$J_{(\frac{a_2}{a_1+a_2},\frac{b_2}{b_1+b_2})} = \begin{bmatrix} 0 & (a_1+a_2)\frac{b_1 b_2}{(b_1+b_2)^2} \\ (b_1+b_2)\frac{a_1 a_2}{(a_1+a_2)^2} & 0 \end{bmatrix}$$

The trace of J is 0, as it is the sum of the diagonal, whereas the determinant is $-\frac{a_1 a_2 b_1 b_2}{(a_1+a_2)(b_1+b_2)}$, as it essentially the trace summed with the multiplication of the other elements of the matrix.

Given this information it is possible to extrapolate information regarding the phase portrait by looking at T^2 - 4D, that is, $4\ \frac{a_1 a_2 b_1 b_2}{(a_1+a_2)(b_1+b_2)}$, since in strategy 11, a_1, b_1, a_2, b_2> 0, it is evident that T^2 - 4D > 0, hence the interior equilibrium is hyperbolic.

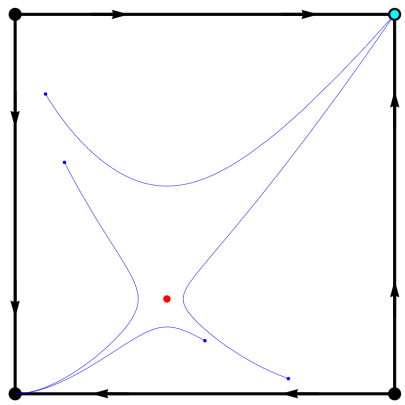

Figure 2.3: Phase portrait of strategy 11

In a matter of fact, since it is a bimatrix replicator there are solely two possible different dynamics when there is an interior equilibrium, as stated in [11], either it is a center and the orbits are periodic or it is hyperbolic and the orbits go to one of the two vertices.

Strategy	a_1	a_2	b_1	b_2	ESS
1	>0	<0	>0	<0	(e_1, e_1)
3	>0	<0	>0	>0	(e_1, e_1)
4	>0	<0	<0	<0	(e_1, e_2)
5	<0	>0	>0	<0	(e_2, e_1)
6	<0	>0	<0	>0	(e_2, e_2)
7	<0	>0	>0	>0	(e_2, e_2)
8	<0	>0	<0	<0	(e_2, e_1)
9	>0	>0	>0	<0	(e_1, e_1)
10	>0	>0	<0	>0	(e_2, e_2)
11	>0	>0	>0	>0	(e_1, e_1) , (e_2, e_2)
12	>0	>0	<0	<0	None
13	<0	<0	>0	<0	(e_2, e_1)
14	<0	<0	<0	>0	(e_1, e_2)
15	<0	<0	>0	>0	None
16	<0	<0	<0	<0	(e_1, e_2) , (e_2, e_1)

Table 2.2: Equilibrium stable solutions under different parameters.

2.4 POLIMATRIX REPLICATORS

The polymatrix replicator, introduced by Alishah, Duarte, and Peixe [2] [4], is a set of ordinary differential equations designed for exploring the dynamics of polymatrix games. It describes how the strategies selected by individuals in a stratified population change over time as they interact with each other. These equations expand upon the class of bimatrix replicator equations previously examined in [11] [13] [1] to encompass replicator dynamics within a population divided into a finite number of groups.

The polymatrix replicator generates a flow within a polytope formed by the finite product of simplices. Alishah et al. [3] introduced a novel approach for analyzing the long-term behavior of flows within polytopes, with polymatrix replicators serving as illustrative instances of these flows. Such dynamic systems naturally emerge in the realm of Evolutionary Game Theory (EGT) developed by Smith and Price [14].

Starting by considering a population divided in p groups, labelled an integer α ranging from 1 to p. Individuals of each group have exactly n_α strategies to interact with other members of the population. The strategies of a group α are labelled by positive integers j in the range,

$$n_1 + \ldots + n_{\alpha-1} < j \leq n_1 + \ldots + n_\alpha$$

So $j \in \alpha$ means that j is a strategy of the group α. So the strategies of all population are labelled by $j = 1, \ldots, n$ where $n = n_1, \ldots, n_p$.

The matrix A is referred to as the payoff matrix. It characterizes the outcomes of interactions between individuals employing strategies i from the set α and individuals employing strategies j from the set β. The element $a_{ij} = a_{ij}^{\alpha\beta}$ signifies the average payoff that an individual achieves when employing the first strategy in interactions with individuals using the second strategy. Consequently, the payoff matrix A can be partitioned into block matrices $A^{\alpha\beta}$, each having dimensions $n_\alpha \times n_\beta$, with elements denoted as $a_{ij}^{\alpha\beta}$, where α and β take values from 1 to p.

The state of the population is described by a point $x = (x_\alpha)^\alpha$ in the prism,

$$\Delta^{n_1-1} \times \cdots \times \Delta^{n_p-1} \subset \mathbb{R}^n$$

where $\Delta^{n_\alpha-1} = \{x \in \mathbb{R}^{n_\alpha} : \sum_{i=1}^{n_\alpha} x_i = 1\}$, and the entry $x_j = x_j^\alpha$ represents the usage frequency of strategy j within the group α.

Hence the polimatrix replicator system is defined on the prism by,

$$\frac{dx_i^\alpha}{dt} = x_i^\alpha \left((Ax)_i - \sum_{\beta=1}^{p} (x^\alpha)^T A^{\alpha\beta} x^\beta \right)$$

Showing that the growth rate of each frequency x_i^α is the difference between its payoff and the average payoff of all strategies in group α.

3. Regulation and innovation in finance

In this chapter we will analyse the model presented in [5]. In this model the stakeholders are financial institutions and regulation institutions. Firstly, financial institutions describes an establishment that completes and facilitates monetary transactions, such as loans, mortgages, and deposits, which are divided into two categories, banking industry (commercial banks, saving banks, loan associations) and non-banking industry (investment banks, insurance institutions). Lastly the other stakeholder are regulation institutions, which essentially supervise the banking industry, securities and futures industry and insurance industry by judging whether innovation exists in order to prevent market crisis. An example of this in the U.S is the Federal Reserve that is responsible for making the monetary policy and supervising its subsidiary banks and financial holding companies.

3.1 Model assumptions

In this model simple assumptions were considered,

Assumption 1: This point regards the possible strategies for each stakeholder, financial institutions can either choose (**Conservation**,**Innovation**), while regulation institutions can either choose (**Regulation**,**Deregulation**). The strategy chosen by either stakeholder influences the other. Like when innovation is chosen by financial institutions evading all regulation, regulation institutions can choose either of the strategies. When it comes to choosing regulation, it means that financial institution profit seeking behavior have put the healthy and orderly development of the financial industry at stake, hence regulation institutions have to take corresponding measures passively. For the deregulation strategy, the costs of regulation are too high. When regulation is rigorous or payoffs of innovation are smaller than that of conservation, financial institutions will choose conservation, otherwise innovation will be chosen, and financial institutions will face the risk of punishment once this behavior is discovered.

Assumption 2: Each stakeholder has limited rationality. In order to reach the expected payoffs, stakeholders must play the game multiple times and constantly learn and adjust their strategies to arrive at the equilibrium point. Hence regulation institutions and financial institutions can combine randomly with multiple games, avoiding the influences of accidental equilibrium effectively.

Assumption 3: Let q denote the probability of a regulation institution choosing a regulation strategy, meaning 1-q is the proportion choosing deregulation. Similarly, for financial institutions, the ratio choosing the innovation strategy is p, while when it comes to the conservation strategy is 1-p. Payoffs for financial institutions when choosing conservation a, while b represents the payoffs of regulation institutions when financial institutions choose the strategy of conservation. Other terms are defined such as a_1 which refers to positive payoffs of financial institutions without regulation under innovation [1] and b_1 refers to payoffs of regulation institutions when financial institutions choose the strategy of innovation [2]. Finally, c is the regulation cost of regulation institutions, while f is the punishment from regulation institutions when financial institutions choose the innovation strategy. Also, it is important to note that, $f > c$ and $a_1 > a$.

3.2 REPLICATOR EQUATION

Since the information is symmetric, the strategy of selection of two stakeholders will have mutual effects and be affected by the results of the former game. This determines the features of inheritance and dynamics of the game. Based on the assumptions of the game model, the payoff matrices between financial institutions and regulation institutions are as follows,

$$F = \begin{bmatrix} a_1 - f & a_1 \\ a & a \end{bmatrix} \qquad R = \begin{bmatrix} b_1 + f - c & b - c \\ b_1 & b \end{bmatrix}$$

which allows to calculate the expectation of each strategy, like the expectation U_I of the payoff value of financial institutions with the strategy of innovation is:

U_I = q $(a_1 - f)$ + (1 - q) a_1 ,

that can be explained by the fact that once financial institutions choose to innovate, regulation institutions can either choose to regulate, in which case financial institutions payoff are essentially the positive payoffs of financial institutions without regulation under innovation, a_1 minus the punishment from regulation institutions when financial institutions choose innovation, f, or can choose to deregulate, in which case the payoff is positive payoff of financial institutions without regulation under innovation, a_1.

[1] The article defines a_1 as the payoff under conservation, which makes it impossible to define the payoff
[2] The article refers to b_1 as after regulation, which stops its purpose, therefore it was removed

The expectation U_C of conservation strategy for financial institutions is:

U_C = q a + (1 - q) a ,

which is rather straightforward as whether regulation institutions choose to regulate or deregulate, if the financial institutions choose conservation, the payoff are the payoffs for financial institutions when choosing the strategy of innovation, a.

The average expectation $\overline{U}_{IC}$ of the payoff value of the two strategies is:

$\overline{U}_{IC}$ = p U_I + (1 - p) U_C

Based on the previous equations, the replicator equation of financial institutions is:

F(p) = $\frac{dp}{dt}$ = p (U_I - $\overline{U}_{IC}$)

$$= p\Big[\big(q(a_1-f)+(1-q)a_1\big)-\big(p(q(a_1-f)+(1-q)a_1)+(1-p)(qa+(1-q)a)\Big]$$

$$= p\Big[(1-p)\big(q(a_1-f)+(1-q)a_1\big)+(1-p)\big(qa+(1-q)a\big)\Big]$$

$$= p(1-p)\Big[q(a_1-f)+(1-q)a_1+qa+(1-q)a\Big]$$

$$= p(1-p)\Big[q(a_1-f+a)+(1-q)(a_1+a)\Big]$$

$$= p(1-p)\Big[qa_1-fq+aq+a_1+a-a_1q-aq\Big]$$

$$= p(1-p)(a_1-a-fq)$$

The replicated dynamics equation of regulation is calculated in the same way, starting by the expectation U_R of the payoff value with the strategy of regulation, that is:

U_R = p (b_1+f-c) + (1 - p) $(b-c)$

which means that once regulation institutions choose to regulate and financial institutions choose to innovate, the payoff of the regulation institutions are the payoffs when financial institutions choose innovation, b_1, plus the punishment applied to the financial institutions for choosing to innovate when there is regulation , f, minus the cost of regulation, c, while they choose to be conservative the payoff will be the payoff of regulation institutions when financial institutions choose the strategy of conservation, b, minus the cost of regulation, c.

