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A PDE APPROACH TO BILATERAL COUNTERPARTY RISKS AND FUNDING COSTS

ANA BEATRIZ LOPES DOS SANTOS

SUPERVISOR:
JOÃO MIGUEL ESPIGUINHA GUERRA

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Em memória aos meus queridos avós
João,
Francisco e Maria do Céu.

Abstract, Keywords, and JEL Codes

This dissertation presents a valuation framework for over-the-counter derivatives that integrates bilateral counterparty credit risk and funding costs by extending the classical Black-Scholes model through a replication-based Partial Differential Equation approach. The model captures the cost of funding required to support a self-financing hedging strategy and introduces the default risk of both counterparties by incorporating positions in each party's own bonds within the replication portfolio. To implement the framework, a Hull-White one-factor short-rate model is used to simulate yield curves and perform the numerical evaluation of an Interest Rate Swap under different scenarios. The analysis considers multiple close-out conventions and funding scenarios, including those in which the derivative cannot be posted as collateral. Ultimately, it is discussed how this approach could be extended to incorporate climate-related risks, such as carbon pricing, by aligning the derivative valuation process with the Carbon Equivalence Principle, which supports a broader, Environmental, Social and Governance (ESG)-aligned view of derivative risk management.

Keywords: Credit Value Adjustment; Bilateral Counterparty Risk; Funding Costs; Partial Differential Equation; Carbon Equivelence Principle.

JEL Codes: G12; G23; G33.

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ACRONYMS

COe Carbon Dioxide Equivalent Emission Adjustment.

ATM At-the-money.

CEP Carbon Equivalence Principle.

CSA Credit Support Annex.

CVA Credit Valuation Adjustment.

ECB European Central Bank.

EE Expected Exposure.

ESG Environmental, Social and Governance.

EURIBOR Euro Interbank Offered Rate.

FNZ Financial Net-Zero.

FVA Funding Valuation Adjustment.

IRS Interest Rate Swap.

ISDA International Swaps and Derivatives Association.

NET Negative Emissions Technology.

NGFS Network for Greening the Financial System.

NPV Net Present Value.

OIS Overnight Index Swap.

OTC Over-the-counter.

P&L Profit and Loss.

PDE Partial Differential Equation.

RMSE Root Mean Squared Error.

SDE Stochastic Differential Equation.

XVA X-Value Adjustment.

ZBP Zero - coupon Bond Put Option.

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1 Introduction

Banks engaged in derivative transactions are exposed to the risk of incurring mark-to-market losses due to the deterioration of their counterparties' creditworthiness. Conversely, counterparties may incur losses if the seller defaults, while the derivative's mark-to-market value is positive from their perspective.

Before the 2008 Global Financial Crisis, most financial institutions priced derivatives under risk-free discounting assumptions, often neglecting counterparty and funding risks. At that time, credit risk mitigation was primarily based on collateral agreements and netting mechanisms outlined in the International Swaps and Derivatives Association (ISDA) 2002 Master Agreement. Although exchange-traded derivatives benefit from centralized clearing and standardized collateral requirements, these cannot be applied to Over-the-counter (OTC) products.

Following the collapse of major institutions, notably Lehman Brothers, many financial entities incurred substantial losses due to counterparty exposure, particularly in OTC derivatives. In addition, the crisis led to increased borrowing costs and severe liquidity shortages, thereby heightening the cost of capital required to maintain derivative positions.

Basel III formalized a comprehensive regulatory framework mandating that banks incorporate capital requirements to mitigate both counterparty and funding risks. Over the years, numerous academic studies have developed methodologies for valuing derivatives while explicitly accounting for these risks.

This dissertation presents a framework for deriving a Partial Differential Equation (PDE) and closed-form formulas as an extension to the Black-Scholes model, incorporating funding costs in scenarios where the derivative can or cannot be posted as collateral and bilateral counterparty risk is present. We introduce a self-financing portfolio replication strategy and then generalize the classical Black-Scholes PDE by incorporating jump-to-default terms and funding spread dynamics, following the approach of Burgard & Kjaer (2011).

We analyze two distinct scenarios: one in which the mark-to-market value of the derivative at default reflects the total risky value, and another in which it corresponds to the riskless value. The latter is the predominant approach in the existing literature. It aligns with contracts under ISDA 2002 Master Agreements, wherein a dealer pool determines the mark-to-market value independently of knowing which party is the defaulting one. Consequently, this leads to pricing based on the riskless value.

Under the first scenario, we derive a generally non-linear PDE. However, under specific payoff conditions, the non-linear terms vanish, resulting in a linear PDE. The Feynman-Kac representation is then applied to express the solution, enabling the decomposition of the risky derivative into three components: a risk-free component, a funding adjustment, and a credit valuation adjustment.

A core component of the replication strategy is the hedging strategy in the self-financing portfolio, which establishes the foundation of our framework. We ensure that the derivative price adequately incorporates the costs associated with all considered risk factors. This hedging strategy is crucial for risk management, as it allows financial institutions to assess and mitigate risk factors within their derivative portfolios. A key strategy, as outlined in Burgard & Kjaer (2011), involves the seller (re)purchasing its own bonds to hedge its credit risk while attempting to achieve delta neutrality. This replication strategy ensures that the seller's funding costs for purchasing its own bonds are covered through the cash account.

Under the setting described above, we aim to present the numerical value adjustments for a plain vanilla Interest Rate Swap (IRS) using the Hull-White model as the Interest Rate model to generate yield curves under various scenarios, including stressed ones, via a Monte Carlo simulation. For the Hull-White calibration, we employed a Co-terminal method and utilized Bloomberg's implied volatility data for European Swaptions.

The dissertation is structured as follows. In Chapter 2, we introduce the valuation methodology, outlining the mathematical foundations, the structure of the cash account, and the derivation of the pricing PDE, which incorporates both funding costs and bilateral credit risk. Chapters 3 and 4 introduce the PDE framework, with each chapter focusing on a different close-out convention. Chapter 5 presents the Hull-White model, followed by Chapter 6, which details its numerical implementation, market calibration, and the generation of simulated yield curve scenarios. Chapter 7 focuses on the computation of credit and funding valuation adjustments, outlining the methodology and presenting results based on the IRS setup. By combining the theoretical strength of PDE - based replication with the practical flexibility of simulation-based exposure modeling, this work provides a robust and computationally tractable solution to the problem of X-Value Adjustment (XVA) estimation.

2 PDE Derivation - Funding Costs and Bilateral Counterparty Risk

In this chapter, we derive the risky PDE to consider bilateral and funding costs using an extended version of the standard Black-Scholes model, aiming for delta neutrality of the portfolio through the use of each seller's own bonds and the counterparty's bonds via the repurchase agreement strategy. We participate as the seller of the derivative, named bank B, and we will consider two approaches to the mark-to-market value upon default. The first will consider the mark-to-market value as the risky price, and the second, as the most common one, will consider it to be the risk-free value. For the first case, the pricing PDE is non-linear. We will impose certain conditions on the payoff to make the resulting PDE linear, and then use the Feynman-Kac formulas.

2.1 Preliminaries

We begin by clarifying the use of the term mark-to-market value at default throughout this paper. It refers specifically to the close-out value of the derivative, i.e, the value of the derivative used to determine the settlement amount when one of the parties defaults. In other words, it is the amount the surviving party can claim. As such, when collateral is involved — whether it is cash, high-quality securities, or other eligible collateral — there is already credit protection against credit risk, which helps reduce losses in the event of default. These collateral mechanisms are commonly enforced in centrally cleared exchanges, where margin calls are frequent and tightly regulated. For OTC derivative transactions, there is a non-mandatory agreement on post-collateral, also part of ISDA, Credit Support Annex (CSA), which allows parties to discuss specifications and tailor terms for the transaction, thereby ensuring credit protection. However, many bilateral OTC trades remain uncollateralized in practice. To reflect real-world trading, we will distinguish between two situations:

- When **collateral is posted**, we assume the bank can fund its position at the **risk-free rate**, with no additional funding spread.
- When **collateral is not posted**, we introduce a **funding spread** s_F, defined as $r_F - r$, to account for the cost the bank incurs above the risk-free rate.

Why do we assume the bank can fund itself at the risk-free rate when collateral is posted? This goes to the heart of the Funding Valuation Adjustment (FVA) debate.

Taking the point of view of the bank's trading desk. The bank's funding desk charges the trading desk according to the bank's actual cost of funding. Let us break this down.

As stated by John Hull and Alan White (Hull & White (2014)), if the bank funds itself at the Overnight Index Swap (OIS) as a proxy for a risk-free funding rate, then the posted collateral will earn interest at the same rate, resulting in no funding costs. However, when the trade is not collateralized or one of the parties funds itself at an OIS on the US Federal Reserve Bank with a spread (e.g *OIS + 50 bps*), an adjustment must be made, as it involves a real cash flow. The question arises about what the "fair" discount rate should be used in valuation. If funding costs are not factored in, the trading desk may incur a loss. Following the 2008 Crisis, this accountability became critical. In many cases, banks could no longer assume they could fund at the risk-free rate, and ignoring these costs often resulted in losses.

Returning to credit risk, for contracts following ISDA agreements, the value of the derivative upon default is determined by a dealer pool, with no mention of the defaulting counterparty. As such, one would expect that the mark-to-market value at close-out would be the riskless value of the derivative. Nevertheless, some models use the risky value instead. This choice leads to fundamental differences in the structure of the resulting PDE, in the construction of the replication portfolio, and in the interpretation of the resulting valuation adjustments.

1. When the mark-to-market value at default is taken to be the *risky value* of the derivative, $\hat{V}$.

2. When the mark-to-market value at default is taken to be the *counterparty-riskless value*, V.

In what follows, we derive the pricing PDE under both conventions and show how each one impacts the final value of the derivative. This sets the foundation for the simulation-based valuation techniques presented in later chapters.

2.2 *Model Setup*

Let $\hat{V}$ denote the risky value of a derivative contract on asset S, between seller B and counterparty C. The underlying asset S is assumed to be independent of the default events of both B and C, and it evolves according to a Markov process with generator $\mathcal{A}_t$. Let V be the derivative contract under the assumption that neither **B** nor **C** defaults.

The bonds P_B and P_C each pay 1 at maturity T if no default has occurred, and 0 otherwise. We also consider the jump-to-default extension to **B** and **C** default state.

