Lisbon School of Economics & Management
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MASTER IN

MATHEMATICAL FINANCE

MASTER´S FINAL WORK

PROJECT

Credit Default Swap valuation with structural models

GUILHERME BAIO FREITAS GONÇALVES

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SUPERVISION:

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ABSTRACT AND KEYWORDS

In this project, the concept of credit risk and, more specifically, the Credit Default Swaps (CDS), will be explored, with the objective of understanding how CDS work and how they are evaluated using different methods that use a great variety of mathematical concepts. To do that, two structural credit risk models were implemented: Merton model and First Passage model.

These structural models are based on the capital structure of a firm, assets and liabilities, and the evolution of assets' value over time. In addition, both models define that a default is endogenous, which means that it occurs when the value of the firm's assets is not sufficient to cover the debt. Then, the same approach was used to compute the CDS spread, to understand the impact of using different models and how they fit with market data.

To model these two approaches, Python was used since it is one of the most programming languages covered during my academic path and one of the most suitable to implement mathematical methods.

Keywords: Credit spread, Credit Default Swaps, Merton model, First Passage model, Probability of default.

RESUMO E PALAVRAS-CHAVE

Neste projeto, será explorado o conceito de risco de crédito e, mais especificamente, os Credit Default Swaps (CDS), com o objetivo de perceber como funcionam os CDS e como são avaliados através de diferentes métodos que utilizam uma grande variedade de conceitos matemáticos. Para o efeito, foram implementados dois modelos estruturais de risco de crédito: Modelo de Merton e Modelo de First-Passage.

Estes modelos estruturais baseiam-se na estrutura de capital de uma empresa, activos e passivos, e na evolução do valor dos activos ao longo do tempo. Além disso, ambos os modelos definem que o incumprimento é endógeno, o que significa que ocorre quando o valor dos activos da empresa não é suficiente para cobrir a dívida. De seguida, a mesma abordagem foi utilizada para calcular o spread do CDS, para compreender o impacto da utilização de diferentes modelos e a forma como se enquadram nos dados de mercado.

Para modelar estas duas abordagens, foi utilizada a linguagem Python, uma vez que é uma das linguagens de programação mais abordadas durante o meu percurso académico e uma das mais adequadas para implementar métodos matemáticos.

Palavras-chave: Spread de crédito, Credit Default Swaps, Modelo de Merton, Modelo de First-Passage, Probabilidade de incumprimento.

GLOSSARY

ATM – At the Money

BA – Boeing

C – Citigroup

CDS – Credit Default Swaps

CLF – Cleveland-Cliffs

DD – Distance to Default

DOC – Down and Out Call

GDP – Gross Domestic Product

LGD – Loss Given Default

LL – Long Term Liabilities

MAE – Mean Absolute Error

MVE – Market Value of Equity

NOK – Nokia

PD – Probability of Default

PE – Prediction Error

PV – Present Value

RMSE – Root Mean Squared Error

RR – Recovery Rate

SL – Short Term Liabilities

TSLA – Tesla

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1. INTRODUCTION

Credit risk is crucial in financial institutions as it represents the possibility of a counterparty defaulting on its obligations, being one important pillar to sustain financial stability. Carcanholo, M., Pinto, E., Filgueiras, L., & Gonçalves, R. (2008) state that the great example of the importance that credit risk has in the world is the Crisis of 2008, when a lot of defaults occurred. It allowed to see the impact that default on credit had in an economy, leading to a global recession, when there was a sharp decline in Gross Domestic Product (GDP) in many countries, massive unemployment and consequently a significant increase in public debt. All this led governments to inject trillions of dollars into the economy and to other consequences as the housing market crash that at the same time was one of the main reasons for the crash, since Subprime mortgages were given to people without a strict control over their credit risk profile which results in undervalued probabilities of default. With all this, United States of America has seen interest rates reduced from 6.25% to 2% in eight months.