The expectation U_D of the payoff value of regulation institutions with the strategy of deregulation is:

U_D = p b_1 + (1 - p) b

that is easily deduced from the definitions, as when regulation institutions choose deregulation, there are two possibilities, in the case of innovation the payoff would be the payoffs when financial institutions choose innovation, b_1, or in the case financial institutions choose conservation, the payoff is the payoff of regulation institutions when financial institutions choose the strategy of conservation, b.

The average expectation $\overline{U}_{RD}$ of the payoff value of regulation institutions is:

$\overline{U}_{RD}$ = q U_R + (1 - q) U_D

Based on the previous equations, the replicator equation of regulation institutions is:

F(q) = $\frac{dq}{dt}$ = q (U_R - $\overline{U}_{RD}$)

$$= q\Big[\big(p(b_1 + f - c) + (1-p)(b-c)\big) - \big(q(p(b_1 + f - c) + (1-p)(b-c)) + (1-q)(pb_1 + (1-p)b\big)\Big]$$

$$= q\Big[pb_1 + pf - pc + b - c - pb + pc - qpb_1 - qpf + qpc - qb + qc + qpb - qpc - pb_1 - (1-p)b + qpb_1 + (1-p)qb\Big]$$

$$= q\Big[pf - c - qpf + qc\Big]$$

$$= q(1-q)(pf - c)$$

3.3 EVOLUTIONARY STABLE ANALYSES

3.3.1 Evolutionary stable strategy for financial institutions

It has been calculated that F(p) = $p(1-p)(a_1 - a - fq)$, therefore,

Proposition 3. *F(p) = 0, only happens when* $q^* = \frac{(a_1 - a)}{f}$ *,* $p_1{}^* = 0$ *and* $p_2{}^* = 1$*.*

Proof. This can be divided into two situations:

1) If $q^* = \frac{(a_1 - a)}{f}$, F(p) is constantly 0, which means all the points p axis that are between 0 and 1 are stable, that means the proportion of regulation institutions choosing the regulation strategy is $q^* = \frac{(a_1 - a)}{f}$ and this choice is not affected by the proportion of companies choosing the innovation strategy, the state is always stable.

2) In the case $q^* \neq \frac{(a_1 - a)}{f}$, $p_1{}^* = 0$ and $p_2{}^* = 1$ may be in a stable state. The evolutionary stable strategy requires that the stable strategy is robust when shocked by small disturbances. The derivative of F(p) must be smaller than 0 in the stable state, and $\dot{F}(p)$ = (1 - 2 p) (a_1 - a - f q). Thus, for $p_1{}^* = 0$ to be a stable state, q > $\frac{(a_1 - a)}{f}$, which means the financial institutions will choose the conservation strategy to realize the stable equilibrium. Then, $p_2{}^* = 1$ is the evolutionary stable strategy when q < $\frac{(a_1 - a)}{f}$, and innovation is a stable strategy for financial institutions. □

The smaller q^* value will lead to a larger proportion choosing the innovation strategy for financial institutions, and vice versa. Therefore, financial institutions tend to choose the conservation strategy when there is a large punishment and a small difference in payoffs between innovation and conservation. Similarly, the innovation strategy will be chosen with small punishment and a large difference in payoffs between innovation and conservation.

3.3.2 Evolutionary stable strategy for regulation institutions

As it has been calculated, F(q) = q (1 - q) (p $f - c$), therefore,

Proposition 4. *F(q) = 0 means that either* $p^* = \frac{c}{f}$ *,* $q_1{}^* = 0$ *and* $q_2{}^* = 1$*.*

Proof. This can be divided into two situations:

1) If $p^* = \frac{c}{f}$, F(q) is constantly 0, which means all the points q that satisfy 0 $\leq$ q $\leq$ 1, which indicated that the probability of choosing the innovation strategy for financial institutions is $p^* = \frac{c}{f}$, the stable state can be achieved regardless of the strategies chosen by regulation institutions.

2) In the case $p^* \neq \frac{c}{f}$, $q_1{}^* = 0$ and $q_2{}^* = 1$ may be in a stable state. The derivative is $\dot{F}(q)$ = (1 - 2 q) (f $p - c$). If $q_1{}^* = 0$ is the evolutionary stable strategy, the condition p < $\frac{c}{f}$ should be satisfied, and deregulation is the stable strategy for regulation institutions. When p > $\frac{c}{f}$, regulation is a stable strategy for regulation institutions. □

The larger the punishment, f for financial institutions with the innovation strategy will lead to smaller regulation costs c, so p^* has a smaller value, when that happens, the proportion of regulation institutions to choose the regulation strategy will be larger, and vice versa. Once innovation is carried out, regulation institutions will increase the punishment increasing the payoffs of regulation institutions so that the regulation strategy will be chosen.

3.3.3 Evolutionary game equilibrium

Based on the analysis of the replicator equation of both institutions, it is evident the system has five local equilibrium points, (p,q), which are, (0 , 0), (1 , 0), (0 , 1), (1 , 1), ($\frac{c}{f}$, $\frac{(a_1-a)}{f}$).

This conclusion can also be achieved by using the general theory developed on the first part of the article, considering the payoff matrices,

$$F = \begin{bmatrix} a_1 - f & a_1 \\ a & a \end{bmatrix} \qquad R = \begin{bmatrix} b_1 + f - c & b - c \\ b_1 & b \end{bmatrix}$$

as explained before it is possible to add a constant to each column and obtain an equivalent system, [11], therefore obtaining the bimatrix replicator with respective payoff matrices,

$$F = \begin{bmatrix} a_1 - a - f & 0 \\ 0 & a - a_1 \end{bmatrix} \qquad R = \begin{bmatrix} f - c & 0 \\ 0 & c \end{bmatrix}$$

So the payoff matrices are of the form used in the general theory that was presented, $a_1 = a_1 - a - f$, $a_2 = a - a_1$, $b_1 = f - c$, b_2 = c , hence the equilibrium point, that is not on the edges is $\left(\frac{c}{f}, \frac{(a_1-a)}{f}\right)$, as previously calculated.

There are 4 different general cases, which conjugate into 8 different cases, as seen in Figure 3.1, where it is not an interior equilibrium, these will be analysed in the context of the model:

① $\frac{(a_1-a)}{f} > 1$
Since, by definition, a_1 > a, it is only possible if 0 < a_1 - a < f.

② $\frac{(a_1-a)}{f} < 0$
This is equivalent to f < 0, since a_1 - a > 0.

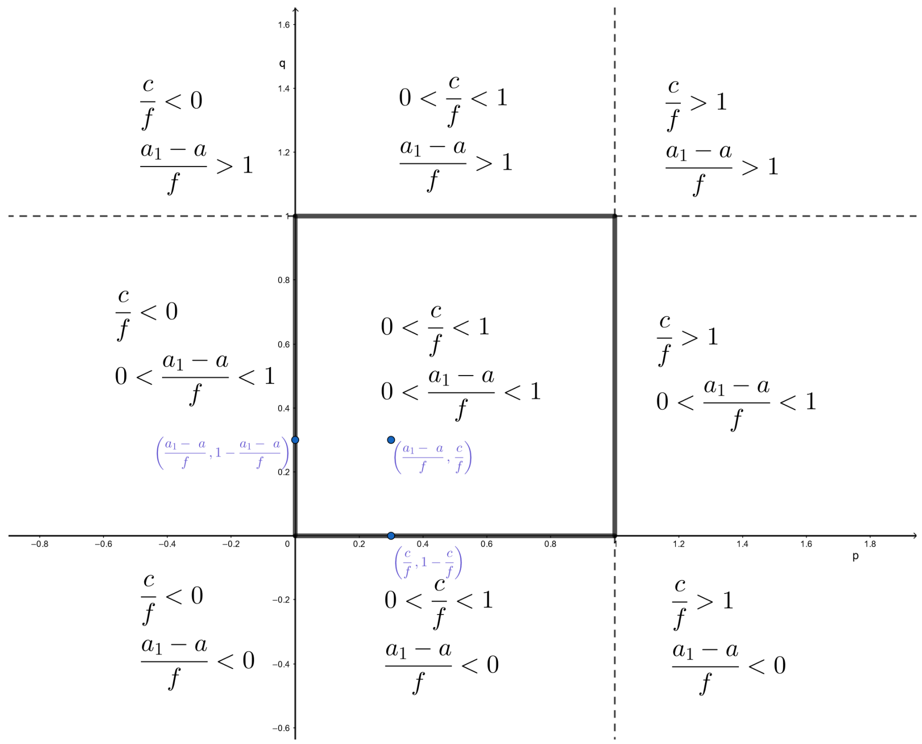

Figure 3.1: Possible positions for the equilibrium and their mathematical meaning

③ $\frac{c}{f} > 1$

There are two possibilities, either c > 0 in which case f > 0 as well, which in reality is not possible as by definition f > c. Or c < 0 in which case f < 0.

④ $\frac{c}{f} < 0$

Since, f > c, the only possible scenario is when c < 0, being valid whether f is greater or smaller than 0.

Proposition 5. *In this specific model there is only one possible case of interior equilibrium where, $a_1 - a - f$, $a - a_1 < 0$ and $f - c$, $c > 0$.*

Proof. In fact there is only an interior equilibrium when $0 \leq \frac{(a_1 - a)}{f} \leq 1$ and $0 \leq \frac{c}{f} \leq 1$, which according to theory presented in the last chapter is equivalent to say that there is only an interior equilibrium if,

$a_1 - a - f$, $a - a_1 > 0$ and $f - c > 0$, $c > 0$

$a_1 - a - f$, $a - a_1 > 0$ and $f - c$, $c < 0$

$a_1 - a - f$, $a - a_1 < 0$ and $f - c$, $c > 0$

$a_1 - a - f$, $a - a_1 < 0$ and $f - c$, $c < 0$.

However since this is a specific model, the restrictions of the model must be taken into account, that is, the fact that by definition $f > c$ and $a_1 > a$, so the only viable cases are the ones that,

$a_1 - a - f$, $a - a_1 < 0$ and $f - c$, $c > 0$. □

However, having in account, that $f > c$ and $a_1 > a$. Further analysis is needed to check whether all positions of the graph are viable, as seen in Figure 3.2,

① $\frac{c}{f} > 1$, $\frac{(a_1-a)}{f} > 1$

As calculated above, $\frac{c}{f} > 1$, means that $c < 0$ and $f < 0$, which makes it impossible for $\frac{(a_1-a)}{f} > 1$, as by definition $a_1 > a$, so the equilibrium point can't be located in this area since it is limited by the definition of the model.

② $\frac{c}{f} > 1$, $0 < \frac{(a_1-a)}{f} < 1$

As stated above, $c < 0$ and $f < 0$ are immediate consequences of the model definition and since $a_1 - a > 0$, it is obvious that in these conditions $\frac{(a_1-a)}{f} < 0$, so this case is also ruled out.

③ $\frac{c}{f} > 1$, $\frac{(a_1-a)}{f} < 0$

Since $c < 0$ and $f < 0$, $\frac{(a_1-a)}{f} < 0$ comes immediately from $a_1 - a > 0$, so it is a possible position for the equilibrium point, and can be represented by $c < 0$ and $f < 0$.

