Master

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Master's Final Work

Dissertation

Optimal Reinsurance maximising the Expected Utility of the Insurer's Surplus in a Cox Process

João Oliveira Ferreira

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Supervision:

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Manuel Guerra

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The most important questions of life are indeed,
for the most part, really only problems of probability.
- Pierre-Simon Laplace

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Abstract

The main objective of this thesis[1] is to study the optimal reinsurance problem, from the ceding insurance company's perspective. The direct insurer has monotonic preferences (more wealth is better), limited resources and rates his wealth by a non-decreasing concave utility function. The wealth of the cedent is modelled considering a modified version of the classical Cramér-Lundberg surplus process. A Cox process with a Poisson shot noise intensity is used to model the claim arrivals, introducing dependencies between inter-arrival times. Proportional reinsurance treaties are considered and the percentage that the first-line insurance company wants to cede for each claim is denoted as α. Therefore, the cedent seeks the optimal percentage $\hat{\alpha}$ that maximises his expected utility of wealth in any given year.
Furthermore, an implicit solution, the optimal condition is obtained. The results show that the optimal level of reinsurance depends on the relationship between the first-line insurer's expected utility of wealth and the level of reinsurance.

Keywords: Cox Process, Reinsurance, Expected Utility, Insurer's Surplus, Quota Share.

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Resumo

O foco principal desta tese[2] é estudar o problema do resseguro ótimo sob a ótica da seguradora cedente. Esta seguradora tem preferências monotónicas (mais riqueza é melhor), recursos limitados e a sua riqueza segue uma função de utilidade côncava não decrescente. O excedente do segurador é modelado considerando uma versão modificada do processo clássico de excedente de *Cramér-Lundberg*. O processo *Cox* com *Poisson* de intensidade *shot noise*, é utilizado para modelar a intensidade de chegadas de sinistros, introduzindo dependências entre os mesmos. São considerados contratos de resseguro proporcional e a percentagem que a seguradora de primeira linha quer ceder para cada sinistro é denotado como α. Portanto, a cedente procura uma percentagem ótima, $\hat{\alpha}$, que maximize a utilidade esperada da sua riqueza, para um determinado ano.
Por último, é obtido uma solução implícita, da qual seguem as condições de otimalidade. Os resultados mostram que o nível ótimo de resseguro depende da relação entre a utilidade esperada da riqueza da seguradora de primeira linha e o nível de resseguro.

Palavras-chave: Processo Cox, Resseguro, Utilidade Esperada, Excedente do Segurador, Resseguro de Quotas.

[2]Esta Tese foi financiada pela FCT - Fundação para a Ciência e Tecnologia através de uma bolsa de investigação que faz parte do projecto de investigação "Optimal-Re: Optimal reinsurance with dependencies", com a referência EXPL/EGE-ECO/0886/2021.

Contents

CHAPTER 1

Introduction

Uncertainty has always been a part our lives. Everyone has dealt with situations whose outcomes were unknown, simply put, the outcomes were random. Nonetheless, uncertainty was once considered undesirable by the scientific community. It was not until the 20th century that, due to a paradigm shift, it stopped being regarded as unscientific and started to be considered something unavoidable and of significant value (see Klir and Yuan (1995)). Hubbard (2014) defined uncertainty merely as the lack of certainty, in other words, a state where it is impossible to describe with certainty the existing and future outcomes. On the other hand, when the possible outcomes are known and can be quantified using probabilities, we use a risk measure.

Insurance is a contract in which an insurance company agrees to reimburse an individual or entity, in case of losses, in exchange of a payment, the insurance premium. Reinsurance is a contract in which the reinsurer gives a reimbursement to the cedent company for specified parts of its insurance risk, in exchange reinsurance premium. To put it another way, the cedent company transfers part of its risk to the reinsurer, improving the cedent's company risk profile, but decreasing its wealth due to the premium payment.

As already stated, reinsurance is an allocation of risk between two insurance companies, so in terms of regulation it also differs. The close relationship between the cedent insurance company and reinsurer allows frequently a long-term association when compared to a plain insurance contract, which subsequently impacts the reinsurance premiums. There are clear benefits for both the insurer and reinsurer from having this long-term relationship such as industry knowledge sharing, usually the reinsurer also acts as a consultant for the insurance company and so on.

The aim of this thesis is to study the optimal reinsurance problem, from the ceding insurance company's point of view. We approach this problem by modelling the uncertainty surrounding the claim arrivals intensity using a Cox process, with a Poisson shot noise intensity, where we consider dependencies between claims, together with an altered version of the classical Cramér-Lundberg surplus process to model the insurer's wealth. The direct insurer has monotonic preferences (more wealth is better), limited resources and rates his wealth by a non-decreasing concave utility function. In addition, the cedent insurance company uses a quota share reinsurance treaty, where it tries to find the percentage it wants to cede for each claim, denoted α, that allows it to maximise the expected utility of its wealth, for a given year. Ceding a percentage of each claim comes at cost for the first-line insurance company, the reinsurance premium, which is the amount that they have to pay to the reinsurer. The ceding insurance company has to assess whether to buy more or less reinsurance, in order to, mostly important have bigger profits and take lesser risk.

The optimal reinsurance-investment problem has been been broadly studied in the academia. Nonetheless, we take a different approach by studying the optimal reinsurance problem without the possibility of investment, considering both externally excited jumps and dependencies between claims and using a quota share reinsurance strategy.

We solve analytically, using the expected value principle as a way of pricing the reinsurance premium, getting an implicit solution of the problem.

This thesis is organised in the following structure. In Chapter 2, it is done a brief overview in reinsurance and it is carried out a literature discussion of research articles on this topic, especially focusing on proportional reinsurance with dependencies. In Chapter 3, the optimisation problem is settled, starting by the modelling of the reinsurance company's surplus process and introducing the mathematical background. In Chapter 4, the quota share proportional reinsurance optimisation problem is solved by designing the optimality conditions. In Chapter 5, the main findings are analysed and conclusions are outlined, leaving new clues for further future research in optimal reinsurance.

CHAPTER 2

Reinsurance and Literature Review

Reinsurance plays a very important role in the insurance industry constituting a flexible and sustainable technique for insurers to mitigate risks.

In reinsurance, insurers transfer part of their risk to the reinsurers. However, that comes at a price, the reinsurance premium. To find a balance between the risk mitigation and the decrease in profits is the purpose of the optimal reinsurance problem. Thus, reinsurance allows for the insurer to reduce his risk exposure and therefore to reduce the amount of capital it must hold to satisfy safety constraints and regulators requirements. The capital requirement protects consumers but limits the amount of each policy an insurer can take on, hence through reinsurers, insurers can take more and larger insurance policies. This is only possible because through reinsurance, insurers cede a part of the risk to the reinsurers against the payment of a premium. The insurance company that issues the policy in the first instance is known as the ceding insurance company. The company that takes the ceded part of insurer's risk is the reinsurer.

Reinsurance contracts are classified into two types: proportional and non-proportional. We have quota share reinsurance and surplus reinsurance as proportional types of reinsurance. The non-proportional reinsurance treaties are the excess of loss and the stop loss reinsurance. Next, we briefly describe these four types of reinsurance.

- **Quota share** reinsurance is a form of reinsurance in which the reinsurer takes a fixed percentage of all the risk insured by the ceding insurer, for that, the reinsurer receives the corresponding share of the direct (cedent) insurance premium. Let $0 < a < 1$, be the percentage of each risk ceded by the insurer, and $1 - a$ the percentage retained. For each claim Y, the insurer cedes $Y_a = aY$ and retains $(1 - a)Y$. To contribute to management and acquisition costs, the reinsurer pays the insurance company a commission that is proportional to the premiums received (see Centeno (2003)).
- **Surplus** reinsurance is a type of reinsurance where the policies whose insured capital exceeds a specified limit M are partially ceded to the reinsurer. Let V be the sum insured in a policy, if an accident occurs causing a reimbursement of Y amount and if $V \leq M$, the ceding company is liable for the reimbursement. If $V > M$ then the ceding company retains the M/V proportion of the compensation (see Centeno (2003)).
- **Excess of loss** reinsurance is a type of non-proportional reinsurance treaty in which the direct insurer chooses the amount to withhold from each indemnity that occurs, called retention limit M. Individual losses higher than the retention limit are the reinsurer's liability. The reinsurer's liability is generally also limited to a specified amount

for each individual loss, called coverage L. That is, the reinsured risk is $Z(M, L) = min(L, (Y - M)_+)$.

The insurer, in designing its reinsurance program, may make use of various Excess of loss reinsurance treaties. Treaties are generally made so that the $M + L$ ceiling of one treaty is the retention limit of the next treaty. When $L = +\infty$ the coverage is said to be unlimited. The reinsurer may limit its liability to a fraction of the excess $Y - M$ (see Centeno (2003)).

- **Stop loss** reinsurance is an aggregate excess of loss reinsurance contract. The reinsurer assumes the liability for all claims arising from claims incurred over a certain period (usually a year), that overrall exceed a limit M previously chosen by the first-line insurer for its retained claims, generally limiting its liability to a certain amount (see Centeno (2003)).

Often, reinsurance agreements require more than one reinsurer joining the same contract at separate percentages of participation, i.e., Multiparty Reinsurance Contracts, in which the cedent insurance company frequently purchases its reinsurance from more than one reinsurer under the same reinsurance agreement. Multiple reinsurers joining the same agreement allows for a greater risk-sharing among all participants, reducing the risk of loss to the reinsured company. This ultimately leads to economic growth boost and to an increase in the overall stability due to an higher risk-sharing between insurance companies. Furthermore, reinsurance also helps insurers to disasters, lower risk by helping to diversify the insurer's portfolio and exchange expertise. These are other relevant elements that prove reinsurance to be a drive for improving market efficiency (see Albrecher, Beirlant, and Teugels (2017)). However, the optimisation problem from the reinsurer's and insurer's point of view are completely different. Conflict of interest arise when trying to find an equilibrium between the profit maximisation of both the reinsurer and insurer. The reinsurer decides how much risk is worth taking per monetary unit of reinsurance premium received. The insurer decided how much to cede to the reinsurer to minimise the risk, but also to not reduce the insurer's profit beyond a point which is no longer profitable to cede. The latter, the optimal reinsurance problem, from the ceding insurance company's perspective, is the focus of our research. The objective is to maximise the insurer's expected utility of wealth using a quota share reinsurance treaty. This goal changes our exercise into a complex mathematical problem and that's when the interplay between mathematics and finance sets-in.