Let us now introduce the following portfolio assets and the respective dynamics:

r : risk-free rate

r_B : yield on recovery-less bond of seller B

r_C : yield on recovery-less bond of counterparty C

P_R : default risk-free zero coupon bond

P_B : default risky, zero-recovery, zero-coupon bond of party B

P_C : default risky, zero-recovery, zero-coupon bond of party C

S : spot asset with no default risk

$$\begin{cases} \dfrac{dP_R(t)}{P_R(t)} = r(t)dt \\ \dfrac{dP_B(t)}{P_B(t)} = r_B(t)dt - dJ_B(t) \\ \dfrac{dP_C(t)}{P_C(t)} = r_C(t)dt - dJ_C(t) \\ \dfrac{dS(t)}{S(t)} = \mu(t)dt + \sigma(t)dW(t) \end{cases} \tag{1}$$

where $W(t)$ is a standard Brownian motion, $r(t) > 0$, $r_B(t) > 0$, $r_C(t) > 0$, $\mu(t) > 0$ are deterministic functions of t, dJ_B and dJ_C are point processes that jump from 0 to 1 in the case of the respective parties' default.

The convention used in the payout scenario involves the seller receiving cash or an asset from the counterparty if $H(S) \geq 0$.

Let's suppose that *B* and *C* enter into a derivative contract on the spot asset *S* with payoff $H(S)$ at maturity T.

The risky value of the derivative from the seller's perspective at time t is denoted by $\hat{V}(t, S, J_B, J_C)$, while $V(t, S)$ represents the corresponding risk-free value.

As stated before, when *B* or *C* defaults, the mark-to-market value of the derivative relies on the close-out or claim on the position. Under the ISDA 2002 Agreement, the mark-to-market value, $M(t, S)$, can follow one of the following scenarios, from the view of the surviving party:

1. The recovery value of the positive mark-to-market value of the derivative just prior to default.

2. The full mark-to-market amount is owed to the defaulting party when the mark-to-market value is negative.

As is standard under the ISDA 2002 Master Agreement, the close-out amount is typically taken to be close to the counterparty-riskless value, even though it is unclear whether funding costs should be included in the close-out amount. As such, we consider $V(t, S) = M(t, S)$.

Alternatively, under different modeling frameworks, the close-out value may be represented as $\hat{V}(t, S, 0, 0) = M(t, S)$, where $\hat{V}(t, S, 0, 0)$ denotes the risky value of the derivative at time t, before any default has occurred.

Let $R_B \in [0, 1]$ and $R_C \in [0, 1]$ denote the recovery-rates, supposed deterministic, of the derivative positions of parties *B* and *C*, respectively.

Let us consider the following boundary conditions based on the arguments above.

$$\begin{aligned} \hat{V}(t, S, 1, 0) &= M^+(t, S) + R_B M^-(t, S) \quad \text{(Seller defaults first)} \\ \hat{V}(t, S, 0, 1) &= R_C M^+(t, S) + M^-(t, S) \quad \text{(Counterparty defaults first)} \end{aligned} \tag{2}$$

where $M^+ = \max(M, 0)$[1], and $M^- = \min(M, 0)$[2].

Considering the standard Black-Scholes framework, the hedge of the derivative is achieved through a self-financing portfolio that covers all the underlying risk factors.

The seller sets up a portfolio Π consisting of $\delta(t)$ units of the underlying asset S, $\alpha_B(t)$ units of the bond P_B, $\alpha_C(t)$ units of the bond P_C, and $\beta(t)$ units of cash. The portfolio is structured such that its value hedges out the value of the derivative contract to the seller at time t, i.e., $\hat{V}(t) + \Pi(t) = 0$. Thus,

$$-\hat{V}(t) = \Pi(t) = \delta(t)S(t) + \alpha_B(t)P_B(t) + \alpha_C(t)P_C(t) + \beta(t) \tag{3}$$

As mentioned earlier, to hedge against counterparty default, we will employ a strategy that involves incorporating bonds from both parties, B and C.

Let us first consider the case where $\hat{V}(t) > 0$ from the seller's perspective. If the counterparty defaults on its obligations to B, the seller is exposed to a loss. To hedge against this loss, the seller enters into a short position in the counterparty's bond P_C, i.e., $\alpha_C(t) < 0$. It is assumed the seller can borrow P_C via a repurchase agreement at a repo rate close to the risk-free rate. If counterparty C defaults, the price of P_C drops, allowing

[1] M^+ denotes the positive part of M.
[2] M^- denotes the negative part of M.

the seller to repurchase the bond at a lower price to close the repo transaction. The cash generated from the initial sale is lent out at the risk-free rate r, while the bond P_C pays a yield r_C. The spread $s_C = r_C - r$ can therefore be interpreted as an effective credit spread of the counterparty. If P_C has zero recovery, this spread approximates the default intensity, i.e., the hazard rate λ_C. If instead the bond has recovery R_C, the spread is given by:

$$s_C = r_C - r = \lambda_C(1 - R_C),$$

Conversely, if $\hat{V}(t) < 0$, the seller would gain from its own default, owing money to the counterparty. To hedge against this gain, the seller buys its own bond P_B, i.e., $\alpha_B(t) > 0$. If party B defaults, the market value of its own bond declines, generating a loss in the replicating portfolio that offsets the gain from the derivative position.

The positions $\alpha_B(t)$ and $\alpha_C(t)$ are updated dynamically over time, depending on the sign and magnitude of $\hat{V}(t)$. This ensures that the replication strategy remains self-financing and neutralizes both market and credit risk exposures. Any excess cash generated in the replication process, after the purchase of P_B, is assumed to be invested at the risk-free rate r, thereby not introducing any additional credit risk to the hedging portfolio.

The self-financing condition implies:

$$-d\hat{V}(t) = \delta(t)dS(t) + \alpha_B(t)dP_B(t) + \alpha_C(t)dP_C(t) + d\bar{\beta}(t). \tag{4}$$

The cash account $\bar{\beta}(t)$ is composed of three components:

$$d\bar{\beta}(t) = d\bar{\beta}_S(t) + d\bar{\beta}_F(t) + d\bar{\beta}_C(t),$$

each with its own economic interpretation:

- **Equity position (dividend and financing costs):**

$$d\bar{\beta}_S(t) = \delta(t)(\gamma_S(t) - q_S(t))S(t)dt,$$

  where $\gamma_S(t)$ is the dividend yield and $q_S(t)$ depends on the repo rate of the underlying asset $S(t)$ and $r(t)$.

- **Funding account (cost of borrowing or surplus investment after own bond purchasing):**

$$d\bar{\beta}_F(t) = r(t)(-\hat{V}(t) - \alpha_B(t)P_B(t))dt + s_F(t)(-\hat{V}(t) - \alpha_B(t)P_B(t))^{-}dt,$$

with the funding spread defined as

- Collateralized: $r_F(t) = r(t)$, assuming no haircut on the collateral
- Unsecured: $r_F(t) = r(t) + (1 - R_B)\lambda_B$

- **Repo cost for counterparty bond:**

$$d\bar{\beta}_C(t) = -\alpha_C(t)r(t)P_C(t)dt$$

The full cash account dynamics are given by:

$$\begin{aligned} d\bar{\beta}(t) = \ & \delta(t)(\gamma_S(t) - q_S(t))S(t)dt \\ & + \left[r(t)(-\hat{V}(t) - \alpha_B(t)P_B(t)) + s_F(t)(-\hat{V}(t) - \alpha_B(t)P_B(t))^- \right] dt \\ & - \alpha_C(t)r(t)P_C(t)dt. \end{aligned} \tag{5}$$

This formulation ensures that:

- The cash account grows passively from returns on invested assets.
- All costs of maintaining hedge positions are properly accounted for.
- Funding costs appear explicitly through s_F.

From (4), it yields

$$-d\hat{V} = \delta\, dS + \alpha_B\, dP_B + \alpha_C\, dP_C + d\bar{\beta}(t) \tag{6}$$

$$\begin{aligned} &= \delta\, dS + \alpha_B P_B(r_B\, dt - dJ_B) + \alpha_C P_C(r_C\, dt - dJ_C) \\ &\quad + r\left(-\hat{V} - \alpha_B P_B\right) dt + s_F\left(-\hat{V} - \alpha_B P_B\right)^- dt \\ &\quad - \alpha_C r P_C\, dt - \delta(q_S - \gamma_S)S(t)\, dt \end{aligned} \tag{7}$$

$$\begin{aligned} &= \Big\{ - r\hat{V} + s_F(-\hat{V} - \alpha_B P_B)^- + (\gamma_S - q_S)\delta S \\ &\quad + (r_B - r)\alpha_B P_B + (r_C - r)\alpha_C P_C \Big\} dt \\ &\quad - \alpha_B P_B\, dJ_B - \alpha_C P_C\, dJ_C + \delta\, dS \end{aligned} \tag{8}$$

On the other hand, by Itô's Lemma for jump diffusions (for more detail please see Cont & Tankov (2003)) with the assumption that a simultaneous jump is a zero probability event, the derivative value moves by:

$$d\hat{V} = \partial_t \hat{V} dt + \partial_S \hat{V} dS + \frac{1}{2}\sigma^2 S^2 \partial_S^2 \hat{V} dt + \Delta\hat{V}_B dJ_B + \Delta\hat{V}_C dJ_C \tag{9}$$

where:

$$\Delta\hat{V}_B = \hat{V}(t, S, 1, 0) - \hat{V}(t, S, 0, 0) \tag{10}$$

$$\Delta\hat{V}_C = \hat{V}(t, S, 0, 1) - \hat{V}(t, S, 0, 0) \tag{11}$$

These parameters can be computed from the boundary condition (2). Replacing $d\hat{V}$ in (8) by (9) shows that we can eliminate all risks in the portfolio by choosing δ, α_B, and α_C as:

$$\delta = -\partial_S \hat{V} \tag{12}$$

$$\alpha_B = \frac{\Delta\hat{V}_B}{P_B} \tag{13}$$

$$= -\frac{\hat{V} - (M^+ + R_B M^-)}{P_B} \tag{14}$$

$$\alpha_C = \frac{\Delta\hat{V}_C}{P_C} \tag{15}$$

$$= -\frac{\hat{V} - (M^- + R_C M^+)}{P_C} \tag{16}$$

Hence, the dynamics of the cash account can be expressed as:

$$d\bar{\beta}_F(t) = \big[- r(t) R_B M^- - r_F(t) M^+ \big] dt \tag{17}$$

where:

- $-R_B M^-$: cash deposited at risk-free rate $r(t)$
- $-M^+$: cash borrowed at funding rate $r_F(t)$

Let us now introduce the parabolic differential operator $\mathcal{A}_t$:

$$\mathcal{A}_t V \triangleq \frac{1}{2}\sigma^2 S^2 \partial_S^2 V + (q_S - \delta_S) S \partial_S V \tag{18}$$

The derivative value $\hat{V}$ satisfies the PDE:

$$\begin{cases} \partial_t \hat{V} + \mathcal{A}_t \hat{V} - r\hat{V} = s_F(\hat{V} + \Delta\hat{V}_B)^+ - \lambda_B \Delta\hat{V}_B - \lambda_C \Delta\hat{V}_C, \\ \hat{V}(T, S) = H(S) \end{cases} \tag{19}$$

where $\lambda_B = r_B - r$ and $\lambda_C = r_C - r$.[3]

Inserting equations (10) and (11), with boundary condition (2), we derive the following refined PDE:

$$\begin{cases} \partial_t \hat{V} + \mathcal{A}_t \hat{V} - r\hat{V} = (\lambda_B + \lambda_C)\hat{V} + s_F M^+ - \lambda_B(R_B M^- + M^+) - \lambda_C(R_C M^+ + M^-) \\ \qquad \hat{V}(T,S) = H(S) \end{cases} \tag{20}$$

where $(\hat{V} + \Delta\hat{V}_B)^+ = (M^+ + R_B M^-)^+ = M^+$.