④ $0 < \frac{c}{f} < 1$, $\frac{(a_1-a)}{f} < 0$

As $\frac{(a_1-a)}{f} < 0$, can be deduced that $f < 0$, and since $f > c$, it also means that $c < 0$, however this makes it impossible for $0 < \frac{c}{f} < 1$, as if both c,$f < 0$ and $f > c$ it means that $\frac{c}{f} > 1$, hence this region is also non obtainable.

⑤ $\frac{c}{f} < 0$, $\frac{(a_1-a)}{f} < 0$

It was calculated previously that $\frac{c}{f} < 0$ it is only possible if $c < 0$ and $f > 0$, and since $\frac{(a_1-a)}{f} < 0$, it is impossible as f can't be both smaller, so the equilibrium can't be located in this position.

(6) $\frac{c}{f} < 0$, $0 < \frac{(a_1-a)}{f} < 1$
Since $c < 0$ and $f > 0$, it makes it totally possible for $0 < \frac{(a_1-a)}{f} < 1$, as long as $a_1 - a < f$, so this region is obtainable for the equilibrium point as long as $a_1 - a < f$ and c < 0.

(7) $\frac{c}{f} < 0$, $\frac{(a_1-a)}{f} > 1$
So, $c < 0$, $f > 0$ and with $\frac{(a_1-a)}{f} > 1$, it means that $0 < f < a_1 - a$.

(8) $0 < \frac{c}{f} < 1$, $\frac{(a_1-a)}{f} > 1$
As shown before, $\frac{(a_1-a)}{f} > 1$ is equivalent to, $0 < f < a_1 - a$, so in this case $0 < c < f < a_1 - a$.

Hence, the only possible cases are (3), (6), (7), (8), as shown in the next graph,

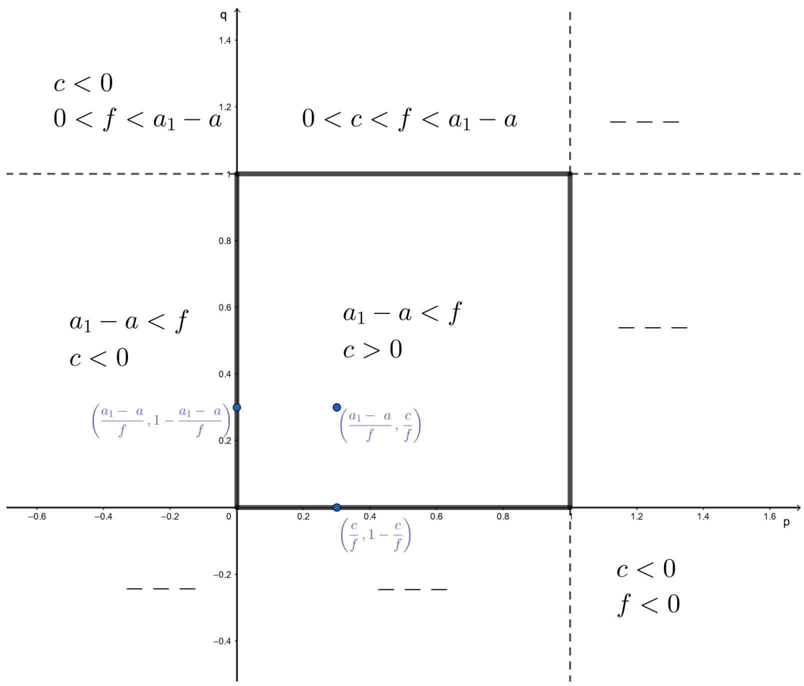

Figure 3.2: Positions for the equilibrium with model under analysis, considering the given assumptions

4. Extended model - Polimatrix Replicator

As the bimatrix game has been deconstructed and thoroughly analysed, an extension of the model will be considered. In fact a bimatrix game is simply a particular case of a polimatrix game, where there are two players and the interactions inside each group are not considered. So following [10], in this chapter we will consider a population divided in two groups, where individuals of each group have two possible strategies for interacting with any other individual in the population, including those from the same group.

Since the only difference is considering payoffs inside a group, there are still only two strategies member of each group can adapt, therefore the payoff matrix can be described as,

$$B = \left[\begin{array}{c|c} B^{1,1} & B^{1,2} \\ \hline B^{2,1} & B^{2,2} \end{array}\right] = \left[\begin{array}{cc|cc} b_{11}^{1,1} & b_{12}^{1,1} & b_{11}^{1,2} & b_{12}^{1,2} \\ b_{21}^{1,1} & b_{22}^{1,1} & b_{21}^{1,2} & b_{22}^{1,2} \\ \hline b_{11}^{2,1} & b_{12}^{2,1} & b_{11}^{2,2} & b_{12}^{2,2} \\ b_{21}^{2,1} & b_{22}^{2,1} & b_{21}^{2,2} & b_{22}^{2,2} \end{array}\right] = \left[\begin{array}{cc|cc} c_{11} & c_{12} & a_{11} & a_{12} \\ c_{21} & c_{22} & a_{21} & a_{22} \\ \hline b_{11} & b_{12} & d_{11} & d_{12} \\ b_{21} & b_{22} & d_{21} & d_{22} \end{array}\right]$$

where each block $B^{\alpha,\beta}$, $\alpha, \beta \in \{1,2\}$, represents the payoff of the individuals of the group α when interacting with individuals of the group β, and each entry of the matrix $B_{ij}^{\alpha,\beta}$ represents the average payoff of an individual of the group α using strategy i when interacting with an individual of the group β using strategy j. In order to have simpler analysis, each block is represented as a different matrix.

According to proposition 2.6. of [4], it is possible to obtain an equivalent game with a payoff matrix whose second row of each group is null, by considering the matrix,

$$C = \left[\begin{array}{cc|cc} c_{21} & c_{22} & a_{21} & a_{22} \\ c_{21} & c_{22} & a_{21} & a_{22} \\ \hline b_{21} & b_{22} & d_{21} & d_{22} \\ b_{21} & b_{22} & d_{21} & d_{22} \end{array}\right]$$

and using it to reach the new payoff matrix, A, that is equivalent to the original,

$$B - C = \left[\begin{array}{cc|cc} c_{11}-c_{21} & c_{12}-c_{22} & a_{11}-a_{21} & a_{12}-a_{22} \\ 0 & 0 & 0 & 0 \\ \hline b_{11}-b_{21} & b_{12}-b_{22} & d_{11}-d_{21} & d_{12}-d_{22} \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{cc|cc} c_1 & c_2 & a_1 & a_2 \\ 0 & 0 & 0 & 0 \\ \hline b_1 & b_2 & d_1 & d_2 \\ 0 & 0 & 0 & 0 \end{array}\right] = A$$

4.1 DEDUCTION OF THE DYNAMIC EQUATIONS

A polimatrix game determines the following system of differential equations,

$$\dot{x_i^\alpha} = x_i^\alpha \left((Ax)_i - \sum_{\beta=1}^{p} (x^\alpha)^T A^{\alpha\beta} x^\beta \right)$$

which is called a polymatrix replicator system, defined on the phase space,

$$\Delta^1 \times \Delta^1 = \{\ (x_1,x_2,x_3,x_4) \in \mathbb{R}^4\text{: } x_i \geq 0\text{, } \textstyle\sum_{i\in\alpha} x_i^\alpha = 1\text{, } \alpha \in 1,2\ \}$$

Considering there are two different groups $X^1 = \{x_1, x_2\}$ and $X^2 = \{x_3, x_4\}$, it considering $x = x_1$ and $y = x_3$, hence,

$$\dot{x_1}^1 = x_1^1 \left((Ax)_1 - \textstyle\sum_{\beta=1}^{2} (x^1)^T A^{1\beta} x^\beta \right)$$

$$= x_1 \left(\left(A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \right)_1 - \left(\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} c_1 & c_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} \right) \right)$$

$$= x_1(c_1x_1+c_2x_2+a_1x_3+a_2x_4) - \left(\begin{bmatrix} x_1c_1 & x_2c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} x_1a_1 & x_2a_2 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} \right)$$

$$= x(c_1x+c_2(1-x)+a_1y+a_2(1-y)) - (x^2c_1 + x(1-x)c_2 + xya_1 + x(1-y)a_2)$$

$$= x(c_1x(1-x) + c_2(1-x)^2 + a_1y(1-x) + a_2(1-y)(1-x))$$

$$= x(1-x)(c_1x + c_2(1-x) + a_1y + a_2(1-y)) = x(1-x)(Ax)_1$$

It is important to note, that the deduction is analogous for $\dot{y}$, with $\alpha = 2$ since it refers to the second group.

The replicator equation are therefore,

$$\begin{cases} \dot{x} = x(1-x)(c_1x + c_2(1-x) + a_1y + a_2(1-y)) \\ \dot{y} = y(1-y)(b_1x + b_2(1-x) + d_1y + d_2(1-y)) \end{cases}$$

The approach to analyse the extended model is identical to the initial model, as it starts by figuring out the equilibria points, thus,

$$\begin{cases} f_1(x,y) = 0 \\ f_2(x,y) = 0 \end{cases} \Leftrightarrow \begin{cases} x(1-x)(c_1x + c_2(1-x) + a_1y + a_2(1-y)) = 0 \\ y(1-y)(b_1x + b_2(1-x) + d_1y + d_2(1-y)) = 0 \end{cases}$$

as in the original case the vertices of the phase space are equilibria, that is,

$$\begin{cases} x = 0 \\ y = 0 \end{cases} \vee \begin{cases} x = 0 \\ y = 1 \end{cases} \vee \begin{cases} x = 1 \\ y = 0 \end{cases} \vee \begin{cases} x = 1 \\ y = 1 \end{cases}$$

however at this point it starts to differentiate itself from the original case, as it contains other equilibria on the border that aren't vertices, see Figure 4.1,

$$\begin{cases} x = 0 \Rightarrow \dot{y} = y(1-y)(b_2 + d_1 + d_2(1-y)) \\ x = 1 \Rightarrow \dot{y} = y(1-y)(b_1 + d_1 + d_2(1-y)) \\ y = 0 \Rightarrow \dot{x} = x(1-x)(c_1x + c_2(1-x) + a_2) \\ y = 1 \Rightarrow \dot{x} = x(1-x)(c_1x + c_2(1-x) + a_1) \end{cases}$$

it is only a equilibrium point if $\dot{x} = 0 \vee \dot{y} = 0$,

$$\begin{cases} y = 0 \Rightarrow y = 0 \vee y = 1 \vee b_2 + d_1y + d_2(1-y) = 0 \\ y = 1 \Rightarrow y = 0 \vee y = 1 \vee b_1 + d_1y + d_2(1-y) = 0 \\ x = 0 \Rightarrow x = 0 \vee x = 1 \vee c_1x + c_2(1-x) + a_2 = 0 \\ x = 1 \Rightarrow x = 0 \vee x = 1 \vee c_1x + c_2(1-x) + a_1 = 0 \end{cases}$$

which in turn is equivalent to,

$$\begin{cases} y = 0 \Rightarrow y = -\dfrac{b_2 + d_2}{d_1 - d_2} = \dfrac{b_2 + d_2}{d_2 - d_1} \\ y = 1 \Rightarrow y = -\dfrac{b_1 + d_2}{d_1 - d_2} = \dfrac{b_1 + d_2}{d_2 - d_1} \\ x = 0 \Rightarrow x = -\dfrac{a_2 + c_2}{c_1 - c_2} = \dfrac{a_2 + c_2}{c_2 - c_1} \\ x = 1 \Rightarrow x = -\dfrac{a_1 + c_2}{c_1 - c_2} = \dfrac{a_1 + c_2}{c_2 - c_1} \end{cases}$$

This equilibrium point is only situated on the edge when $0 < \frac{b_2+d_2}{d_2-d_1} < 1$, which is equivalent to $0 < b_2 + d_2 < d_2 - d_1 \vee d_2 - d_1 < b_2 + d_2 < 0$, similarly obtained for the other equilibria on the edge.

