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INTERNSHIP REPORT

MODELLING A REAL ESTATE DEVELOPMENT AS A FINANCIAL OPTION

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Abstract

This Thesis, developed in the scope of an internship in the enterprise technology and performance team at Deloitte, aims to develop Williams (1991) real option pricing methodology, into a discrete time model implemented in python code with the purpose of using that model – along with the company's historical data – to valuate a real option case study. The term Real options, first emerged in the late 1970s, capturing the value of waiting and adapting to new information. This methodology overcomes investment timing and irreversibility limitations associated with more traditional methods, such as the net present value, where we consider a now-or-never/all-or-nothing investment analysis. Therefore, it is possible for an investment that was once viewed as not viable through a more conventional method, to appear as viable when considering a real option approach, and vice-versa. This is because decision making is more complex than a one point in time decision, and it strongly depends on the current market conditions which evolve over time. The case study in this Thesis is related to real estate and considers a company's expansion investment where, through land purchasing, the company can decide either to start development and/or abandon the project. We take into consideration both developed and undeveloped properties' market value, resulting from the respective net cash inflows and development costs. By computing the optimal abandonment and development point, through various scenarios, we examine all the components that influence the optimal final decision and conclude the company's expansion is overall a viable investment with no significant drawbacks.

Keywords: Real Estate Development; Real Options; Optimal Development; Optimal Abandonment; Optimal timing; Optimal Density, Decision Making.

Resumo

Esta Tese visa desenvolver a metodologia de analise de opções reais de Williams (1991), num modelo de tempo discreto implementado em código python com o objetivo de o utilizar – juntamente com os dados históricos da empresa – para avaliar um estudo de caso sobre opções reais. O termo “opções reais” surgiu no final da década de 1970 realçando, na análise de investimentos, o valor de esperarmos e de nos adaptarmos à medida que novas informações vão surgindo. Esta metodologia supera as limitações de *timing* de investimento e irreversibilidade associadas a métodos mais tradicionais, como o valor atual líquido, onde consideramos uma análise de investimento “agora ou nunca” ou “tudo ou nada”. Portanto, um investimento que antes parecia inviável, por meio de um método mais convencional, torna-se viável ao considerarmos uma metodologia de opções reais, e vice-versa. Isto porque a tomada de decisão é complexa e depende fortemente das condições atuais de mercado. O estudo de caso desta Tese está relacionado com o mercado imobiliário e analisa a expansão de uma empresa que, ao adquirir um terreno, pode decidir entre iniciar o seu desenvolvimento e/ou abandonar o projeto. Tendo em consideração para o valor de mercado da propriedade em desenvolvimento e da propriedade por desenvolver, os respetivos fluxos de caixa e custos de desenvolvimento. Ao calcular, os pontos ótimos de abandono e desenvolvimento para diversos cenários, analisamos todos os componentes que influenciam a decisão ótima final, e concluímos que a expansão da empresa é, no geral, um investimento viável sem desvantagens significativas.

Palavras-chave: Desenvolvimento Imobiliário; Opções Reais; Desenvolvimento Ótimo; Abandono Ótimo; Tempo Ótimo; Densidade Ótima, Tomada de Decisão.

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1. Introduction

This thesis forms part of my master's degree in Mathematical Finance at the Lisbon School of Economics and Management. As part of my degree, I participated in a three-month internship at Deloitte, a leading global provider of audit and assurance, consulting, financial advisory, risk advisory, tax, and related services.

I did my internship in Business Consulting, more specifically related to the area of Enterprise Technology and Performance, where I participated in a project aiming the financial processes' uniformization of a Real Estate Company with recent acquisitions in the hospitality sector. This project sparked my interest to analyse business expansion investments. Therefore, by combining the challenges of the project I was integrated in with the subjects taught in my master's program, I saw an interesting opportunity to evaluate the investment in Deloitte Portugal's most recent office expansion as a real option (RO). However, due to the lack of necessary data access, I ended up applying the model to an alternative case study for which data was available. As a result, we consider a home decor and furniture company, called Bazar Dili, and analyse a potential expansion scenario.

In 2014, Bazar Dili acquired a 739-square-meter property and developed it into a new store. This expansion created new opportunities for the company, with new products and better brand visibility. Using this past expansion data, we now assume the company plans to expand its business again by opening a new store in a different market. Subject to legal limitations, once the property is purchased, the company can choose when to begin development and at what scale or density. Once construction starts and the irreversible investment is made, the company exercises the option. Additionally, if market conditions become unfavourable at any point, the company has the possibility to abandon the project. RO framework provides important and flexible results on investments, such as company expansions, creating future opportunities where it is advantageous to analyse them using option pricing, contrary to conventional discounted cash flow (DCF) and net present value (NPV) analyses. As such, Bazar Dili's decision to open a new store will be viewed as a RO because it creates strategic flexibility for future growth.

All computational implementations, either for simulated and real case study scenarios, are carried out in python. The code is presented in Appendices C to G.

RO's history and significant contributions to financial investment analysis are presented in chapter 2, along with an overall summary on the methodology chosen for the expansion scenario analysis – Williams 1991 methodology. In chapter 3, Williams' methodology is further explained and in section 3.2 its discrete time model implementation in python is discussed, followed by its numerical illustration in section 3.3. Finally, using the company's development costs and revenue data, in chapter 4 we use Williams' model to examine the company's expansion and access the investment viability and final optimal decision, through different scenarios. We close drawing conclusions in chapter 5.

2. A Review on Real Options

Although the exact origins of contingent claims are unclear, their use dates back to the Romans and Phoenicians, when dealing with commercial shipping trades. In ancient Greece, the mathematician and philosopher Thales used such contracts to secure a low price for olive presses, in advance of olive harvest, as he believed demand would be particularly strong that year. He acquired the rights when the demand for the equipment was very low and, during the harvest season, he exercised his option and rented the equipment at a higher price. Another famous early use of contingent claims happened in Holland, during the early 1600s. While tulip dealers used call options to secure a low price to meet demand, tulip growers used put options to ensure a reasonable selling price. However, as speculators joined the mix and traded these contracts for profit, when the market crashed, many failed to honour their agreements, leading to a devastating impact on the economy (Schoutens, 2003).

For centuries these contracts have played a critical role in commercial strategy, offering both opportunities for gain and risk management. Nowadays, contingent claim contracts have evolved into standardized financial instruments widely used in various markets, including commodities, equities, and derivatives. Options, futures, and other derivatives are now essential tools for investors and companies to hedge against risks and speculate on price movements.

An option is a financial instrument giving one the right, but not the obligation, to make a specified transaction at (or by) a specified date at a specified price. Options are thus privileges sold by one party to another. The right is granted by the person who sells the option – referred to as the seller or writer of the option. And the person who buys the option is called the option buyer (Schoutens, 2003).

There are different types of options with each serving a different purpose. The basic types include call options, which grant the right to buy, and put options, which grant the right to sell. Generally, the payoff function of a call option at expiry is given by $\phi = (S_T - K)^+$, i.e. $\max(S_T - K, 0)$, where $t = 0$ is the initial time, $t = T$ is the maturity, K is the strike or exercise price and S_T is the stock price at maturity. Conversely, the payoff function at expiry of a put option is given by $\phi = (K - S_T)^+$. For some t, we say that the call option is at the money if $S_t = K$, in the money if $S_t > K$, and out of the money for $S_t < K$ (Schoutens, 2003). In the case of a put option, the opposite is true.

Additionally, options are further categorized as either plain vanilla or exotic options. European options, considered plain vanilla, allow the right to buy or to sell only on the specified expiration date. In contrast, exotic options have more complex structures. The most commonly known are the American options which allow the holder to buy or sell at any time until the expiration date. Other examples are barrier options, where the contract is only active if the stock reaches a predetermined level, or even range options, where their payoff at expiry is determined by the spread between maximum and minimum prices of the underlying asset during the options lifetime (Schoutens, 2003).

The pricing of option contracts was an unresolved issue in finance until 1973, when Fischer Black and Myron Scholes introduced their now widely accepted option pricing formula (Black, and Scholes, 1973). Their model is based on the principle of creating a replicating portfolio, which mimics the payoff of a European call option using a combination of common stock and a risk-free bond under some assumptions, such as the normality of the returns of the underlying asset. This formula revolutionized options trading and remains a cornerstone of financial markets.

Not long after, in 1977, the term "real options" was coined (Myers, 1977) as its study attracted interests among researchers and practitioners that followed the defining developments in options pricing theory (Adetunji et al., 2016). It was Stewart Myers (Myers, 1977) who first conceived the option-based idea of assessing future opportunities that are inherent in projects and started studying its potential in the field of real estate property investments.

In the late 1970s, finance researchers were looking forward to another breakthrough in RO pricing technique and its wide application in the field of finance. In 1984, Myers pointed out that conventional valuation methods based on DCF are not able to handle projects encompassing both production and strategic options, and therefore proposed option pricing to be used for valuation purposes (Rózsa, 2016). So, in the same way option pricing formula started to be used by option traders daily on the floors of option exchanges, theorists were expecting that the RO technique would replace the traditional DCF in the valuation of investment projects (Adetunji et al., 2016). However, while RO analysis has indeed added valuable insights, the DCF method was not completely replaced.

Nowadays, to decide whether to invest while facing uncertainty over future market conditions, the consensus is still to determine the NPV, by calculating the present value of the expected stream of cash that the investment is expected to generate, computing the present value of the stream of expenditures required to undertake the project, and determining the difference between the two. The rule being: while positive, invest; otherwise, do not invest.

A common problem with the conventional NPV rule is that it ignores the value of creating options. Sometimes an investment that appears uneconomical when viewed in isolation may, in fact, create options that enable the company to undertake other investments in the future, should market conditions turn favourable. The simple NPV rule could be modified to a new approach where instead of just imposing it is positive, the present value of the expected stream of cash from a project must also exceed the cost of the project by an amount equal to the value of keeping the investment option alive (Dixit et al., 1995). This means that the rule for NVP could still apply if we subtract from the conventional calculation the opportunity cost of exercising the option to invest. But despite being relatively easy to apply, the NPV rule is usually built on faulty assumptions (Dixit et al., 1995).

Namely, it assumes one of two things: (i) either that the investment is irreversible, i.e. it is an industry-specific capital project that is viewed as a sunk cost so if the company does not make the investment now, it will lose the opportunity forever; (ii) or that the investment is reversible, can somehow be undone, and the expenditures recovered, should market conditions turn out to be worse than anticipated (Dixit et al., 1995).

Through the RO methodology (see Table 1) we are able to address issues regarding irreversibility, uncertainty, and timing. Thus, viewing this investment as an opportunity creates something much like a call option, having the right but not the obligation to buy an asset (to invest) at a future date of its choosing (Luehrman, 1998).

For a call option, the exercise price or strike price is the value paid which grants the holder the right but not the obligation to exercise the option. Therefore, the expenditure required to acquire the projects' assets, given its irreversible nature, represents our strike price, K. And the market value of the property, based on its cash-inflows, will represent the stock price, S. Therefore, the company will have the incentive to abandon the property if the costs of carrying an undeveloped property exceed sufficiently its operating revenues, losing only what it has spent to obtain the investment opportunity. As long as there are some contingencies under which the company would prefer not to invest, the opportunity to delay the decision – and thus to keep the option alive – has value (Dixit et al., 1995). The question, then, is when to exercise the option.

Table 1: Real options analogy to an investment opportunity (Luehrman, 1998)

Investment Opportunity	Variable	Call option
Present value of the project opening assets to be acquired	S	Stock Price
Expenditure to acquire the project assets	K	Exercise price
Length of time the decision may be deferred	t	Time to expiration
Time value of money	r_f	Risk free rate of return
Riskiness of the projects assets	σ^2	Variance of returns on a stock

Since the option to develop never expires, the owner optimally exercises his option when his property's developed value – determined by its operating cash inflows – exceeds its costs of development (Williams, 1991). Subsequently, the owner optimally abandons the property, should its costs sufficiently exceed its operating revenue. So, the option is exercised when the irreversible investment expenditure is made. By deciding to go ahead with an expenditure, the company gives up the possibility of waiting for new information that might affect the desirability or timing of the investment. It cannot disinvest, should market conditions change adversely (Dixit et al., 1995).

To determine what influences the builder's investment decision, in 2003, Mayer, Bulan and Somerville (Bulan et al., 2009) examined the relationship between uncertainty, competition, and irreversible investment using a sample of 1214 condominium developments in Vancouver, Canada, built from 1979 to 1998. They found that the volatility of returns, exposure to market risk, and competition play important roles in the timing of investment. If competition is less pronounced in recessions, RO behaviour may lead developers to delay irreversible investments in structures longer than they would in booms when markets are more competitive. In highly competitive markets, sensitivity of investment timing to volatility is reduced, and firms are incentivised to invest sooner. Even though "some theoretical papers have argued that RO models have limited power to predict investment in competitive markets" (Bulan et al., 2009), these results offer stronger support in favour of RO models and its applications.