On the other hand, the risk-free value $V(t,S)$ satisfies the standard Black-Scholes PDE:

$$\begin{cases} \partial_t V + \mathcal{A}_t V - rV = 0 \\ V(T,S) = H(S) \end{cases} \tag{21}$$

Defining the effective default spreads as: $\lambda_B = r_B - r$ and $\lambda_C = r_C - r$, the differences between (20) and (21) can be inferred:

- **First term:** $(\lambda_B + \lambda_C)\hat{V}$, represents the additional return required by the seller on the risky asset $\hat{V}$ to compensate the risk that the contract might suddenly terminate due to either the seller's own default or counterparty's default.
- **Second term:** $s_F M^+$, captures the incremental cost of funding arising from negative balances in the cash account under the hedging strategy, i.e., when the bank must borrow money to replicate the position.
- **Third term:** $\lambda_B(R_B M^- + M^+)$, represents the adjustment to the growth rate to account for the effect of the seller's own default.
- **Fourth term:** $\lambda_C(R_C M^+ + M^-)$, represents the adjustment to the growth rate to account for the effect of the counterparty's default.

The first, third, and fourth terms are associated with **counterparty credit risk**, whereas the second term reflects the **funding cost**.

Based on this interpretation, the PDE corresponding to an *extinguisher trade*—a contract in which it is contractually agreed that no party receives any payoff in the event of default—is derived by omitting the third and fourth terms from (20).

[3] Let us recall that P_B and P_C are assumed to be *recovery-less* bonds.

In the subsequent sections, we will examine the PDE (20) under the following four scenarios:

1. $M(t,S) = \hat{V}(t,S,0,0)$ and $r_F = r$
2. $M(t,S) = \hat{V}(t,S,0,0)$ and $r_F = r + s_F$
3. $M(t,S) = V(t,S)$ and $r_F = r$
4. $M(t,S) = V(t,S)$ and $r_F = r + s_F$

3 Using $\hat{V}$ as the Mark-to-Market Value at Default

Let us focus on the case where the close-out payment at default is based on the risky value of the derivative. That is, we assume the mark-to-market value satisfies: $M(t,S) = \hat{V}(t,S)$ in the boundary conditions (2). This assumption simplifies the analysis. If the defaulting party is the in-the-money with respect to the derivative, no profit and loss arises. On the other hand, if the surviving party is in-the-money, its loss is given by $(1-R)\hat{V}$.

From PDE (20), we derive:

$$\begin{cases} \partial_t \hat{V} + \mathcal{A}\hat{V} - r\hat{V} = (1-R_B)\lambda_B \hat{V}^- + (1-R_C)\lambda_C \hat{V}^+ + s_F \hat{V}^+ \\ \hat{V}(T,S) = H(S), \end{cases} \tag{22}$$

where $\hat{V}^+ = \max(\hat{V},0)$[4] and $\hat{V}^- = \min(\hat{V},0)$[5].

The above PDE can be further examined in two particular subcases:

1. $s_F = 0$, which corresponds to $r_F = r$, i.e., when the derivative can be used as collateral.
2. $s_F = (1-R_B)\lambda_B$, where the derivative cannot be posted as collateral, and the bank must fund its position at its own funding rate.

Moreover, the hedge ratios can be obtained as follows:

$$\alpha_B = -\frac{(1-R_B)\hat{V}^-}{P_B}, \tag{23}$$

$$\alpha_C = -\frac{(1-R_C)\hat{V}^+}{P_C} \tag{24}$$

[4] $\hat{V}^+$ denotes the positive part of $\hat{V}$.
[5] $\hat{V}^-$ denotes the negative part of $\hat{V}$.

As such, $\alpha_B \geq 0$ and $\alpha_C \leq 0$, the replication strategy ensures that sufficient cash is generated, $(-\hat{V}^-)$, so the seller can purchase back its own bonds. To the cash generated $(-\hat{V}^-)$, $(1 - R_B)$ can be allocated to repurchasing the seller's own bonds while R_B is invested at the risk-free rate. This is equivalent to investing the total amount $(-\hat{V}^-)$ into purchasing back a seller bond, B^*, with recovery-rate R_B.

In Credit Valuation Adjustment (CVA) literature, it is common to express the decomposition of the risky value $\hat{V}$ as the sum of the risk-free value V and an adjustment term U.

$$\hat{V} = V + U \tag{25}$$

Inserting the decomposition into the PDE (22) and recalling that V satisfies the PDE (21), we obtain:

$$\begin{cases} \partial_t U + A_t U - rU = (1 - R_B)\lambda_B (V + U)^- + (1 - R_C)\lambda_C (V + U)^+ + s_F (V + U)^+ \\ \qquad\qquad U(T, S) = 0 \end{cases} \tag{26}$$

Applying the Feynman-Kac representation formula (for detail please see Karatzas & Shreve (1998)) to PDE (26) under the assumption of deterministic rates, it yields the following non-linear integral equation:

$$\begin{aligned} U(t, S) = &- (1 - R_B) \int_t^T \lambda_B(u) D_r(t, u) \mathbb{E}_t \left[(V(u, S(u)) + U(u, S(u)))^- \right] du \\ &- (1 - R_C) \int_t^T \lambda_C(u) D_r(t, u) \mathbb{E}_t \left[(V(u, S(u)) + U(u, S(u)))^+ \right] du \\ &- \int_t^T s_F(u) D_r(t, u) \mathbb{E}_t \left[(V(u, S(u)) + U(u, S(u)))^+ \right] du \end{aligned} \tag{27}$$

This formulation enables the computation of U once V is known, either by solving the PDE or using the integral equation.

Before delving into particular funding spread assumptions, it is helpful to analyze simple illustrative case studies in which $\hat{V}$ corresponds to a defaultable bond issued by B and C, either with or without recovery. These examples provide intuition and serve as a consistency check for the general framework.

3.1 *Case Study: Seller sells P_B to the Counterparty C*

We consider, in this first case, that a risky, recovery-less bond is sold by the seller B to the counterparty C. Therefore, we consider $\hat{V} = \hat{V}^- = -P_B$ and $R_B = 0$. Since we consider deterministic rates and credit spreads, we do not encounter any risk referencing any market factors, and as such, the term $A_t\hat{V}$ vanishes and the PDE (22) reduces to:

$$\begin{cases} \partial_t\hat{V} = (r+\lambda_B)\hat{V} = r_B\hat{V} \\ \hat{V}(T,S) = -1 \end{cases} \tag{28}$$

As a result, we obtain the solution:

$$\hat{V}(t) = -\exp\left(-\int_t^T r_B(s)ds\right), \tag{29}$$

which is expected for $\hat{V}^- = -P_B$.

If instead the bond has recovery R_B, the PDE (22) becomes:

$$\begin{cases} \partial_t\hat{V} = [r+\lambda_B(1-R_B)]\ \hat{V} \\ \hat{V}(T,S) = -1 \end{cases} \tag{30}$$

with solution:

$$\hat{V}(t) = -\exp\left(-\int_t^T [r(s)+(1-R_B)\lambda_B(s)]\, ds\right) \tag{31}$$

As expected, the adjusted drift matches the unsecured funding rate, $r_F = r + (1 - R_B)\lambda_B$, representing the cost incurred by the seller on negative cash balances when the derivative cannot be posted as collateral.

3.2 Case Study: Seller Purchases P_C from Counterparty C

In this case, we will assume $\hat{V} = \hat{V}^+ = P_C$, with $R_C = 0$. The PDE (22) becomes:

$$\begin{cases} \partial_t\hat{V} = (r_F+\lambda_C)\hat{V} = (r_F+(r_C-r))\hat{V} \\ \hat{V}(T) = 1 \end{cases} \tag{32}$$

If the derivative (i.e. the loan asset) can be used as collateral by the seller to fund its short cash position within the replication portfolio strategy, and, neglecting haircuts, $r_F = r$.

Hence,

$$\partial_t\hat{V} = r_C\hat{V} \tag{33}$$

therefore,

$$\hat{V}(t) = -\exp\left(-\int_t^T r_C ds\right) \tag{34}$$

as would be expected for $\hat{V} = P_C$. For a bond with recovery R_C, we obtain:

$$\hat{V}(t) = \exp\left(-\int_t^T [r(s) + (1 - R_C)\lambda_C(s)]\, ds\right) \tag{35}$$

as also expected.

3.3 Case 1: $M = \hat{V}$ and $s_F = r_F - r = 0$

Let us now focus on a more general framework. When the risky value of the derivative is the mark-to-market value, and it can be posted as collateral, then the PDE (22) becomes:

$$\begin{cases} \partial_t \hat{V} + A_t \hat{V} - r\hat{V} & = (1 - R_B)\lambda_B \hat{V}^- + (1 - R_C)\lambda_C \hat{V}^+ \\ \hat{V}(T, S) & = H(S) \end{cases} \tag{36}$$

This is a non-linear PDE that must be solved numerically unless it can be assumed that $\hat{V} \leq 0$ or $\hat{V} \geq 0$.

Let us first assume that $\hat{V} \leq 0$, corresponding to the seller writing an option to the counterparty. Under the assumption that all rates are deterministic, the PDE becomes linear and admits the Feynman-Kac representation of $\hat{V}$ given by:

$$\hat{V}(t, S) = \mathbb{E}_t\left[D_{r+(1-R_B)\lambda_B}(t, T) H(S_T)\right] \tag{37}$$

where $D_k(t;T) = \exp\left(-\int_t^T k(s)\, ds\right)$ is the discount factor over $[t, T]$ given the rate k.