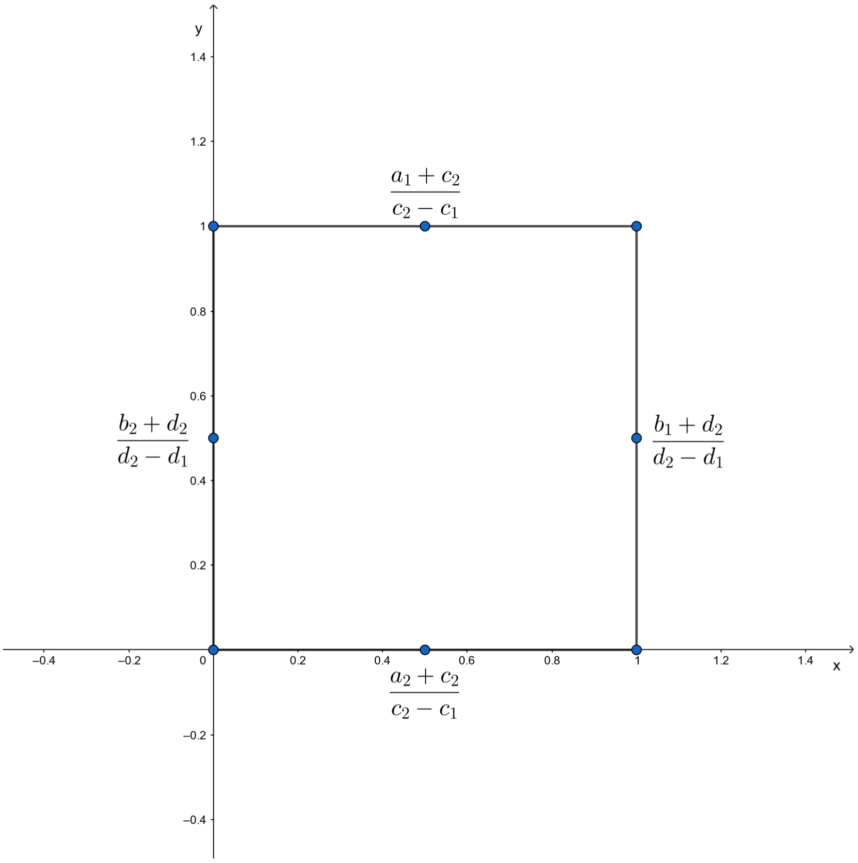

Figure 4.1: Possible positions for equilibria on the edge of the phase space

Logically, the next step is to calculate the equilibria that are not on the edges of the phase space. So, we want to solve the following system,

$$\begin{cases} x(1-x)(x(c_1-c_2)+y(a_1-a_2)+c_2+a_2)=0 \\ y(1-y)(x(b_1-b_2)+y(d_1-d_2)+b_2+d_2)=0 \end{cases}$$

Since the goal is to find the equilibria outside the edges of the phase space, it is equivalent to solve the system,

$$\begin{cases} (c_1-c_2)x+(a_1-a_2)y=-c_2-a_2 \\ (b_1-b_2)x+(d_1-d_2)y=-b_2-d_2 \end{cases} \tag{4.1}$$

Depending on the determinant of the matrix and the rank of the expanded matrix, this system can be divided into 3 possible different types: possible and determined (a unique solution), possible and undetermined (infinite solutions) or impossible (no solution). This means that there is either a single equilibrium point, infinite equilibria, or no equilibrium, respectively. It is in fact impossible to have solely two, or any other finite number of equilibria points, as any linear combination of two equilibria points would be an equilibrium point in itself.

4.2 ANALYSIS OF THE INTERIOR EQUILIBRIA

In order for the system to be possible and determined,

$$\begin{vmatrix} c_1 - c_2 & a_1 - a_2 \\ b_1 - b_2 & d_1 - d_2 \end{vmatrix} \neq 0 \Leftrightarrow (c_1 - c_2)(d_1 - d_2) - (b_1 - b_2)(a_1 - a_2) \neq 0$$

Proposition 6. *The interior equilibrium of system* (4.1) *exists and is unique if one of the following conditions is verified:*

① $\begin{cases} c_1 = c_2 \\ a_1 \neq a_2 \wedge b_1 \neq b_2 \end{cases}$

② $\begin{cases} d_1 = d_2 \\ a_1 \neq a_2 \wedge b_1 \neq b_2 \end{cases}$

③ $\begin{cases} b_1 = b_2 \\ c_1 \neq c_2 \wedge d_1 \neq d_2 \end{cases}$

④ $\begin{cases} a_1 = a_2 \\ c_1 \neq c_2 \wedge d_1 \neq d_2 \end{cases}$

⑤ $\frac{(c_1-c_2)(d_1-d_2)}{(b_1-b_2)(a_1-a_2)} \neq 1$

each different case leads to a different equilibrium point.

Proof. As the determinant of the matrix is different of zero in each case the equilibrium is unique,

① $\begin{cases} c_1 = c_2 \\ a_1 \neq a_2 \wedge b_1 \neq b_2 \end{cases}$, translating into the model as following

$$\begin{cases} (a_1 - a_2)y = -c_2 - a_2 \\ (b_1 - b_2)x + (d_1 - d_2)y = -b_2 - d_2 \end{cases} \Leftrightarrow \begin{cases} y = -\dfrac{c_2 + a_2}{a_1 - a_2} \\ x = \dfrac{-b_2 - d_2}{b_1 - b_2} + \dfrac{(d_1 - d_2)(c_2 + a_2)}{(a_1 - a_2)(b_1 - b_2)} \end{cases}$$

It is similar for the cases ②, ③, ④ so this will be used as a example, hence only ⑤ is left,

⑤ $\frac{(c_1-c_2)(d_1-d_2)}{(b_1-b_2)(a_1-a_2)} \neq 1$

$$\begin{cases} y = \dfrac{c_2 - c_1}{a_1 - a_2}x - \dfrac{c_2 + a_2}{a_1 - a_2} \\ (b_1 - b_2)x + \dfrac{(d_1 - d_2)(c_2 - c_1)}{a_1 - a_2}x - \dfrac{(d_1 - d_2)(c_2 + a_2)}{a_1 - a_2} = -b_2 - d_2 \end{cases}$$

that is equivalent to,

$$\begin{cases} y = \dfrac{c_2 - c_1}{a_1 - a_2}x - \dfrac{c_2 + a_2}{a_1 - a_2} \\ x = \dfrac{(d_1 - d_2)(c_2 + a_2) - (b_2 + d_2)(a_1 - a_2)}{(b_1 - b_2)(a_1 - a_2) - (d_1 - d_2)(a_1 - a_2)} \end{cases}$$

Hence, covering all possible scenarios where the equilibrium point is unique. □

The conditions for a system to be possible undetermined or impossible are very similar, that is why they are analysed together, however there is a slight difference.

In this case the determinant of the matrix of the coefficients has to be null,

$$\begin{vmatrix} c_1 - c_2 & a_1 - a_2 \\ b_1 - b_2 & d_1 - d_2 \end{vmatrix} \neq 0 \Leftrightarrow (c_1 - c_2)(d_1 - d_2) - (b_1 - b_2)(a_1 - a_2) = 0$$

Proposition 7. *If the system* (4.1) *follows one of these conditions,*

① $c_1 = c_2 \wedge b_1 = b_2$

② $c_1 = c_2 \wedge a_1 = a_2$

③ $d_1 = d_2 \wedge b_1 = b_2$

④ $d_1 = d_2 \wedge a_1 = a_2$

⑤ $\frac{(c_1-c_2)(d_1-d_2)}{(b_1-b_2)(a_1-a_2)} = 1$

the system can be either impossible or possible undetermined, in which case there is a continuum of equilibria, each case leading to different equilibria.

Proof. What differentiates a matrix being impossible or possible undetermined is a comparison between the rank of the matrix of the coefficients and the extended matrix, each will be analysed in order for to define the different cases,

① $c_1 = c_2 \wedge b_1 = b_2$

$$\left[\begin{array}{cc|c} 0 & a_1 - a_2 & -c_2 - a_2 \\ 0 & d_1 - d_2 & -b_2 - d_2 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 0 & a_1 - a_2 & -c_2 - a_2 \\ 0 & 0 & (-b_2 - d_2) - \frac{(-c_2 - a_2)(d_1 - d_2)}{a_1 - a_2} \end{array}\right]$$

Therefore the system is impossible for all values, except $(-b_2 - d_2) = \frac{(-c_2-a_2)(d_1-d_2)}{a_1-a_2}$ in which case it is possible undetermined, since if $(-b_2 - d_2) \neq \frac{(-c_2-a_2)(d_1-d_2)}{a_1-a_2}$, the rank of the coefficient matrix, r(A), is smaller than the rank of the expanded matrix, r[A|B]. Hence only the situation where the system is possible undetermined is of interest.

So the equilibria points when $c_1 = c_2 \wedge b_1 = b_2$ are,

$$\begin{cases} (a_1 - a_2)y = -c_2 - a_2 \\ (d_1 - d_2)y = -b_2 - d_2 \end{cases} \Leftrightarrow \begin{cases} y = -\dfrac{c_2 + a_2}{a_1 - a_2} \\ y = -\dfrac{b_2 + d_2}{d_1 - d_2} \end{cases} \Leftrightarrow y = -\frac{c_2 + a_2}{a_1 - a_2} = -\frac{b_2 + d_2}{d_1 - d_2}$$

which lines up with the initial condition for the system to be possible undetermined,

$$(-b_2 - d_2) = \frac{(-c_2 - a_2)(d_1 - d_2)}{a_1 - a_2} \Leftrightarrow \frac{-b_2 - d_2}{d_1 - d_2} = \frac{-c_2 - a_2}{a_1 - a_2}$$

so in this case the interior equilibria points are in fact a continuum of points represented by either $y = -\frac{c_2+a_2}{a_1-a_2} = -\frac{b_2+d_2}{d_1-d_2}$ and any value of x.

The analysis of (4) $d_1 = d_2 \wedge a_1 = a_2$ is analogous, so it will be omitted.