2.1. State of the art

As discussed above, insurance companies buy reinsurance to reduce their risk. One way to measure their likelihood of reaching a "near default" state is to look at their classical a surplus process, S_t, as a function of time. The classical surplus process known as the Cramér-Lundberg model is defined as (Albrecher et al. (2017)),

$$S_t = u + ct - \sum_{i=1}^{N_t} Y_i$$

where, u is the amount of initial reserves, c is the constant premium intensity, N_t is a Poisson process with intensity λ and the random variables $\{Y_i\}_{i\geq 1}$ represent the size of claims and are considered to be i.i.d.. For the sake of simplicity, this model ignores many important aspects of the surplus process of an insurance company, for example, inflation and interest rates, investments, dividend payments, dependence between risks and so on are neglected. In recent studies, new variants of the Cramér-Lundberg model have been developed to take into account these aspects, leading to more complex mathematical problems (see Albrecher et al. (2017) and (Asmussen & Albrecher, 2010)).

In this Thesis, we consider a modification to the classical surplus process, mentioned above, where a Cox process with a Poisson shot noise intensity is used to model the claim arrivals intensity, instead of a Poisson process, similar to what is introduced by Dassios and Zhao (2011) and then developed in Cao, Landriault, and Li (2020). However, unlike the latter, we are not interested in studying an optimal-investment reinsurance problem. We focus our attention just on an optimal reinsurance problem using a quota share reinsurance treaty that only allows to capture the externally excited jumps, such as earthquakes, drought and floods. The Cox process is first proposed by Cox (1955). In this seminar paper, it is discussed that through space and time there exists a link between events occurring in a random manner, but it is only later on that researchers suggest using a Cox process to model the claim arrivals (see Cox (1955)).

In the existing literature, apart from Dassios and Zhao (2011) and Cao et al. (2020), we find a few papers using also counting processes, such as Poisson and Cox processes, for their insurance models. Namely, Björk and Grandell (1988), Embrechts, Schmidli, and Grandell (1993), Albrecher and Asmussen (2006) and Delong and Gerrard (2007). Björk and Grandell (1988) and Embrechts et al. (1993) both use a Cox process to model the number of claims. The main difference between the two is that in the first, the authors use diffusion processes to obtain the finite-time Lundberg inequalities while in the second, the authors apply piecewise-deterministic Markov processes. Albrecher and Asmussen (2006) consider a superposition of an homogeneous Poisson process and a Cox process (Cox, 1955) with a Poisson shot noise intensity to model the claim arrivals, taking into account the externally excited factors. Delong and Gerrard (2007) use a compound Cox process to model the claim arrivals intensity whose evolution is drawn by a stochastic differential equation driven by a Brownian motion. Dassios and Zhao (2011) introduce for the first time the dynamic contagion process to model the claim arrivals dynamics. This process incorporates both the self-excited factors described by the Hawkes process (Hawkes, 1971) and the externally excited factors by the Cox process (Cox, 1955) with shot noise intensity.

Cao et al. (2020) study optimal reinsurance-investment strategy using compound dynamic contagion process to model the claim arrivals. In that case, the excess of loss reinsurance treaty is the optimal treaty using a time-consistent mean-variance criterion. With this type of strategy, it was possible for the insurer to allocate the increased return from the risk-free bond on the increase of the reinsurance coverage through the decreasing of the retention limit. This dynamic contagion claims model included a Cox process (Cox, 1955) with shot noise intensity and a Hawkes process (Hawkes, 1971), capturing self-exciting and externally exciting jumps.

Taking a different approach from the literature mentioned above, Schmidli (2002) studies optimal reinsurance-investment problem using a Poisson process to model the claim arrivals and the Black-Scholes equations to model the investment on the risky asset. The solution to the problem is found using Hamilton-Jacobi-Bellman equation. Guerra and Centeno (2008) tries to reach an optimal-form of reinsurance by looking to maximise the adjustment coefficient of the retained risk of the first-line insurer. When reinsurance is priced by an expected value principle, the stop loss reinsurance treaty is found to be optimal and when it is priced by the a variance premium principle, the nonlinear function is optimal form.

In this work, we analyse the optimal reinsurance problem, from the ceding insurance company's point of view. We model the claims intensity process, λ_t using a Cox process with externally excited jumps, with the special case of exponential decay and the insurer's wealth using a modified version of the classical Cramér-Lundberg surplus process. Dependencies between claims are considered. Furthermore, the cedent insurance company uses a proportional reinsurance treaty - the quota share reinsurance treaty. The first-line insurer tries to maximise the expected utility of its wealth in respect to the percentage the insurer cedes for each claim, α. Even though, an increase in α means a lesser risk to take for each claim by the cede, the bigger the α, the bigger the reinsurance premium paid to the reinsurer. The reinsurance premium is priced using the expected value principle. In the following chapter, we dive deeper into this problem and show the whole process of solving it.

CHAPTER 3

The Cox Process

In this chapter, before introducing the stochastic intensity claim process it is useful and important to first study and understand the properties of the Poisson and Cox counting processes, as well as the properties of the exponential distribution.

3.1. Counting processes

The Poisson and Cox processes are counting processes that have been used with multiple applications in insurance and credit risk modelling (see Jang and Oh (2021)). Further on, we will describe the insurer's surplus process including the stochastic claim intensity process.
The (homogeneous) Poisson process is a counting process that has a (constant) deterministic intensity rate ρ and possesses the memoryless property, that is the arrival of an event is independent of previous events. The average number of occurrences in the interval is known and given by ρ, but the exact time at which each one occurs is unknown (see Jang and Oh (2021)). When the inter-arrival times are independent exponentially distributed, a random variable X has a exponential distribution with parameter $\rho > 0$ and its distribution function is given by $F(x) = 1 - e^{-\rho x}$ for $x \geq 0$, with a density function $f(x) = \rho e^{-\rho x}$ for $x > 0$ (see Dickson (2016)).
The probability function of an homogeneous Poisson process K, with intensity ρ is

$$Pr(K_t = n) = e^{-t\rho}\frac{t^n \rho^n}{n!}. \tag{3.1}$$

An inhomogeneous Poisson process is a Poisson process with a time-varying rate $\rho(t)$. Therefore, it allows the intensity of the process λ, to vary over time. The intensity is a locally integrable function, λ: $[0, +\infty[\mapsto [0, +\infty[$. Let T_i be the i^{th} arrival time and $t_0 \in [0, +\infty[$. If the intensity function λ is known then, the distribution function of an inhomogeneous Poisson process is

$$F_{T_i|T_{i-1}\leq t_0<T_i} = P\{T_i \leq t|T_{i-1} \leq t_0 < T_i\} = 1 - \exp\Big(-\int_{t_0}^{t} \lambda_s \, ds\Big), \tag{3.2}$$

with a density function

$$f_{T_i|T_{i-1}\leq t_0<T_i} = \lambda_t \exp\Big(-\int_{t_0}^{t} \lambda_s \, ds\Big). \tag{3.3}$$

REMARK 3.1. *If λ is integrable, then*

$$F_{T_i|T_{i-1}\le t_0<T_i}(+\infty) = P\{T_i < +\infty | T_{i-1} \le t_0 < T_i\} = 1 - \exp\Big(-\int_{t_0}^{+\infty} \lambda_s \, ds\Big) < 1$$

i.e. the probability of a trajectory without jumps after time t_0 is strictly positive.

The following proposition extends the Equation (3.1) to inhomogeneous Poisson processes.

PROPOSITION 3.1. *The probability function of an inhomogeneous Poisson process $\{N_t\}_{t\in[0,+\infty[}$, with intensity function λ is*

$$(3.4)\quad Pr\{N_t = N_{t_0} + n\} = \exp\Big(-\int_{t_0}^{t} \lambda_u \, du\Big) \frac{\Big(\int_{t_0}^{t} \lambda_u \, du\Big)^n}{n!} \quad \textit{whenever} \quad 0 \le t_0 \le t < +\infty.$$

Proof: see Appendix.

The (3.2), (3.3) and (3.4) Equations above are also valid for $P\{T_i \le t | T_i > t_0, \{\lambda_s\}_{s\in[t_0,t]}\}$.

3.2. The Cox Surplus process

A Cox process is a time-inhomogeneous Poisson process with stochastic intensity. In particular, Jang and Oh (2021) considered a Cox process of the following type.

DEFINITION 3.1 (Cox process with externally excited jumps). *A Cox process $\{N_t\}_{t\ge 0}$, with shot noise intensity is a time-inhomogeneous Poisson process where the intensity λ_t is a stochastic process of type*

$$(3.5)\qquad \lambda_t = a + (\lambda_0 - a)e^{-\delta t} + \sum_{j=1}^{K_t} V_j e^{-\delta(t-T_j)},$$

where

- *$\lambda_0 \ge a$ is the initial intensity at time $t = 0$*
- *$a \ge 0$ is a constant (base line) level of intensity*
- *$\delta > 0$ is a rate of exponential decay of intensity above the base line level*
- *$\{V_j\}_{j=1,2,...}$ is a sequence of i.i.d positive random variables, independent of the process K, representing the severities of external shocks*
- *K is a homogeneous Poisson process with intensity ρ and arrival times $\{T_j\}_{j=1,2,...}$.*

The conditional distribution function of a Cox process given the intensity sample path λ is given by

$$F_{T_i|T_{i-1}\le t_0<T_i,\lambda} = P\{T_i \le t | T_{i-1} \le t_0 < T_i, \lambda\} = 1 - \exp\Big(-\int_{t_0}^{t} a + (\lambda_0 - a)e^{-\delta s} + \sum_{j=1}^{K_s} V_j e^{-\delta(s-T_j)} \, ds\Big)$$

$$= 1 - \exp\Big(-a(t - t_0) - (\lambda_0 - a)e^{-\delta t_0} \frac{1 - e^{-\delta(t-t_0)}}{\delta} - \sum_{j=1}^{i-1} V_j \frac{1 - e^{-\delta(t-T_j)}}{\delta}\Big),$$

with a conditional density function given by

$$f^{(t)}_{T_i|T_{i-1}\le t_0<T_i,\lambda} = \Big(a + (\lambda_0 - a)e^{-\delta t} + \sum_{j=1}^{i-1} V_j e^{-\delta(t-T_j)}\Big)\times$$
$$\times \exp\Big(-a(t-t_0) - (\lambda_0 - a)e^{-\delta t_0}\frac{1-e^{-\delta(t-t_0)}}{\delta} - \sum_{j=1}^{i-1} V_j \frac{1-e^{-\delta(t-T_j)}}{\delta}\Big).$$

Taking into consideration Proposition 3.1, we get that the conditional probability function of the Cox process $\{N_t\}_{t\ge 0}$ is

$$Pr\{N_t = N_{t_0} + n | T_{i-1} \le t_0 < T_i, \lambda\} =$$
$$= \exp\Big(-a(t-t_0) - (\lambda_0 - a)e^{-\delta t_0}\frac{1-e^{-\delta(t-t_0)}}{\delta} - \sum_{j=1}^{i-1} V_j \frac{1-e^{-\delta(t-T_j)}}{\delta}\Big)\times$$
$$\times \frac{\Big(a(t-t_0) + (\lambda_0 - a)e^{-\delta t_0}\frac{1-e^{-\delta(t-t_0)}}{\delta} + \sum_{j=1}^{i-1} V_j \frac{1-e^{-\delta(t-T_j)}}{\delta}\Big)^n}{n!}.$$

The surplus process of the insurance company without investment used here is the Classical Cramér-Lundberg model (explained in Section 2.1.).