The first application of RO research was in the natural resource extraction industry during the 1980s and has since been applied across various fields. As an example, more recently in 2017, the RO methodology was applied in renewable energy investments due to its high-risk profile and irreversibility (Kozlova, 2017). In that paper, apart from providing many valuable insights for investors and policymakers, the reviewed studies illustrate the relevance of the RO approach and demonstrate its superiority over traditional capital budgeting techniques, highlighting its ability to capture uncertainty and flexibility.

Given its relevance, ever since RO first introduction in 1977 (Myers, 1977), a diverse range of methodologies that extend from financial option pricing models were introduced. There is not a universally recognized methodology but rather a set of widely accepted approaches and models.

Lenos Trigeorgis, a RO theorist, identified several types of common RO which could be incorporated into firm's investment projects (Trigeorgis, 1993a). According to Trigeorgis, these options include:

- **Wait or Defer Option**: where firm managers delay projects, by holding a lease on (or an option to buy) valuable land resources, until demand uncertainty or project costs justify the investment. These options are particularly relevant in natural resource extraction industries, real estate development, farming and paper products. Several studies have valued options to wait or defer in investment projects (Titman ,1985; Tourinho, 1979; Titman, 1985; McDonald and Siegel, 1986; Paddock, Siegel and Smith, 1988; Ingersoll and Ross, 1992, Yongma, 2014).

- **Time-to-Build Option**: where projects can be staged over time as series, allowing managers to pause and reassess at various stages. Depending on the stages outcome, if new unfavourable information arises, the abandonment option is considered. Each stage is viewed as an option on the subsequent stages' value and valued as a compound option. These options are particularly relevant in R&D intensive industries, specifically pharmaceuticals, and in long development capital intensive projects. Several studies have valued time-to-build options (Majd and Pindyck, 1987; Carr, 1998; Trigeorgis, 1993b).

- **Option to Alter Operating Scale**: firm managers can expand/reduce the project's scale of production, or accelerate/decelerate resource usage, in response to demand fluctuations. This option is especially useful in the natural resource industries and facilitates planning and construction in cyclical industries. Several RO studies have examined options to expand or contract operating scales in capital investment (Brennan and Schwartz, 1985a; McDonald and Siegel, 1985, Trigeorgis and Manson, 1998; Pindyck, 1998; Lawryshyn and Jaimungal, 2014).

- **Option to Switch**: where projects are designed with the flexibility to switch, inputs or outputs, according to demand or price changes. "There are some plants that can be built with the option of switching between inputs depending on the prices of raw materials or inputs used in the production of finished products by the plants. For example a power generating plant can be built with the option to switch between oil and gas depending on the prevailing prices of these input products" (Adetunji et al., 2016). Several RO studies have examined options to switch in capital investments (Henseler and Roemer, 2013; Kulatilaka and Marks, 1988).

- **Growth Option**: where initial investments create the opportunity for future investments, allowing firms to capitalize on favourable conditions as they arise. "Some key investments are needed by firms to enjoy the opportunities for future investments. If these key investments are not made, firms may not enjoy the future revenues from follow-on investments that will be made based on the assets that are already in place" (Adetunji et al., 2016). Several RO studies have examined growth options in capital investments (Abel and Eberly, 2012; Albuquerque, 2014; Taudes, 1998).

- **Multiple Interacting Options**: this type of options can be found in multiple industries where projects contain several compound options that cannot be valued independently, as they impact each other. Several studies have examined multiple interacting options application (Brennan and Schwartz, 1985b; Trigeorgis 1993b; Kulatilaka, 1988)

- **Option to Abandon**: firm managers can choose, depending on the market conditions which influence the project's future cash flows, to start development and/or permanently abandon current operations. If necessary, they can resale their investments in second-hand markets. This option is especially useful when determining the economic continual usage viability of certain assets. The importance of abandonment options refers more specifically to capital intensive industries, financial services, and new product introductions in uncertain markets. Several RO studies have examined options to abandon (Williams, 1991; Huang and Chou, 2006; Quigg, 1993; Myers and Majd, 1990).

Considering the previously discussed option types and given the specifics of the case study under analysis in this work, we can characterize Bazar Dili's RO decision to expand – by developing a new store in the purchased property – as a development option with the possibility of abandonment. By entering in a new market, the company poses a certain level of uncertainty over consumer demand and, consequently, over future revenues. The abandonment option gives the company more flexibility, allowing it to cancel the project if market conditions become unfavourable.

From the analysis through various abandonment option methodologies, (Williams 1991) stood out as it is not only well explained but also highly relevant to the case study under analysis. "When the real options in the investment projects tend to become more complex, the option pricing theory approach also becomes complicated and computations become increasingly difficult." (Kemna,1993). As a result, despite being an older approach, because we also consider a relatively simple case study, Willaims (1991) clarity and practical application make it particularly effective in this context. Thus, this Thesis aims to provide a comprehensive analysis of the 1991 Joseph T. Williams' methodology through its computational implementation, analysis, and its application to our case study.

According to this methodology, when mapping this investment opportunity onto a Call option we consider 3 main points in time (see Figure 1):

- **Date of acquirement:** the initial moment, $t = 0$, is the date at which the company acquired the property.

- **Development point:** at any $t \geq 0$ the company can opt to develop the facilities at a feasible density q , such that $1 \leq q \leq \delta$, where δ is the maximum density of development.

- **Abandonment point:** there exists a $t = T$, where the company has an incentive to abandon the property.

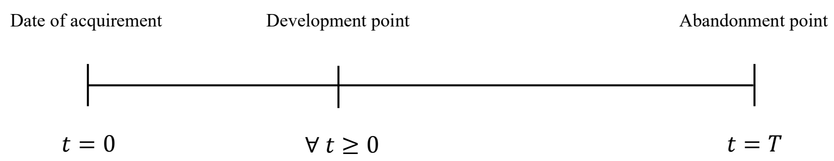

Figure 1: Williams' time horizon and optimal points

So, according to Williams, the investor has the flexibility to choose not only the timing when to begin developing the property or potentially abandon it, but also the scale or density of the development itself. Therefore, it's interesting to note that more recently in 2021, Yongqiang and Tien Foo (Yongqiang and Sing, 2021) combined the RO models of Pindyck (1988), Williams (1991) and Capozza and Li (1994) to model the optimal intensity and timing of a development as joint decisions. They found that this combination significantly increases the value of waiting during periods of rising demand shocks. Therefore, real estate development activities slow down, and city growth decelerates in highly volatile markets. However, for the purpose of this Thesis, and for the previously stated reasons, both decisions will be considered separately, following (Williams, 1991).

In summary, given the extensive history and significant contributions of RO theory to financial investment analysis, this Thesis aims to implement Williams approach by developing its computational framework and determining both the optimal abandonment and development points in time, in simulated examples and real case scenarios, along with its real efficiency and applicability.

3. Real Options and Real Estate Investments

In this chapter, Williams' methodology (Williams, 1991) for abandonment options will be closely followed in Section 3.1. This methodology provides a framework to determine the optimal investment strategy through an optimal abandonment and development ratio which considers both undeveloped and developed property values, along with the projects' expected cash inflows and development costs. In Section 3.2 we present the framework used to simulate the development cost and cash inflow trajectories which, according to Williams, follow a geometric Brownian motion, and present Williams' framework used to compute the optimal ratios. Finally, in Section 3.3, we present a numerical illustration of the discrete time model implemented in python, based on Williams' methodology.

3.1. Williams' Methodology

In this section we delve deeper into Williams RO analysis, where we aim to understand which key factors contribute for the investors timing decision, consequently establishing which variables and parameters contribute for the valuation of both developed and undeveloped properties that will latter influence the optimal investment points. The PDE solutions presented in this chapter will be better explained in Section 3.2. with the computational framework implementation.

According to Williams, if the company decides to develop the properties, the cost of the development is given by $q^{\gamma}x_1$, where q is the density of development chosen, γ is the constant cost of scale or the elasticity coefficient, and x_1 is the cost per unit of development. The cost of scale implies that the density of the development, at higher densities, is more costly per unit of development, $\gamma \geq 1$. Moreover, the net cash inflow per unit of time of the developed property is given by qx_2, where q is the density of development chosen, and x_2 is the cash inflow per unit of density, measured per unit of time.

On the other hand, if the company decides to not develop the properties, then the net cash inflow per unit of time of the undeveloped property is given by βx_2, where β is a constant such that $0 \leq \beta < 1$, and x_2 the cash inflow per unit of density, measured per unit of time. Note that, given the inequalities $0 \leq \beta < 1 \leq q < \delta$, we conclude that $\beta < q$, thus $\beta x_2 < qx_2$, meaning that for the same cash inflow, the net cash inflow of the underdeveloped property will always be higher than for the undeveloped property.

Moreover, both the development cost and the net cash inflow evolve stochastically through time. By assumption, the unit development cost x_1 and the unit cash inflows x_2 follow geometric Wiener processes:

$$dx_i = \mu_i x_i dt + \sigma_i x_i dz_i,$$

where $i = 1, 2$, dt represents infinitesimal time increments, and dz_i are stochastic increments. Each variable with the respective constant expected rate of growth μ_i and the constat variance of the growth rate σ_i^2, both measured per unit of time. Additionally, the covariance between x_1 and x_2 , σ_{12}, is constant and measured per unit of time, and, therefore, the correlation coefficient $\rho = {}^{\sigma_{12}}/_{\sigma_1\sigma_2}$, is also constant.

To determine the values of both the undeveloped and the developed properties, Williams presents the following assumptions:

1. The stochastic evolution of dx_i can be replicated by two portfolios of continuously traded securities, $i = 1, 2$, without transaction costs, in a perfectly competitive capital market.

2. The returns of portfolio i are perfectly locally correlated with the stochastic increments, dz_i. For each portfolio, the excess mean return per unit of standard deviation equals some constant, λ_i. So, the risk-adjusted expected growth rates $v_i = \mu_i - \lambda_i\sigma_i$, $i = 1, 2$, are also constant.

3. By assuming that a constant riskless rate of interest per unit of time, r, satisfies the inequalities $v_2 < r \leq 1 + v_2$, then the developed and undeveloped properties will have finite values, as will be further demonstrated in this report.

The current price of the developed property, $P(x_2)$, evolves in response to the stochastic evolution of the net cash inflow. We consider a market with one risk free asset r, and two risky assets $i = 1,2$, as previously explained. Because the price of the developed property only depends on x_2, and other various parameters, an instantaneously riskless portfolio can be constructed from the developed property and the second portfolio of substitute securities, $i = 2$. To preclude riskless arbitrage, this instantaneous riskless portfolio must yield the riskless rate r.

Following the general pricing formula under the Black-Scholes framework, for a $G(S_T)$ sufficiently integrable payoff function, then the price is given by $V_t = F(t, S_t)$, where F solves the Black–Scholes partial differential equation (see, for instance, Schoutens 2003),

$$F_t + (r - q)sF_s + \frac{1}{2}\sigma^2 s^2 F_{ss} - rF = 0,$$

$$F(T, s) = G(s).$$

This follows from the Feynman-Kac representation for Brownian motion, where $F(t, s)$ is the price of the option, r is the risk-free interest rate, σ is the volatility of the log returns of the underlying security and $(r - q)$ is the risk adjusted growth rate (Schoutens, 2003).

Similarly, Williams arrived at the following valuation equation that the price of the developed property $P(x_2)$ must satisfy:

$$qx_2 + v_2 x_2 P' + \frac{1}{2}\sigma_2^2 x_2^2 P'' - rP = 0, \tag{1}$$

where we recall that q is the density of development in the net cash inflow per unit of time, i.e. qx_2. Given the irreversible nature of development, as previously discussed, all feasible net cash inflows must satisfy $x_2 > 0$. The price must also satisfy two boundary conditions: (i) if, for some t, the developed property as no net cash inflow, i.e. $x_2 = 0$, then from that time forward the property must become worthless $P(0) = 0$; (ii) if the property is to have a well defined income multiplier, then its price per unit of cash inflow $P(x_2)/x_2$, must be bounded above by some constant $0 < \xi < \infty$; i.e. $P(x_2) \leq \xi x_2$. PDE (1) with these boundary conditions has a unique solution given by

$$P(x_2) = \pi q x_2, \tag{2}$$

where π is a constant such that $\pi = 1/r - v_2$.