Before the 2008 crisis, CDS, which will later be introduced and explained, were one of the most demanded financial products, since they allow investors to transfer the risk of default. The main problem was that there was a great belief that credit events, such as default on payments, were not as probable as they were in reality, and when default occurred, there were a lot of CDS issuers that were not able to pay the other counterparty.

Increasingly, institutions care about their risk exposure, whether they are a market maker, an insurance company, or other financial institution, since it is one of the main pillars of business nowadays.

On the specific side of CDS, they are not just a financial derivative that can be traded for hedging purposes, but they are also a viable risk indicator in the market, allowing investors to check the risk profile of a company or a country.

2. LITERATURE REVIEW AND MODELS DESCRIPTION

2.1 Theoretical background

Before going into the modelling part of this project, it is important to give some theoretical context.

Hull (2006) provides an in-depth exploration of credit risk and credit derivatives. In the world of finance, we have a lot of risks associated with investments, being they related to the risk of the assets themselves, also known as systematic risk, measured by alpha, risk of the market, also known as unsystematic risk, measured by beta. However, when we dive into credit investments, there is another risk that appears: the risk of counterparty default. Credit, in a brief way, is the lending of money, expecting to be paid the notional value plus interest, reflecting the risk that one party takes on lending. With this risk is essential to understand what the borrower characteristics are, and to do that, there are a lot of financial institutions that rate the credibility of the borrower. In the context of this project, we will focus more on corporate bonds, since although they are not so risky as some stocks, they are not so risk-free, at least most of them, as government bonds.

Hull (2006) explains the rating concept, showing that different rating agencies, as Moody’s, S&P and Fitch, reflect the creditworthiness of corporate bonds in different ways. For example, Moody’s best rating is Aaa, followed by Aa, A, Baa, Ba, B, Caa, Ca and C and also breaks down some categories further, like Aa1, Aa2 and Aa3. S&P and Fitch are more similar since they have AAA, AA, A, BBB, BB, B, CCC, CC and C, being AAA equivalent to Aaa of Moody’s. In both cases, the higher rate is the first one, decreasing until the last one. S&P also have more categories, such as AA+, AA and AA-. Although the three institutions can rate the same corporate bonds, there can be cases where they rate them on different scales, depending on their way of doing it.

These ratings are a good and simple way, at least in a brief way, for the investor to filter some investments, being in line with the investor´s attitude towards risk (risk profile). Then, someone who is more risk-averse will not choose a CCC rated bond but will also try to hedge the position to which they have exposure in order to mitigate the risk. For that, there are a wide variety of financial instruments, such as derivatives, including options, swaps, futures, forwards and many more.

Within the framework of this project, the one that I will work on is CDS. The name of this financial derivative is self-explained: allows investors to exchange (swap) the losses that they are exposed to (credit) in case of a credit event (default).

Stulz (2010) states that CDS can be compared with usual insurance, since they are a way of protecting the buyer against the loss that they will have if the other party fails. Like any derivative product, CDS has an underlying asset, which can be a bond, indices or tranches of structured credit products. Also, some contracts can be related to not only one (Single-name CDS) but a basket of assets (Multi-name CDS). Only the first one will be the focus of this work.

Before understanding how a CDS is modelled, it is important to know the stages of a CDS contract. As can be seen in Terzi, N., & Uluçay, K. (2011), CDS are traded in the over-the-counter market and have three parties involved: the protection buyer (short on risk), someone that wants to protect a credit exposure, like holding a bond of a company; the protection seller (long on risk), the counterparty of the trade and the reference entity, the party which risk credit is being negotiated. A CDS contract also has a maturity and a notional amount, defining the time horizon until when the contract is activated and the amount to which the protection is being paid for, respectively. During the time until maturity, the protection buyer pays the protection seller an annual percentage of the notional, known as the CDS spread. A credit spread is mainly the surplus, in general cases, between a risky investment and an equivalent risk-free investment. Also, CDS spread can be seen as the required return to transform the risky investment into a risk-free investment. It follows that if we have a riskier investment, the higher the CDS spread and vice versa.