DEFINITION 3.2. *The insurer's surplus process* $\{\omega_t\}_{t\ge 0}$ *is defined as*

$$\omega_t = u + ct - \sum_{i=1}^{N_t} Y_i \tag{3.6}$$

where

- u *is the amount of initial reserves to cover potential losses*
- c *is the amount of premiums the insurer receives, per unit of time*
- $\{Y_i\}_{i\ge 1}$ *is a sequence of independent and identically distributed nonnegative random variables representing claim severities*
- $\{N_t\}_{t\ge 0}$ *is a Cox process with shot noise intensity (3.5) and arrival times* $\{\theta_i\}_{i\ge 1}$
- $\{Y_i\}_{i\ge 1}$ *is independent of* N, K *and* $\{V_j\}_{j=1,2,\ldots}$.

3.3. Expectations

Consider the $\mathbb{R}^2$ - valued process $\{\lambda_t, w_t\}_{t\ge 0}$, where w is the surplus process (3.6) and λ is the corresponding intensity process (3.5). We wish to characterise the function $G : [0,1] \times \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R}$, defined as

$$G(t, v, \omega) = \mathrm{E}[\varphi(\lambda_1, \omega_1) \mid \lambda_t = a + v, \omega_t = \omega], \tag{3.7}$$

where $\varphi : \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R}$ is a measurable function satisfying suitable regularity conditions.

PROPOSITION 3.2. *The following estimates hold when $h \mapsto 0^+$:*

(1) $Pr(K_{t+h} = K_t, N_{t+h} = N_t | \lambda_t = a + v, \omega_t = \omega) = 1 - (\rho + a + v)h + o(h)$

(2) $Pr(K_{t+h} = K_t + 1, N_{t+h} = N_t | \lambda_t = a + v, \omega_t = \omega) = \rho h + o(h)$

(3) $Pr(K_{t+h} = K_t, N_{t+h} = N_t + 1 | \lambda_t = a + v, \omega_t = \omega) = (a + v)h + o(h)$

Proof: see Appendix.

Due to the above proposition, we can say that,
$Pr\{(K_{t+h}, N_{t+h}) \notin \{(K_t, N_t), (K_t + 1, N_t), (K_t, N_t + 1)\} | \lambda_t, \omega_t\} = o(h)$.

Using Proposition 3.2 and assuming continuity of G, we obtain

$$\begin{aligned}
G(t, v, \omega) &= \mathrm{E}[G(t + h, \lambda_{t+h} - a, \omega_{t+h}) \mid \lambda_t = a + v, \omega_t = \omega] \\
&= G(t + h, ve^{-\delta h}, \omega + ch)(1 - (\rho + a + v)h) \\
&+ \mathrm{E}[G(t, v + V, \omega)]\rho h + \mathrm{E}[G(t, v, \omega - Y)](a + v)h + o(h) \\
&= G(t + h, ve^{-\delta h}, \omega + ch) \\
&+ \Big(\rho\, \mathrm{E}[G(t, v + V, \omega)] + (a + v)\, \mathrm{E}[G(t, v, \omega - Y)] - (\rho + a + v)G(t, v, \omega)\Big)h + o(h),
\end{aligned}$$

where the first equality follows from the iterative property of conditional expectations. Assuming that G is differentiable, dividing by h and making $h \to 0^+$, we get

$$\begin{aligned}
\lim_{h\to 0} \frac{G(t + h, ve^{-\delta h}, \omega + ch) - G(t, v, \omega)}{h} &+ \rho\, \mathrm{E}[G(t, v + V, \omega)] \\
+(a + v)\, \mathrm{E}[G(t, v, \omega - Y)] - (\rho + a + v)G(t, v, \omega) &= 0.
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{\partial G(t, v, \omega)}{\partial t} - \delta v \frac{\partial G(t, v, \omega)}{\partial v} + c \frac{\partial G(t, v, \omega)}{\partial \omega} &+ \rho\, \mathrm{E}[G(t, v + V, \omega)] \\
+(a + v)\, \mathrm{E}[G(t, v, \omega - Y)] - (\rho + a + v)G(t, v, \omega) &= 0.
\end{aligned}$$

For any $0 \le t_1 \le t_2 \le 1$:

$$\begin{aligned}
&G(t_2, ve^{-\delta(t_2 - t_1)}, \omega + c(t_2 - t_1)) - G(t_1, v, \omega) \\
&= \int_{t_1}^{t_2} \frac{d}{ds} G(s, ve^{-\delta(s - t_1)}, \omega + c(s - t_1))\, ds \\
&= \int_{t_1}^{t_2} \Big(\frac{\partial G}{\partial t}(s, ve^{-\delta(s - t_1)}, \omega + c(s - t_1)) \\
&- \delta ve^{-\delta(s - t_1)} \frac{\partial G}{\partial v}(s, ve^{-\delta(s - t_1)}, \omega + c(s - t_1))
\end{aligned}$$

$$
\begin{aligned}
&+ c\frac{\partial G}{\partial \omega}(s, ve^{-\delta(s-t_1)}, \omega + c(s-t_1))\Big)\, ds\\
&= -\int_{t_1}^{t_2} \Big(\rho\, \mathrm{E}[G(s, ve^{-\delta(s-t_1)} + V, \omega + c(s-t_1))]\\
&+ (a + ve^{-\delta(s-t_1)})\, \mathrm{E}[G(s, ve^{-\delta(s-t_1)}, \omega + c(s-t_1) - Y)]\\
&- (\rho + a + ve^{-\delta(s-t_1)}) G(s, ve^{-\delta(s-t_1)}, \omega + c(s-t_1))\Big)\, ds \Leftrightarrow
\end{aligned}
$$

$$
\begin{aligned}
\Leftrightarrow G(t_1, \lambda, \omega) &= G(t_2, ve^{-\delta(t_2-t_1)}, \omega + c(t_2-t_1))\\
&+ \int_{t_2}^{t_1} \Big(\rho\, \mathrm{E}[G(s, ve^{-\delta(s-t_1)} + V, \omega + c(s-t_1))]\\
&+ (a + ve^{-\delta(s-t_1)})\, \mathrm{E}[G(s, ve^{-\delta(s-t_1)}, \omega + c(s-t_1) - Y)]\\
&- (\rho + a + ve^{-\delta(s-t_1)}) G(s, ve^{-\delta(s-t_1)}, \omega + c(s-t_1))\Big)\, ds.
\end{aligned} \tag{3.8}
$$

For each $\tau \in [0,1]$, we consider the operator $g \mapsto \psi^\tau g$, which assigns to each measurable and bounded function $g : [0,1] \times \mathbb{R}^2 \mapsto \mathbb{R}$ a new function $(\psi^\tau g) : [0,1] \times \mathbb{R}^2 \mapsto \mathbb{R}$, defined by

$$
\begin{aligned}
(\psi^\tau g)(t, v, \omega) = \int_t^\tau &\Big(\rho\, \mathrm{E}[g(s, ve^{-\delta(s-t)} + V, \omega + c(s-t))]\\
&+ (a + ve^{-\delta(s-t)})\, \mathrm{E}[g(s, ve^{-\delta(s-t)}, \omega + c(s-t) - Y)]\\
&- (\rho + a + ve^{-\delta(s-t)}) g(s, ve^{-\delta(s-t)}, \omega + c(s-t))\Big)\, ds.
\end{aligned}
$$

Thus, Equality (3.8) reduces to

$$
G(t, v, \omega) = G(\tau, ve^{-\delta(\tau-t)}, \omega + c(\tau - t)) + \psi^\tau G(t, v, \omega) \tag{3.9}
$$

In particular, given that $G(1, ve^{-\delta(1-t)}, \omega + c(1-t)) = \varphi(a + ve^{-\delta(1-t)}, \omega + c(1-t))$, we obtain

$$
G(t, v, \omega) = \varphi(a + ve^{-\delta(1-t)}, \omega + c(1-t)) + \psi^1 G(t, v, \omega) \tag{3.10}
$$

Now we prove the existence and unicity of the solution for Equation (3.10).
To do this, we need to properly define the domain where we are seeking solutions for (3.10).
For any $0 \le a < b \le 1$, we consider the space $Lip[a,b]$ consisting of measurable functions $g : [a,b] \times \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R}$ which are bounded with respect to the first variable, and uniformly Lipschitz with respect to the second and third variables.
For any $M \in]0, +\infty[$, we consider a function $\gamma_M : \mathbb{R} \mapsto \mathbb{R}$ satisfying the following conditions,

- $\gamma_M(v) = v \quad \forall\, v \le M$
- $\gamma_M(v) \le M + 1 \quad \forall\, v \in \mathbb{R}$
- $0 < \gamma'_M(v) \le 1 \quad \forall\, v \in \mathbb{R}$

and consider the operator $g \mapsto \psi^\tau_{\gamma_M} g$ defined by

$$\begin{aligned}(\psi^\tau_{\gamma_M} g)(t,v,\omega) = \int_t^\tau \Big(&\rho\,\mathrm{E}[g(s,\gamma_M(v)e^{-\delta(s-t)}+V,\omega+c(s-t))]\\ &+(a+\gamma_M(v)e^{-\delta(s-t)})\,\mathrm{E}[g(s,\gamma_M(v)e^{-\delta(s-t)},\omega+c(s-t)-Y)]\\ &-(\rho+a+\gamma_M(v)e^{-\delta(s-t)})g(s,\gamma_M(v)e^{-\delta(s-t)},\omega+c(s-t))\Big)\,ds.\end{aligned}$$

Substituting the operator $\psi^\tau_{\gamma_M}$ for ψ^τ, we obtain the following analogous to Equation (3.9):

$$G(t,v,\omega) = G(\tau,\gamma_M(v)e^{-\delta(\tau-t)},\omega+c(\tau-t)) + \psi^\tau_{\gamma_M} G(t,v,\omega) \tag{3.11}$$

This leads us to two important remarks.