As for the current price of the undeveloped property $V(x)$, it evolves in response to the random evolution of both the unit construction cost and the unit cash inflow, $x = (x_1,\ x_2)$. An instantaneously riskless portfolio is constructed by combining the undeveloped property with two portfolios of substitute securities, $i = 1,2$. To preclude riskless arbitrage, the instantaneous riskless portfolio must yield the riskless rate r. Williams arrived at the following valuation equation that the price of the undeveloped property V must satisfy:

$$\beta x_2 + v_1 x_1 V_1 + v_2 x_2 V_2 + \frac{1}{2}\sigma_1^2 x_1^2 V_{11} + \frac{1}{2}\sigma_2^2 x_2^2 V_{22} + \sigma_{12} x_1 x_2 V_{12} - rV = 0,$$

where we recall that β is a constant in βx_2, i.e. the net cash inflow per unit of time, such that $0 \leq \beta < 1$. PDE (2) must be satisfied for all values $x = (x_1, x_2)$ for which development of the property is not optimal. Additionally, the price must also satisfy two boundary conditions. Neither the unit development cost x_1 nor the unit cash inflow x_2 can be negative, $0 \leq V(x) \leq P(x_2)$. And, similarly to the developed property, if the undeveloped property has no current cash inflow, $x_2 = 0$, then from that time forward the property must become worthless, $V(x_1, 0) = 0$. Using the price of the developed property and an appropriate change of variables, where $W(y) = V(x)/x_1$, Williams derives the valuation function V for the undeveloped property measured per unit of development cost, as the solution of equation (2), resulting in the following unique solution:

$$W(y) = \pi\beta y + [\pi(q^* - \beta)y^* - q^*]\left(\frac{y}{y^*}\right)^{\eta}.$$

Now that both property values are well defined, because the investor can choose both the density and the starting date of the development, it is important to determine the optimal points that maximize the market value of his undeveloped property. At the optimal density $q = q^*$ the value of the undeveloped property $V(x^*)$ must be equal to the price of the developed property $P(x_2^*)$ minus the cost of development, $V(x^*) = P(x_2^*) - {q^*}^{\gamma} x_1^*$.

To determine the optimal abandonment point Williams introduces negative cash inflows. This concept allows to deal with situations when, for a given property, its maintenance costs sufficiently exceed its revenues, at which point it is optimal to abandon the property. Net cash inflow of the undeveloped property is introduced as $\alpha x_1 + \beta x_2$, for $\alpha < 0$, where the net cash outflow of the undeveloped property is, due to maintenance, αx_1. Moreover, according to Williams, the option to abandon the property can affect its optimal development point, as undeveloped properties that are more costly to maintain are on average abandoned and developed sooner.

The valuation function for the undeveloped property now becomes

$$\alpha x_1 + \beta x_1 + v_1 V_1 + v_2 V_2 + \frac{1}{2}\sigma_1^2 x_1^2 V_{11} + \frac{1}{2}\sigma_2^2 x_2^2 V_{22} + \sigma_{12} x_1 x_2 V_{12} - rV = 0 \qquad (3)$$

Abandonment is optimal at $x = x_a^*$, only if $V_1(x_a^*) = 0$, and $V_2(x_a^*) = 0$. As for the boundary conditions, if for some t the undeveloped property is abandoned at x_a^* then its value will be $V(x_a^*) = 0$. Applying a change of variables, Williams derives the new equation for the undeveloped property as

$$0 = \frac{1}{2}\omega^2 y^2 W'' + (v_2 - v_1) y W' - (r - v_1) W + \alpha + \beta y, \qquad (4)$$

with the boundary conditions at the optimal abandonment y_a^* being $W(y_a^*) = 0$ and $W'(y_a^*) = 0$. The general solution of PDE (4) is

$$W(y) = \frac{\alpha}{r} + \pi\beta y + A\left(\frac{y}{y_a^*}\right)^{\zeta} - B\left(\frac{y}{y_a^*}\right)^{\eta},$$

for $y_a^* \leq y \leq y_d^*$ with

$$A = \frac{\frac{\alpha}{r} + \beta\pi y_d^* - \left[\frac{\alpha}{r} + \delta^\gamma + (\beta - \delta)\pi y_d^*\right]\left(\frac{y_d^*}{y_a^*}\right)^\eta}{\left(\frac{y_d^*}{y_a^*}\right)^\eta - \left(\frac{y_d^*}{y_a^*}\right)},$$

$$B = \frac{\frac{\alpha}{r} + \beta\pi y_d^* - \left[\frac{\alpha}{r} + \delta^\gamma + (\beta - \delta)\pi y_d^*\right]\left(\frac{y_d^*}{y_a^*}\right)^\zeta}{\left(\frac{y_d^*}{y_a^*}\right)^\eta - \left(\frac{y_d^*}{y_a^*}\right)}.$$

In short, according to Williams, at the optimal density of development q^* the developed property has the net cash inflow q^*y, the resulting market value $\pi q^* y$ and the development cost $q^{*\gamma}$, all measured relative to the unit development cost x_1. The developed property as the net cash inflow βy and the resulting market value $W(y)$, also measured relative to the unit development cost x_1. As for the optimal abandonment, now considering negative cash inflows to account for maintenance costs, the undeveloped property has the net cash inflow $\alpha x_1 + \beta x_2$, where, if operational costs sufficiently exceed revenues, the investor has now the possibility to abandon the development, therefore abandoning the option and consequently the property.

3.2. Computational Framework

In this section we present the (Williams, 1991) approach, to simulate and compute the optimal abandonment and development values. With an overall understanding of the Williams methodology and RO importance in Finance, the goal is now to detail and substantiate each step of the computational implementation, while providing a deeper understanding of this methodology.

3.2.1. Development Costs and Cash Inflows Simulation

As previously outlined, since x_1 and x_2 follow a Geometric Brownian motion, where $dx_i = \mu_i x_i dt + \sigma_i x_i dz_i$, $i = 1, 2$, it is crucial to understand how to derive these parameters and consequently how to simulate these trajectories. Inevitably, as in many models' empirical tests, Williams' methodology encounters the problem of parameter estimation. As a result, following (Williams, 1991) in Chapter 4, three methods will be considered to estimate these variables' risk-adjusted drift and volatility. However, for now, the purpose is simply to assume these parameters, to generate illustrative numerical results, through simulation. In Chapter 4 we examine the impact of these parameters in the model.

Therefore, as we start the computational framework by defining each variables' parameters, it is now crucial to understand how to simulate their trajectories through a Geometric Brownian motion (see Reddy and Clinton, 2016). In theory, we know that a Geometric Brownian motion is given by

$$S_t = S_0 e^{X_t}\ ,\ \ t \geq 0,$$

where $X_t = \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma\, W_t$, $t > 0$ is a Brownian Motion following a Normal$(t(\mu - \frac{1}{2}\sigma^2), \sigma^2 t)$ distribution and S_t has a lognormal distribution (Schoutens, 2003). Generally, if $Y = e^X$ is a lognormal with $X \sim N(0,1)$, then we can simulate Y by setting $Y = e^{\,\mu\, -\, \sigma^2/2\, +\, \sigma Z}$, with $Z \sim N(0,1)$. Additionally, for any $0 \leq s < t$ it holds that

$$S_t = S_0\, \frac{S_s}{S_0} \times \frac{S_t}{S_s} = \ S_0\, e^{X_s} \times e^{X_t - X_s}\ ,$$

because the increment X_s is independent of the increment $X_t - X_s$, and thus consecutive ratios S_s/S_0 and S_t/S_s are independent lognormal. As a result, we can simulate them by generating two iid $N(0,1)$ random variables, Z_1, Z_2, and setting

$$S_s = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)s\, +\, \sigma\sqrt{s}Z_1}\ ,\ \ S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)(t-s)\, +\, \sigma\sqrt{t-s}Z_1}.$$

In general, for $0 = t_0 < t_1 < t_2 < \cdots < t_k$, and defining $Y_i = S(t_i)/S(t_{i-1})$, $i \in \{1,2, \ldots, k\}$ we can write

$$S(t_k) = S(t_{k-1})Y_k = S_0 Y_1 \times Y_2 \times \ldots \times Y_k\ .$$

Since we have that Y_i are independent lognormal random variables that can be constructed by generating k iid $N(0,1)$ random variables, $Z_1, Z_2, \ldots Z_k$, they can be defined as

$$Y_i = e^{\left(\mu - \frac{\sigma^2}{2}\right)(t_i - t_{i-1})\, +\, \sigma\sqrt{t_i - t_{i-1}}\, Z_i},\ i\ \in \{1, 2, \ldots k\}.$$

Therefore, the simulation of both the development costs x_1 and cash inflows x_2, for some initial value S_0 , simplifies to

$$S(t_{i+1}) = S(t_i)Y_i.$$

The GBM implementation code, in python, can be found in Appendix D, Figure 33 and 34.

3.2.2. Optimal Development

Using the cash inflows simulated trajectories per unit of density, x_2, we can determine the developed property value, given by $P(x_2) = \pi q x_2$ (see figure 34 – Appendix E for the net cash inflow implementation code in python – and figure 36 – for the developed property value computation). Furthermore, assuming the ratio $y = x_2/x_1$ and implementing a change of variable $W(y) = V(x)/x_1$, Williams derives the valuation function V for the undeveloped property, defining two additional parameters:

$$\eta = \ -\left[\frac{v_2 - v_1}{\omega^2} - \frac{1}{2}\right] + \sqrt{\left[\frac{v_2 - v_1}{\omega^2} - \frac{1}{2}\right]^2 + 2\frac{r - v_1}{\omega^2}}\ , \qquad \psi = \frac{\eta}{\eta - 1}\ ,$$

where $\omega^2 = \sigma_1^2 - 2\sigma_{12} + \sigma_2^2$, and $\eta, \psi > 1$ (see figure 37 – Appendix F for python code). Consequently, the undeveloped property PDE (2) simplifies to

$$\frac{1}{2}\omega^2 y^2 W'' + (v_2 - v_1)yW' - (r - v_1)W + \ \beta y = 0\ ,$$

for all $0 \le y \le y_d$. With the initial and boundary conditions $W(0) = 0$, $W(y^*) = \pi q^* y^* - q^{*\gamma}$ and the optimal condition $W'(y^*) = \pi q^*$. Therefore, subject to these the boundary conditions, the latter differential equation has the following unique solution

$$W(y) = \pi\beta y + [\pi(q^* - \beta)y^* - q^*]\left(\frac{y}{y^*}\right)^{\eta},$$

where y^* is the optimal development ratio and $q^*$is the optimal density of development. Finally, the optimal development ratio y^* is given by

$$y^* = \begin{cases} \frac{\psi}{\pi}\frac{1}{1-\beta\psi}, & \frac{\psi}{1-\beta\psi} \le \gamma, \\ \frac{\gamma}{\pi}\left[\frac{\beta\gamma\psi}{\gamma-\psi}\right]^{\gamma-1}, & \frac{\delta\psi}{\delta-\beta\psi} \le \gamma < \frac{\psi}{1-\beta\psi}, \\ \frac{\psi}{\pi}\frac{\delta\gamma}{\delta-\beta\psi}, & \gamma < \frac{\delta\psi}{\delta-\beta\psi}, \end{cases} \tag{5}$$

and the optimal density of development $q^*$is

$$q^* = \begin{cases} 1, & \frac{\psi}{1-\beta\psi} \le \gamma, \\ \frac{\beta\gamma\psi}{\gamma-\psi}, & \frac{\delta\psi}{\delta-\beta\psi} \le \gamma \le \frac{\psi}{1-\beta\psi}, \\ \delta, & \gamma < \frac{\delta\psi}{\delta-\beta\psi}. \end{cases} \tag{6}$$

Optimal development ratio and density calculations, in python, can be found in Appendix F, Figure 38 and 39 respectively.

Therefore, the value of the undeveloped property relative to its construction costs is increasing and convex in the ratio $y = {}^{x_2}/_{x_1}$. Now that we have both solutions for property values, $W(y)$ and $P(x_2)$, we can determine the optimal point for development. At $x = x^*$ and $q = q^*$ we have that the undeveloped property is equal to the developed property minus the costs of construction, i.e. $V(x^*) = P(x_2^*) - q^{*\gamma} x_1^*$, therefore, the value of undeveloped property at the optimal point measured per unit of development cost is given by (figure 40 – see Appendix F)

$$W(y^*) = \frac{\pi q x_2^*}{x_1^*} - q^{*\gamma} = \pi q^* y^* - q^{*\gamma}.$$

So, the optimal development is given by the tangent $W(y) = W(y^*)$ and the relative value of the undeveloped property will be

$$W = \begin{cases} \pi\beta y + [\pi(q^* - \beta)y^* - q^*]\left(\frac{y}{y^*}\right)^{\eta}, & y < y^*, \\ \pi q^* y - q^{*\gamma}, & y \geq y^*, \end{cases}$$

as demonstrated in the graph below, taken from (Willaims, 1991), where we can see that the value of the undeveloped property increases as the unit cash inflow relative to the unit construction cost, y , increases.