This type of contract, where there are periodic payments, is called a running CDS, however, there is a case when there is only one up-front premium payment at the beginning. Those amounts can be seen as the value to pay for having protection, like in a simple insurance. Additionally, the contract will see his market value change during the time, since if the risk of default of a company, for example, increases, the amount to pay for protection for a new CDS contract will need to be more, leading the CDS spread to increase and with that the value of the previous contract will increase, since it has a smaller spread, so you can pay less for protection.

Then, the contract can be triggered if some defined credit events occur. These events needed to be defined in the contract, and they can be bankruptcy, failing in coupon payment, a restructure of the reference entity or another event.

There are two main endings for a CDS contract: default does not occur until the maturity of the contract, or it happens at maturity or before. If the reference entity does not default, the only cash flows that will happen will be the premiums paid from the protection buyer until the end of the contract, resulting in a loss for the protection buyer and a profit for the protection seller. Apart from these two scenarios, the contract can also finish if we have a counterparty step out, leading to the end of the contract.

When trading CDS, at the beginning, the buyer pays an upfront fee to the seller, even before the first protection payment, so in the case when there is a counterparty step-out, that payment will not be reimbursed. However, in the context of this project, some of these trading situations are left aside since they do not heavily impact the calculation of the Probability of Default (PD) from the structural models.

In case of a credit event, the protection seller will need to pay the protection buyer. When the protection buyer holds a bond of the reference entity and it defaults, there is a percentage of the notional value that can be recovered, which is defined by the International Swaps and Derivatives Association (ISDA), even without having a CDS contract for that credit exposure, and it is called Recovery Rate.

There are two main ways of settlement for a contract, either physical, if the protection buyer, in the case that he holds the credit asset, can sell it for the notional value even after default to the protection seller, or a cash settlement, when the protection buyer receives the difference between the notional value and the Recovery Rate (RR). This difference is called Loss Given Default (LGD). Basically, this value is the amount that any investor would lose if he hold the credit asset without any protection, and its value is equal to 1- RR.

Another important concept that will be the main focus of this project is the PD. Not every company has the same capacity to pay their debt or to grow the way they want, and this is something to be aware of when investing. We can look through some companies' balance sheets and use some ratios to understand how they are in terms of liquidity, generating profit, turnover and other metrics, but that alone does not allow us to model our expectation of a default. Using different methods, that will be covered later, we can estimate the probability that a company will enter default. This default event, also called a credit event, will also be defined to meet the objective of this project, since it can be a wide variety of things and needs to be well defined.

2.2 Structural models

To evaluate CDS contracts, the main thing that we need to model is the PD, the approach that will be followed to do that is by using structural models. After that, another approach is followed to estimate the CDS.

For structural models, the default is endogenous, which means that it happens when the market value of a company's assets reaches a level below their liabilities, and that is why they are called structural, since they consider the capital structure of the firm.

This project aims to understand how different default modelling strategies, and so different models, influence the CDS valuation. To do that, two models were implemented.

According to Merton (1974), some key assumptions of the two structural models should be considered:

1. Transaction costs, taxes, or problems with indivisible assets are not taken into account.
2. Every investor can buy and sell the amount they want at the market price.
3. Interest rates for borrowing and lending are the same.
4. Investors can short-sell all assets.
5. Trading happens in continuous time.
6. The Modigliani-Miller theorem holds, so the value of the firm is independent of its capital structure.
7. The price of a riskless zero-coupon bond is given by a constant risk-free rate of return.
8. There is a positive barrier that defines when default occurs.
9. Stocks do not pay dividends.
10. The firm's value process follows a geometric Brownian motion with drift:

$$dV_t = \mu V_t dt + \sigma V_t dW_t, \tag{1}$$

being μ the drift rate of the firm's assets, σ the volatility rate of the assets and W a standard Brownian motion.