REMARK 3.2. *Suppose $a, b \in \mathbb{R}$ such that $a \le \tau \le b$. G is the solution of Equation (3.9) in the $[a,b]\times[0,M]\times\mathbb{R}$ domain iff it is also solution of Equation (3.11) in the same domain.*

REMARK 3.3. *Suppose $a, b \in \mathbb{R}$ such that $a \le \tau \le b$. If $0 < M_1 \le M_2 < +\infty$, then G is the solution of Equation (3.11) with $M = M_2$ in the $[a,b]\times[0,M_1]\times\mathbb{R}$ domain iff it is also solution of the same equation with $M = M_1$ in the same domain.*

Given the above remarks, we prove that for any $M \in]0,+\infty[$, the Equation (3.10) has one unique solution in $Lip[0,1]$.
We will prove that under suitable assumptions, (3.10) has one unique solution in $Lip[0,1]$. $Lip[a,b]$ is a linear space, and it can be provided with the norm

$$\|g\|_{Lip[a,b]} = \sup_{t\in[a,b]} |g(t,0,0)| + \sup_{\substack{t\in[a,b]\\(v,\omega)\neq(\tilde v,\tilde\omega)}} \frac{|g(t,v,\omega)-g(t,\tilde v,\tilde w|}{|v-\tilde v|+|\omega-\tilde\omega|}.$$

Due to the Arzelà–Ascoli theorem, we can say that $Lip[a,b]$ is topologically complete, i.e., it is a Banach space.

LEMMA 3.1. *Suppose that the random variables V and Y are integrable (i.e. $\mathrm{E}[V] < +\infty$ and $\mathrm{E}[Y] < +\infty$). For any $M, L \in]0,+\infty[$, there is some $\epsilon > 0$, such that for every Lipschitz function $\eta : \mathbb{R}^+ \mapsto [0,M]$ with Lipschitz constant less or equal to L and every $\tau \in \mathbb{R}$, the operator $g \mapsto \psi^\tau_\eta g$ is defined by*

$$\begin{aligned}\psi^\tau_\eta g(t,v,\omega) = \int_t^\tau &\rho\,\mathrm{E}[g(s,\eta(v)e^{-\delta(s-t)}+V,\omega+c(s-t)) - g(s,\eta(v)e^{-\delta(s-t)},\omega+c(s-t))]\\ &+(a+\eta(v)e^{-\delta(s-t)})\,\mathrm{E}[g(s,\eta(v)e^{-\delta(s-t)},\omega+c(s-t)-Y) - g(s,\eta(v)e^{-\delta(s-t)},\omega+c(s-t))]\,ds.\end{aligned}$$

is a contraction in $Lip[\tau-\epsilon,\tau]$.

Proof: see Appendix.

The Lemma 3.1 leads us to an important remark and theorem, as we can see below.

REMARK 3.4. *C_1 and C_2 are dependent of M and of the constant L, but they do not depend on τ.*

Using Lemma 3.1, we can obtain the theorem in this section:

THEOREM 3.1. *If the random variables V and Y are integrable, $h \in Lip[0,1]$, $M \in]0, +\infty[$, Then the equation*

$$G(t, v, \omega) = h(t, v, \omega) + \psi^1_{\gamma_M} G(t, v, \omega) \tag{3.12}$$

admits one unique solution in $Lip[0,1]$.

Proof: see Appendix.

Theorem 3.1 has the following corollary.

COROLLARY 3.1. *If $\varphi : \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R}$ is a Lipschitzian function then the function G defined in Equation (3.7) is the unique solution of Equation (3.10).*

CHAPTER 4

The Optimal Reinsurance Problem

In this chapter, we present optimality conditions for our quota share reinsurance optimisation problem.

4.1. Quota share reinsurance

As described in Chapter 2, quota share reinsurance is a reinsurance treaty which allows the ceding insurer to share a percentage, α, of every risk to which it is exposed, with one or more reinsurers.
In the following, we assume that the reinsurance premium is proportional to the size of the ceded risk and that it is payed continuously in time. Thus, we obtain the following (reinsured) surplus process.

DEFINITION 4.1. *The insurer's surplus process using a quota share reinsurance treaty* $\{\omega_t\}_{t\geq 0}$ *is defined as*

$$\omega_t^{(\alpha)} = u + (1 - q\alpha)ct - (1 - \alpha)\sum_{i=1}^{N_t} Y_i \tag{4.1}$$

where

- q *is the percentage of commission to be paid that is proportional to the premiums received,*
- α *is the percentage of claim amounts ceded by the insurer.*

REMARK 4.1. *The reinsured part of each claim is denoted by* $Z(Y_i) = \alpha Y_i$ *and the amount of premiums paid to the reinsurer, per unit of time. is given by* $P(Z) = q\alpha c$ *with* $\alpha \in [0, 1]$.

The process (4.1) is of the same type as process (3.6), with data depending on the parameters q and $\alpha \in [0, 1]$. We introduce the corresponding version of function (3.7),

$$G_\alpha(t, \lambda, \omega) = \mathrm{E}[\varphi(\lambda_1, \omega_1^{(\alpha)}) \mid \lambda_t = a + v, \omega_t^{(\alpha)} = \omega]$$

4.2. Optimality conditions

In this section, we start by presenting the optimisation problem at hand.

PROBLEM 4.1. *Find the optimal quota share from the point of view of the direct insurer, assuming that he rates his wealth by a non-decreasing concave utility function* $U\colon \mathbb{R} \mapsto \mathbb{R}$. *In other words , for every* $\lambda \geq a$, *and every* $\omega \in \mathbb{R}$, *find* $\hat{\alpha} \in [0,1]$ *such that*

$$\mathrm{E}[U(\omega_1^{(\hat{\alpha})}) \mid \lambda_0 = \lambda, \omega_0^{(\hat{\alpha})} = \omega] = \sup_{\alpha \in [0,1]} \mathrm{E}[U(\omega_1^{(\alpha)}) \mid \lambda_0 = \lambda, \omega_0^{(\alpha)} = \omega]$$

Using the same procedure as in Section 3.3, we define the function $\hat{\varphi}_\alpha$ as

$$\hat{\varphi}_\alpha(t,v,\omega) = \varphi(a + ve^{-\delta(1-t)}, \omega + (1-q\alpha)c(1-t))$$

For $\eta : \mathbb{R}^+ \mapsto \mathbb{R}^+$ of class $\mathbb{C}^\infty$, limited such that $\eta(0) = 0$:

$$\begin{aligned}\psi_{\eta,\alpha} g(t,v,\omega) = &\int_t^1 \Big(\rho\,\mathrm{E}[\,g(s,\eta(v)e^{-\delta(s-t)} + V, \omega + (1-q\alpha)c(s-t))\\
&- g(s,\eta(v)e^{-\delta(s-t)}, \omega + (1-q\alpha)c(s-t))\,]\\
&+ (a + \eta(v)e^{-\delta(s-t)})\,\mathrm{E}[\,g(s,\eta(v)e^{-\delta(s-t)}, \omega - (1-\alpha)Y + (1-q\alpha)c(s-t))\\
&- g(s,\eta(v)e^{-\delta(s-t)}, \omega + (1-q\alpha)c(s-t))\,]\Big)\,ds\end{aligned}$$

G_α is the unique solution of $G = \hat{\varphi_\alpha} + \psi_{\eta,\alpha} G$ in $Lip[0,1]$.

Next, it follows two relevant lemmas.

LEMMA 4.1. *If* $\varphi : \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R}$ *is Lipschitz ,* $\mathbb{C}^2$*, with bounded* 2^{nd} *partial derivatives, then exists a constant,* C *(dependent on* φ *but independent of* α*) such that*

$$\|\hat{\varphi_\beta} - \hat{\varphi_\alpha}\|_{Lip[0,1]} \leq C|\beta - \alpha| \quad \forall \alpha, \beta \in [0,1]$$

Proof: see Appendix.

LEMMA 4.2. *Suppose that* $\varphi : \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R}$ *satisfies the hypothesis of Lemma 4.1 and the random variables* V, Y *are integrable.*
For all $\alpha, \beta \in [0,1]$ *there exists a constant* $C \in \mathbb{R}$ *(dependent of* α *but independent of* β*) such that*

$$\|G_\beta - G_\alpha\|_{Lip[t,1]} \leq C\Big(|\beta - \alpha| + \int_t^1 \|G_\beta - G_\alpha\|_{Lip[s,1]}\, ds\Big) \quad \forall \beta \in [0,1], t \in [0,1]$$

So, Grönwall's lemma guarantees that

$$\|G_\beta - G_\alpha\|_{Lip[t,1]} \leq e^{C(1-t)} C|\beta - \alpha|$$

Proof: see Appendix.