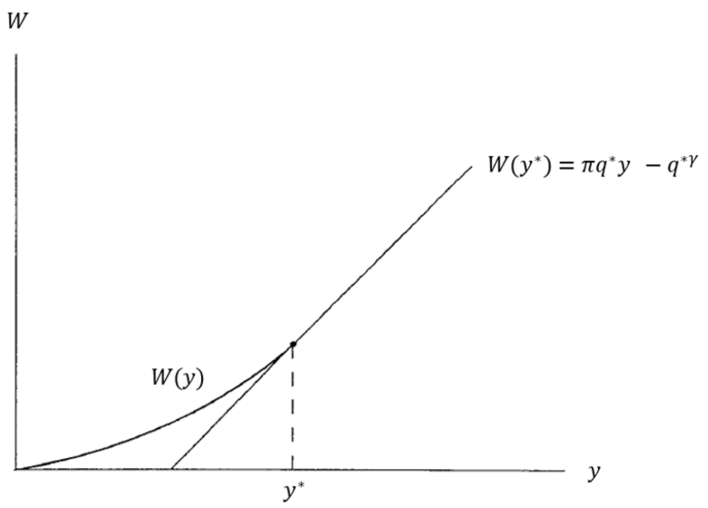

Figure 2: Value of the Undeveloped Property (Williams, 1991)

3.2.3. Optimal Abandonment

Continuing to closely follow (Williams, 1991) to derive the optimal abandonment point, negative cash inflows are introduced along with a change of notation, as described in section 3.1:

$$\zeta = -\left[\frac{v_2 - v_1}{\omega^2} - \frac{1}{2}\right] - \sqrt{\left[\frac{v_2 - v_1}{\omega^2} - \frac{1}{2}\right]^2 + 2\frac{r - v_1}{\omega^2}} \quad \text{and} \quad \phi = \frac{\zeta}{\zeta - 1},$$

with $\zeta < 0 < \phi$ and assuming that $r > v_1$ (Figure 41 – see Appendix G). To simplify, we consider a scenario where the maximum density of development is in fact the optimal density of development. Additionally, the optimal points can be shown to satisfy the following conditions:

$$\left(\frac{y_d^*}{y_a^*}\right)^{\zeta} = \frac{\psi\left(\frac{\alpha}{r} + \delta^{\gamma}\right) + \pi(\beta - \delta)y_d^*}{\psi\frac{\alpha}{r} + \pi\beta y_a^*},$$

and

$$\left(\frac{y_d^*}{y_a^*}\right)^\eta = \frac{\phi\left(\frac{\alpha}{r} + \delta^\gamma\right) + \pi(\beta - \delta)y_d^*}{\phi\frac{\alpha}{r} + \pi\beta y_a^*}.$$

By assuming that $\beta = 0$, i.e. the undeveloped property does not generate any cash inflows, for any $y_d^* \geq 0$, the optimal problem simplifies to

$$y^* = F(y^*) = \frac{\alpha\psi}{\delta r\pi}\left[1 + \frac{r}{\alpha}\delta^\gamma - \left[1 + \frac{r}{\alpha}\delta - \frac{\delta r\pi}{\alpha\phi}y_d^*\right]^{\zeta/\eta}\right],$$

where $F(y)$ is an increasing and convex function with $F(0) > 0 = F(\infty)$, hence it has a unique root y_d^*. To derive an approximation of the optimal development point, y_d^* can be obtained as the limit of the fixed point iterative process $y_{j+1} = F(y_j)$, for $j = 1,2, \ldots$, as $j \to \infty$. Beginning with the lower bound, we start with the values

$$y_0 = \frac{\alpha\phi}{\delta r\pi}\left[1 + \frac{r}{\alpha}\delta^\gamma - \left[-\frac{\zeta - 1}{\eta - 1}\right]^{\eta/\eta - \zeta}\right],$$

and

$$y_1 = \frac{y_0 + F(y_0)}{2}.$$

Using these values, the subsequent points can be computed through the iterative formula, $y_{j+1} = F(y_i)$. Once the optimal development point is approximated as $y_d^* = F(y_d^*)$ (Figure 43 – see Appendix G), we can finally derive the optimal abandonment point given by the now simplified expression (figure 44 – see Appendix G):

$$y_a^* = y_d^*\left[1 + \frac{r}{\alpha}\delta^\gamma - \frac{\delta r\pi}{\alpha\psi}y_d^*\right]^{-1/\zeta}.$$

3.3. Numerical Illustration Using Simulation

We consider for our implementation the same parameter values used by Williams in his paper (Williams, 1991), apart from q, δ, and $x_i, i = 1,2$, initial values, and drift and variance (Table 10 – see Appendix A). Therefore, these parameters are not a representation of reality. Indeed they should depend on the data, and consequently on the investment scenario under analysis, and are here chosen to be equal to (Williams, 1991) only for this illustration purpose.

Nonetheless, based on these parameters, simulations of the cash inflows and development costs are generated, as illustrated in Figures 3 and 4. As mentioned, the computation implementation is carried out in python and the code can be found in Appendices from D to G. At a first glance, both trajectories exhibit an increasing tendency, coherent with both having positive drift parameters.

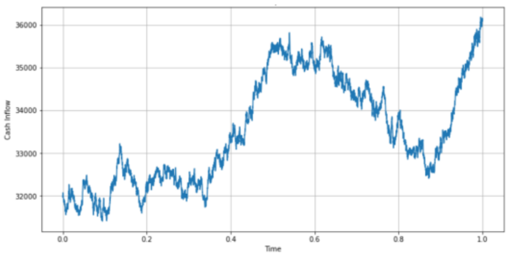

Figure 3: Development Cost simulated example

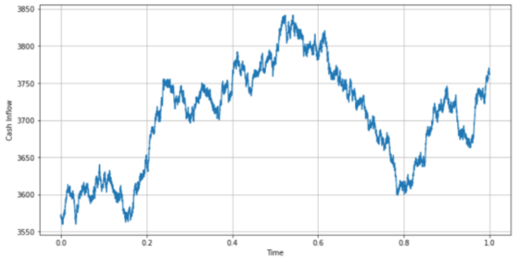

Figure 4: Cash Inflow simulated example

Using the simulated cash inflow values, we can model and compare the net cash inflows for both the developed and undeveloped properties. In theory, for the same cash inflow, the net cash inflow of the developed property will always be higher than that of the undeveloped property, $\beta x_2 < q x_2$. This occurs because the developed property, having already undergone some level of investment, is positioned to generate returns more efficiently, whereas the undeveloped property faces higher initial costs. Using the previously simulated trajectories, this relationship is better resented in Figure 7.

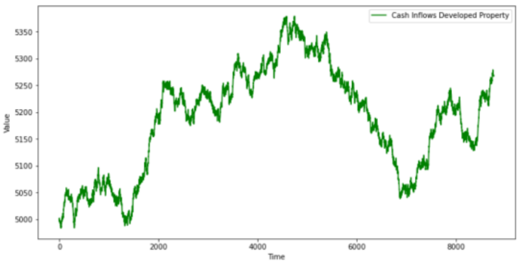

Figure 5: Developed Property's Net Cash Inflow simulated example

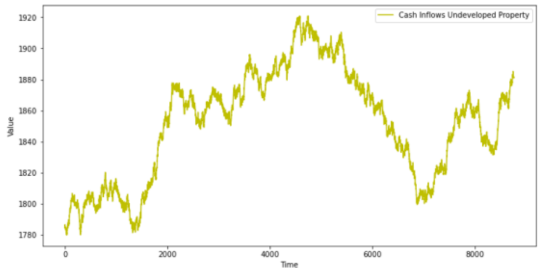

Figure 6: Undeveloped Property's Net Cash Inflow simulated example

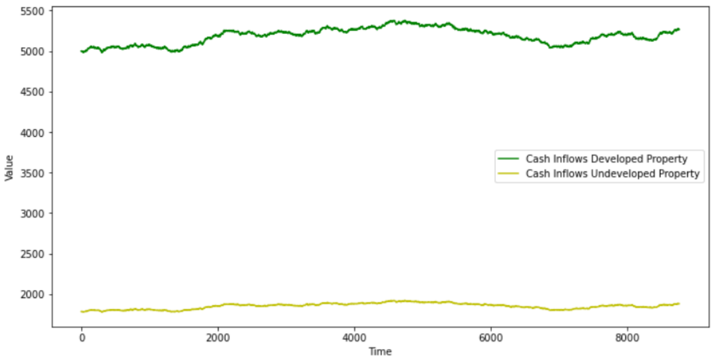

Figure 7: Net Cash Inflow Relationship simulated example

After successfully simulating the net cash inflows, we compute both the optimal development ratio $y^* = 0.280287$ and the optimal development density $q^* = \delta = 1.5$, in this case the maximum density. Since the ratio $y = x_2/x_1$ is always below the optimal development point, represented in Figure 8, the company will not choose to develop their property when considering these parameters and for this time horizon. If, however, the cash inflows would increase, the company would likely be more inclined to develop the property, as it would bring the ratio closer to, or above, the optimal point, thereby justifying the investment decision.

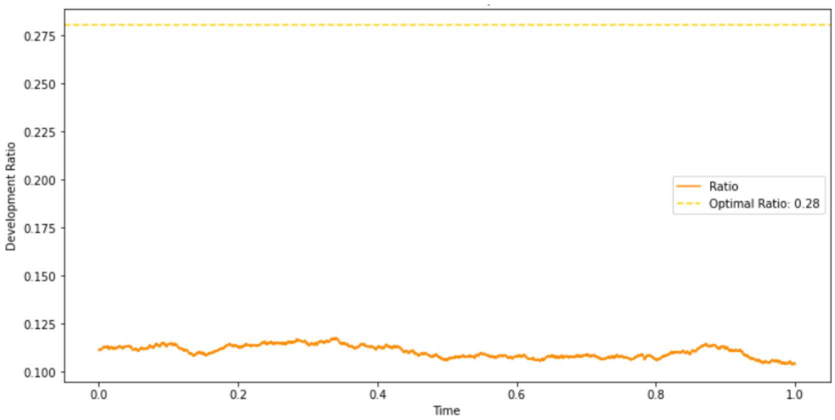

Figure 8: Optimal Development Decision simulated example

Once negative cash inflows are introduced, we assume a special scenario, in order to compute the optimal ratios y_d^* and y_a^*. In this scenario the undeveloped property does not generate any positive cash inflows, and the optimal density of development is equal to the maximum density, i.e. $\beta = 0$ and $q^* = \delta$, respectively. Despite not changing the net cash inflow of the developed property, the negative cash inflows significantly impact the net cash inflows of the undeveloped property.

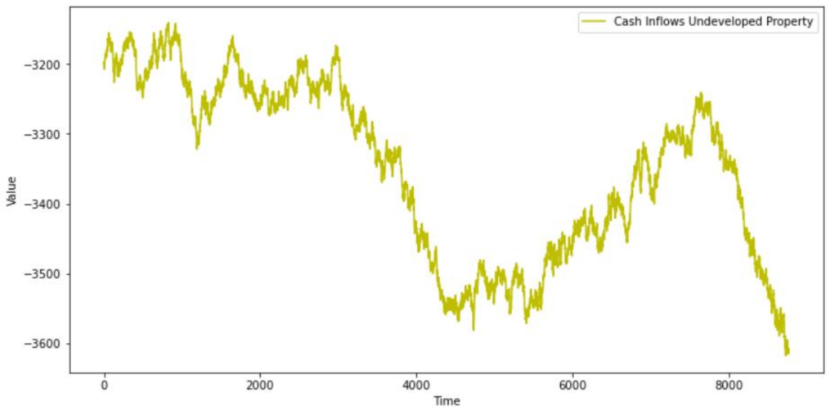

Figure 9: Developed Property's Net Cash Inflow with $\alpha = -0.01$ *simulated example*

Finally, the model arrives at the optimal development $y_d^* = 0.101133$ and abandonment $y_a^* = 0.113459$ ratios. Once again, comparing with the y ratio, development starts once y_d^* line is surpassed, and the property is abandoned once y_a^* is crossed. In the specific case represented below, for the simulated values in Figures 3 and 4, the company optimally starts development shortly after the purchased date and it doesn't surpass the abandonment threshold for the considered time-horizon.

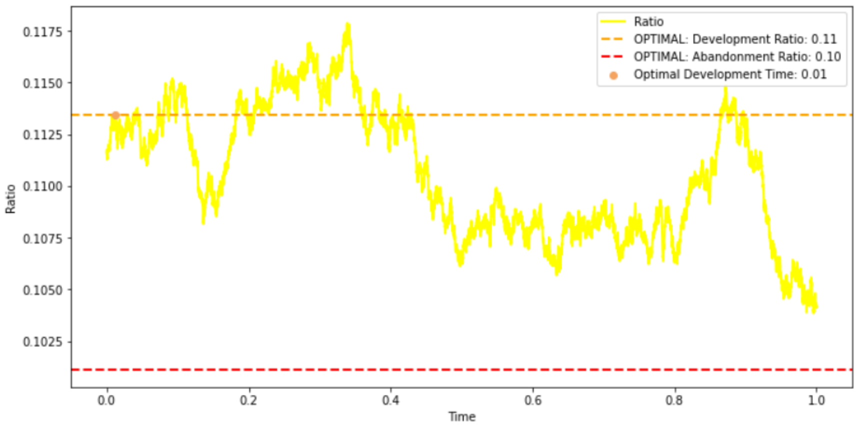

Figure 10: Optimal Decisions simulated example

However, when considering multiple x_1 and x_2 simulations, different optimal decision scenarios emerge. If the initial development ratio is such that $y_0 \in \,]\, y_a^*, y_d^* \,[$, then, as cash inflows progressively decrease, while construction costs increase, the company has greater incentive to abandon the property and, consequently, cancel the project. Conversely, for a development cost progressive decrease followed by a cash inflows increase, the company becomes more incentivised to start development.