(2) $c_1 = c_2 \wedge a_1 = a_2$

$$\left[\begin{array}{cc|c} 0 & 0 & -c_2 - a_2 \\ b_1 - b_2 & d_1 - d_2 & -b_2 - d_2 \end{array}\right]$$

In this case the system is impossible unless $-c_2 = a_2$, where r[A] = r[A|B], hence it is possible undetermined, in this case the continuum of equilibria is the following,

$$(b_1 - b_2)x + (d_1 - d_2)y = -b_2 - d_2 \Leftrightarrow y = \frac{b_2 - b_1}{d_1 - d_2}x - \frac{b_2 + d_2}{d_1 - d_2}$$

This case is identical to (3), hence it will be the one taken into account.

(5) $\frac{(c_1-c_2)(d_1-d_2)}{(b_1-b_2)(a_1-a_2)} = 1 \Leftrightarrow d_1 - d_2 = \frac{(b_1-b_2)(a_1-a_2)}{c_1-c_2} \Leftrightarrow d_1 - d_2 - \frac{(b_1-b_2)(a_1-a_2)}{c_1-c_2} = 0$

$$\left[\begin{array}{cc|c} c_1-c_2 & a_1-a_2 & -c_2-a_2 \\ b_1-b_2 & d_1-d_2 & -b_2-d_2 \end{array}\right]$$

that is equivalent to,

$$\left[\begin{array}{cc|c} c_1-c_2 & a_1-a_2 & -c_2-a_2 \\ 0 & d_1-d_2-\frac{(b_1-b_2)(a_1-a_2)}{c_1-c_2} & (-b_2-d_2)-\frac{(-c_2-a_2)(b_1-b_2)}{c_1-c_2} \end{array}\right]$$

having in account the initial restriction,

$$\left[\begin{array}{cc|c} c_1-c_2 & a_1-a_2 & -c_2-a_2 \\ 0 & 0 & (-b_2-d_2)-\frac{(-c_2-a_2)(b_1-b_2)}{c_1-c_2} \end{array}\right]$$

This system is impossible unless $-b_2-d_2=\frac{(-c_2-a_2)(b_1-b_2)}{c_1-c_2}$. In this case it is possible undetermined, so the continuum of equilibria, is given by,

$$y=\frac{c_2-c_1}{a_1-a_2}x-\frac{c_2+a_2}{a_1-a_2}.$$

Like this, all possible equilibria of the model have been represented, and what is left is the analysis of the phase portraits in each case. □

4.3 PHASE PORTRAIT ANALYSIS

Each equilibrium leads to different possible phase portraits, so a thorough analysis is due. This analysis, that follows the same procedure applied to bimatrix game equilibria. As it must be done individually, starting by the single equilibria.

4.3.1 Single equilibria

When it comes to the single equilibria, there are two cases that must be analysed,

$$\begin{cases} c_1=c_2 \\ a_1\neq a_2\wedge b_1\neq b_2 \end{cases} \quad \text{and} \quad \frac{(c_1-c_2)(d_1-d_2)}{(b_1-b_2)(a_1-a_2)}\neq 1$$

as the first case is a good representation of the other possible single equilibria.

As seen before, when it comes to bimatrix games this was the only possible type of equilibrium point, it lead to solely two types of equilibria, either it was a center or an hyperbolic equilbria. The analysis of the polimatrix replicator single equilibria lead to various types of phase portraits, as expected a center and hyperbolic but also spiral source, spiral sink and even real source or sink. Examples of each will be presented and analysed.

As the situation is more complex than the bimatrix game, some computation was done in order to obtain each type of phase portrait, by controlling the values of the payoff matrix, it is possible to make the equilibrium point be in the interior of the phase and to control the determinant and trace of the Jacobian matrix on that point.

So to start, the case where,

$$\begin{cases} c_1 = c_2 \\ a_1 \neq a_2 \wedge b_1 \neq b_2 \end{cases}$$

the equilibrium point is represented by,

$$\begin{cases} x^* = \dfrac{-b_2 - d_2}{b_1 - b_2} + \dfrac{(d_1 - d_2)(c_2 + a_2)}{(a_1 - a_2)(b_1 - b_2)} \\ y^* = -\dfrac{c_2 + a_2}{a_1 - a_2} \end{cases}$$

hence the Jacobian matrix is calculated at this point,

$$J_{(x^*,y^*)} = \begin{bmatrix} 0 & f_1 \\ f_2 & f_3 \end{bmatrix}$$

where

$$f_1 := -\frac{(-a_2 (b_1 + d_1) + a_1 (b_1 + d_2) + c_2 (-d_1 + d_2)) (-a_2 (b_2 + d_1) + a_1 (b_2 + d_2) + c_2 (-d_1 + d_2))}{(a_1 - a_2) (b_1 - b_2)^2}$$

$$f_2 := \frac{-(b_1 - b_2) (a_1 + c_2) (a_2 + c_2)}{(a_1 - a_2)^2}$$

$$f_3 := \frac{-(a_1 + c_2) (a_2 + c_2) (d_1 - d_2)}{(a_1 - a_2)^2}$$

allowing for the calculation of the determinant and the trace of the matrix which are as follows,

$$\text{Det}(J_{(x^*,y^*)}) = -\frac{(a_1+c_2)(a_2+c_2)(-a_2(b_1+d_1)+a_1(b_1+d_2)+c_2(-d_1+d_2))(-a_2(b_2+d_1)+a_1(b_2+d_2)+c_2(-d_1+d_2))}{(a_1-a_2)^3(b_1-b_2)}$$

$$\text{Tr}(J_{(x^*,y^*)}) = -\frac{(a_1+c_2)(a_2+c_2)(d_1-d_2)}{(a_1-a_2)^2}$$

Therefore, all that is left is identifying the several different types of phase portraits. In the following images, the red dot represents the interior equilibrium while the blue dot is the initial condition and the line how it evolves through time.

The center equilibrium arises when a system possesses exclusively two eigenvalues positioned along the imaginary axis, specifically, a single pair of purely imaginary eigenvalues. In linear systems, centers exhibit families of concentric periodic orbits encircling them. The following example illustrates a case of a polymatrix replicator where the interior equilibrium is a center.

Example 4.3.1. *If we consider the parameters values,*

$$a_1 = -1,\ a_2 = 0,\ c_1 = c_2 = \frac{1}{2},\ b_1 = 1,\ b_2 = -1,\ d_1 = 0,\ d_2 = 0$$

the corresponding polymatrix replicator has a unique interior equilibrium that is a center, as seen in Figure 4.2.

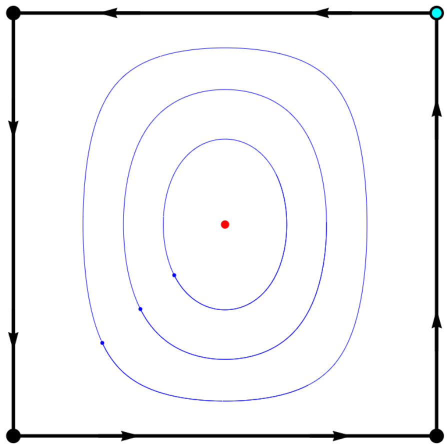

Figure 4.2: Phase portrait of the example 4.3.1

There are two other possible solutions, the interior equilibrium can be a spiral sink, whose orbits in a neighborhood approach the equilibrium point in a spiral motion, as we can see in the following example.

Example 4.3.2. *If we consider the parameter values,*

$$a_1 = -98,\ a_2 = 58,\ c_1 = c_2 = 3,\ \ b_1 = -\frac{33}{10},\ b_2 = -7,\ d_1 = -\frac{191}{28},\ d_2 = -\frac{2398}{345}$$

the corresponding polimatrix replicator has a unique interior equilibrium that is a spiral sink, as can be seen in Figure 4.3.

Or it can be a spiral source, whose orbits in a neighborhood drfit away from the equilibrium point in a spiral motion.

Example 4.3.3. *If we consider the parameter values,*

$$a_1 = 82,\ a_2 = -41,\ c_1 = c_2 = -\frac{89}{10},\ b_1 = -\frac{16}{5},\ b_2 = 5,\ d_1 = -\frac{11}{13},\ d_2 = -\frac{97}{15}$$

the corresponding polimatrix replicator has a unique interior equilibrium that is a spiral source, as seen in Figure 4.4.

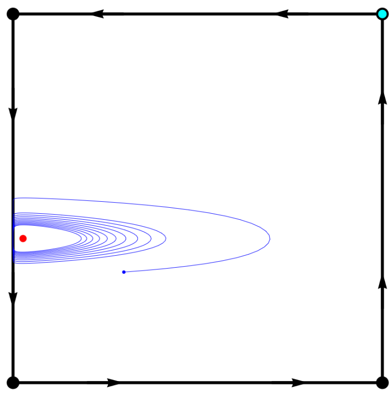

Figure 4.3: Phase portrait of example 4.3.2

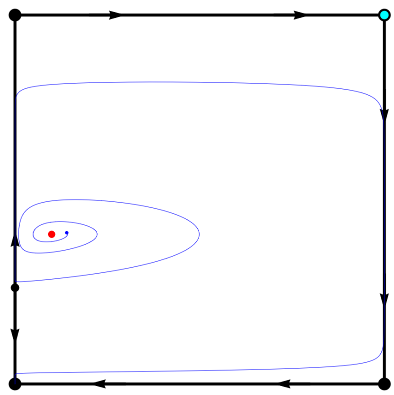

Figure 4.4: Phase portrait of example 4.3.3

The only thing these three different types of solutions have in common is the fact that $T^2 - 4D < 0$, that is the eigenvalues are complex with non-zero imaginary part. The only thing that varies between them is the sign of the trace of the Jacobian matrix, which allows for simple ways to find an instance for each case.

If $T^2 - 4D > 0$, there are three different types of equilibria, where both eigenvalues are positive.

The hyperbolic type of equilibrium point is present on the bimatrix games as one of the two possible situations.

Example 4.3.4. *If we consider the parameter values,*

$$a_1 = -37,\ a_2 = 68,\ c_1 = c_2 = \frac{3}{5},\ b_1 = \frac{8}{5},\ b_2 = 51,\ d_1 = -\frac{55}{3},\ d_2 = -\frac{55}{3}$$

the corresponding polymatrix replicator has a unique interior equilibrium that is hyperbolic, as we can see in Figure 4.5.

By keeping both the eigenvalue positive there are two other possible types of equilibria: Real sink, and Real source.

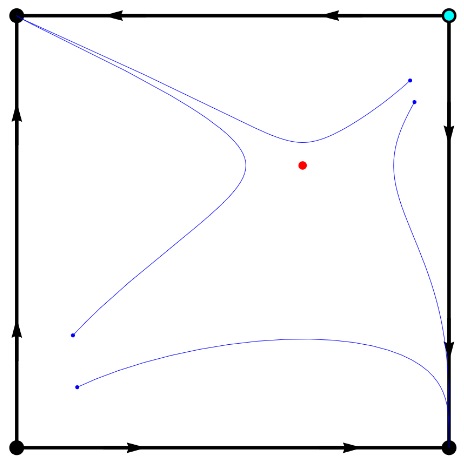

Figure 4.5: Phase portrait of example 4.3.4

Example 4.3.5. *If we consider the parameter values,*

$$a_1 = -37,\ a_2 = 52,\ c_1 = c_2 = \frac{3}{5},\ b_1 = \frac{8}{5},\ b_2 = -\frac{38391}{1315},\ d_1 = -\frac{42599}{1315},\ d_2 = \frac{397}{5}$$

the corresponding polymatrix replicator has a unique interior equilibrium that is a Real sink, as seen in Figure 4.6.