We go now on to show that the first order derivative of $\hat{\varphi}_\alpha$ in respect to α is as follows,

$$\frac{\partial \hat{\varphi}_\alpha(t,v,\omega)}{\partial \alpha} = -\frac{\partial \varphi(a+ve^{-\delta(1-t)}, \omega + (1-q\alpha)c(1-t))}{\partial \omega} qc(1-t) = -\frac{\partial \hat{\varphi}_\alpha(t,v,\omega)}{\partial \omega} qc(1-t)$$

$$\begin{aligned}
\hat{\varphi}_{\alpha+h}(t,v,\omega) - \hat{\varphi}_\alpha(t,v,\omega) &= \varphi(a+ve^{-\delta(1-t)}, \omega + (1-q(\alpha+h))c(1-t)) \\
&\quad - \varphi(a+ve^{-\delta(1-t)}, \omega + (1-q\alpha)c(1-t)) \\
&= \varphi(a+ve^{-\delta(1-t)}, \omega + (1-q\alpha)c(1-t) - qc(1-t)h) \\
&\quad - \varphi(a+ve^{-\delta(1-t)}, \omega + (1-q\alpha)c(1-t)) \\
&= -\frac{\partial \varphi(a+ve^{-\delta(1-t)}, \omega + (1-q\alpha)c(1-t))}{\partial \omega} qc(1-t)h + o(h)
\end{aligned}$$

Consider the operator $\frac{\partial \psi_\alpha}{\partial \alpha} : Lip[0,1] \mapsto L_\infty[0,1]$, defined by

$$\Big(\frac{\partial \psi_\alpha}{\partial \alpha}\Big) g(t,v,\omega) = \lim_{\beta \to \alpha} \frac{1}{\beta - \alpha} (\psi_\beta g(t,v,\omega) - \psi_\alpha g(t,v,\omega))$$

It follows,

$$\begin{aligned}
\Big(\frac{\partial \psi_\alpha}{\partial \alpha}\Big) g(t,v,\omega) = \int_t^1 \Big(&\rho\, \mathrm{E}\Big[\frac{\partial g}{\partial \omega}(s, ve^{-\delta(s-t)} + V, \omega + (1-q\alpha)c(s-t)) \\
&- \frac{\partial g}{\partial \omega}(s, ve^{-\delta(s-t)}, \omega + (1-q\alpha)c(s-t))\Big] \\
&+ (a+ve^{-\delta(s-t)})\, \mathrm{E}\Big[\frac{\partial g}{\partial \omega}(s, ve^{-\delta(s-t)}, \omega + (1-q\alpha)c(s-t) - Y) \\
&- \frac{\partial g}{\partial \omega}(s, ve^{-\delta(s-t)}, \omega + (1-q\alpha)c(s-t))\Big]\Big) qc(s-t)\, ds
\end{aligned}$$

Then,

$$\begin{aligned}
&(\psi_{\alpha+h} - \psi_\alpha) g(t,v,\omega) = \\
&= \int_t^1 \Big(\rho\, \mathrm{E}[g(s, ve^{-\delta(s-t)} + V, \omega + (1-q\alpha)c(s-t) - qc(s-t)h) \\
&- g(s, ve^{-\delta(s-t)}, \omega + (1-q\alpha)c(s-t) - qc(s-t)h) - g(s, ve^{-\delta(s-t)} + V, \omega + (1-q\alpha)c(s-t)) \\
&+ g(s, ve^{-\delta(s-t)}, \omega + (1-q\alpha)c(s-t))] \\
&+ (a+ve^{-\delta(s-t)})\, \mathrm{E}[g(s, ve^{-\delta(s-t)}, \omega + (1-q\alpha)c(s-t) - Y - qc(s-t)h) \\
&- g(s, ve^{-\delta(s-t)}, \omega + (1-q\alpha)c(s-t) - qc(s-t)h) - g(s, ve^{-\delta(s-t)}, \omega + (1-q\alpha)c(s-t) - Y) \\
&+ g(s, ve^{-\delta(s-t)}, \omega + (1-q\alpha)c(s-t))]\Big)\, ds \\
&= \int_t^1 \Big(\rho\, \mathrm{E}\Big[\int_0^1 \frac{\partial g}{\partial \omega}(s, ve^{-\delta(s-t)} + V, \omega + (1-q\alpha)c(s-t) - uqc(s-t)h) qc(s-t)h\, du \\
&- \int_0^1 \frac{\partial g}{\partial \omega}(s, ve^{-\delta(s-t)}, \omega + (1-q\alpha)c(s-t) - uqc(s-t)h) qc(s-t)h\, du\Big]
\end{aligned}$$

$$+ (a + ve^{-\delta(s-t)})\,\mathrm{E}\Big[\int_0^1 \frac{\partial g}{\partial \omega}(s, ve^{-\delta(s-t)}, \omega + (1 - q\alpha)c(s-t) - Y - uqc(s-t)h)qc(s-t)h\,du$$

$$- \int_0^1 \frac{\partial g}{\partial \omega}(s, ve^{-\delta(s-t)}, \omega + (1 - q\alpha)c(s-t) - uqc(s-t)h)qc(s-t)h\,du\Big]\Big)\,ds$$

$$= \int_t^1 \Big(\rho\,\mathrm{E}\Big[\frac{\partial g}{\partial \omega}(s, ve^{-\delta(s-t)} + V, \omega + (1 - q\alpha)c(s-t))$$

$$- \frac{\partial g}{\partial \omega}(s, ve^{-\delta(s-t)}, \omega + (1 - q\alpha)c(s-t))\Big]$$

$$+ (a + ve^{-\delta(s-t)})\,\mathrm{E}\Big[\frac{\partial g}{\partial \omega}(s, ve^{-\delta(s-t)}, \omega + (1 - q\alpha)c(s-t) - Y)$$

$$- \frac{\partial g}{\partial \omega}(s, ve^{-\delta(s-t)}, \omega + (1 - q\alpha)c(s-t))\Big]\Big)qc(s-t)\,ds\,h + o(h)$$

Now, we present a key theorem towards getting the optimality conditions.

THEOREM 4.1. *For $\alpha \in [0,1]$, let G_α be the unique solution in $Lip[0,1]$ of the equation,*

$$G = \hat{\varphi}_\alpha + \psi_\alpha G,$$

and let M_α be the unique solution in $Lip[0,1]$ of the equation

$$M = \frac{\partial \hat{\varphi}_\alpha}{\partial \alpha} + \Big(\frac{\partial \psi_\alpha}{\partial \alpha}\Big)G_\alpha + \psi_\alpha M$$

Then:

$$\lim_{\beta \to \alpha} \frac{G_\beta(t,v,\omega) - G_\alpha(t,v,\omega) - (\beta - \alpha)M_\alpha(t,v,\omega)}{\beta - \alpha} = 0$$

i.e., $M_\alpha(t,v,\omega) = \frac{\partial G_\alpha(t,v,\omega)}{\partial \alpha}$.

Proof: see Appendix.

Thanks to the Theorem 4.1, we can now show the optimality conditions that solve our quota share proportional reinsurance optimisation problem.

COROLLARY 4.1 (Optimality conditions). *For every $\alpha \in [0,1]$, G_α, M_α are the functions defined in Theorem 4.1. Let $(v,\omega) \mapsto \hat{\alpha}(v,\omega)$ be a function that each pair of values from*

$\lambda_0 = a + v,\ \ \omega_0 = \omega$ *corresponds a maximiser of the following function*

$$\alpha \mapsto \mathrm{E}[\varphi(0, \lambda_1, \omega_1^{(\alpha)}) \mid \lambda_0 = a + v,\, \omega_0 = \omega]$$

The function $\hat{\alpha}$ satisfies the following conditions:

- $\hat{\alpha}(v, \omega) = 0$ *for all* $(v, \omega) \in \mathbb{R}^+ \times \mathbb{R}$ *such that* $M_\alpha(0, v, \omega) < 0 \quad \forall \alpha \in [0, 1]$
- $\hat{\alpha}(v, \omega) = 1$ *for all* $(v, \omega) \in \mathbb{R}^+ \times \mathbb{R}$ *such that* $M_\alpha(0, v, \omega) > 0 \quad \forall \alpha \in [0, 1]$
- $M_{\hat{\alpha}(v,\omega)}(0, v, \omega) = 0$ *for all remaining points* $(v, \omega) \in \mathbb{R}^+ \times \mathbb{R}$.

Proof: Theorem 4.1 means that for every $(v, \omega) \in \mathbb{R}^+ \times \mathbb{R}$ *the function* $\alpha \mapsto G_\alpha(v, \omega)$ *is differentiable and its derivative is equal to* $M_\alpha(v, \omega)$.

Given the assumptions that the direct insurer has monotonic preferences (more wealth is better), limited resources and his wealth is valued by a non-decreasing concave utility function, reinsurance reduces the risk exposure but it also involves costs, namely reinsurance premiums. The optimality conditions lead to three cases. In the first and second cases (extreme scenarios):

(1) the expected utility of wealth of the ceding insurance company decreases with each value of α, in which case the insurer is better off without reinsurance. The direct insurer does not have any incentive to purchase reinsurance and,

(2) the expected utility of wealth of the ceding insurance company increases with each value of α, in which case, the insurer is better off purchasing reinsurance. The direct insurer has incentive to cede every claim in full.

The first case is counterintuitive since there are clear benefits to the use of reinsurance, such as, risk transfer, economic growth, increase stability due to an higher risk-sharing and so on. The second case is senseless because when the reinsurance is purchased, the direct insurer must pay a reinsurance premium to the reinsurer and this premium paid increases with the percentage the insurer cedes for each claim. There is a tipping point where it it no longer profitable to cede. The third case is the scenario that best describes the decision-making of a ceding insurance company. The direct insurer has an optimal level of reinsurance, $\hat{\alpha} \in]0, 1[$, which maximises the expected value of his wealth. Any increase or decrease of the $\hat{\alpha}$ will lead to worse results.

CHAPTER 5

Conclusions

The optimal reinsurance problem has been widely studied by many researchers. In this thesis, we studied the optimal reinsurance problem, from the ceding insurance company's point of view without the prospect of investment. Unlike previous papers that use a Dynamic contagion model capturing both externally and internally jumps to model the claims intensity process, we worked with a Cox process, with a Poisson shot noise intensity, that captures only externally excited jumps. Considered dependencies between claims and a modified version of the classical Cramér-Lundberg surplus process to model the ceding insurer's wealth. Furthermore, some important assumptions were outlined. The first-line insurer has monotonic preferences (more wealth is better), limited resources and rates his wealth by a non-decreasing concave utility function. Additionally, we examined the problem using a proportional reinsurance treaty, the quota share reinsurance treaty. In this reinsurance type, the reinsurer gets paid a reinsurance premium to take a fixed percentage of all the risk insured by the first-line insurer. The reinsurance premium is priced using the expected value principle. The challenge was to find the optimal level of reinsurance, $\hat{\alpha}$, for the ceding insurance that maximises the expected utility of its wealth, for a given year. A bigger α means a greater risk-sharing but also a bigger reinsurance premium paid to the reinsurer. The problem was solved analytically, from which we got an implicit solution - the optimality conditions.