Moreover, if the initial development ratio y_0 begins above the optimal development threshold, then the model suggests that development is financially justifiable immediately after purchase. However, if costs significantly increase and revenues from the developed property decline, the company still has the financially viable option to abandon the project. On the other hand, if the initial ratio starts below the optimal abandonment threshold – though this scenario is unlikely – the model indicates that the company should refrain from beginning development and immediately consider abandoning the project.

Additionally, since the optimal ratios are highly sensitive to the chosen parameters, varying parameter sets results on different optimal ratios, which in turn lead to distinct optimal investment decisions. This highlights the importance of carefully selecting or estimating, as well as analysing, these parameters to ensure that the resulting decisions align with the company's strategic objectives and market conditions. Therefore, in the next chapter, a real case implementation will be presented, along with an analysis of the model's behaviour under various parameter sets.

4. Practical Case Study

In light of this discussion, it is important to consider how the RO theory specifically applies to the market and case under study. In this chapter, we examine a home decor and furniture company, called Bazar Dili. The company first opened a store in Évora in 2000 and has ever since expanded, opening multiple stores across the country. In 2014, it changed its installations in Évora by acquiring a 739-square-meter plot of land. The company's available data captures development costs over a 15-month period and revenue figures for approximately 8.5 years after the store's opening (see the actual outputs in the Appendix B). In this scenario we assume the company has purchased a new plot of land and plans to develop into a new store.

Table 2: Case Study Fixed Parameters

Parameter	Value
$\boldsymbol{\gamma}$	2.00
$\boldsymbol{q}$	1.50
$\boldsymbol{\delta}$	1.50
$\boldsymbol{v_1}$	0.00
$\boldsymbol{v_2}$	0.00
$\boldsymbol{\iota}$	0.10
$\boldsymbol{\beta}$	0.00

4.1. Case Study Overview and Results

Fixing, for implementation purposes, the parameters showed in Table 2 we estimate the drift and variance of x_1 and x_2 using 3 methods (Croghan, et al. 2017). The first method is based on the difference between successive observations (x_t and x_{t-1}). As a result, the drift and volatility parameters are given by

$$\hat{\mu} = \sum_{t=1}^{n} \frac{X_t - X_{t-1}}{X_{t-1}}, \text{ and} \qquad \hat{\sigma} = \sqrt{\frac{1}{n-1} \sum_{t=1}^{n} \left(\frac{X_t - X_{t-1}}{X_{t-1}} - \hat{\mu} \right)^2}.$$

The second method is the maximum likelihood method, such that

$$\hat{\mu} = 1 - \sum_{t=1}^{n} \frac{X_t}{X_{t-1}}, \text{ and} \qquad \hat{\sigma} = \sqrt{\frac{1}{n} \sum_{t=1}^{n} \left(\frac{X_t}{X_{t-1}} + \hat{\mu} - 1 \right)^2}.$$

And finally, in the third method, we compute the sample mean of the normal distribution given by

$$\bar{X} = \frac{1}{n}\sum_{t=1}^{n} log\left(\frac{X_t}{X_{t-1}}\right),$$

and estimate the drift and volatility parameters as

$$\hat{\mu} = \bar{X} + \frac{\hat{\sigma}^2}{2} \text{ , where } \qquad \hat{\sigma} = \sqrt{\frac{1}{n-1}\sum_{t=1}^{n}\left(log\left(\frac{X_t}{X_{t-1}}\right) - \bar{X}\right)^2} \;.$$

The implementation code in python is presented in the Appendix C and the results obtained for the development costs and cash inflows estimates are shown in Tables 3 and Table 4, respectively. Contrary to the results obtained by Croghan (Croghan, et al. 2017), looking at the cash inflow estimates, for a sample of $n = 100$ observations, we observe significantly different results among the 3 methods for estimating drift and volatility. This discrepancy could be explained by the fact that Methods 1 and 2 have better performance considering data with small increments, i.e., with a high granularity which is not our case. On the other hand, Method 3 assumes a lognormal distribution for the ratio x_{t-1} to x_t. As a result, the third method provides the most balanced parameter estimates in our case, while the first and the second methods appear unreliable due to their extreme values. Hence, the chosen method to estimate both x_1 and x_2 trajectories is the third method.

Table 3: Development Costs Parameter Estimation

	First Method	**Second Method**	**Third Method**
$\hat{\boldsymbol{\mu}}_1$	-0.894	-12.106	-0.061
$\hat{\boldsymbol{\sigma}}_1$	0.861	11.759	0.226

Table 4: Cash Inflows Parameter Estimation

	First Method	**Second Method**	**Third Method**
$\hat{\boldsymbol{\mu}}_2$	9.211	-107.211	0.068
$\hat{\boldsymbol{\sigma}}_2$	9.153	106.584	0.364

4.2. Numerical Results

The initial values, used to simulate the trajectories, were chosen assuming the company's historical values. The starting development cost is set to $x_{1_0} = 31\ 327.86$ € assuming the company spendings during the first month of construction would remain the same. Note that, by definition, "x_2 is the cash inflow per unit of density, also measured per unit of time" (Willams, 1991). As a result, applying the same reasoning, the starting cash inflow is set to $x_{2_0} = 17\,339.41$ €, given the company's first month opening revenue, 26 009.11 €, divided by the density of development chosen $q = 1.50$. The results obtained for the development costs and cash inflows simulated trajectories are demonstrated in Figure 11 and 12, respectively.

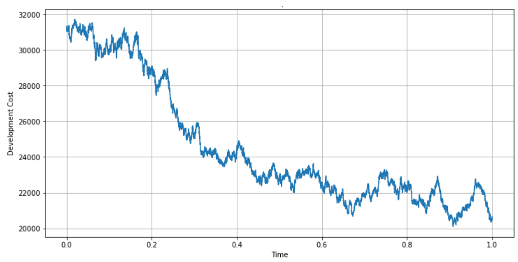

Figure 11: Development Cost Trajectory

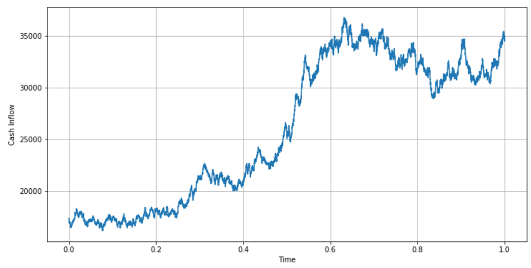

Figure 12: Cash Inflows Trajectory

Examining the simulated trajectories, the cash inflows exhibit an upward trend, as expected due to their positive drift parameter. In contrast, with a negative drift parameter, the simulated development costs show a downward trend. Once x_1 and x_2 are simulated, the model computes both properties, undeveloped and developed, net cash inflows.

Note we assume $\beta = 0$ because, prior to development, the store is not open to the public and therefore does not produce any revenues. Thus, the net cash inflow of the undeveloped property is null, as showed in Figure 14. The inequality $\beta x_2 < q x_2$ is still satisfied, i.e. the net cash inflows of the developed property are always higher than those of the undeveloped property.

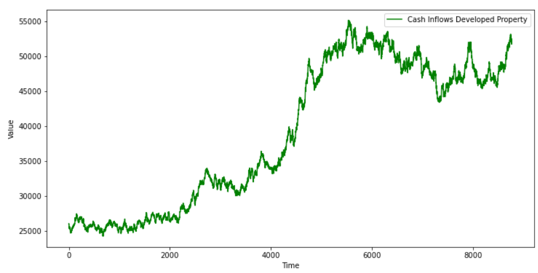

Figure 13: Developed Property's Net Cash Inflows

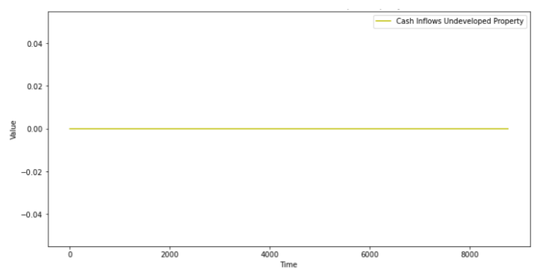

Figure 14: Undeveloped Property's Net Cash Inflows

Figure 15: Undeveloped vs Developed Net Cash Inflows

Additionally, as we had previously set $\gamma = 2$ and $\beta = 0$, the optimal development point and density expressions (5) and (6) are simplified (see Table 5). Therefore, the results are an optimal development ratio of $y^* = 0.606570$ and an optimal density of $q^* = 1.5$. Hence, the undeveloped property value, measured per unit of development cost, is equal to $W^* = 6.848553$. The ratio y first crosses the optimal development threshold at $t = 0.173744$, as posted in Figure 16, suggesting the company should start development a month shortly after the property's purchase date. Note that we are still assuming only positive cash inflows.

Table 5: Optimal Developed Point and Density

y^*	q^*	Interval
0.30329	0	$\gamma \geq 3.0329$
0.60657	$\delta = 1.5$	$\gamma < 3.0329$

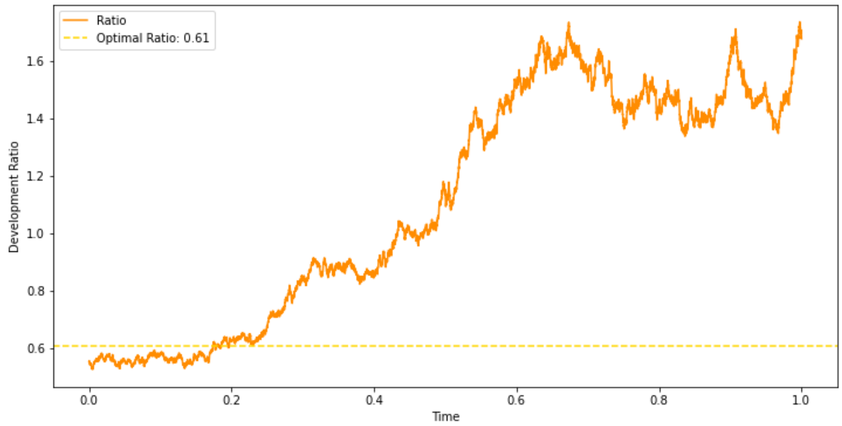

Figure 16: Optimal Development Decision restricted to positive Cash Flows

The introduction of negative cash inflows to account for the costs of carrying an undeveloped property is established by $\alpha = -0.1$. In this new setting, if the company opts to delay development during the period under analysis, instead of having null net cash inflows (see Figure 14) the company now has continuously increasing costs from carrying an undeveloped property (see Figure 17). As a result, the optimal abandonment point is introduced, and the optimal development point is recalculated to analyse whether to start development or to abandon the project.

Because we assume that the initial cash inflow and development cost are the same as our historical data, our initial ratio will be equal to approximately $y_0 = 0.55$, way above the models' optimal points $y_a^* = 0.055232$ and $y_d^* = 0.335927$ (see Figure 18). As a result, in this scenario, the model suggests that it would be optimal to immediately start development once the property is purchased. Additionally, for a 2-year time horizon the abandonment option is never optimal.

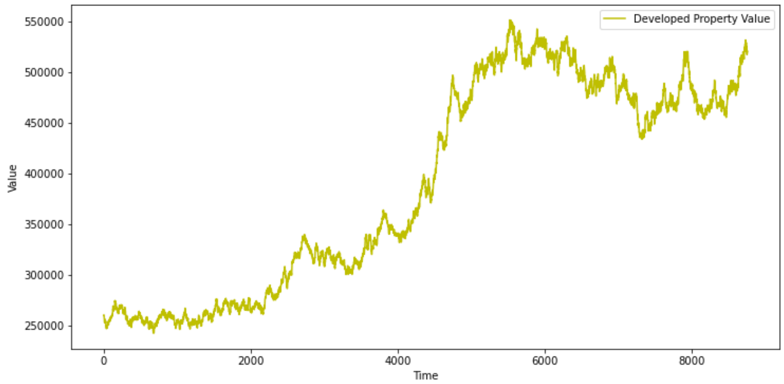

Figure 17: Introducing negative cash inflows

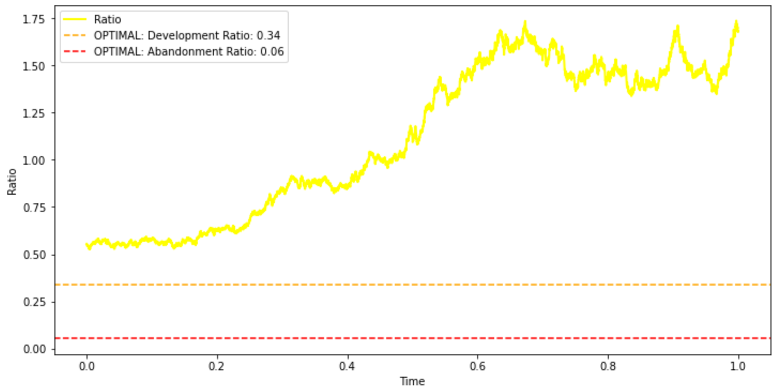

Figure 18: Optimal Decision

4.3. Comparing Different Scenarios

To assess the investment viability and arrive to a final optimal decision, the model's behaviour under different scenarios and across various parameter levels is examined. As such, the numerical values for the optimal ratios at which the company should develop or abandon the property are presented in Tables 6 to 9. Note that for each parameter combination, the upper value refers to the optimal abandonment point, y_a^*, and the lower value denotes the optimal development point, y_d^*.