2.2.1 Merton Model

Robert Merton, a pioneer American economist in his work in financial economics, mainly in option pricing and corporate finance, established the foundations for structural models applying Black-Scholes option pricing strategies to the capital structure. Following Merton (1974), the main idea of Merton Model is that if the bonds are zero-coupon, shareholders hold a call option on assets (V) with exercise price equal to the facial value of debt (D) with maturity (T) and debtholders have a risk-free asset, a bond, and a put option on the company assets with the same features. Then, the payoff of shareholders is given by:

$$P_E = \max(V_T - D, 0) \tag{2}$$

and to debtholders it is given by:

$$P_D = \min(D, V_T) = D - \max(0, D - V_T) \tag{3}$$

One important assumption of the model is that default can only occur at the maturity date if and only if the asset value is lower than the debt value. A company first pays their debt and then if there is any amount left, shareholders will get an amount and that is why at maturity if the asset value is higher than debt, the shareholders will receive some money and debtholders will only get the value of debt as we can see by looking at (3). However, if it is the other way around, shareholders will get nothing, so they do not exercise the call option on assets, but debtholders will exercise the put option on assets, allowing them to have a payoff in the amount of the face value of debt.

Applying Black-Scholes formulas (see Black and Scholes (1973)), with the strike price being equal to the value of the debt, the value of a European call option on the firm's assets is given by:

$$E_A = VN(d_1) - De^{-rT}N(d_2) \tag{4}$$

where:

$$d_1 = \frac{\ln\left(\frac{V}{D}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \tag{5}$$

$$d_2 = d_1 - \sigma\sqrt{T} \tag{6}$$

with r being the risk-free rate and N the cumulative density function of the standard normal distribution.

For the value of a European put option on the assets, we have:

$$E_D = De^{-rT} - [De^{-rT}N(-d_2) - VN(-d_1)] \tag{7}$$

with d_1 and d_2 defined the same way.

As we defined above, default only occurs at maturity if the debt value is below the asset value. Then, following Vassalou and Xing (2004) it is concluded that the risk-neutral default probability is derived as follows:

$$Probability\ of\ Default = Prob(V_{t+T} \leq H|V_t) = Prob(ln(V_{t+T}) \leq \ln(H)|\ln(V_t)) \tag{8}$$

and, by (1), because the value of assets follows a Geometric Brownian motion GBM:

$$\ln(V_{t+T}) = \ln(V_t) + \left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}\,\varepsilon_{(t+T)} \tag{9}$$

with,

$$\varepsilon_{t+T} = \frac{W(t+T) - W(t)}{\sqrt{T}} \tag{10}$$

where, $\varepsilon_{t+T} \sim N(0,1)$.

With this, substituting $ln(V_{t+T})$ in (8), we get:

$$Prob\left(\ln(V_t) + \left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}\,\varepsilon_{t+T} - \ln(H) \leq 0\right) \tag{11}$$

and reorganising:

$$Prob\left(-\frac{\ln\left(\frac{V_t}{H}\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \geq \varepsilon_{t+T}\right) \tag{12}$$

Distance to Default (DD), defined as:

$$DD = \frac{\ln\left(\frac{V_t}{H}\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \tag{13}$$

that is, the number of deviations that the logarithm of assets per debt ratio needs to be away from the mean, allowing default to occur. Following this reasoning, it is understandable that default occurs when the ratio $\frac{V_t}{H}$ is less than 1, which is equivalent to the logarithm of it being negative.

So, in Vassalou and Xing (2004), PD for the Merton model is defined as:

$$PD = N\,(-DD) \tag{14}$$

Later, when applying this model, it will be seen that the D that here is the value of debt, will be changed to the default barrier (H) that will also be introduced and explained why it is used on this project.