The results showed that the decision-making of the insurer purely depends on the behaviour of the expected utility function of its wealth. We have seen two extreme cases. In the first case, expected utility function of wealth is a decreasing function in respect to α, the insurer is better off not buying reinsurance. In the second case, expected utility function of wealth is an increasing function in respect to α, the insurer is better off ceding all the risk to the reinsurer, independent of how much the reinsurance premium costs. There are clear flaws to both cases when compared to how the real insurance industry works. On a brighter note, apart from the mentioned cases, there was also a third case that described very well the reality we observe. The ceding insurer decides to use an optimal level of reinsurance, $\hat{\alpha}$ between 0 and 1 but never 0 nor 1, that maximises the expected value of its wealth. Any positive or negative deviation from the optimal point leads to a worse position.

To fulfil the continuity of this research, it would be interesting to apply the findings of this work to get an explicit solution using numerical methods.

CHAPTER 6

Appendix

6.1. Proof of Proposition 3.1.

It follows from (3.2) that,

$$P\{N_t = N_{t_0}\} = P\{T_1 > t | T_1 > t_0\} = \exp\Big(-\int_{t_0}^{t} \lambda_s \, ds\Big) \frac{\Big(\int_{t_0}^{t} \lambda_s \, ds\Big)^0}{0!} = \exp\Big(-\int_{t_0}^{t} \lambda_s \, ds\Big).$$

Assuming that, for some n $\in \mathbb{N}_0$,

$$P\{N_t = N_{t_0} + n\} = \exp\Big(-\int_{t_0}^{t} \lambda_u \, du\Big) \frac{\Big(\int_{t_0}^{t} \lambda_u \, du\Big)^n}{n!}, \quad \text{holds for every} \quad 0 \le t_0 \le t < +\infty,$$

we wish to show that the analogous equality holds for $n + 1$.

$$\begin{aligned}
P\{N_t = N_{t_0} + n + 1\} &= \int_{t_0}^{t} P\{N_t = N_s + n\} \, F_{T_1 | T_1 > t_0}(ds) \\
&= \int_{t_0}^{t} \exp\Big(-\int_{s}^{t} \lambda_u \, du\Big) \frac{\Big(\int_{s}^{t} \lambda_u \, du\Big)^n}{n!} \lambda_s \exp\Big(-\int_{t_0}^{s} \lambda_u \, du\Big) \, ds \\
&= \exp\Big(-\int_{t_0}^{t} \lambda_u \, du\Big) \int_{t_0}^{t} \lambda_s \frac{\Big(\int_{s}^{t} \lambda_u \, du\Big)^n}{n!} \, ds \\
&= \exp\Big(-\int_{t_0}^{t} \lambda_u \, du\Big) \left[-\frac{\Big(\int_{s}^{t} \lambda_u \, du\Big)^{n+1}}{(n+1)!} \right]_{s=t_0}^{s=t} \\
&= \exp\Big(-\int_{t_0}^{t} \lambda_u \, du\Big) \frac{\Big(\int_{t_0}^{t} \lambda_u \, du\Big)^{n+1}}{(n+1)!}. \quad \square
\end{aligned}$$

6.2. Proof of Proposition 3.2.

Let T_1 be the moment when occurs the first shock after $t = 0$ and, θ_1 the moment when occurs the first claim after $t = 0$.

(1) To prove that $Pr(K_{t+h} = K_t, N_{t+h} = N_t|\lambda_t = a + v,\, \omega_t = \omega) = 1 - (\rho + a + v)h + o(h)$.

This is the event where no claim and no shock occur during the time interval $[t, t + h]$.

$$
\begin{aligned}
Pr(K_{t+h} = K_t, N_{t+h} = N_t|\lambda_t = a + v,\, \omega_t = \omega) &= Pr(K_h = K_0, N_h = N_0|\lambda_0 = a + v,\, \omega_0 = \omega) \\
&= Pr(T_1 > h, \theta_1 > h|\lambda_0 = a + v) \\
&= 1 - Pr(T_1 \le h) - Pr(T_1 > h, \theta_1 \le h|\lambda_0 = a + v) \\
&= e^{-\rho h} - \int_h^{+\infty}\int_0^h (a + v)e^{-\int_0^\theta (a+\lambda_u)\,du}\, d\theta \rho e^{-\rho t}\, dt \\
&= e^{-\rho h} - \int_h^{+\infty} (1 - e^{-\int_0^h a+ve^{-\delta u}\,du})\rho e^{-\rho t}\, dt \\
&= e^{-\rho h} - (1 - e^{-\int_0^h a+ve^{-\delta u}\,du})e^{-\rho h} \\
&= e^{-\rho h-\int_0^h a+ve^{-\delta u}\,du} \\
&= e^{-\rho h-h(a+v)+o(h)} \\
&= 1 - (\rho + a + v)h + o(h). \quad \square
\end{aligned}
$$

(2) To prove that $Pr(K_{t+h} = K_t + 1, N_{t+h} = N_t|\lambda_t = a + v, \omega_t = \omega) = \rho h + o(h)$.

This is the event where there is exactly one shock but there are no claims in the interval $[t, t+h]$:

$$
\begin{aligned}
Pr(K_{t+h} = K_t + 1, N_{t+h} = N_t|\lambda_t = a + v, \omega_t = \omega) &= Pr(K_h = K_0 + 1, N_h = N_0|\lambda_0 = a + v, \omega_0 = \omega) \\
&= Pr(T_1 \le h, T_2 > h, \theta_1 > h|\lambda_0 = a + v).
\end{aligned}
$$

By the iterative property of conditional expectation:

$$
\begin{aligned}
&Pr(T_1 \le h, T_2 > h, \theta_1 > h|\lambda_0 = a + v) = \mathrm{E}\Big[\mathrm{E}\Big[\chi_{\theta_1>h,T_1\le h,T_2>h}\Big|T_1, T_2, \lambda_0 = a + v\Big]\Big|\lambda_0 = a + v\Big] \\
&= \mathrm{E}\Big[e^{-(\int_0^{T_1} a+ve^{-\delta s}\,ds+\int_{T_1}^h a+ve^{-\delta s}+V_1e^{-\delta(s-T_1)}\,ds)}\chi_{T_1\le h<T_2}\Big|\lambda_0 = a + v\Big] \\
&= \mathrm{E}\Big[e^{-\int_0^h a+ve^{-\delta s}\,ds-V_1\int_{T_1}^h e^{-\delta(s-T_1)}\,ds}\chi_{T_1\le h<T_2}\Big|\lambda_0 = a + v\Big] \\
&= \mathrm{E}\Big[e^{-(ah+v\frac{1-e^{-\delta h}}{\delta})-V_1\frac{1-e^{-\delta(h-T_1)}}{\delta}}\chi_{T_1\le h<T_2}\Big|\lambda_0 = a + v\Big] \\
&= e^{-((a+v)h+o(h))}\mathrm{E}\Big[e^{-V_1\frac{1-e^{-\delta(h-T_1)}}{\delta}}\chi_{T_1\le h<T_2}\Big|\lambda_0 = a + v\Big] \\
&= e^{-((a+v)h+o(h))}\int_0^{+\infty}\int_0^h\int_{h-t_1}^{+\infty} e^{-y\frac{1-e^{-\delta(h-t_1)}}{\delta}}\rho e^{-\rho t_2}\, dt_2\rho e^{-\rho t_1}\, dt_1\, dF_v(y) \\
&= (1 - O(h))\int_0^{+\infty}\int_0^h e^{-y\frac{1-e^{-\delta(h-t_1)}}{\delta}} e^{-\rho(h-t_1)}\rho e^{-\rho t_1}\, dt_1\, dF_v(y)
\end{aligned}
$$

$$= (1 - O(h))\rho e^{-\rho h} \int_0^h \int_0^{+\infty} e^{-y\frac{1-e^{-\delta(h-t_1)}}{\delta}}\, dF_v(y)\, dt_1$$

By Lebesgue's dominated convergence theorem,

$$\lim_{h \to 0^+} \int_0^{+\infty} e^{-y\frac{1-e^{-\delta(h-t_1)}}{\delta}}\, dF_v(y) = 1$$

and therefore

$$Pr(T_1 \le h, T_2 > h, \theta_1 > h | \lambda_0 = a + v) = (1 - O(h))\rho e^{-\rho h}(h + o(h)) = \rho h + o(h) \quad \square$$

(3) To prove that $Pr(K_{t+h} = K_t, N_{t+h} = N_t + 1 | \lambda_t = a + v, \omega_t = \omega) = (a + v)h + o(h)$.