Much like in Williams' paper (Williams, 1991), we see that a higher development cost variance, σ_1^2 or a lower correlation coefficient ρ, results in a smaller optimal abandonment and a larger optimal development. Therefore, if $y_a^* < y_0 < y_d^*$, as risk becomes higher the company has the tendency to delay both development and abandonment decisions. This is explained as "the greater the uncertainty over the potential profitability of the investment, the greater the value of the opportunity and the greater the incentive to wait and to keep the opportunity alive rather than exercise it by investing at once" (Dixit et al., 1995). As such, the company's decision to delay both options serves as strategy to monitor market conditions. Once they turn favourable and $y_d^* \geq y$, the company invests, otherwise, if $y_a^* \geq y$, the property is abandoned.

Additionally, for larger values of either risk adjusted growth rates, v_1 or v_2, or smaller values of the riskless rate of interest, r, both optimal points become smaller. As a result, on average, the company tends to start development sooner and abandon the property later. On the other hand, for larger cost of scale γ or maximum density δ values both optimal points become larger. In this case, on average, the company tends to start development latter and abandon the property sooner.

Finally, for a smaller α, i.e., larger values of the absolute value of α, the abandonment ratio becomes larger as the development ratio becomes smaller. This is explained by the fact that a smaller α results in higher maintenance costs, which translates in a sooner property abandonment.

Taking all these factors into account, let us assume a pessimistic scenario where development costs grew 30% over the past 10 years, and revenues are projected to be only 50% of the initial historical data. The model is set to re-simulate these trajectories over 100 times and determine the average optimal abandonment and development times, with some of the results being displayed in Figures 19 and 20.

In fact, only 12% of the simulations crossed the development threshold resulting on an average optimal time equal to $t = 0.99212$, while none crossed the optimal abandonment threshold. Under these conditions, the model suggests that the company should wait on average almost 2 years, before starting development once the market conditions turn favourable. Overall, even under challenging conditions – with increased costs, reduced revenues and fixing the parameters established in Table 2 – the model suggests that the company's expansion is a promising investment, with timing having an interesting value in the investment analysis.

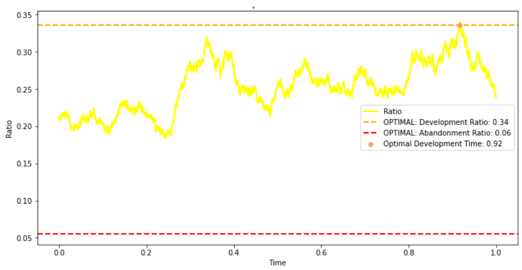

Figure 19: Pessimistic Scenario Optimal Decision 1

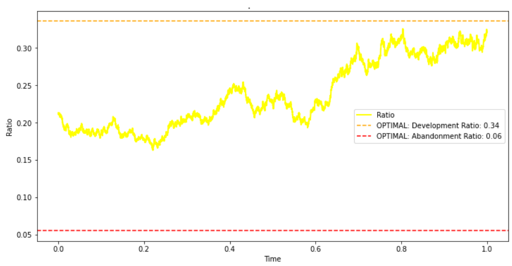

Figure 20: Pessimistic Scenario Optimal Decision 2

Moreover, let us consider a riskier scenario where we assume that the company incurs greater costs to maintain an undeveloped property and that development costs have a higher volatility. In this scenario we assume $\alpha = -0.2$ and $\sigma_1 = 0.6$, with the remaining parameters kept fixed as established in Table 2. The model is set once again to re-simulate these trajectories over 100 times and determine the average optimal abandonment and development times. With neither thresholds crossed, for the horizon under analysis, the model consistently suggests abandonment is not an optimal decision. Therefore, we can conclude that the company's expansion represents a viable investment opportunity.

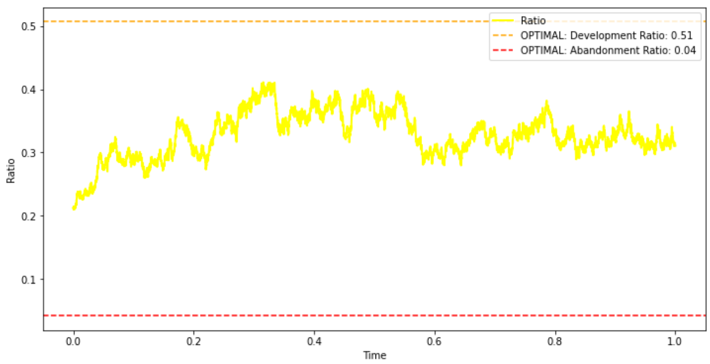

Figure 21: Optimal decision assuming a riskier scenario

Table 6: Optimal abandonment and development points per variance and α

σ_1^2 \ α	-0.100	-0.050	-0.010	-0.005	-0.001
0.001	0.077053 0.252462	0.056784 0.289514	0.024624 0.331026	0.016813 0.337374	0.006833 0.342813
0.010	0.069522 0.276410	0.049530 0.317438	0.020073 0.362387	0.013341 0.369220	0.005100 0.375083
0.020	0.064781 0.293735	0.045194 0.337460	0.017570 0.384754	0.011486 0.391920	0.004230 0.398076
0.100	0.046041 0.390224	0.029578 0.447022	0.009786 0.505785	0.006000 0.514598	0.001911 0.522208
0.200	0.034945 0.487967	0.021355 0.555745	0.006412 0.624197	0.003782 0.634411	0.001103 0.643262

Table 7: Optimal abandonment and development points per δ and γ

δ \ γ	1.01	1.10	1.25	1.50	1.75
1.01	0.056921 0.175573	0.056943 0.175778	0.056980 0.176121	0.057040 0.176694	0.057101 0.177268
1.10	0.054848 0.181076	0.055044 0.183116	0.055373 0.186572	0.055927 0.192484	0.056488 0.198592
1.25	0.051748 0.189471	0.052164 0.194496	0.052867 0.203176	0.054066 0.218513	0.055297 0.234990
1.50	0.047388 0.201526	0.048043 0.211232	0.049163 0.228423	0.051101 0.260068	0.053126 0.295771
1.75	0.043807 0.211554	0.044608 0.225528	0.045989 0.250764	0.048409 0.298704	0.050971 0.354869

Table 8:Optimal abandonment and development points per ρ and r

ρ \ ι	**0.010**	**0.050**	**0.075**	**0.100**	**0.150**
-0.50	0.004928 0.062752	0.026550 0.201343	0.041013 0.269346	0.055998 0.332102	0.087212 0.448793
-0.25	0.005596 0.051505	0.029654 0.179853	0.045555 0.244772	0.061943 0.305137	0.095887 0.418080
0.00	0.006429 0.041690	0.033431 0.158806	0.051044 0.220238	0.069091 0.277902	0.106250 0.386623
0.10	0.006820 0.038167	0.035175 0.150513	0.053568 0.210416	0.072369 0.266893	0.110982 0.373761
0.25	0.007481 0.033306	0.038090 0.138209	0.057778 0.195645	0.077828 0.250204	0.118846 0.354076

Table 9: Optimal abandonment and development points per risk adjusted growth rates

v_2 \ v_1	**-0.050**	**-0.025**	**0.000**	**0.025**	**0.050**
-0.050	0.104336 0.414430	0.100211 0.411724	0.094818 0.410167	0.087496 0.409924	0.076946 0.411405
-0.025	0.082446 0.379261	0.078900 0.375380	0.074303 0.372877	0.068110 0.371928	0.059258 0.372974
0	0.061932 0.345214	0.058999 0.339742	0.055232 0.335927	0.050208 0.333975	0.043106 0.334376
0.025	0.043012 0.312847	0.040740 0.305190	0.037851 0.299518	0.034041 0.296100	0.028735 0.295519
0.050	0.025984 0.283076	0.024431 0.272373	0.022474 0.263984	0.019927 0.258330	0.016447 0.256175

5. Conclusions

Given RO's history and significant contributions to financial investment analysis, we follow closely (Williams, 1991) approach to abandonment options – as characterized by (Trigeorgis, 1993a) and applying it to a real case study. With an overall understanding of the Williams methodology each step of the computational implementation is detailed and substantiated. The simulations of various scenarios for the development costs, x_1, and cash inflows, x_2, reveal distinct optimal decisions not only depending on the development ratio, y, and consequently, on both cash inflows and development costs, but also on the parameters used considering the investment scenario's characteristics. Cash inflow's and development cost's volatility, drift, and correlation parameters depend on the data, as the optimal density of development is more specifically related to the project's legal limitations.

For multiple x_1 and x_2 simulations, different scenarios and optimal decisions emerge. If the initial development ratio is such that $y_0 \in \,]\, y_a^*, y_d^* \,[$, decreasing cash inflows and rising construction costs lead on average to a sooner abandonment, while decreasing development costs and increasing cash inflows incentivize a sooner development. If y_0 is above the optimal development threshold, development is financially justifiable immediately, but with high costs and low revenues, abandonment remains a viable option for the company. Conversely, a very low initial ratio suggests immediate abandonment. Since optimal ratios depend on parameter selection, varying parameters sets results on different optimal ratios, which in turn lead to distinct optimal investment decisions. As a result, to analyse our case study investment viability, different scenarios are considered.

A pessimistic scenario is considered – where development costs grew 30% over the past 10 years and revenues are projected to be only 50% of the initial historical data – resulting on 12% of the simulations crossing the development threshold, with an average optimal time equal to $t = 0.99212$. While none crossed the optimal abandonment threshold, the model suggests that the company should wait on average almost 2 years, before starting development once the market conditions turn favourable.

Moreover, much like in Williams' paper, as risk becomes higher the company has the tendency to delay both development and abandonment decisions. Conversely, higher maintenance costs result, on average, in a sooner property abandonment. Therefore, a riskier scenario is assumed where the company incurs greater costs to maintain an undeveloped property and that development costs have a higher volatility. With neither thresholds crossed, for the horizon under analysis, the model consistently suggests abandonment is not an optimal decision.

In conclusion, even under challenging conditions, for the case study here analysed, the model consistently suggests that the company's expansion is a promising investment. Therefore, we can assume that Bazar Dili's expansion investment is a viable investment and that the company could follow – depending on the company's chosen parameters – the optimal abandonment and development ratios provided by the model and, consequently, start development depending on those parameters average optimal time.

Bibliography

Abel, A. B., and Eberly, J. C., 2012. Investment, Valuation, and Growth Options. Quarterly Journal of Finance, 2(1), pp. 1-32.

Adetunji, O., and Owolabi, A., 2016. The Evolution of Real Options and its Applications in Management: A Review of Literature. The International Journal of Business and Management, 4(1), pp. 98-118.

Albuquerque, A. M., 2014. Do Growth-Option Firms Use Less Relative Performance Evaluation? Accounting Review., 89(1), pp. 27-60.

Black, F., and Scholes, M., 1973. The pricing of options and corporate liabilities. Journal of political economy, 81(3), pp. 637–654.

Brennan, M., and Schwartz, E., 1985a. Evaluating Natural Resource Investments. Journal of Business, pp. 135-157.

Brennan, M., and Schwartz, E., 1985b. A New Approach to Evaluating Natural Resource Investments. Midland Corporate Finance Journal, pp. 37-47.

Bulan, L., Mayer C. and Somerville T., 2009. Irreversible Investment, Real Options, and Competition: Evidence from Real Estate Development. Journal of Urban Economics, Vol. 65, pp. 237-251.

Carr, P., 1988. The Valuation of Sequential Exchange Opportunities. Journal of Finance, pp. 1235-1256.

Dixit, A. K., and Pindyck, R. S., 1995. The New Option View of Investment. Working Papers 3794-95, Massachusetts Institute of Technology (MIT), Sloan School of Management.