These are exactly opposite cases where one can be obtained from the other by doing the symmetric payoff matrix.

Example 4.3.6. *If we consider the parameter values,*

$$a_1 = 37,\ a_2 = -52,\ c_1 = c_2 = -\frac{3}{5},\ b_1 = -\frac{8}{5},\ b_2 = \frac{38391}{1315},\ d_1 = \frac{42599}{1315},\ d_2 = -\frac{397}{5}$$

the corresponding polymatrix replicator has a unique interior equilibrium that is a Real source, as seen in Figure 4.7.

The three other similar cases are analogous,

$$\textcircled{2} \begin{cases} d_1 = d_2 \\ a_1 \neq a_2 \wedge b_1 \neq b_2 \end{cases},\ \textcircled{3} \begin{cases} b_1 = b_2 \\ c_1 \neq c_2 \wedge d_1 \neq d_2 \end{cases},\ \textcircled{4} \begin{cases} a_1 = a_2 \\ c_1 \neq c_2 \wedge d_1 \neq d_2 \end{cases}$$

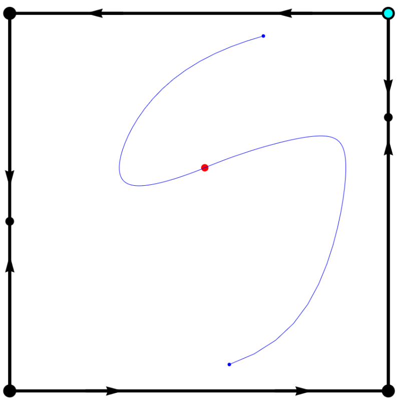

Figure 4.6: Phase portrait of example 4.3.5

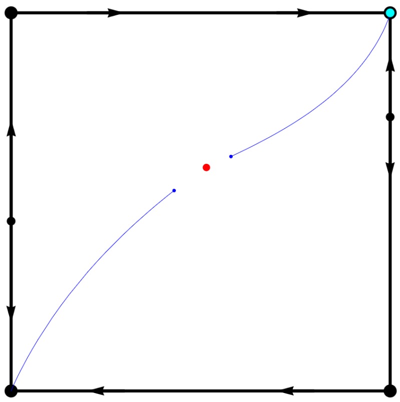

Figure 4.7: Phase portrait of example 4.3.6

So all that is left is to check for the last possible case of a single interior equilibrium which is when $\frac{(c_1-c_2)(d_1-d_2)}{(b_1-b_2)(a_1-a_2)} \neq 1$, this is the most general case.

In this case the equilibrium point is represented by,

$$\begin{cases} x^* = \dfrac{((a_2+c_2)(d_1-d_2)-(a_1-a_2)(b_2+d_2))}{((a_1-a_2)(b_1-b_2)+(-c_1+c_2)(d_1-d_2))} \\ y^* = -\dfrac{a_2+c_2}{a_1-a_2} + \dfrac{(-c_1+c_2)((a_2+c_2)(d_1-d_2)-(a_1-a_2)(b_2+d_2))}{(a_1-a_2)((a_1-a_2)(b_1-b_2)+(-c_1+c_2)(d_1-d_2))} \end{cases}$$

which leads to a very complex Jacobian matrix and respective determinant and trace, and computations using T^2-4D become rather heavy so they are avoided.

One way to circumvent this was to find a center without using T^2-4D, was to simply use the determinant and trace, since the latter has to necessarily be null in order to be a center, and just make the determinant be positive. It is important to have in account that since the program used to build the phase portaits is numeric, errors will increase exponentially as the initial condition is farther away from the equilibrium point, however it is simply an approximation error.

Example 4.3.7. *If we consider the parameter values,*

$$a_1 = \frac{1}{2},\ a_2 = 0,\ c_1 = \frac{1}{4},\ c_2 = -\frac{3}{8},\ b_1 = -1,\ b_2 = 0,\ d_1 = 0,\ d_2 = \frac{2}{5}$$

the corresponding polymatrix replicator has a unique interior equilibrium that is a center, as seen in Figure 4.8.

An interesting peculiarity regarding this system, allows to avoid all the complex calculations since when the same payoff values are kept but b_1 is altered, there is a bifurcation, to types of solutions that were not possible in the bimatrix game, they appear when, as happens with the center, $T^2-4D<0$, such that if,

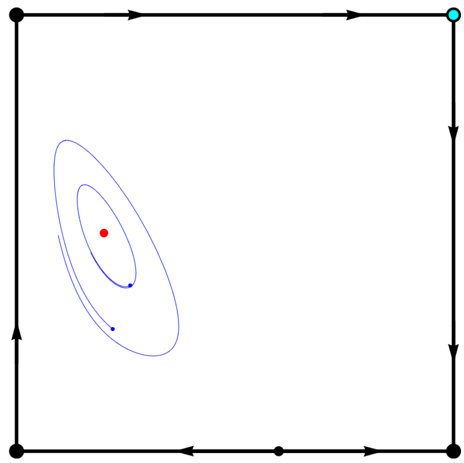

Figure 4.8: Phase portrait of example 4.3.7

b_1 is decreased, then the solution goes from a center to a spiral sink.

Example 4.3.8. *If we consider the parameter values,*

$$a_1 = \frac{1}{2},\ a_2 = 0,\ c_1 = \frac{1}{4},\ c_2 = -\frac{3}{8},\ b_1 = -1.4,\ b_2 = 0,\ d_1 = 0,\ d_2 = \frac{2}{5}$$

the corresponding polymatrix replicator has a unique interior equilibrium that is a spiral sink, as seen in Figure 4.9.

On the other way, b_1 is increased, then the solution goes from a center to a spiral source, as seen in the following example.

Example 4.3.9. *If we consider the parameter values,*

$$a_1 = \frac{1}{2},\ a_2 = 0,\ c_1 = \frac{1}{4},\ c_2 = -\frac{3}{8},\ b_1 = -0.9,\ b_2 = 0,\ d_1 = 0,\ d_2 = \frac{2}{5}$$

the corresponding polymatrix replicator has a unique interior equilibrium that is a spiral source, as seen in Figure 4.10.

the difference between these three is the value of the trace, so by keeping the original payoff values, and decreasing b_1 the trace of the Jacobian matrix will turn negative. Hence allowing to find other types of phase portaits that were not present on bimatrix games.

Similarly, it is possible to find an example for a hyperbolic phase portrait without complex situations as it was done to find the center, simply by forcing the determinant to be negative instead of positive.

Simply by forcing the determinant to be negative instead of positive, an hyperbolic type of solution appears

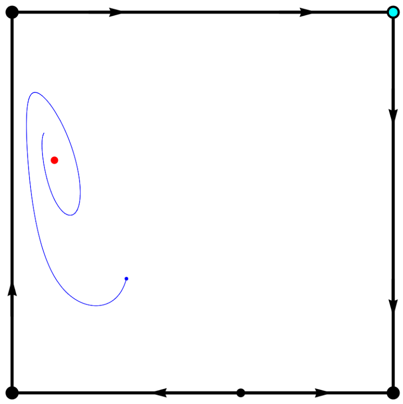

Figure 4.9: Phase portrait of example 4.3.8

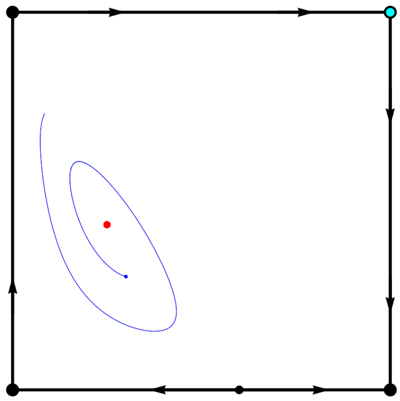

Figure 4.10: Phase portrait of example 4.3.9

Example 4.3.10. *If we consider the parameter values,*

$$a_1 = 0,\ a_2 = \frac{7}{8},\ c_1 = 0,\ c_2 = -1,\ b_1 = -1,\ b_2 = 0,\ d_1 = 0,\ d_2 = 1$$

the corresponding polymatrix replicator has a unique interior equilibrium that is hyperbolic, as seen in Figure 4.11.

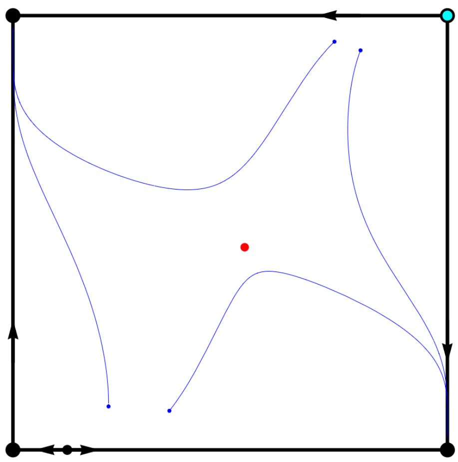

Figure 4.11: Phase portrait of example 4.3.10

By keeping the eigenvalues of the matrix positive, it is possible to find a real sink.

Example 4.3.11. *If we consider the parameter value,*

$$a_1 = -2,\ a_2 = -\frac{1}{4},\ c_1 = -\frac{1}{4},\ c_2 = \frac{3}{4},\ b_1 = \frac{1}{8},\ b_2 = \frac{1}{4},\ d_1 = -1,\ d_2 = 0$$

the corresponding polymatrix replicator has a unique interior equilibrium that is a real sink, as seen in Figure 4.12.

A real source, is obtained by doing the symmetric payoff matrix.

Example 4.3.12. *If we consider the parameter value,*

$$a_1 = 2,\ a_2 = \frac{1}{4},\ c_1 = \frac{1}{4},\ c_2 = -\frac{3}{4},\ b_1 = -\frac{1}{8},\ b_2 = -\frac{1}{4},\ d_1 = 1,\ d_2 = 0$$

the corresponding polymatrix replicator has a unique interior equilibrium that is a real source, as seen in Figure 4.13.