This is the event where no shock occurs and exactly one claim occurs in the period $[0, h]$:

$$\begin{aligned}
Pr(K_{t+h} = K_t, N_{t+h} = N_t + 1 | \lambda_t = a + v, \omega_t = \omega) &= Pr(K_h = K_0, N_h = N_0 + 1 | \lambda_0 = a + v, \omega_0 = \omega) \\
&= Pr(T_1 > h, \theta_1 \le h | \lambda_0 = a + v) \\
&= \int_h^{+\infty} \int_0^h (a + ve^{-\delta\theta}) e^{-\int_0^\theta a + ve^{-\delta u}\, du}\, d\theta \rho e^{-\rho t}\, dt \\
&= \int_h^{+\infty} (1 - e^{-\int_0^h a + ve^{-\delta u}\, du}) \rho e^{-\rho t}\, dt \\
&= (1 - e^{-\int_0^h a + ve^{-\delta u}\, du}) e^{-\rho h} \\
&= e^{-\rho h} - e^{-\rho h - \int_0^h a + ve^{-\delta u}\, du} \\
&= e^{-\rho h} - e^{-\rho h - h(a+v) + o(h)} \\
&= 1 - \rho h + o(h) - (1 - (\rho + a + v)h) + o(h) \\
&= (a + v)h + o(h). \quad \square
\end{aligned}$$

6.3. Proof of Lemma 3.1.

For each $g \in Lip[0, \tau], t \in [\tau - \epsilon, \tau]$, we have

$$\begin{aligned}
|\psi_\eta^\tau g(t, 0, 0)| \le& \int_t^\tau \Big(\rho\, \mathrm{E}[\, |\, g(s, \eta(v)e^{-\delta(s-t)} + V, \omega + c(s - t)) - g(s, \eta(v)e^{-\delta(s-t)}, \omega + c(s - t))\, |\,] \\
&+ (a + \eta(v)e^{-\delta(s-t)})\, \mathrm{E}[\, |\, g(s, \eta(v)e^{-\delta(s-t)}, \omega + c(s - t) - Y) - g(s, \eta(v)e^{-\delta(s-t)}, \omega + c(s - t))\, |\,] \Big)\, ds \\
\le& \int_t^\tau \Big(\rho\, \mathrm{E}[\|g\|_{Lip[\tau-\epsilon,\tau]} LV] + (a + M + 1)\, \mathrm{E}[\|g\|_{Lip[\tau-\epsilon,\tau]} LY] \Big)\, ds \\
=& (\tau - t)L(\rho\, \mathrm{E}[V] + (a + M + 1)\, \mathrm{E}[Y])\|g\|_{Lip[\tau-\epsilon,\tau]} \\
\le& \epsilon L(\rho\, \mathrm{E}[V] + (a + M + 1)\, \mathrm{E}[Y])\|g\|_{Lip[\tau-\epsilon,\tau]}
\end{aligned}$$

$$= \epsilon C_1 \|g\|_{Lip[\tau-\epsilon,\tau]}$$

Further,

$$|\psi_\eta^\tau g(t,v,\omega) - \psi_\eta^\tau g(t,\tilde{v},\tilde{\omega})|$$
$$\leq \int_t^\tau \Big(\rho \,\mathrm{E}[\mid g(s,\eta(v)e^{-\delta(s-t)}+V,\omega+c(s-t)) - g(s,\eta(\tilde{v})e^{-\delta(s-t)}+V,\tilde{\omega}+c(s-t)) \mid$$
$$+ \mid g(s,\eta(v)e^{-\delta(s-t)},\omega+c(s-t)) \mid - \mid g(s,\eta(\tilde{v})e^{-\delta(s-t)},\tilde{\omega}+c(s-t)) \mid]$$
$$+ (a+\eta(v)e^{-\delta(s-t)})\,\mathrm{E}[\mid g(s,\eta(v)e^{-\delta(s-t)},\omega+c(s-t)-Y) - g(s,\eta(\tilde{v})e^{-\delta(s-t)},\tilde{\omega}+c(s-t)-Y) \mid$$
$$+ \mid g(s,\eta(v)e^{-\delta(s-t)},\omega+c(s-t)) \mid - \mid g(s,\eta(\tilde{v})e^{-\delta(s-t)},\tilde{\omega}+c(s-t)) \mid]\Big)\, ds$$
$$\leq \int_t^\tau \Big(2\rho\|g\|_{Lip[\tau-\epsilon,\tau]}(|\eta(v)-\eta(\tilde{v})| + |\tilde{w}-w|)$$
$$+ (a+M+1)2\|g\|_{Lip[\tau-\epsilon,\tau]}(|\eta(v)-\eta(\tilde{v})| + |\tilde{w}-w|)\Big)\, ds$$
$$\leq 2\epsilon(\rho+a+M+1)\|g\|_{Lip[\tau-\epsilon,\tau]}(L|v-\tilde{v}| + |w-\tilde{w})$$
$$\leq \epsilon C_2\|g\|_{Lip[\tau-\epsilon,\tau]}(|v-\tilde{v}| + |w-\tilde{w})$$

So,

$$\|\psi_\eta^\tau g\|_{Lip[\tau-\epsilon,\tau]} \leq \epsilon(C_1+C_2)\|g\|_{Lip[\tau-\epsilon,\tau]} \quad \forall g \in Lip[\tau-\epsilon,\tau]$$

and, if $0<\epsilon<\frac{1}{C_1+C_2}$, then ψ_η^τ is a contraction in $Lip[\tau-\epsilon,\tau]$. $\square$

6.4. Proof of Theorem 3.1.

Given Lemma 3.1, we can infer that exists $\epsilon<0$ such that, for all $\tau\in]0,1]$, $\psi_{\gamma_M}^\tau$ is a contraction in $Lip[\tau-\epsilon,\tau]$. Therefore, the affine transformation $g\mapsto h+\psi_{\gamma_M}^1 g$ is also a contraction in $Lip[1-\epsilon,1]$, so that it has one unique fixed point $\hat{g}\in Lip[1-\epsilon,1]$ and that fixed point verifies $\hat{g}=h+\psi_{\gamma_M}^1\hat{g}$.

Futhermore, for $0\leq t\leq\tau_1\leq\tau_2\leq 1$,

$$\psi_{\gamma_M}^{\tau_2} g(t,v,\omega) = \int_t^{\tau_1} \Big(\rho\ \mathrm{E}[g(s,\gamma_M(v)e^{-\delta(s-t)}+V,\omega+c(s-t)) - g(s,\gamma_M(v)e^{-\delta(s-t)},\omega+c(s-t))]$$
$$+ (a+\gamma_M(v)e^{-\delta(s-t)})\,\mathrm{E}[g(s,\gamma_M(v)e^{-\delta(s-t)},\omega+c(s-t)-Y) - g(s,\gamma_M(v)e^{-\delta(s-t)},\omega+c(s-t))]\Big)\, ds$$
$$+ \int_{\tau_1}^{\tau_2} \Big(\rho\ \mathrm{E}[g(s,\gamma_M(v)e^{-\delta(s-\tau_1+\tau_1-t)}+V,\omega+c(s-t)) - g(s,\gamma_M(v)e^{-\delta(s-\tau_1+\tau_1-t)},\omega+c(s-t))]$$
$$+ (a+\gamma_M(v)e^{-\delta(s-\tau_1+\tau_1-t)})\times$$

$$\times \mathrm{E}[g(s,\gamma_M(v)e^{-\delta(s-\tau_1+\tau_1-t)},\omega+c(s-t)-Y)-g(s,\gamma_M(v)e^{-\delta(s-\tau_1+\tau_1-t)},\omega+c(s-t))]\Big)\,ds$$
$$=\psi^{\tau_1}_{\gamma_M}g(t,v,\omega)+\psi^{\tau_2}_{\gamma_M e^{-\delta(\tau_1-t)}}g(\tau_1,v,\omega).$$

Due to Lemma 3.1, the function $(t,v,\omega)\mapsto\psi^{\tau_2}_{\gamma_M e^{-\delta(\tau_1-t)}}g(\tau_1,v,\omega)$ is of class $Lip[0,\tau_1]$. We define the function $h_\epsilon:[0,1-\epsilon]\times\mathbb{R}^+\times\mathbb{R}\mapsto\mathbb{R}$ as

$$h_\epsilon(t,v,\omega)=h(t,v,\omega)+\psi^{1}_{\gamma_M e^{-\delta(1-\epsilon-t)}}\hat{g}(1-\epsilon,v,\omega)$$

Then, Equation (3.12) reduces to

$$G(t,v,\omega)=h_\epsilon(t,v,\omega)+\psi^{1-\epsilon}_{\gamma_M}G(t,v,\omega),\quad (t,v,\omega)\in[0,1-\epsilon]\times\mathbb{R}^+\times\mathbb{R}$$

Repeating the same argument used above, we conclude that this equation has a unique solution $\tilde{g}\in Lip[1-2\epsilon,1-\epsilon]$ and that the function

$$g(t,v,\omega)=\begin{cases}\hat{g}(t,v,\omega), & t\in[1-\epsilon,1]\\ \tilde{g}(t,v,\omega), & t\in[1-2\epsilon,1-\epsilon]\end{cases}$$

is the unique solution of Equation (3.12) in $Lip[1-2\epsilon,1]$. Repeating the same idea, $n\le\frac{1}{\epsilon}$ times, we reach the conclusion that the Equation (3.12) has one unique solution in $Lip[0,1]$. □

6.5. Proof of Lemma 4.1.

We have that,

$$|(\hat{\varphi}_\beta-\hat{\varphi}_\alpha)(t,0,0)|=|\varphi(a,(1-q\beta)c(1-t))-\varphi(a,(1-q\alpha)c(1-t))|\le C|\beta-\alpha|.$$

Also,

$$\begin{aligned}&|(\hat{\varphi}_\beta-\hat{\varphi}_\alpha)(t,v,\omega)-(\hat{\varphi}_\beta-\hat{\varphi}_\alpha)(t,\tilde{v},\tilde{\omega})|=\\ &=|\varphi(a+ve^{-\delta(1-t)},\omega+(1-q\beta)c(1-t))-\varphi(a+ve^{-\delta(1-t)},\omega+(1-q\alpha)c(1-t))\\ &-\varphi(a+\tilde{v}e^{-\delta(1-t)},\tilde{\omega}+(1-q\beta)c(1-t))+\varphi(a+\tilde{v}e^{-\delta(1-t)},\tilde{\omega}+(1-q\alpha)c(1-t))|\\ &=\Big|\int_0^1\Big\langle\nabla\varphi(a+(\tilde{v}+u(v-\tilde{v}))e^{-\delta(1-t)}+\tilde{\omega}+u(\omega-\tilde{\omega})\\ &-(1-q\beta)c(1-t))-\nabla\varphi(a+(\tilde{v}+u(v-\tilde{v}))e^{-\delta(1-t)}+\tilde{\omega}+u(\omega-\tilde{\omega})\\ &-(1-q\alpha)c(1-t)),(v-\tilde{v},w-\tilde{w})\Big\rangle\,du\Big|\\ &\le C|\beta-\alpha|(|v-\tilde{v}|+|\omega-\tilde{\omega}|).\quad\square\end{aligned}$$