Alternatively, $\hat{V} = V + U_0$[6], where V solves the risk-free PDE given by (21), the adjustment term becomes:

$$U_0(t, S) = -V(t, S) \int_t^T (1 - R_B)\lambda_B(u) D_{(1-R_B)\lambda_B}(t, u)\, du \tag{38}$$

Symmetrically, when $\hat{V} \geq 0$, i.e, the seller bought an option from the counterparty, the adjustment term yields:

$$U_0(t, S) = -V(t, S) \int_t^T (1 - R_C)\lambda_C(u) D_{(1-R_C)\lambda_C}(t, u)\, du \tag{39}$$

[6]We consider U_0 as U under the case where there is no funding spread, i.e, $r_F = r$

In conclusion, when $\hat{V} \leq 0$, U_0 depends entirely on the credit of the seller on the other hand, when $\hat{V} \geq 0$, U_0 depends only on the credit of the counterparty.

3.4 Case 2: $M = \hat{V}$ and $s_F = r + (1 - R_B)\lambda_B$

We will now consider that the derivative cannot be posted as collateral, the bank funds itself at a spread of $s_F = (1 - R_B)\lambda_B$. Therefore, the PDE (22) becomes:

$$\begin{cases} \partial_t \hat{V} + A_t \hat{V} - r\hat{V} = (1 - R_B)\lambda_B \hat{V}^- + \left[(1 - R_B)\lambda_B + (1 - R_C)\lambda_C\right] \hat{V}^+ \\ \qquad \hat{V}(T, S) = H(S) \end{cases} \tag{40}$$

The equation is also non-linear. If we assume $\hat{V} \leq 0$ and employ the decomposition $\hat{V} = V + U_0$, we find the same case defined in (38) (where $s_F = 0$), and consequently $U = U_0$. When $\hat{V} \geq 0$, the value of the derivative is given by:

$$\hat{V}(t, S) = \mathbb{E}_t \left[D_{r+(1-R_B)\lambda_B+(1-R_C)\lambda_C}(t, T) H(S_T)\right] \tag{41}$$

Aligning with the same decomposition $\hat{V} = V + U$ we show,

$$U(t, S) = -V(t, S) \int_t^T k(u) D_k(t, u)\, du \tag{42}$$

where $k = r + (1 - R_B)\lambda_B + (1 - R_C)\lambda_C$.

4 Using $V(t, S)$ as Mark-to-Market Value at Default

We now consider the case where the mark-to-market value at default is taken to be the risk-free value $V(t, S)$ of the derivative. We take $M(t, S) = V(t, S)$ in the boundary conditions (2), the general PDE (20) is given by:

$$\begin{cases} \partial_t \hat{V} + A_t \hat{V} - (r + \lambda_B + \lambda_C)\hat{V} = -(R_B\lambda_B + \lambda_C)V^- - (\lambda_B + R_C\lambda_C)V^+ + s_F V^+ \\ \qquad \hat{V}(T, S) = H(S) \end{cases} \tag{43}$$

where $V^+ = \max(V, 0)$[7], and $V^- = \min(V, 0)$[8].

This is a linear PDE with V acting as a known source term. Writing $\hat{V} = V + U$, the

[7] V^+ denotes the positive part of V.
[8] V^- denotes the negative part of V.

hedge ratios become:

$$\alpha_B = \frac{U + (1 - R_B)V^-}{P_B}, \tag{44}$$

$$\alpha_C = \frac{U + (1 - R_C)V^+}{P_C} \tag{45}$$

By comparing equations (23) and (44), we observe that in the latter case, a default event results in a sudden cash flow equal to U. This jump must be accounted for within the replication strategy to ensure accurate hedging.

Inserting $\hat{V} = V + U$, we obtain a linear PDE for U:

$$\begin{cases} \partial_t U + A_t U - (r + \lambda_B + \lambda_C)U = (1 - R_B)\lambda_B V^- + (1 - R_C)\lambda_C V^+ + s_F V^+ \\ U(T, S) = 0 \end{cases} \tag{46}$$

Using the Feynman-Kac representation, we derive:

$$\begin{aligned} U(t, S) = & -(1 - R_B) \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t \left[V^-(u, S(u)) \right] du \\ & -(1 - R_C) \int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t \left[V^+(u, S(u)) \right] du \\ & - \int_t^T s_F(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t \left[V^+(u, S(u)) \right] du \end{aligned} \tag{47}$$

The adjustment U can be computed by solving the PDE (46) or evaluating the integrals above.

When the derivative **can be posted as collateral**, i.e, $s_F = 0$, the last term in PDE (46) drops out. This reduces to the standard bilateral CVA form widely known in the literature (e.g. Gregory (2009)). The bilateral benefit does not stem from own default but from the ability to use the cash generated by the hedging strategy to buy back the bank's own bonds, earning an excess return of $(1 - R_B)\lambda_B$.

However, in practice, the derivative usually **cannot** be used as collateral. In this case, the full adjustment U (47) must be computed. When the funding spread corresponds to the same as the unsecured bond B with recovery-rate R_B, i.e., $s_F = (1 - R_B)\lambda_B$, the first and third terms in (47) can be merged:

$$\begin{aligned} U(t,S) = &-(1-R_B)\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V^-(u,S(u))\right] du \\ &-(1-R_C)\int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V^+(u,S(u))\right] du \end{aligned} \tag{48}$$

We considered the valuation of derivatives under bilateral counterparty risk and funding costs using an extended version of the Black-Scholes model. The approach is based on dynamic replication strategies, where the derivative is hedged using a self-financing portfolio that may include the underlying asset, cash, and defaultable bonds issued by both the bank (B, the seller) and its counterparty (C). The model accounts for the possibility of default from either party and incorporates the reality that unsecured funding often carries a spread over the risk-free rate.

Two different close-out conventions for the mark-to-market value at default were studied:

Risky close-out: where the value at default is the risky value of the derivative itself. This choice is internally consistent with the replication strategy, and the resulting pricing equation is a non-linear PDE that captures both funding costs and default losses. In this setting, the derivative's value evolves continuously up to default, and no artificial valuation jumps occur at the default time.

Risk-free close-out: where the close-out value is the counterparty-riskless value V. This is commonly assumed in legal agreements, such as the ISDA Master Agreement 2002, and leads to a linear PDE for the credit valuation adjustment and funding costs.

Throughout the analysis, special attention was paid to the economic interpretation of each term in the PDEs. These include components representing:

- compensation for the risk of contract termination due to default;
- funding costs for uncollateralized exposures;
- adjustments reflecting the impact of the seller's own credit risk;

Illustrative examples were used to verify the model's consistency, showing how simplified bond payoffs lead to known pricing expressions, such as the value of a zero-coupon risky bond.

This framework forms the basis for the numerical experiments presented in Chapter 7, where we calibrate the model and quantify the impact of bilateral credit risk and funding costs on derivative pricing.

5 Affine Interest Rate Model: Hull-White Model

Affine models such as the one-factor Hull-White model are widely used in interest rate modeling due to their analytical tractability and flexibility. A key feature of these models is that zero-coupon bond prices depend exponentially on the short rate, which allows for closed-form solutions and efficient calibration to market data. The Hull-White framework extends the Vasicek model by allowing time-dependent parameters. It provides analytical pricing formulas for a variety of interest rate derivatives, including caplets, swaptions, and bond options.

In this section, we outline the pricing methodology for caplets and swaptions under the Hull-White model, leveraging its affine term structure. The model enables the derivation of closed-form expressions for options on zero-coupon bonds, which can be utilized to price caplets and swaptions via decomposition techniques. However, since the Hull-White model is not a direct model of the swap rate — and the swap rate itself is not log-normally distributed — it is common in the literature to approximate the swap rate as the ratio of zero-coupon bonds, which can then be assumed to be log-normal. This enables pricing swaptions using the Black model, which requires a log-normal assumption for volatility.

Prior to the Global Financial Crisis, the Black model remained ubiquitous largely because prevailing market conditions ensured strictly positive interest rates. However, the emergence of negative interest rates in the post-2008 period, particularly in European markets, has made the Bachelier model increasingly relevant.

Although the literature on the Bachelier model remains sparse, we present a hybrid approach to address this gap. We present the swaption pricing framework under the Black model, but implement the Bachelier model in our numerical implementation.

The present chapter draws on the work of Gurrieri et al. (2009) as its main support.

5.1 The Hull-White Short Rate Model

The Hull-White model assumes that under the risk-neutral measure, denoted as $\mathbb{Q}$, the short rate $r(t)$ evolves according to the Stochastic Differential Equation (SDE):

$$dr(t) = [\theta(t) - a(t)r(t)]dt + \sigma(t)dW(t), \tag{49}$$

where:

- $a(t)$ is the time-dependent mean reversion speed.

- $\sigma(t)$ is the time-dependent volatility.
- $W(t)$ is a standard Brownian motion under the risk-neutral measure, $\mathbb{Q}$.
- $\theta(t)$ is the time-dependent drift function, which ensures the model fits the initial term structure of the discount curve.

The short rate process $r(t)$ under the Hull-White model has the following conditional expectation and conditional variance:

$$\mathbb{E}[r(t) \mid \mathcal{F}_s] = \frac{E(s)}{E(t)} r(s) + \alpha(t) - \frac{E(s)}{E(t)} \alpha(s) \tag{50}$$

$$\mathrm{Var}[r(t) \mid \mathcal{F}_s] = V_r(s,t) \tag{51}$$

where $\mathcal{F}_s$ is the filtration generated by $r(t)$ up to time s:

$$\begin{aligned} E(t) &= \exp\left(\int_0^t a(u)\, du\right), \\ \alpha(t) &= f(0,t) + \frac{1}{E(t)} \int_0^t E(u)\, \sigma^2(u)\, B(u,t)\, du, \\ V_r(s,t) &= \frac{1}{E^2(t)} \int_s^t E^2(u)\, \sigma^2(u)\, du. \end{aligned}$$

The derivation of $E(t)$, $\alpha(t)$, and $V_r(s,t)$ can be found in detail in Appendix A of Gurrieri et al. (2009).

This model is part of the family of affine term structure models, meaning that zero-coupon bond prices have an exponential affine form.

5.2 Zero-Coupon Bond Pricing

Under the Hull-White model, the price at time t of a zero-coupon bond maturing at time T is given by:

$$P(t,T) = A(t,T) e^{-B(t,T) r(t)}, \tag{52}$$

where (from Brigo & Mercurio (2006)):

$$\begin{aligned} B(t,T) &= \frac{1 - e^{-a(T-t)}}{a}, \\ A(t,T) &= \frac{P(0,T)}{P(0,t)} \exp\left(B(t,T) f(0,t) - \frac{\sigma^2}{4a}(1 - e^{-2a(T-t)}) B(t,T)^2\right), \end{aligned}$$

and $f(0,t)$ is the instantaneous forward rate at time t.