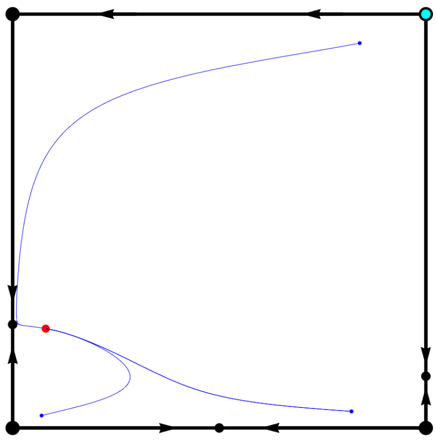

Figure 4.12: Phase portrait of example 4.3.11

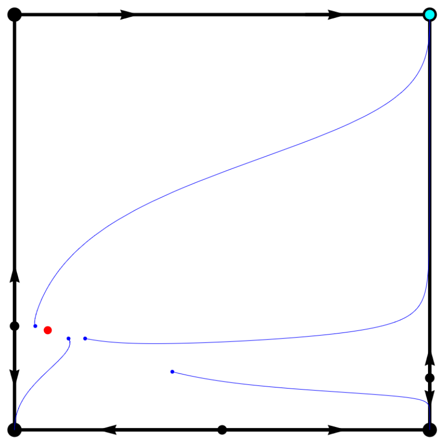

Figure 4.13: Phase portrait of example 4.3.12

4.3.2 Limit cycle

However these are not all the possible phase portraits for a single equilibrium, in fact there is yet another, rather interesting, possible phase portrait. That is, a limit cycle, which is a closed trajectory, hence by the Jordan curve theorem divides the phase plane in two different regions, the interior and the exterior, where one will be a spiral source, approaching the limit cycle as time moves forward and on the outside a spiral sink approaching the limit cycle as time moves forward. This statement holds true for a spiral sink on the interior and a spiral source on the exterior of the limit cycle. In fact it is possible to prove the existence of this without drawing the phase space.

Example 4.3.13. *Consider the following payoff matrix,*

$$\left[\begin{array}{cc|cc} \frac{5}{2} & 0 & 29 & -32 \\ 0 & 0 & 0 & 0 \\ \hline -10 & 10 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the corresponding polimatrix replicator is,

$$\begin{cases} \dot{x} = x(1-x)\left(\frac{5}{2}x + 29y - 32(1-y)\right) \\ \dot{y} = y(1-y)(-10x + 10(1-x) - y + (1-y)) \end{cases}$$

The Jacobian matrix is again calculated using the partial derivatives,

$$J = \begin{bmatrix} g_1(x,y) & g_2(x,y) \\ g_3(x,y) & g_4(x,y) \end{bmatrix}$$

where,

$$\begin{aligned} g_1(x,y) &:= -32 - \frac{15x^2}{2} + x(69 - 122y) + 61y \\ g_2(x,y) &:= -61(-1+x)x \\ g_3(x,y) &:= 20(-1+y)y \\ g_4(x,y) &:= 11 - 26y + 6y^2 + 20x(-1+2y) \end{aligned}$$

Having this in account and by calculating the eigenvalues of this matrix at each of the four vertices of the phase space, the behaviour inside of the phase space can be determined, hence,

$$J_{(0,0)} = \begin{bmatrix} -32 & 0 \\ 0 & 11 \end{bmatrix}$$

so the corresponding eigenvalues are {-32 , 11}, which means that on (0,0) the eigenvalue on the x axis is -32 and it is 11 on the y axis, meaning it is contracting on the x axis and expanding on the y axis.

All the other vertices eigenvalues are calculated in an analogous way, being,

$$\{29, -9\} \text{ at (0,1) , } \{-\frac{63}{2}, 11\} \text{ at (1,1) and } \{\frac{59}{2}, -9\} \text{ at (1,0) ,}$$

it is observable that each vertex has both a contracting and expanding eigenvalue.

Let C_i be the contracting eigenvalue on the vertex v_i with $i \in \{1,2,3,4\}$ and E_i be the expanding eigenvalue on the vertex v_i, as illustrated on the graph 4.14.

By [15], it is possible to deduce the stability of the heteroclinic cycle, in the following way:

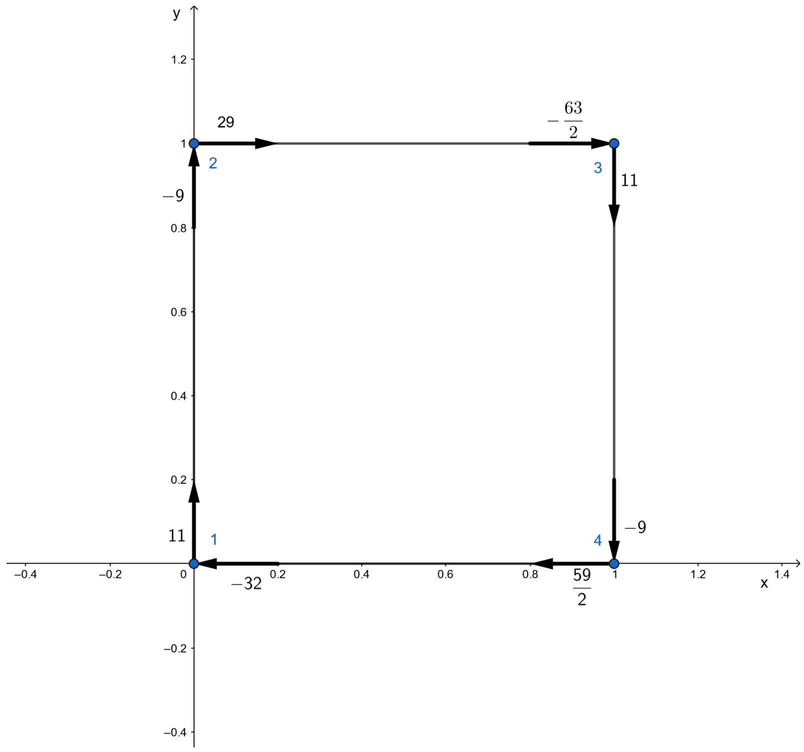

Figure 4.14: Eigenvalues at the vertices and their corresponding labels

Proposition 8. *If* $\prod_{i=1}^{4} C_i < \prod_{i=1}^{4} E_i$ *, then the heteroclinic cycle is unstable;*

On the other hand, if $\prod_{i=1}^{4} C_i > \prod_{i=1}^{4} E_i$ *, then the heteroclinic cycle is stable.*

So, considering example 4.3.13, since

$$C_1C_2C_3C_4 = 81648 \leq 103515,5 = E_1E_2E_3E_4$$

we can deduce by Proposition 8 that the heteroclinic cycle (that corresponds to the boundary of the phase space) is unstable.

Now to analyse the interior equilibrium point, having in account that the general case equilibrium point is,

$$\begin{cases} x = \dfrac{((a_2+c_2)(d_1-d_2)-(a_1-a_2)(b_2+d_2))}{((a_1-a_2)(b_1-b_2)+(-c_1+c_2)(d_1-d_2))} \\ y = -\dfrac{a_2+c_2}{a_1-a_2} + \dfrac{(-c_1+c_2)((a_2+c_2)(d_1-d_2)-(a_1-a_2)(b_2+d_2))}{(a_1-a_2)((a_1-a_2)(b_1-b_2)+(-c_1+c_2)(d_1-d_2))} \end{cases}$$

In the present example the interior equilibrium point is,

$$\begin{cases} x = \dfrac{607}{1215} \\ y = \dfrac{245}{486} \end{cases}$$

Hence the eigenvalues of $J_{\left(\frac{607}{1215}, \frac{245}{486}\right)}$ are

$$\lambda_{\pm} = \frac{73831 \pm i\sqrt{105898378970639}}{1180980}.$$

Since that they have a non zero imaginary part and the real part is positive the interior equilibrium is a spiral source, see Figure 4.15.

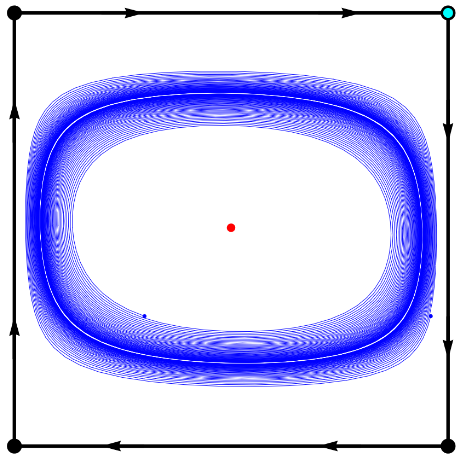

Figure 4.15: Phase portrait of example 4.3.13

In conclusion, having in account that there is a single equilibrium point inside the phase space and that both the heteroclinic orbit and that equilibrium point are sources, there must exist something in between where the orbits accumulate. As per 2.1.1, there can only be a periodic orbit, which is verified numerically, see Figure 4.15.

4.3.3 Multiple equilibria

As calculated before there are three different cases that ought to be analysed that lead to a continuum of equilibria,

$$\begin{cases} c_1 = c_2 \wedge b_1 = b_2 \\ \dfrac{-b_2 - d_2}{d_1 - d_2} = \dfrac{-c_2 - a_2}{a_1 - a_2} \end{cases}, \begin{cases} c_1 = c_2 \wedge a_1 = a_2 \\ -c_2 = a_2 \end{cases} \text{and} \begin{cases} \dfrac{(c_1 - c_2)(d_1 - d_2)}{(b_1 - b_2)(a_1 - a_2)} = 1 \\ -b_2 - d_2 = \dfrac{(-c_2 - a_2)(b_1 - b_2)}{c_1 - c_2} \end{cases}$$

Eventhough such situations did not exist on bimatrix games, the analysis is done in a similar way.

Starting by,

$$\begin{cases} c_1 = c_2 \wedge b_1 = b_2 \\ \dfrac{-b_2 - d_2}{d_1 - d_2} = \dfrac{-c_2 - a_2}{a_1 - a_2} \end{cases}$$

the continuum of equilibria is represented by $y^* = -\frac{c_2+a_2}{a_1-a_2} = -\frac{b_2+d_2}{d_1-d_2}$ and any value of x.

So, the Jacobian matrix can be calculated at any point of this line, such as $\left(\frac{1}{2}, -\frac{c_2+a_2}{a_1-a_2}\right)$, that is,

$$J_{\left(\frac{1}{2},y^*\right)} = \begin{bmatrix} 0 & h_1 \\ 0 & h_2 \end{bmatrix}$$

$$h_1 := \frac{a_1 - a_2}{4}$$
$$h_2 := \frac{-a_2^2(b_2 + d_1) - 2a_2c_2(b_2 + 2d_1 - d_2) + a_1^2(b_2 + d_2) + 3c_2^2(-d_1 + d_2) + 2a_1(b_2c_2 - a_2d_1 - c_2d_1 + a_2d_2 + 2c_2d_2)}{(a_1 - a_2)^2}$$

which makes the calculation of both the trace and the determinant straigthforward,

$$\text{Det}\left(J_{\left(\frac{1}{2},y^*\right)}\right) = 0$$
$$\text{Tr}\left(J_{\left(\frac{1}{2},y^*\right)}\right) = \frac{-a_2^2(b_2+d_1)-2a_2c_2(b_2+2d_1-d_2)+a_1^2(b_2+d_2)+3c_2^2(-d_1+d_2)+2a_1(b_2c_2-a_2d_1-c_2d_1+a_2d_2+2c_2d_2)}{(a_1-a_2)^2}$$

therefore it is easier to work with the eigenvalues of the matrix at this point, as one is null, since the determinant is zero, hence, the eigenvalues of the Jacobian matrix are

$$\left\{0, -\frac{-a_1^2b_2+a_2^2b_2-2a_1b_2c_2+2a_2b_2c_2+2a_1a_2d_1+a_2^2d_1+2a_1c_2d_1+4a_2c_2d_1+3c_2^2d_1-a_1^2d_2-2a_1a_2d_2-4a_1c_2d_2-2a_2c_2d_2-3c_2^2d_2}{(a_1-a_2)^2}\right\}$$

The second eigenvalue can either be negative, in which case it means that the solution will approach the continuum line of equilibria.