Henseler, J., and Roemer, E. 2013. Let's Wait and See! The Real Option to Switch as a New Element of Customer Value. Schmalenbach Business Review.

Ingersoll, J., and Ross, S., 1992. Waiting to Invest: Investment and Uncertainty. Journal of Business, pp. 1-19.

Lawryshyn, Y., and Jaimungal, S., 2014. Optimising a modular expansion of a wastewater treatment plant. Civil Engineering & Environmental Systems, 31(3), pp. 243-259.

Luerrman, A.T., 1998. Investment Opportunities as Real Options: Getting Started on the Numbers. The Harvard Business Review, pp. 3-15.

Majd, S., and Pindyck, R., 1987. Time to Build, Option Value, and Investment Decisions. Journal of Financial Economics, pp. 7-27.

McDonald, R., and Siegel, D., 1985. Investment and the Valuation of Firms When There is an Option to Shut Down. International Economic Review, pp. 331-349.

McDonald, R. and Siegel, D., 1986. The Value of Waiting to Invest. Quarterly Journal of Economics, pp. 707-727.

Myers, S. C., 1977. Determinants of Corporate Borrowing. Journal of Financial Economics, 5(2), pp. 147-176.

Myers, S. C. 1984. Finance Theory and Financial Strategy, Interfaces 14, pp. 126.

Myers, S.C., and Majd, S., 1990. Abandonment Value and Project Life. Advances in Futures and Options Research, pp. 1-21.

Paddock, J., Siegel, D., and Smith, J., 1988. Option Valuation of Claims on Physical Assets: The Case of Offshore Petroleum Leases. Quarterly Journal of Economics, pp. 479-508.

Pindyck, R., 1988. Irreversible Investment, Capacity Choice, and the Value of the Firm. American Economic Review, pp. 969-985.

Quigg, L., 1993. Empirical testing of real option-pricing models. Journal of Finance, 48(2), pp. 621–40.

Reddy, K., and Clinton, V., 2016. Simulating stock prices using geometric Brownian motion: Evidence from Australian companies. Australasian Accounting, Business and Finance Journal, 10(3), pp. 23-47.

Schoutens, W., 2003. Lévy processes in finance: pricing financial derivatives, pp. 1-180. J. Wiley.

Titman, S., 1985. Urban Land Prices Under Uncertainty. American Economic Review, pp. 505-514.

Taudes, A., 1998. Software Growth Options. Journal of Management Information Systems., 15(1), pp. 165-185.

Tourinho, O., 1979. The Option Value of Reserves of Natural Resources. Working Paper No. 94, University of California at Berkeley.

Trigeorgis, L., 1993a. Real Options and Interactions with Financial Flexibility. Financial Management, 22(3), 202-224.

Trigeorgis, L., 1993b. The Nature of Option Interactions and the Valuation of Investments with Multiple Real Options. Journal of Financial and Quantitative Analysis, pp. 1-20.

Trigeorgis, L., and Mason, S.P., 1987. Valuing Managerial Flexibility. Midland Corporate Finance Journal, pp. 14-21.

Williams, J.T., 1991. Real Estate Development as an Option. The Journal of Real Estate Finance and Economics, 4(2), pp. 191–208.

Kozlova, M., 2017. Real option valuation in renewable energy literature: Research focus, trends and design. Renewable and Sustainable Energy Reviews, Vol. 80, pp. 180-196.

Kulatilaka, N., and Marcus, A., 1988. A General Formulation of Corporate Operating Options. Research in Finance, JAI Press, 1988, pp. 183-200.

Kulatilaka, N., and Marks, S. G. 1988. The Strategic Value of Flexibility: Reducing the Ability to Compromise. American Economic Review., 78(3), pp. 574-580.

Yongma, M., 2014. Optimal Time to Invest Energy Storage System under Uncertainty Conditions. Energies (19961073), 7(4), pp. 2701-2719.

Yongqiang, C. H. U., and Sing, T. F., 2021. Intensity and Timing Options in Real Estate Developments. International Real Estate Review, Vol. 24, pp. 1-17.

Appendix

A Numerical Illustration Parameters and Initial Values

Table 10: Parameter values for the simulated example

Parameter	Value
x_1^0	32 000
μ_1	0.05
σ_1	0.10
x_2^0	3571
μ_2	0.05
σ_2	0.05
γ	1.25
q	1.40
ι	0.10
δ	1.50
ρ	0.00
β	0.50
α	-0.01
v_1	0.00
v_2	0.00

B Correlation Parameter Computation

Please note that the table below shows the first 15 observations used to compute the model's correlation parameter through Excel's Data Analysis tool. While there are only 15 observations available for development costs, 100 observations are used to estimate the cash inflow's GBM parameters.

Table 11: Sample of 15 observations

Month	Development Costs	Cah Inflows
1	31 327,86 €	26 009,11 €
2	46 317,79 €	26 315,87 €
3	37 155,06 €	27 391,57 €
4	42 204,96 €	28 028,98 €
5	29 436,93 €	24 140,68 €
6	30 403,45 €	30 495,11 €
7	23 751,09 €	31 711,74 €
8	21 705,07 €	34 200,88 €
9	14 914,24 €	29 682,83 €
10	19 903,65 €	32 887,33 €
11	16 175,73 €	40 806,15 €
12	11 439,10 €	54 130,80 €
13	10 761,59 €	29 926,59 €
14	9 229,41 €	33 493,46 €
15	8 530,23 €	31 111,80 €

C GBM Parameter Estimation Code

Setting Simulation's Parameters and Initial Values

```
#Convert the costs and cash inflows into Lists:

dev_cost = df['Developmemt Costs'].tolist()
infl = df['Cash Inflows'].tolist()

#Setting parameters:
q = 1.5     #density of development chosen, where 1 <= q < d
g = 2    #(gamma) constant cost of scale or the elasticity coefficient
d = 1.5      #(delta) maximum density of development
b = 0       #(beta) constant such that 0 < b < 1
r = 0.10
v2 = 0
v1 = 0
correlation = -0.53560525488162 #Computed using Excel Data Analysis (considering the first 15 obs)

#Varibles:
pi= 1/(r-v2)

#Setting Initial Values:
len_dev = 15
len_infl = len(infl)
Wdc_0 = dev_cost[0]
Wci_0 = infl[0]
```

Figure 22: Code used to set the necessary parameters

Parameter Estimation Method 1

```
##Drift parameter:
v=[]

for i in range(1, len_dev):
    v.append((dev_cost[i]-dev_cost[i-1])/dev_cost[i-1])
d_fm = sum(v)

#Volatility parameter:
v=[]
for i in range(1, len_dev):
    v.append((((dev_cost[i]-dev_cost[i-1])/dev_cost[i-1])-d_fm)**2)
v_fm = np.sqrt(sum(v)/(len_dev-1))

print(" Development Cost Drift =", d_fm)
print(" Development Cost Volatility =", v_fm)
```

```
 Development Cost Drift = -0.8938695054118859
 Development Cost Volatility = 0.8611715031160586
```

Figure 23: Code used to estimate the development cost's GBM drift and volatility according to Method 1

```
##Drift parameter:
v=[]
for i in range(1, len_infl):
    v.append((infl[i]-infl[i-1])/infl[i-1])
di_fm = sum(v)

#Volatility parameter:
v=[]
for i in range(1, len_infl):
    v.append(((infl[i]-infl[i-1])/infl[i-1]-di_fm)**2)

vi_fm = np.sqrt(sum(v)/(len_infl-1))

print(" Cash Inflow Drift =", di_fm)
print(" Cash Inflow Volatility =", vi_fm)
```

```
 Cash Inflow Drift = 9.211357158497583
 Cash Inflow Volatility = 9.153437080770413
```

Figure 24: Code used to estimate the cash inflow's GBM drift and volatility according to Method 1

Parameter Estimation Method 2

```
#Drift parameter:
v=[]
for i in range(1, len_dev):
    v.append(dev_cost[i]/dev_cost[i-1])
d_sm = 1-sum(v)


#Volatility parameter:
v=[]
for i in range(1, len_dev):
    v.append((dev_cost[i]/dev_cost[i-1]+d_sm-1)**2)
v_sm = np.sqrt(sum(v)/len_dev)

print(" Development Cost Drift =", d_sm)
print(" Development Cost Volatility =", v_sm)
```

```
 Development Cost Drift = -12.106130494588115
 Development Cost Volatility = 11.759406833622439
```

Figure 25: Code used to estimate the development cost's GBM drift and volatility according to Method 2

```
#Drift parameter:
v=[]
for i in range(1, len_infl):
    v.append(infl[i]/infl[i-1])
di_sm = 1-sum(v)


#Volatility parameter:
v=[]
for i in range(1, len_infl):
    v.append((infl[i]/infl[i-1]+di_sm-1)**2)
vi_sm = np.sqrt(sum(v)/len_infl)

print(" Cash Inflow Drift =", di_sm)
print(" Cash Inflow Volatility =", vi_sm)
```

```
 Cash Inflow Drift = -107.21135715849756
 Cash Inflow Volatility = 106.58435644733198
```

Figure 26: Code used to estimate the cash inflow's GBM drift and volatility according to Method 2

Parameter Estimation Method 3

```
#Sample Mean:
v=[]
for i in range(1, len_dev):
    v.append(math.log(dev_cost[i]/dev_cost[i-1]))
mean_sample = sum(v)/len_dev

#Volatility parameter:
v=[]
for i in range(1, len_dev):
    v.append((math.log(dev_cost[i]/dev_cost[i-1])-mean_sample)**2)
v_tm = np.sqrt(sum(v)/(len_dev-1))

#Drift parameter:
d_tm = mean_sample + v_tm**2 / 2

print(" Development Cost Drift =", d_tm)
print(" Development Cost Volatility =", v_tm)
```

```
 Development Cost Drift = -0.0610840484989911
 Development Cost Volatility = 0.22645998096390474
```

Figure 27: Code used to estimate the development cost's GBM drift and volatility according to Method 3

```
#Sample Mean:
v=[]
for i in range(1, len_infl):
    v.append(math.log(infl[i]/infl[i-1]))
mean_sample = sum(v)/len_infl

#Volatility parameter:
v=[]
for i in range(1, len_infl):
    v.append((math.log(infl[i]/infl[i-1])-mean_sample)**2)
vi_tm = np.sqrt(sum(v)/(len_infl-1))

#Drift parameter:
di_tm = mean_sample + vi_tm**2 / 2

print(" Cash Inflow Drift =", di_tm)
print(" Cash Inflow Volatility =", vi_tm)
```

```
 Cash Inflow Drift = 0.06754020021996243
 Cash Inflow Volatility = 0.3644461464921945
```

Figure 28: Code used to estimate the cash inflow's GBM drift and volatility according to Method 3

D GBM Simulation Code

Development Cost Simulation

```
def Development_Cost_Trajectory(method,n, delta_t,m,s):

    GBM = np.zeros(n+1)

    print("Drift: ",m)
    print("Volatility: ",s)

    for i in range(1, n+1):
        # v is series of standard normal random nunbers
        v = np.random.normal()

        #Initial value
        GBM[0] = Wdc_0

        Y = np.exp((m-s**2/2) * delta_t +  s * np.sqrt(delta_t) * v)

        # Update Time-Changed Brownian Motion
        GBM[i] = GBM[i-1] * Y


    # Time points
    time_points = np.arange(0, n+1) * delta_t
    plt.figure(figsize=(12, 6))
    plt.plot(time_points, GBM)
    plt.title(method+'Development Cost Simulation')
    plt.xlabel('Time')
    plt.ylabel('Cash Inflow')
    plt.grid(True)
    plt.show()

    return GBM

x1 = Development_Cost_Trajectory('THIRD METHOD: ',n, delta_t, d_tm, v_tm, )
```

Figure 29: Code used to generate the development cost's GBM for all time steps

Cash Inflows Simulation

```
def Cash_Inflow_Trajectory(method,n, delta_t,m,s):

    GBM = np.zeros(n+1)
    print("Drift: ",m)
    print("Volatility: ",s)

    for i in range(1, n+1):
        # v is series of standard normal random nunbers
        v = np.random.normal()

        #Initial value
        GBM[0] = Wdc_0

        Y = np.exp((m-s**2/2) * delta_t +  s * np.sqrt(delta_t) * v)

        # Update Time-Changed Brownian Motion
        GBM[i] = GBM[i-1] * Y

    GBM = [x / q for x in GBM]  #x2 is the Cash Inflow per unit of density

    # Time points
    time_points = np.arange(0, n+1) * delta_t
    plt.figure(figsize=(12, 6))
    plt.plot(time_points, GBM)
    plt.title(method+'Cash Inflow Simulation')
    plt.xlabel('Time')
    plt.ylabel('Cash Inflow')
    plt.grid(True)
    plt.show()

    return GBM

x2 = Cash_Inflow_Trajectory('THIRD METHOD: ',n, delta_t, di_tm, vi_tm)
```