Introducing the log-normal SDE of the zero-coupon bond ratio:

$$\frac{dP(t,T)}{P(t,T)} = r(t)dt - \sigma(t)B(t,T)dW^{\mathbb{Q}}(t) \tag{53}$$

To compute closed-form expressions for derivative pricing, we are particularly interested in the ratio of two bond prices with fixing and payment times T_F and T_P ($t \leq T_F \leq T_P$), whose dynamics under the T_P-forward measure is:

$$d\left(\frac{P(t,T_F)}{P(t,T_P)}\right) = \frac{P(t,T_F)}{P(t,T_P)}\sigma(t)\left(B(t,T_P) - B(t,T_F)\right)dW^{T_P}(t) \tag{54}$$

with integrated variance:

$$V_p(t,T_F,T_P) = \int_t^{T_F} \sigma^2(u)\left(B(u,T_P) - B(u,T_F)\right)^2 du \tag{55}$$

$$= V_r(t,T_F) \cdot B(T_F,T_P)^2 \tag{56}$$

5.3 Caplet Pricing via Zero-Coupon Bond Options

Let us introduce a caplet as an option on a short-term interest rate, where the payoff depends on whether the underlying rate exceeds a predetermined strike. Let K denote the strike, T_F the fixing date, and T_P the payment date of the caplet. The payoff of the caplet can be rewritten as a scaled Zero - coupon Bond Put Option (ZBP), where the volatility is derived from the variance of the bond price ratio. Under the Black model, the price of a caplet is given by:

$$\text{Caplet}(K,T_F,T_P) = (1+K\delta)\,\text{ZBP}(T_F,T_P,\tfrac{1}{1+K\delta}) \tag{57}$$

where δ is the accrual fraction between T_F and T_P, and the ZBP is priced via:

$$\text{ZBP}(T_F,T_P,X) = X\,P(0,T_F)\,\mathcal{N}(d_+) - P(0,T_P)\,\mathcal{N}(d_-) \tag{58}$$

with

$$d_\pm = \frac{\ln\left(\frac{P(0,T_F)\,X}{P(0,T_P)}\right)}{\sqrt{V_p(0,T_F,T_P)}} \pm \tfrac{1}{2}\sqrt{V_p(0,T_F,T_P)}, \tag{59}$$

where $\mathcal{N}$ is the standard Normal cumulative function and X the strike price of the zero-coupon bond put option.

5.4 *Swaption Pricing via Jamshidian Decomposition*

Let us now introduce a swaption as an option that grants the holder the right, but not the obligation, to enter into an interest rate swap at a future date and pre-agreed swap terms. Considering a payer swaption maturing at T_0, swap tenor T_P and swap cash-flows at $\{T_i\}_{i=1}^{n}$, with $T_n = T_P$. Introducing Jamshidian's decomposition (for more details, please see Jamshidian (1989)), the payer swaption is represented as a sum of zero-coupon bond options:

$$\text{Swaption}(K, T_0, T_P) = \sum_{i=1}^{n} c_i \, \text{ZBP}(T_0, T_i, X_i) \tag{60}$$

with the cash flow weights c_i given by:

$$c_i = K \cdot \delta_i \quad \text{for } i = 1, \ldots, n-1 \tag{61}$$

$$c_n = 1 + K \cdot \delta_n \tag{62}$$

and X_i representing the strike of the zero-coupon bond put options, defined by:

$$X_i = \exp\left(A(T_0, T_i) - B(T_0, T_i)\, r^*\right) \tag{63}$$

The rate r^* solves the equation:

$$\sum_{i=1}^{n} c_i \exp\left(A(T_0, T_i) - B(T_0, T_i)\, r^*\right) = 1 \tag{64}$$

5.5 *Swaption Implied Volatility via Bond Price Ratio*

To establish a more direct link between market-implied volatility and model parameters for calibration purposes, an approximation framework is introduced. The key idea arises from the fact that the Hull-White model is not inherently a swap rate model and, the swap rate itself is not log-normally distributed, as stated earlier. However, the framework leverages the property that the ratio of zero-coupon bond prices follows a log-normal distribution, which enables the approximation of swap rate dynamics through the bond price ratio.

Let $S(t, T_0, T_n)$ denote the forward swap rate observed at time t for the swap starting at T_0 and maturing on T_n. The swap rate is defined by:

$$S(t, T_0, T_n) = \frac{P(t, T_0) - P(t, T_n)}{\sum_{i=1}^{n} \delta_i \, P(t, T_i)} \tag{65}$$

The true swap rate, under the annuity measure $\mathcal{A}$, is log-normally distributed and it is given by:

$$\widetilde{S}(t, T_0, T_n) \approx \frac{P(0, T_n)}{\sum_{i=1}^{n} \delta_i \, P(0, T_i)} \left(\frac{P(t, T_0)}{P(t, T_n)} - 1 \right) \tag{66}$$

By applying Itô's Lemma to the approximation $\widetilde{S}(t, T_0, T_n)$, switching from the T_n - forward measure to the annuity measure $\mathcal{A}$ and replacing $\widetilde{S}(t, T_0, T_n)$ and $\frac{P(t,T_0)}{P(t,T_n)}$ by their initial values ($t = 0$), the swaption variance can be approximated as follows:

$$V_{\text{swap}}(T_0, T_n) \approx \left(\frac{P(0, T_0)}{P(0, T_0) - P(0, T_n)} \right)^2 V_p(0, T_0, T_n) \tag{67}$$

where $V_p(0, T_0, T_n)$ is the variance of the log-normal bond price ratio given by (56).

This approximation serves as a practical calibration tool used in industry and is implemented in pricing libraries, such as `QuantLib`.

The above dynamics rely on log-normal dynamics and employ the Black formula for pricing. However, in current market environments, particularly in negative interest rate regimes, the log-normal assumption is no longer suitable. Instead, the normal Bachelier model should be adopted, assuming the bond ratio follows a normal distribution.

In the next section, we use the `QuantLib` implementation, which supports the Bachelier (normal) model via `ql.Normal` volatility specification, thus matching market conventions for Euro swaption markets. Nevertheless, the log-normal variance formulation remains essential for understanding the theoretical linkage to classical affine term-structure models. As stated in Schachermayer & Teichmann (2008), when pricing At-the-money (ATM) options, the models produce very close prices and volatilities for $\sigma\sqrt{T} \ll 1$, which typically holds in real-life applications.

6 Hull-White One-Factor Model Calibration and Interpretation

6.1 *Calibration / Implementation Framework Using QuantLib*

The calibration of the Hull-White one-factor model was conducted using 228 European-style vanilla interest rate swaptions with normal implied volatilities obtained from Bloomberg (2025) as of 28-Feb-2025. The swaptions are priced ATM using the Bachelier (normal) model. The Euro Interbank Offered Rate (EURIBOR)-6M index was used as the floating leg benchmark, while discounting was performed using the overnight indexed swap OIS curve.

TABLE I: SWAPTION VOLATILITY CALIBRATION SUMMARY

Swaption Tenor	Market Implied Volatility	Model Implied Volatility	Relative Error (%)
1M/5Y	0.00698	0.00742	6.35
3M/5Y	0.00701	0.00730	4.16
9M/4Y	0.00747	0.00755	1.18
1Y/4Y	0.00748	0.00740	-1.08
2Y/3Y	0.00771	0.00739	-4.14
3Y/2Y	0.00778	0.00744	-4.39
4Y/1Y	0.00779	0.00754	-3.15

Note: Market source data from Bloomberg (2025) as of 28-Feb-2025. Model calibrated using QuantLib with 5Y co-terminal swaptions. Parameters: $a = 0.017344$, $\sigma = 0.01075$.

The initial yield curve used in this study was derived from the zero-coupon spot yield curve of AAA-rated euro area government bonds, published as of 28-Feb-2025 by European Central Bank (2025)[9]. These spot rates were transformed into discount factors using the standard continuous compounding formula:

$$P(t,T) = \exp\left(-r(t,T) \cdot (T-t)\right) \tag{68}$$

In line with the pricing framework developed by Burgard & Kjaer (2011), where the interest rate, r, is interpreted as a general risk-free rate, we adopt a single-curve setup. Both discounting and pricing curves are performed using the European Central Bank (2025) AAA-rated government bond yield curve. This curve is widely regarded as a reasonable proxy for the euro area's risk-free discounting rate.

To preserve internal consistency and avoid distortions at the short end of the curve, we replaced the simulated short rate at $t = 0$ with the observed EURIBOR-6M (2.389%) fixing as of 26-Feb-2025, from EU (2025), since this date corresponds to the fixing used for the first valuation of the swap on 28-Feb-2025.

The calibration was implemented in Python using the `QuantLib` library. Our approach is closely aligned with the theoretical structure outlined in Gurrieri et al. (2009)[10].

First, we constructed the initial term structure using `QuantLib`'s `DiscountCurve` class, enabling extrapolation for long-dated maturities. We then load swaption volatilities from Bloomberg (2025) and filter for Co-terminal swaptions - those whose underlying swaps all end on the same maturity date, regardless of their option expiry. This reduces the dimensionality of the calibration problem, thereby limiting the risk of overfitting.

Each selected swaption was mapped to a `SwaptionHelper` object using the `Normal`

[9] The first point on the curve corresponds to the forward rate starting on 28-May-2025.
[10] See Sections 2.1 and 2.2 of Gurrieri et al. (2009).

volatility type. For pricing, we use the `JamshidianSwaptionEngine`, which decomposes the swaption into a portfolio of bond options. Calibration was performed using the Levenberg-Marquardt algorithm, as also mentioned in Gurrieri et al. (2009), which minimizes the relative price error between the model and market prices. The parameters calibrated are the mean reversion speed, a, and the volatility σ.

Similarly to Gurrieri et al. (2009), three calibration strategies were employed:

- Constant mean reversion and constant volatility
- Constant mean reversion and time-dependent volatility
- Time-dependent mean reversion and volatility

To control model complexity and mitigate overfitting, we calibrate the Hull-White model using a set of 7 Co-terminal swaptions that all terminate within a 5-year maturity. This approach is grounded in both theoretical rigor and practical relevance. In particular, Puetter & Renzitti (2020) emphasize that:

"For a single swap portfolio, Co-terminal swaptions matching the swap's maturity and struck at the swap's fixed rate are the ideal choice, providing smile and maturity aware xVAs.".