6.6. Proof of Lemma 4.2.

Given that,

$$G_\beta - G_\alpha = \hat{\varphi}_\beta - \hat{\varphi}_\alpha + \psi_\beta G_\beta - \psi_\alpha G_\alpha,$$

and taking into account Lemma 4.1, it only remains to prove that

$$\|\psi_\beta G_\beta - \psi_\alpha G_\alpha\|_{Lip[t,1]} \leq C\Big(|\beta - \alpha| + \int_t^1 \|G_\beta - G_\alpha\|_{Lip[s,1]}\, ds\Big)$$

$$\psi_\beta G_\beta - \psi_\alpha G_\alpha = \psi_\beta(G_\beta - G_\alpha) + (\psi_\beta - \psi_\alpha)G_\alpha$$

For any $g \in Lip[0,1]$:

$$\begin{aligned}
&|\psi_\beta g(t,0,0)| = \Big|\int_t^1 \Big(\rho\,\mathrm{E}[g(s,V,(1-q\beta)c(s-t)) - g(s,0,(1-q\beta)c(s-t))] \\
&+ a\,\mathrm{E}[g(s,0,(1-q\beta)c(s-t)-Y) - g(s,0,(1-q\beta)c(s-t))]\Big)\, ds\Big| \\
&\leq \int_t^1 (\rho\,\mathrm{E}[V] + a\,\mathrm{E}[Y])\|g\|_{Lip[s,1]}\, ds
\end{aligned}$$

Also,

$$\begin{aligned}
&|\psi_\beta g(t,v,\omega) - \psi_\beta g(t,\tilde{v},\tilde{\omega})| = \\
&= \Big|\int_t^1 \Big(\rho\,\mathrm{E}[g(s,\eta(v)e^{-\delta(s-t)} + V, \omega + (1-q\beta)c(s-t)) - g(s,\eta(\tilde{v})e^{-\delta(s-t)} + V, \tilde{\omega} + (1-q\beta)c(s-t)) \\
&- g(s,\eta(v)e^{-\delta(s-t)}, \omega + (1-q\beta)c(s-t)) + g(s,\eta(\tilde{v})e^{-\delta(s-t)}, \tilde{\omega} + (1-q\beta)c(s-t))] \\
&+ (a + \eta(v)e^{-\delta(s-t)})\,\mathrm{E}[g(s,\eta(v)e^{-\delta(s-t)}, \omega + (1-q\beta)c(s-t) - Y) \\
&- g(s,\eta(\tilde{v})e^{-\delta(s-t)}, \tilde{\omega} + (1-q\beta)c(s-t) - Y) \\
&- g(s,\eta(v)e^{-\delta(s-t)}, \omega + (1-q\beta)c(s-t)) + g(s,\eta(\tilde{v})e^{-\delta(s-t)}, \tilde{\omega} + (1-q\beta)c(s-t))] \\
&+ (\eta(v) - \eta(\tilde{v}))e^{-\delta(s-t)}\,\mathrm{E}[g(s,\eta(\tilde{v})e^{-\delta(s-t)}, \tilde{\omega} + (1-q\beta)c(s-t) - Y) \\
&- g(s,\eta(\tilde{v})e^{-\delta(s-t)}, \tilde{\omega} + (1-q\beta)c(s-t))\Big)\, ds\Big| \\
&\leq \int_t^1 2(\rho + a + M)\|g\|_{Lip[s,1]}(|v - \tilde{v}| + |w - \tilde{w}|) + |\eta(v) - \eta(\tilde{v})|\,\mathrm{E}[Y]\|g\|_{Lip[s,1]}\, ds \\
&\leq C\int_t^1 \|g\|_{Lip[s,1]}\, ds(|v - \tilde{v}| + |w - \tilde{w}|)
\end{aligned}$$

It follows that,

$$|(\psi_\beta - \psi_\alpha)g(t,0,0)| = \Big|\int_t^1 \Big(\rho\,\mathrm{E}[g(s,V,(1-q\beta)c(s-t)) - g(s,V,(1-q\alpha)c(s-t))$$

$$
\begin{aligned}
&- g(s,0,(1-q\beta)c(s-t)) + g(s,0,(1-q\alpha)c(s-t))] \\
&+ a\mathrm{E}[g(s,0,(1-q\beta)c(s-t)-Y) - g(s,0,(1-q\alpha)c(s-t)-Y) \\
&- g(s,0,(1-q\beta)c(s-t)) + g(s,0,(1-q\alpha)c(s-t))]\Big)\, ds\Big| \\
&\leq 2qC(\rho+a)\int_t^1 \|g\|_{Lip[s,1]}\, ds|\beta-\alpha|
\end{aligned}
$$

Then,

$$
\begin{aligned}
&|(\psi_\beta-\psi_\alpha)g(t,v,\omega) - (\psi_\beta-\psi_\alpha)g(t,\tilde{v},\tilde{\omega})| = \\
&= \Big|\int_t^1 \Big(\rho\,\mathrm{E}[g(s,\eta(v)e^{-\delta(s-t)}+V,\omega+(1-q\beta)c(s-t)) - g(s,\eta(v)e^{-\delta(s-t)}+V,\omega+(1-q\alpha)c(s-t)) \\
&- g(s,\eta(v)e^{-\delta(s-t)},\omega+(1-q\beta)c(s-t)) + g(s,\eta(v)e^{-\delta(s-t)},\omega+(1-q\alpha)c(s-t)) \\
&- g(s,\eta(\tilde{v})e^{-\delta(s-t)}+V,\tilde{\omega}+(1-q\beta)c(s-t)) + g(s,\eta(\tilde{v})e^{-\delta(s-t)}+V,\tilde{\omega}+(1-q\alpha)c(s-t)) \\
&+ g(s,\eta(\tilde{v})e^{-\delta(s-t)},\tilde{\omega}-(1-q\beta)c(s-t)) - g(s,\eta(\tilde{v})e^{-\delta(s-t)},\tilde{\omega}+(1-q\alpha)c(s-t)) \\
&+ (a+\eta(v)e^{-\delta(s-t)})\,\mathrm{E}[g(s,\eta(v)e^{-\delta(s-t)},\omega+(1-q\beta)c(s-t)-Y) \\
&- g(s,\eta(v)e^{-\delta(s-t)},\omega+(1-q\alpha)c(s-t)-Y) - g(s,\eta(v)e^{-\delta(s-t)},\omega+(1-q\beta)c(s-t)) \\
&+ g(s,\eta(v)e^{-\delta(s-t)},\omega+(1-q\alpha)c(s-t)) \\
&- g(s,\eta(\tilde{v})e^{-\delta(s-t)},\tilde{\omega}+(1-q\beta)c(s-t)-Y) + g(s,\eta(\tilde{v})e^{-\delta(s-t)},\tilde{\omega}+(1-q\alpha)c(s-t)-Y) \\
&+ g(s,\eta(\tilde{v})e^{-\delta(s-t)},\tilde{\omega}-(1-q\beta)c(s-t)) - g(s,\eta(\tilde{v})e^{-\delta(s-t)},\tilde{\omega}+(1-q\alpha)c(s-t)) \\
&+ (\eta(v)-\eta(\tilde{v}))e^{-\delta(s-t)}\,\mathrm{E}[g(s,\eta(\tilde{v})e^{-\delta(s-t)},\tilde{\omega}+(1-q\beta)c(s-t)-Y) \\
&- g(s,\eta(\tilde{v})e^{-\delta(s-t)},\tilde{\omega}+(1-q\alpha)c(s-t)-Y) - g(s,\eta(\tilde{v})e^{-\delta(s-t)},\tilde{\omega}-(1-q\beta)c(s-t)) \\
&+ g(s,\eta(\tilde{v})e^{-\delta(s-t)},\tilde{\omega}+(1-q\alpha)c(s-t))]\Big)\, ds\Big| \\
&\leq \int_t^1 \Big(4\rho\|g\|_{Lip[s,1]}qC|\beta-\alpha| + 4(a+M)\|g\|_{Lip[s,1]}qC|\beta-\alpha| + 2M\|g\|_{Lip[s,1]}|\beta-\alpha|\Big)\, ds \quad \square
\end{aligned}
$$

6.7. Proof of Theorem 4.1.

We have that,

$$
\begin{aligned}
G_\beta - G_\alpha - (\beta-\alpha)M_\alpha &= \hat{\varphi}_\beta + \psi_\beta G_\beta - \hat{\varphi}_\alpha - \psi_\alpha G_\alpha - (\beta-\alpha)\Big(\frac{\partial\hat{\varphi}_\alpha}{\partial\alpha} + (\frac{\partial\psi_\alpha}{\partial\alpha})G_\alpha + \psi_\alpha M_\alpha\Big) \\
&= \hat{\varphi}_\beta - \hat{\varphi}_\alpha + (\psi_\beta-\psi_\alpha)G_\alpha + \psi_\alpha(G_\beta - G_\alpha) \\
&- (\beta-\alpha)\Big(\frac{\partial\hat{\varphi}_\alpha}{\partial\alpha} + (\frac{\partial\psi_\alpha}{\partial\alpha})G_\alpha + \psi_\alpha M_\alpha\Big) + (\psi_\beta-\psi_\alpha)(G_\beta-G_\alpha)
\end{aligned}
$$

By definition,

$$
\hat{\varphi}_\beta - \hat{\varphi}_\alpha = (\beta-\alpha)\frac{\partial\hat{\varphi}_\alpha}{\partial\alpha} + o(\beta-\alpha)
$$

$$(\psi_\beta - \psi_\alpha)G_\alpha = (\beta-\alpha)(\frac{\partial \psi_\alpha}{\partial \alpha})G_\alpha + o(\beta-\alpha)$$

Due to Lemma 4.2, we get: $(\psi_\beta - \psi_\alpha)(G_\beta - G_\alpha) = o(\beta-\alpha)$. So,

$$G_\beta - G_\alpha - (\beta-\alpha)M_\alpha = \psi_\alpha(G_\beta - G_\alpha - (\beta-\alpha)M_\alpha) + o(\beta-\alpha)$$

If we use the same argument as in the proof of Lemma 4.2:

$$\|G_\beta - G_\alpha - (\beta-\alpha)M_\alpha\|_{Lip[t,1]} \leq c\int_t^1 \|G_\beta - G_\alpha - (\beta-\alpha)M_\alpha\|_{Lip[t,1]}\, ds + o(\beta-\alpha)$$

Then, Grönwall's lemma implies that,

$$\|G_\beta - G_\alpha - (\beta-\alpha)M_\alpha\|_{Lip[0,1]} = o(\beta-\alpha). \quad \square$$

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