Figure 30: Code used to generate the cash inflow's GBM for all time steps

E Williams' Initial Model Implementation Code

```
def Cost_of_Development(method,q,g,GBM):

    C = np.zeros(n+1)
    SM = q**g

    for i in range(n+1):
        C[i]= GBM[i]*SM

    # Time points
    time_points = np.arange(0, n+1) * delta_t
    plt.figure(figsize=(12, 6))
    plt.plot(C, 'r-', label='Development Costs')

    # Add labels and title
    plt.xlabel('Time')
    plt.ylabel('Value')
    plt.title(method+'Development Costs')
    plt.legend()

    return C

CD_d = Cost_of_Development("THIRD METHOD: ",q,g,x1)
```

Figure 31: Code used to simulate development cost if the company choses to develop

```
def Cash_Inflow_DEV(method,q,GBM):

    CI = np.zeros(n+1)

    for i in range(n+1):
        CI[i]= GBM[i]*q

    # Time points
    time_points = np.arange(0, n+1) * delta_t
    plt.figure(figsize=(12, 6))
    plt.plot(CI, 'g-', label='Cash Inflows Developed Property')

    # Add labels and title
    plt.xlabel('Time')
    plt.ylabel('Value')
    plt.title(method + 'Cahs Inflows')
    plt.legend()

    return CI

CI_d = Cash_Inflow_DEV("THIRD METHOD: ",q,x2)
```

Figure 32: Code used to simulate the developed property net cash inflows

```
def Cash_Inflow_UNDEV(method,b,GBM):

    CI = np.zeros(n+1)

    for i in range(n+1):
        CI[i]= GBM[i]*b

    # Time points
    time_points = np.arange(0, n+1) * delta_t
    plt.figure(figsize=(12, 6))
    plt.plot(CI, 'y-', label='Cash Inflows Undeveloped Property')


    # Add labels and title
    plt.xlabel('Time')
    plt.ylabel('Value')
    plt.title(method+'Cahs Inflows Undeveloped Property')
    plt.legend()

    return CI

CI_u = Cash_Inflow_UNDEV("THIRD METHOD: ",b,x2)
```

Figure 33: Code used to simulate the undeveloped property net cash inflows

F Williams' Optimal Development Implementation Code

```
def DEV_Price(method,pi,CI):

    DP = np.zeros(n+1)

    for i in range(n+1):

        if CI[i]<=0:

            DP[i:] = 0
            break
        else:
            DP[i]= CI[i]*pi

    # Time points
    time_points = np.arange(0, n+1) * delta_t
    plt.figure(figsize=(12, 6))
    plt.plot(DP, 'y-', label='Developed Property Value')

    # Add labels and title
    plt.xlabel('Time')
    plt.ylabel('Value')
    plt.title(method+'Developed Property')
    plt.legend()

    return DP

DP = DEV_Price("THIRD METHOD: ",pi,CI_d)
```

Figure 34: Code used to compute, for each time step, the developed property value

```
#Variables
var1 = v_tm **2                              #variance x1 third method
var2 = vi_tm **2                             #variance x2 third method
cov_x1x2 = correlation * v_tm * vi_tm        #covariance third method
w_squared = var1 - 2*cov_x1x2 + var2         #(omega) third method

#(Eta) has to be >1
h = - ((v2-v1)/w_squared - 1/2) + np.sqrt(((v2-v1)/w_squared - 1/2)**2+ 2* (r-v1)/w_squared)
psi = h/(h-1)    #has to be >1

y_ratio = np.zeros(n+1)
for i in range(n+1):
    y_ratio[i] = x2[i] / x1[i]    #development ration

#Control:
print("h > 1: ", h>1)
print("h = ", h)
print("psi > 1: ", psi>1)
print("psi = ", psi)
print("w squared = ", w_squared)
```

```
h > 1:  True
h =  1.4919199532151688
psi > 1:  True
psi =  3.0328510633976946
w squared =  0.2725147631435306
```

Figure 35: Code used to set the necessary parameters and compute the development ratio

```
def Optimal_Development_Ratio(method,psi):

    ub = psi/(1-b*psi)
    lb = d*psi/(d-b*psi)

    print("Lower Bound:", lb, "Upper Bound", ub,"Gamma:",g)

    if g >= ub:

        y_optimal = psi/pi*1/(1-b*psi)
        print(method,"Optimal Development Ration y*= ",y_optimal)

    elif g < ub and g >= lb:

        y_optimal = g/pi * (b*g*psi/(g-psi))**(g-1)
        print(method,'Optimal Development Ration y*= ',y_optimal)

    else :

        y_optimal = psi/pi * d*g/(d-b*psi)
        print(method,'Optimal Development Ration y*= ',y_optimal)

    return y_optimal

y_optimal = Optimal_Development_Ratio("Third Method: ",psi)
```

```
Lower Bound: 3.0328510633976946 Upper Bound 3.0328510633976946 Gamma: 2
Third Method:  Optimal Development Ration y*=  0.6065702126795389
```

Figure 36: Code used to compute the optimal development ratio

```python
def Optimal_Development_Density(method,psi):

    ub = psi/(1-b*psi)
    lb = d*psi/(d-b*psi)

    if g >= ub:

        q_optimal = 1
        print(method,"Optimal Development Density q*= ",q_optimal)

    elif g < ub and g >= lb:

        q_optimal = b*g*psi/(g-psi)
        print(method,"Optimal Development Density q*= ",q_optimal)

    else :

        q_optimal = d
        print(method,"Optimal Development Density q*= ",q_optimal)

    return q_optimal

q_optimal = Optimal_Development_Density("THIRD METHOD: ",psi)
```

```
THIRD METHOD:  Optimal Development Density q*=  1.5
```

Figure 37: Code used to compute the optimal development density

```python
def UNDEV_Price_W(method, y_optimal,q_optimal):

    W_optimal = pi * q_optimal * y_optimal - q_optimal**g

    print(method,"W* =", W_optimal)


    return W_optimal

W_optimal = UNDEV_Price_W("THIRD METHOD: ",y_optimal,q_optimal)
```

```
THIRD METHOD:  W* = 6.848553190193083
```

Figure 38: Code used to compute the optimal undeveloped property value measured per unit of development cost

G Williams' Optimal Abandonment Implementation Code

```python
alpha = -0.1
q = d #We consider a density of development equal to the maximum density of development

#New parameter notation:
zeta =  -((v2-v1)/w_squared - 1/2) - np.sqrt(((v2-v1)/w_squared - 1/2)**2+ 2* (r-v1)/w_squared)
phi = zeta / (zeta-1)    #(Phi) new parameter notation
```

Figure 39: Code used to set the necessary abandonment parameters

```python
def Cash_Inflow_UNDEV(method,b,GBM_x1,GBM_x2):

    CI = np.zeros(n+1)

    for i in range(n+1):
        CI[i]= GBM_x1[i]*alpha + GBM_x2[i]*b

    # Time points
    time_points = np.arange(0, n+1) * delta_t
    plt.figure(figsize=(12, 6))
    plt.plot(CI, 'y-', label='Cash Inflows Undeveloped Property')

    # Add labels and title
    plt.xlabel('Time')
    plt.ylabel('Value')
    plt.title(method+'Cahs Inflows Undeveloped Property')
    plt.legend()

    return CI


CI_second_u = Cash_Inflow_UNDEV("THIRD METHOD: ",b,x1,x2)
```

Figure 40: Code used to re-simulate the undeveloped property net cash inflows

```python
def Optimal_DEV(w_squared):
    #New parameter notation
    zeta =  -((v2-v1)/w_squared - 1/2) - np.sqrt(((v2-v1)/w_squared - 1/2)**2+ 2* (r-v1)/w_squared)
    phi = zeta / (zeta-1)   #(Phi) new parameter notation
    y_0 = alpha*phi/(d*r*pi)*(1+r/alpha*d**g-(-(zeta-1)/(h-1))**(h/(h-zeta)))

    F_y_0 = alpha*psi/(d*r*pi) * (1 + r*d**g/alpha - (1 + r/alpha*d - d*r*pi/(alpha*phi)* y_0)**(zeta/h))
    y_1 = (y_0 + F_y_0)/2

    stop_point = 1000000
    y_i=np.zeros(stop_point)
    F= np.zeros(stop_point)
    y_i[0] = y_0

    for i in range(1, stop_point):
        F[i-1] = alpha*psi/(d*r*pi) * (1 + r*d**g/alpha - (1 + r/alpha*d - d*r*pi/(alpha*phi)* y_i[i-1])**(zeta/h))
        y_i[i] = F[i-1]

        if abs(y_i[i-1] - y_i[i]) <= 1e-8:

            y_optimal_DEV = y_i[i]
            break

    return y_optimal_DEV
```

Figure 41: Code used to re-compute the optimal development ratio

```python
def Optimal_ABN(y_optimal_DEV):

    #Optimal Abandonment
    y_optimal_ABN = y_optimal_DEV*( 1 + r/alpha*d**g - d*r*pi/(alpha * psi) * y_optimal_DEV )**(-1/zeta)

    return y_optimal_ABN
```

Figure 42: Code used to compute the optimal abandonment ratio

```python
def Optimal_Values(y_optimal_DEV, y_optimal_ABN,x2):
    DEV_POINT_i = [] #Development Point?
    for i in range(n):
        if y_ratio[i] >= y_optimal_DEV and y_ratio[i+1] <= y_optimal_DEV :
            DEV_POINT_i.append(i* delta_t)
        else:
            0
    ABN_POINT_i = [] #Abandonment Point?
    for i in range(n):
        if y_ratio[i] >= y_optimal_ABN and y_ratio[i+1] <= y_optimal_ABN :
            ABN_POINT_i.append(i* delta_t)
        else:
            0
    time_points = np.arange(0, n+1) * delta_t
    plt.figure(figsize=(12, 6))
    plt.plot(time_points,y_ratio, label='Ratio', color='yellow', lw=2,zorder=1)

    if DEV_POINT_i != [] and ABN_POINT_i != []: #Development and abandonment points
        Optimal_Development_Time = DEV_POINT_i[0]
        Optimal_Abandonment_Time = ABN_POINT_i[0]
        plt.axhline(y=y_optimal_DEV, color='orange', linestyle='--', label=f'OPTIMAL: Development Ratio: {y_optimal_DEV:.2f}',lw=2, zorder=1)
        plt.axhline(y=y_optimal_ABN, color='red', linestyle='--', label=f'OPTIMAL: Abandonment Ratio: {y_optimal_ABN:.2f}',lw=2, zorder=1)
        plt.scatter([Optimal_Development_Time], [y_optimal_DEV], color='sandybrown',zorder=2,  label= f'Optimal Development Time: {Optimal_Development_Time:.2f}')
        plt.scatter([Optimal_Abandonment_Time],[y_optimal_ABN], color='orangered',zorder=2,  label= f'Optimal Abandonment Time: {Optimal_Abandonment_Time:.2f}')

    elif  DEV_POINT_i != []: #Only a development, no abandonment
        Optimal_Development_Time = DEV_POINT_i[0]
        plt.axhline(y=y_optimal_DEV, color='orange', linestyle='--', label=f'OPTIMAL: Development Ratio: {y_optimal_DEV:.2f}',lw=2, zorder=1)
        plt.axhline(y=y_optimal_ABN, color='red', linestyle='--', label=f'OPTIMAL: Abandonment Ratio: {y_optimal_ABN:.2f}',lw=2, zorder=1)
        plt.scatter([Optimal_Development_Time], [y_optimal_DEV], color='sandybrown',zorder=2,  label= f'Optimal Development Time: {Optimal_Development_Time:.2f}')

    elif  ABN_POINT_i != []: #Only a abandonment, no development
        Optimal_Abandonment_Time = ABN_POINT_i[0]
        plt.axhline(y=y_optimal_DEV, color='orange', linestyle='--', label=f'OPTIMAL: Development Ratio: {y_optimal_DEV:.2f}',lw=2, zorder=1)
        plt.axhline(y=y_optimal_ABN, color='red', linestyle='--', label=f'OPTIMAL: Abandonment Ratio: {y_optimal_ABN:.2f}',lw=2, zorder=1)
        plt.scatter([Optimal_Abandonment_Time],[y_optimal_ABN], color='orangered', zorder=2,  label= f'Optimal Abandonment Time: {Optimal_Abandonment_Time:.2f}')

    else: # We don't have both
        plt.axhline(y=y_optimal_DEV, color='orange', linestyle='--', label=f'OPTIMAL: Development Ratio: {y_optimal_DEV:.2f}')
        plt.axhline(y=y_optimal_ABN, color='red', linestyle='--', label=f'OPTIMAL: Abandonment Ratio: {y_optimal_ABN:.2f}')
    plt.xlabel('Time')
    plt.ylabel('Ratio')
    plt.title('Optimal Ratios')
    plt.legend()
    plt.show()
    return  y_optimal_DEV , y_optimal_ABN
```

Figure 43: Code used to plot the development ratio trajectory along with the optimal boundaries