Since our target portfolio comprises swaps maturing in 5 years, selecting swaptions that also expire at this horizon ensures consistency in modeling the relevant exposure dynamics. Furthermore, this calibration setup enhances the model stability. When calibrating to longer-dated instruments, such as 10- or 20-year swaptions, mean reversion parameters tend to become unreasonably low, lacking meaningful economic interpretation, as it would imply that the yield curve assumes random walk behavior, which is unacceptable under the Eurozone economic outlook.

Allowing time-dependent parameters offers flexibility, but increases the number of degrees of freedom, introducing the risk of overfitting. By selecting a constant mean reversion and time-dependent volatility, the model maintains a balance between tractability and empirical fit. Co-terminal swaption selection further reduces parameter noise and improves numerical stability.

We compared the three calibration strategies by plotting model-implied volatilities against observed market volatilities and evaluating the Root Mean Squared Error (RMSE). Although the full-time-dependent parameters provided the lowest RMSE, the marginal improvement did not justify the added complexity. Therefore, the model with constant mean reversion and time-dependent volatility was selected for final implementation.

6.2 Economic Interpretation of Calibrated Parameters

The Hull-White model was ultimately calibrated with the following parameters:

- **Mean reversion:** $a = 0.17344$
- **Volatility:** $\sigma = 0.01075$

The mean reversion parameter a measures how quickly the short rate returns to its long-term mean. Although the value of $a = 0.17344$ indicates relatively fast mean reversion, it is consistent with the Eurozone's monetary policy, where the European Central Bank (ECB) actively intervenes to anchor short-term interest rates.

The volatility parameter σ reflects the magnitude of fluctuations in the short rate. The average value of $\sigma = 0.01075$ (1.075%) aligns with historical volatility levels observed in relatively stable macroeconomic outlooks.

These parameters yield a calibration RMSE of approximately 3.9%, indicating a good match between model-implied and market swaption volatilities, particularly given the Hull-White model's parsimony and analytical tractability.

This alignment is visually confirmed in the figure below, where model-implied volatilities exhibit minimal deviation from observed market quotes.

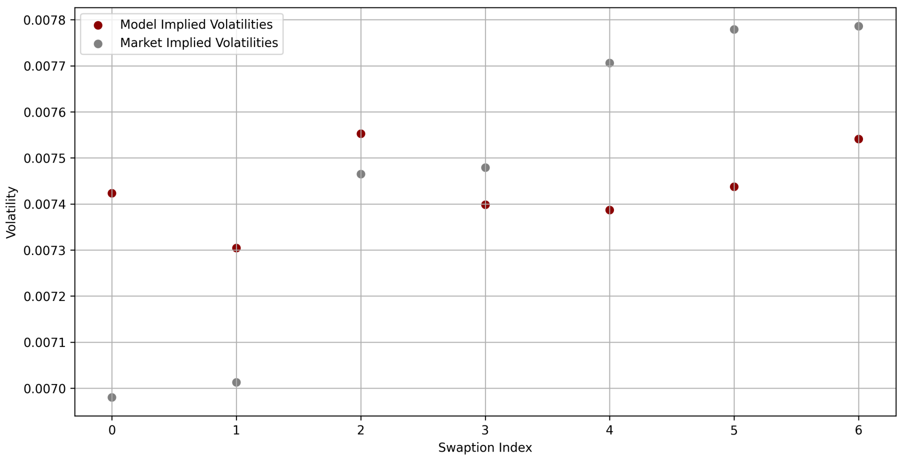

FIGURE 1: Model implied volatilities vs Observed market implied volatilities for each swaption using Co-terminal 5Y setting.

It is essential to note that the Hull-White model ensures an exact fit to the initial term structure of bond prices. This is achieved via internal calibration of the drift adjustment function within the `QuantLib` framework.

To validate this setting, we simulated 10,000 Monte Carlo paths of the short rate using the calibrated parameters. We computed the average bond prices across these scenarios for each tenor of the yield curve. The resulting curve of simulated bond prices closely aligns with the market-observed zero-coupon bond prices, with deviations not exceeding 0.5% across all maturities, confirming the model's consistency with the affine term structure of the initial yield curve.

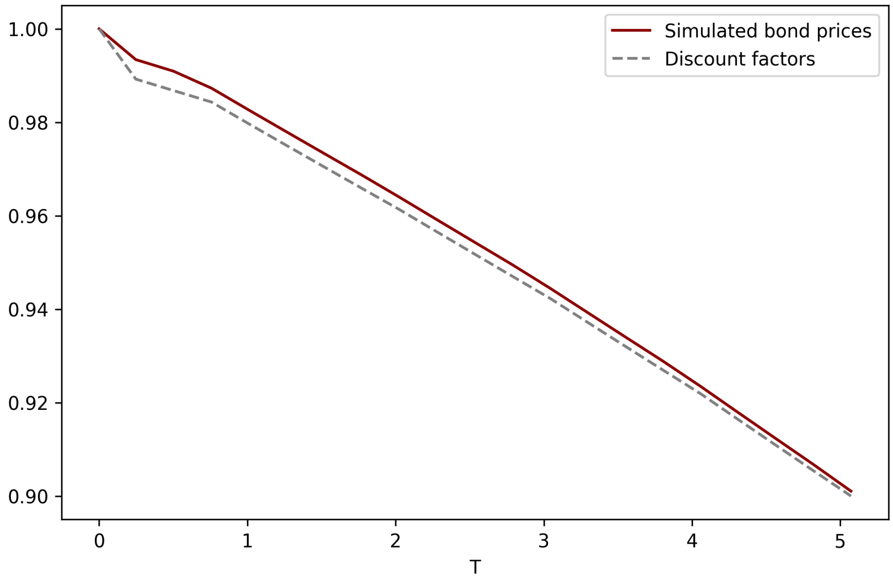

FIGURE 2: Estimated bond prices using Hull-White model by Monte Carlo simulation vs Observed Bond Prices as of 28-02-2025.

The Hull–White one-factor model was selected for its analytical tractability, closed-form solutions for bond and swaption pricing, and its ability to fit the initial yield curve exactly. However, the constant-parameter specification adopted in this work also comes with known limitations. It is often argued that the Hull–White model cannot accurately fit the full swaption volatility matrix, and that introducing time-dependent parameters may lead to numerical instability. However, Gurrieri et al. (2009) show that choosing a time-dependent version of the model parameters can achieve both robust fit and numerical stability. While that approach offers greater flexibility, it introduces additional complexity. In this work, we adopt a more parsimonious specification with constant parameters, which allows for stable calibration and a reasonable RMSE when fitting ATM swaption volatilities. A stability analysis of the time-dependent case is beyond the scope of this thesis.

7 Numerical Estimation of Bilateral Counterparty Risk and Funding Costs

Following the calibration of the Hull-White model with constant volatility and a constant mean-reversion parameter ($\sigma = 0.01075$ and $= a = 0.17344$), we proceeded to a simulation-based estimation of bilateral counterparty risk and funding costs adjustment. This numerical implementation aligns with the theoretical framework introduced in Chapters 2,3, and 4. The evolution of interest rates is captured using Monte Carlo simulations (10,000 paths) of the Hull-White process. These simulations enabled us to generate future yield curve scenarios for swap pricing, allowing for the assessment of expected exposures under various credit and funding conditions, including stressed market environments.

Our goal is to present numerical results in the following four scenarios:

1. $M = \hat{V}, \quad s_F = 0$
2. $M = \hat{V}, \quad s_F = (1 - R_B)\lambda_B$
3. $M = V, \quad s_F = 0$
4. $M = V, \quad s_F = (1 - R_B)\lambda_B$

We presented a coherent pricing framework for interest rate derivatives that incorporates bilateral counterparty credit risk and funding costs, providing a comprehensive approach to pricing these financial instruments. Starting from the partial differential equation replication methodology proposed by Burgard & Kjaer (2011), we extended the traditional risk-neutral valuation approach to reflect the cost of funding hedging strategies and the credit risk of both counterparties. The model was implemented using the Hull-White one-factor short-rate model, calibrated to market-implied volatilities from Bloomberg (2025), which enabled the simulation of realistic yield curves and the computation of valuation adjustments under various risk scenarios.

One of the key advantages of the Burgard & Kjaer (2011) framework is its ability to consistently treat funding and credit risks, as the value adjustments are derived endogenously within a unified replication-based structure. Rather than applying separate valuation adjustments, the framework models funding costs as they naturally arise from hedging, specifically through the dynamic rebalancing of positions in the seller's and counterparty's bonds. This avoids the conceptual and practical issue of double-counting, a concern prominently raised by Hull & White (2014), who argue that funding costs and own-credit adjustments must be clearly distinguished to prevent overstating derivative values.

Before delving into the specifics of the modeled swap, let us introduce an interest rate swap as a bilateral contract in which two parties agree to exchange cash flows based on interest payments over a specified period, based on a notional principal amount. We considered a plain vanilla swap from the perspective of the seller (bank B), who acts as the fixed-leg payer and the floating-leg receiver. The swap has a notional of 100M EUR, starting on 28-Fev-2025 and maturing on 28-Fev-2030 (5 years). Two different rates were considered for the fixed-rate: 2.1% (representing a market-aligned scenario, traded at par) and 2.5% (used as an implementation setup to enforce positive exposures over time). The floating leg is indexed to EURIBOR-6M, with semi-annual payments.

For each case, we computed the Expected Exposure (EE) from B's point of view for each time step, by evaluating the simulated mean Net Present Value (NPV) of the swap across the 10,000 Monte Carlo scenarios. The same approach was applied to the simulated short rates to obtain the mean forward rate curve, which is then used to construct the discount factors for each of the four cases listed above. The discount rates together with the hazard rates, λ_B and λ_C, are used in Riemann sum approximations of the integrals that arise from the Feynman-Kac representation of the valuation adjustments.

Recovery-rates were assumed to be 40% for both parties. The hazard rate λ_C of the counterparty was set at 0%, 2.5%, and 5%, while the hazard rate λ_B of the bank varies between 0% and 5%.

The PDEs associated with Case 1 and 2 are non-linear unless the sign of the risky value $\hat{V}$ is known and constant throughout the life of the derivative. In Burgard & Kjaer (2011) original application to European options, $\hat{V} \geq 0$ holds due to the unilateral nature of the payoff, allowing for a linear PDE and then a closed-form solution given by (39) and (42), respectively. This assumption does not extend to interest rate swaps, where the sign of $\hat{V}$ may vary over time.