Example 4.3.14. *If we consider the parameter values,*

$$a_1 = 37,\ a_2 = -68,\ c_1 = c_2 = -\frac{3}{5},\ b_1 = b_2 = -\frac{8}{5},\ d_1 = -48,\ d_2 = \frac{1236}{13}$$

the corresponding polymatrix replicator has a continuum of interior equilibria that is approached by the solution, as seen in Figure 4.16.

Or it can be positive,in which case it means that it will drift away from the continuum of equilibria and approach the edge of the phase space.

Example 4.3.15. *If we consider the parameter values,*

$$a_1 = -37,\ a_2 = 68,\ c_1 = c_2 = \frac{3}{5},\ b_1 = b_2 = \frac{8}{5},\ d_1 = 48,\ d_2 = -\frac{1236}{13}$$

the corresponding polymatrix replicator has a continuum of interior equilibria such the solution drifts away from, as seen in Figure 4.17.

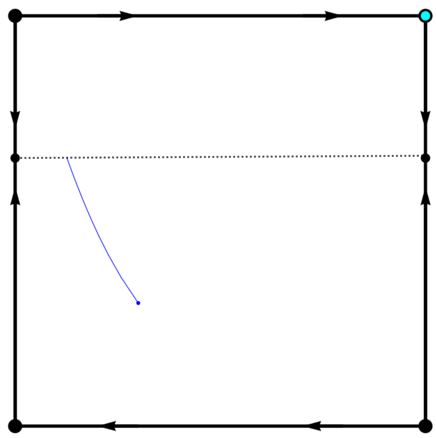

Figure 4.16: Phase portrait of example 4.3.14

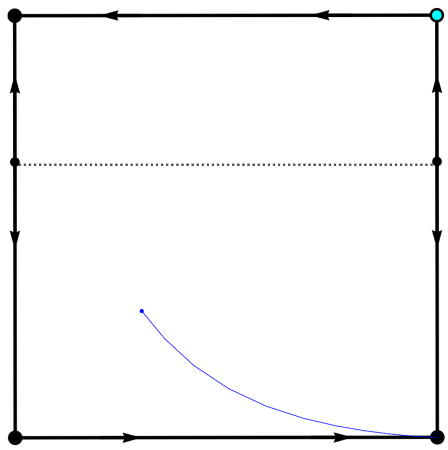

Figure 4.17: Phase portrait of example 4.3.15

This case is analogous to (4) $d_1 = d_2 \wedge a_1 = a_2$.

The continuum of equilibria is represented by the dotted line.

It is important to note that if $\frac{-b_2-d_2}{d_1-d_2} \neq \frac{-c_2-a_2}{a_1-a_2}$ instead of a continuum of equilibria, there will be no equilibria, hence the dynamic of the phase space will be totally determined by the border of the phase space.

On to the second case of multiple equilibria,

$$\begin{cases} c_1 = c_2 \wedge a_1 = a_2 \\ -c_2 = a_2 \end{cases}$$

This particular case is interesting since, given that the replicator equation are,

$$\begin{cases} \dot{x} = x(1-x)(c_1x + c_2(1-x) + a_1y + a_2(1-y)) \\ \dot{y} = y(1-y)(b_1x + b_2(1-x) + d_1y + d_2(1-y)) \end{cases}$$

having in account the restrictions means that the equations are in fact,

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = y(1-y)(b_1x + b_2(1-x) + d_1y + d_2(1-y)) \end{cases}$$

this means that the $x = 0 \wedge x = 1$ axis on the edge of the phase are a continuumn of equilibria and any orbit on the interior will be vertical.

The interior equilibria points can be represented by $y = \frac{b_2-b_1}{d_1-d_2}x - \frac{b_2+d_2}{d_1-d_2}$, so the Jacobian matrix can be calculated at any point described like this, such as $\left(\frac{1}{2}, \frac{-(b_1+b_2)}{2(d_1-d_2)} - \frac{b_2+d_2}{d_1-d_2}\right)$

$$J_{\left(\frac{1}{2}, \frac{-(b_1+b_2)}{2(d_1-d_2)} - \frac{b_2+d_2}{d_1-d_2}\right)} = \begin{bmatrix} 0 & 0 \\ j_1 & j_2 \end{bmatrix}$$

where,

$$j_1 := -\frac{(b_1-b_2)(b_1+b_2+2d_1)(b_1+b_2+2d_2)}{4(d_1-d_2)^2}$$
$$j_2 := -\frac{(b_1+b_2+2d_1)(b_1+b_2+2d_2)}{4(d_1-d_2)}$$

as the previous case, it is better to use the eigenvalues of this matrix in order to reach the different examples, that is $\left\{0, \frac{(b_1+b_2+2d_1)(-d_1+d_2)(b_1+b_2+2d_2)}{4(d_1-d_2)^2}\right\}$

The non null eigenvalue can be negative, and the solution will approach the continuumn of equilibria or be positive, and it will approach the border of the phase space.

Example 4.3.16. *If we consider the parameter values,*

$$a_1 = a_2 = -c_1 = -c_2 = \frac{3}{5},\ b_1 = \frac{1}{2},\ b_2 = \frac{93}{10},\ d_1 = -30,\ d_2 = 16$$

the corresponding polymatrix replicator has a continuum of interior equilibria that is approached by the solution, as seen in Figure 4.18.

Example 4.3.17. *If we consider the parameter values,*

$$a_1 = a_2 = -c_1 = -c_2 = -\frac{3}{5},\ b_1 = -\frac{1}{2},\ b_2 = \frac{93}{10},\ d_1 = 30,\ d_2 = -16$$

the corresponding polymatrix replicator has a continuum of interior equilibria such the solution drifts away from, as seen in Figure 4.19.

This case is analogous to (3) $d_1 = d_2 \wedge b_1 = b_2$.

As it happens with the previous case, when $-c_2 \neq a_2$ there are no equilibria outside the border of the phase space.

So all that is left is the analysis of the last possible case for a continuumn of equilibria,

$$\frac{(c_1-c_2)(d_1-d_2)}{(b_1-b_2)(a_1-a_2)} = 1 \wedge -b_2 - d_2 = \frac{(-c_2-a_2)(b_1-b_2)}{c_1-c_2}$$

here the equilibrium is defined by,

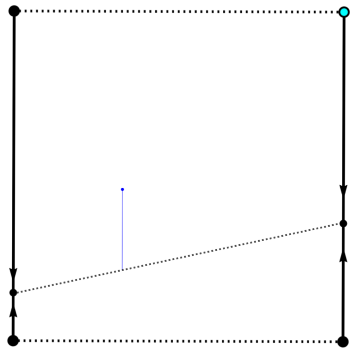

Figure 4.18: Phase portrait of example 4.3.16

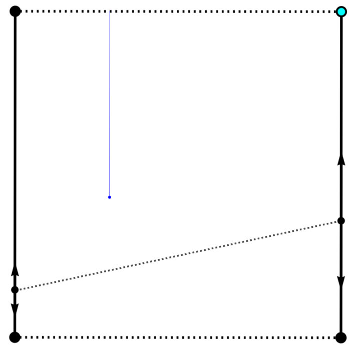

Figure 4.19: Phase portrait of example 4.3.17

$$y = \frac{c_2 - c_1}{a_1 - a_2}x - \frac{c_2 + a_2}{a_1 - a_2}.$$

Two different types of phase portraits are found, one where the eigenvalue is negative, and one where the eigenvalue is positive.

Example 4.3.18. *If we consider the parameter values,*

$$a_1 = -30,\ a_2 = 95,\ c_1 = \frac{83}{10},\ c_2 = -59,\ b_1 = \frac{4}{5},\ b_2 = -128,\ d_1 = -\frac{28488}{673},\ d_2 = \frac{132512}{673}$$

the corresponding polymatrix replicator has a continuum of interior equilibria that is approached by the solution, as seen in Figure 4.20.

Example 4.3.19. *If we consider the parameter values,*

$$a_1 = 30,\ a_2 = -95,\ c_1 = -\frac{83}{10},\ c_2 = 59,\ b_1 = -\frac{4}{5},\ b_2 = 128,\ d_1 = \frac{28488}{673},\ d_2 = -\frac{132512}{673}$$

the corresponding polymatrix replicator has a continuum of interior equilibria such the solution drifts away from, as seen in Figure 4.21.

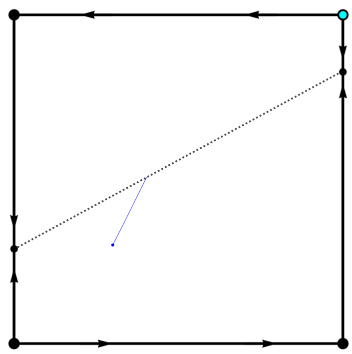

Figure 4.20: Phase portrait of example 4.3.18

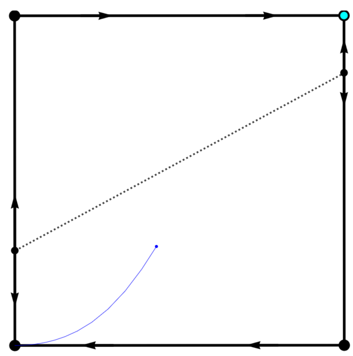

Figure 4.21: Phase portrait of example 4.3.19

5. Conclusion and Future work

This work consists of studying the bimatrix game theory and see how it can be used to analyse the replicator dynamics of a specific model presented in [5]. This model attempts to describe the actions of financial institutions and regulation institutions interacting with each other, finding a balance between these. Moreover, in this work we aim to study the possibilities to extend this model to a more generalized version.

When it comes to bimatrix game theory it is seen that one can immediately describe how the game behaves solely by looking at the payoff matrices.

There are exclusively unique equilibria points in bimatrix games, presented in two different forms, either a center or a hyperbolic equilbrium point, which come from four different strategies. However when applied to a specific model, the considered assumptions might not allow the existence of these equilibria. The example showcased only has one interior equilibrium point in one of these cases.

The extension of this model, allowing interactions between financial institutions and/or interactions between regulation institutions, leads to a panoply of new different types of dynamics. For example, from single equilibria behaving in new ways, such as spiral sink/source or real sink/source, to a continuum of equilibria.

Mathematically, something interesting that has been found in this work is the existense of bifurcations for different parameter values in the polimatrix replicator, as seen in examples 4.3.8 and 4.3.9.

Considering polimatrix replicators allows for infinite possibilities be it in terms of players or in the number of strategies that each of these players can take. Even on the simplest cases, like the one considered on this thesis, its dynamic has shown to be considerably complex. One can choose to try and exhaustively deconstruct such cases as it has been done for bimatrix games, however the added benefit of such would not be worthwile considering the number of cases it would create. The next natural step would be to introduce other players, such as costumers as suggested on the original article, which would complicate the analysis as it involves higher dimensions phase spaces, however there is already some compelling research on this area.

One can also choose to apply the model studied in this paper to a real financial model as it was done on the article. In this case we need the hard work to find the corresponding payoffs for the associated game.

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