To address this issue without solving the full non-linear PDE—which would be numerically demanding, for example, by requiring piecewise PDE solutions—we constructed an alternative swap with a fixed-leg rate of 2.5%, ensuring that the default-free value $V(t)$ remains positive for all $t \in [0, T]$. Under this setup, we verified numerically that the adjusted value $\hat{V}(t) = V(t) + U(t)$ remains non-negative for all $t \in [0, T]$ across the simulated paths. This setup preserves consistency with the assumptions required to linearize the PDE, enabling us to compute value adjustments as proposed. As such, for the swap with a fixed-leg rate of 2.1%, we present results only for Cases 3 and 4.

The exposure profiles $V(t)$ are estimated as the mean across simulated NPVs and are used directly in the numerical approximation of the integrals using Riemann sums. This ensures that the numerical implementation is aligned with the Feynman-Kac representation and the theoretical results.

In our analysis, we considered two interest rate swaps. The first swap, with a fixed rate of 2.5%, is not traded at par, which means that its NPV at inception is not zero. This implies that the swap is traded at a premium. The choice of this fixed rate is primarily motivated by implementation purposes, as explained previously. To model a more market-aligned scenario, we calibrated a second swap using the par rate implied by the modeled yield curve. By discounting the expected floating cash flows to the present, we estimated a par fixed rate of approximately 2.1% (2.056412%).

For the 2.1% fixed-rate swap, value adjustments changes are expressed as a percentage of the mean NPV over the swap's lifetime. Conversely, for the 2.5% fixed-rate swap, we express the value adjustments as a percentage of the NPV at inception.

The results are presented for both swap configurations (2.1% and 2.5% fixed-leg rates) under hazard rates for counterparty C (0%, 2.5%, and 5%).

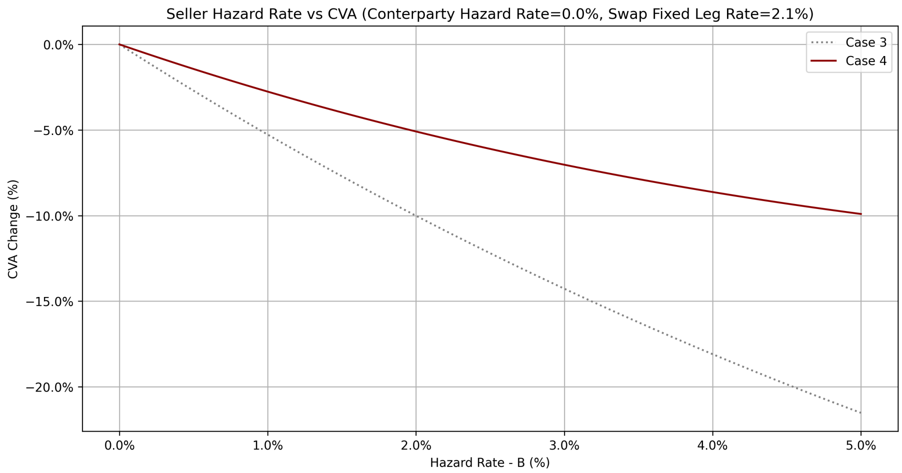

FIGURE 3: Value adjustments changes vs Seller B hazard Rate for swap fixed-leg rate at 2.1% and Counterparty C hazard rate set to 0%. Case 3: $M = V, \quad s_F = 0$; Case 4: $M = V, \quad s_F = (1 - R_B)\lambda_B$.

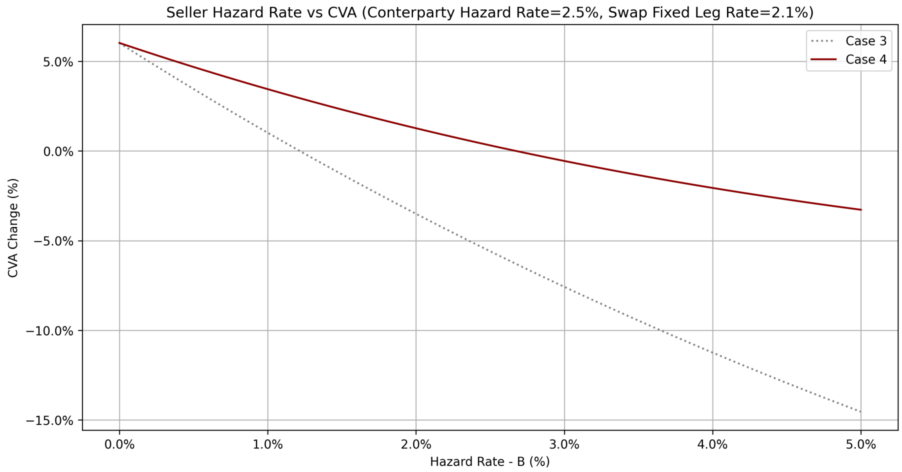

FIGURE 4: Value adjustments changes vs Seller B hazard Rate for swap fixed-leg rate at 2.1% and Counterparty C hazard rate set to 2.5%. Case 3: $M = V, \quad s_F = 0$; Case 4: $M = V, \quad s_F = (1 - R_B)\lambda_B$.

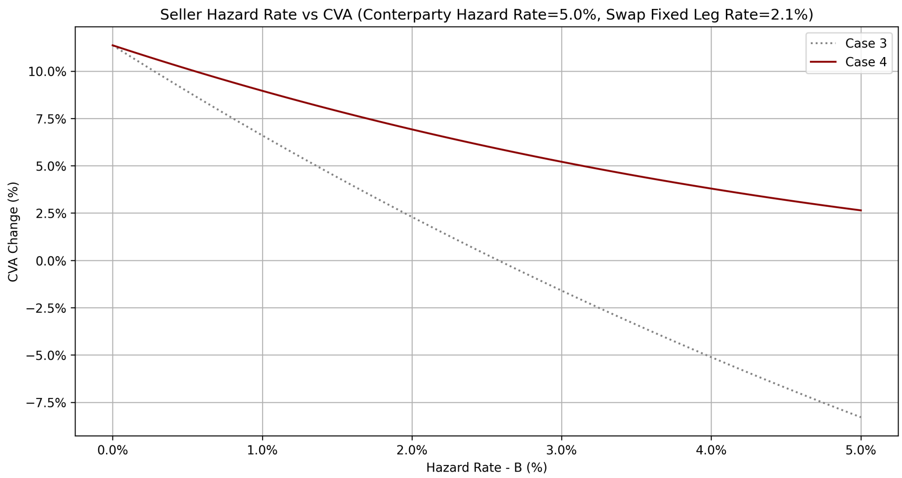

FIGURE 5: Value adjustments changes vs Seller B hazard Rate for swap fixed-leg rate at 2.1% and Counterparty C hazard rate set to 5%. Case 3: $M = V, \quad s_F = 0$; Case 4: $M = V, \quad s_F = (1 - R_B)\lambda_B$.

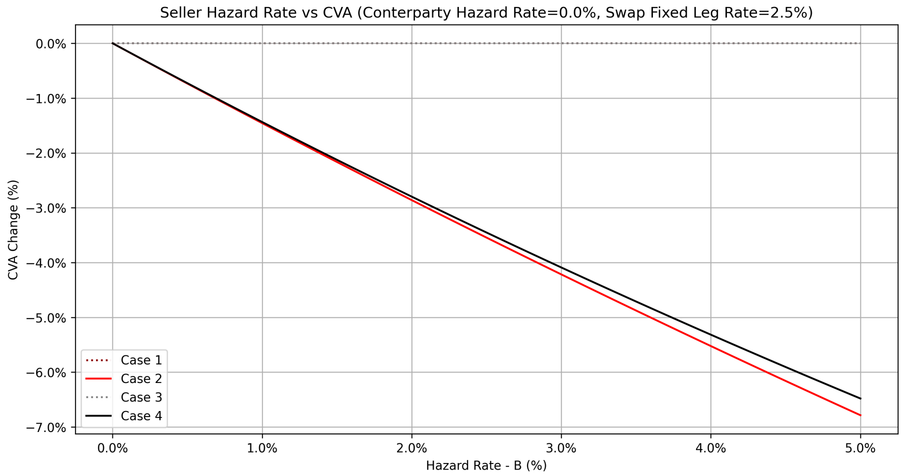

FIGURE 6: Value adjustments changes vs Seller B hazard Rate for swap fixed-leg rate at 2.5% and Counterparty C hazard rate set to 0%. Case 1: $M = \hat{V}, \quad s_F = 0$; Case 2: $M = \hat{V}, \quad s_F = (1 - R_B)\lambda_B$; Case 3: $M = V, \quad s_F = 0$; Case 4: $M = V, \quad s_F = (1 - R_B)\lambda_B$

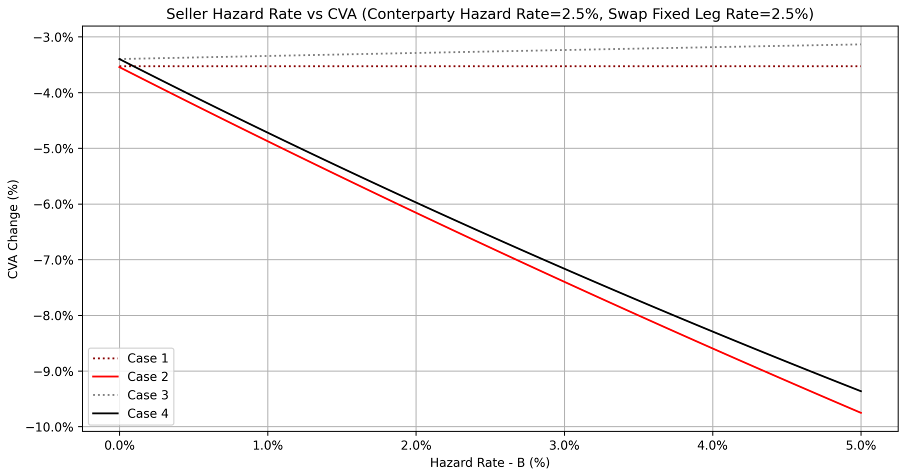

FIGURE 7: Value adjustments changes vs Seller B hazard Rate for swap fixed-leg rate at 2.5% and Counterparty C hazard rate set to 2.5%. Case 1: $M = \hat{V}, \quad s_F = 0$; Case 2: $M = \hat{V}, \quad s_F = (1 - R_B)\lambda_B$; Case 3: $M = V, \quad s_F = 0$; Case 4: $M = V, \quad s_F = (1 - R_B)\lambda_B$.

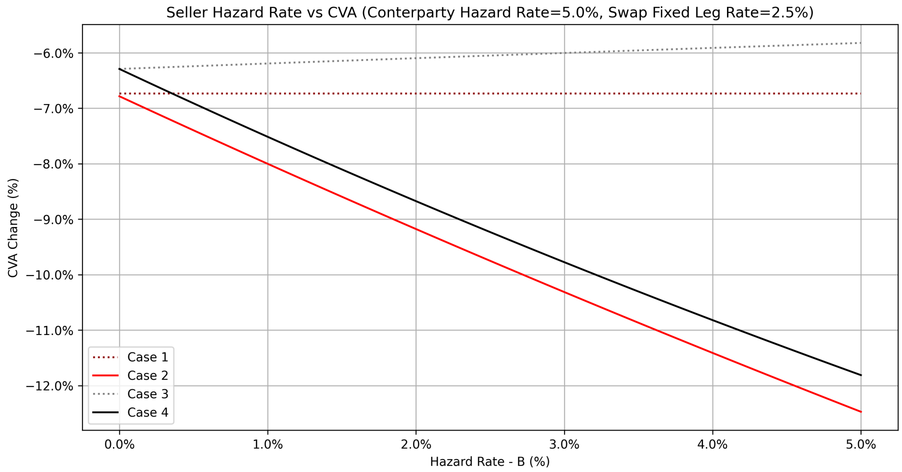

FIGURE 8: Value adjustments changes vs Seller B hazard Rate for swap fixed-leg rate at 2.5% and Counterparty C hazard rate set to 5%. Case 1: $M = \hat{V}, \quad s_F = 0$; Case 2: $M = \hat{V}, \quad s_F = (1 - R_B)\lambda_B$; Case 3: $M = V, \quad s_F = 0$; Case 4: $M = V, \quad s_F = (1 - R_B)\lambda_B$.

8 Conclusion and Future Developments

Hull & White (2014) emphasize that funding costs represent a real economic expense for the derivatives desk — a cost that must be funded and is reflected in the bank's economic Profit and Loss (P&L). Conversely, the reduction in liabilities due to the seller's own potential default is not a realized economic benefit, but rather an accounting adjustment. In their view, including both effects without care may lead to inflated or misleading derivative valuations. As a result, many practitioners adopt a pragmatic approach: they include funding costs but exclude own-credit benefits from pricing.

By contrast, the replication strategy of Burgard & Kjaer (2011) offers a structurally cleaner solution. Funding costs are embedded directly in the pricing PDE by adjusting the risky bond positions dynamically to mimic the underlying derivative's cashflows and account for the risky bond prices. Because both elements are derived endogenously within a single self-financing portfolio, their contributions are automatically aligned and do not require arbitrary separation. This internal consistency is a significant theoretical advantage, especially when pricing uncollateralized derivatives.

Our numerical results are consistent with these theoretical distinctions. For the 2.1% fixed-rate swap, which has a negative mean NPV, we analyze the behavior of the total valuation adjustment under varying levels of the seller's hazard rate. When the hazard rate of bank B is 0, the seller is only exposed to counterparty credit risk, and the value adjustment is negative to the bank. As the hazard rate of the seller increases, the adjustment becomes increasingly positive. This reflects a reduction in the present value of expected liabilities due to the bank's own credit risk. Although the adjustment becomes more positive with increasing default risk, this should not be interpreted as an economic gain. Instead, it represents a reduction in expected obligations, consistent with Hull & White (2014)'s view that such effects are accounting adjustments rather than realizable profits. Furthermore, when funding costs are introduced (as in Case 4), they partially offset the upward adjustment due to own-credit risk. This effect reduces the steepness of the adjustment curve, highlighting the cost of financing the derivative position.

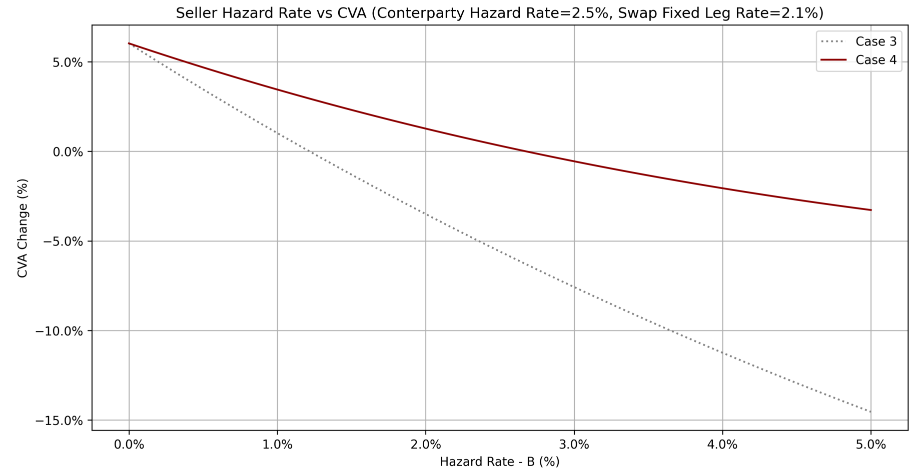

FIGURE 9: Value adjustments changes vs Seller B hazard Rate for swap fixed-leg rate at 2.1% and Counterparty C hazard rate set to 2.5%. Case 4 englobes funding costs, whereas Case 3 depends only on the credit risk.

For the 2.5% fixed-rate swap, which maintains a positive NPV throughout its lifetime, the seller is only exposed to counterparty and funding risks. In this setting, the incorporation of funding costs leads to a significant increase in the total valuation adjustment, highlighting the sensitivity of uncollateralized positions to funding spreads.

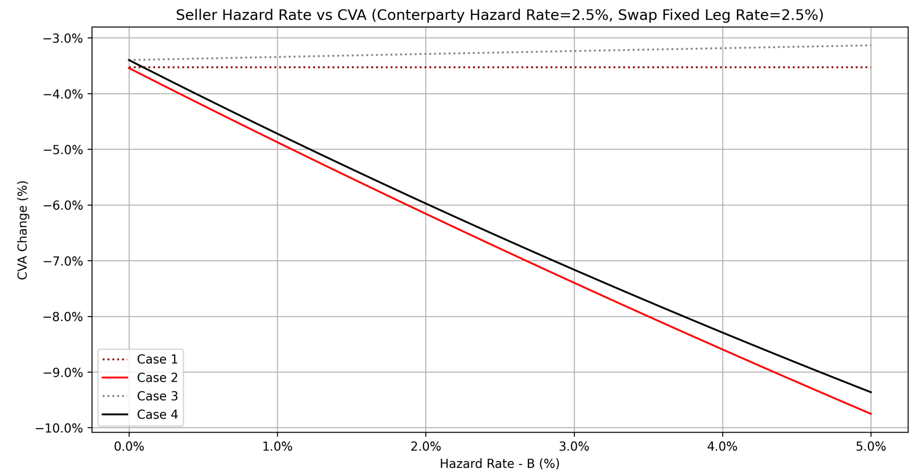

FIGURE 10: Value adjustments changes vs Seller B hazard Rate for swap fixed-leg rate at 2.5% and Counterparty C hazard rate set to 2.5%. Case 2 and Case 4 englobe funding costs, whereas Case 1 and Case 3 depend only on the credit risk.

Across both results for each swap, we found that using the risk-free close-out value V instead of the risky value $\hat{V}$ had a negligible effect on the final valuation, supporting industry practice and aligning with ISDA protocols that favor risk-free settlement conventions.

In summary, this work contributes to a consistent, theoretically grounded, and practically relevant approach to pricing counterparty and funding risks. By embedding both effects within a replication-based PDE framework, the approach avoids double-counting, respects economic reality, and maintains mathematical tractability. It bridges the practical caution of Hull & White (2014) with the structural completeness of the Burgard & Kjaer (2011) approach.

Although the proposed framework captures key features of counterparty and funding risk, it relies on the risk-neutral measure $\mathbb{Q}$ and includes certain simplifying assumptions. In particular, recovery rates and hazard rates are modeled as deterministic and independent of market dynamics. Future work could extend the framework to incorporate stochastic credit intensities, collateral haircuts, and joint exposure–default modeling, thereby better reflecting real-world complexity.

8.1 Directions for Further Research

As environmental concerns grow and climate regulations become more stringent, carbon-related costs—such as carbon taxes, emissions trading schemes, and regulatory penalties—are no longer distant policy risks. They are becoming real, measurable financial risks that directly affect the valuation of projects and financial instruments. As climate regulations tighten and carbon prices rise, financial products that fail to account for their environmental impact risk becoming stranded, meaning they may struggle to attract investment, repay debt, or deliver expected returns due to increased climate-related costs and restrictions.

To address this emerging challenge, recent literature has proposed new frameworks for incorporating environmental risks into financial valuation. A notable recent development is the Carbon Equivalence Principle (CEP), proposed by Kenyon et al. (2022). The CEP proposes that every financial product—whether a loan, derivative, or bond—should disclose its "carbon-equivalent" exposure: the emissions it causes or enables. This is formalized in a secondary carbon term sheet, which operates in parallel to the traditional financial one. By placing carbon flows on equal footing with cash flows, the CEP enables carbon risk to be monitored, managed, and priced alongside conventional metrics, such as CVA and FVA.

An essential extension of this principle is the concept of Financial Net-Zero (FNZ).

Unlike traditional carbon net-zero goals that focus solely on reducing physical emissions, FNZ aims to neutralize the financial cost of those emissions, such as carbon taxes, penalties, or the cost of offsetting. Achieving FNZ may involve the use of Negative Emissions Technology (NET)s, which are techniques designed to remove carbon dioxide from the atmosphere actively, as well as restructuring financial transactions or leveraging carbon credits. Much like hedging interest rate or credit exposures, FNZ seeks to hedge the cost of carbon across a product's life.

To make this idea more concrete, consider a plain vanilla IRS between a bank and a counterparty in the fossil fuel sector. The swap itself does not emit carbon, but it facilitates emissions by helping finance a carbon-intensive project. Under the CEP, this swap would carry a carbon term sheet disclosing its share of the emissions generated by the project. Based on carbon price projections—such as those published by the Network for Greening the Financial System (NGFS)—the expected carbon liability over the swap's life could be priced using a Carbon Dioxide Equivalent Emission Adjustment (COe) just as we do with CVA or FVA. For example, suppose the IRS supports a €100 million oil project expected to emit 500,000 tonnes of carbon dioxide. The swap's share of those emissions can be estimated, priced using NGFS carbon price scenarios, discounted, and added to the valuation model. This enables market participants to assess the financial viability of the deal under different climate policy paths, such as a Net Zero 2050 scenario or a Delayed Transition.

Looking forward, a natural extension of this thesis would be to incorporate COe into the framework presented here.

The tools developed in this thesis for CVA and funding costs provide a foundation for building more comprehensive models. As the integration of carbon costs into financial valuation is not just a theoretical possibility, it is quickly becoming a practical necessity.

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