Although Merton model is a known model, it has some key limitations:

- default only occurs at maturity of debt;
- only takes into account debt in the form of zero-coupon bonds;
- assets' value and volatility are known and follow a Geometric Brownian Motion.

2.2.2 First Passage Model

In the Merton model, during the life of the debt, the asset's value can be below the debt level and that does not trigger a default, since the relation between the asset's value and the debt's value is only evaluated at maturity (path-independent option). However, with Brockman and Turtle (2003), the First Passage Model defines that default can occur not only at the maturity of debt. In this model, default happens whenever the value of assets reaches the default barrier, which is the value of debt, during the life of the CDS contract (path-dependent option). Consequently, this model excludes the possibility of the company recovering after crossing the defined barrier.

With this, the default condition is given by:

$$\text{Default if: } V_t \leq H, \;\; t \in [0, T] \tag{15}$$

Consequently, in the First Passage Model, the PD, or the survival probability, which is complementary, considers the probability of the stochastic process of the asset's values to reach the default barrier that triggers a default event.

To define how the option is exercised or not, there are four main ways. First, the value of the default barrier can be fixed above ("up") or below ("down") the initial value of the assets. Then, the option can be at-the-money (ATM) and so be exercised ("in"), or go beyond the default barrier, and become worthless ("out"). The scope of this project will be the down-and-out call (DOC) option pricing model, since it reflects the default of the company when the asset's value goes below the default barrier. Also, the case that will be approached is when D is higher than or equal to H, which means that the asset's value can reach the debt value and does not trigger any default if it does not cross the default barrier.

The European DOC value is given by:

$$E_{DOC} = VN(a) - De^{-rT}N\left(a - \sigma\sqrt{T}\right) - V\left(\frac{H}{V}\right)^{2\eta} N(b) + De^{-rT}\left(\frac{H}{V}\right)^{2\eta-2} N\left(b - \sigma\sqrt{T}\right) + R\left(\frac{H}{V}\right)^{2\eta-1} N(c) + R\left(\frac{V}{H}\right) N\left(c - 2\eta\sigma\sqrt{T}\right) \tag{16}$$

with the same defined variables as above for the Merton model, H the value of the default barrier and R the value that equity holders get when the option gets worthless. Also:

$$a = \frac{\ln\left(\frac{V}{D}\right) + \left(r + \left(\frac{\sigma^2}{2}\right)\right)T}{\sigma\sqrt{T}} \tag{17}$$

$$b = \frac{\ln\left(\frac{H^2}{VD}\right)+\left(r+\left(\frac{\sigma^2}{2}\right)\right)T}{\sigma\sqrt{T}} \tag{18}$$

$$c = \frac{\ln\left(\frac{H}{V}\right)+\left(r+\left(\frac{\sigma^2}{2}\right)\right)T}{\sigma\sqrt{T}} \tag{19}$$

$$\eta = \frac{r}{\sigma^2} + \frac{1}{2} \tag{20}$$

In the original formulas, there is $(T - t)$ instead of only T , in this project, the goal is only to calculate the CDS spread of a new contract, so at time $t = 0$.

Brockman and Turtle (2003) then defined the risk-neutral PD as:

$$PD = N\left(\frac{\ln\left(\frac{H}{V}\right)-\left(r-\left(\frac{\sigma^2}{2}\right)\right)T}{\sigma\sqrt{T}}\right) + \exp\left(\frac{2\left(r-\left(\frac{\sigma^2}{2}\right)\right)\ln\left(\frac{H}{V}\right)}{\sigma^2}\right)\left[1 - N\left(\frac{-\ln\left(\frac{H}{V}\right)-\left(r-\left(\frac{\sigma^2}{2}\right)\right)T}{\sigma\sqrt{T}}\right)\right] \tag{21}$$

2.3 Discount model to calculate the breakeven CDS spread

After modelling the probabilities of default using two different structural models, I used the same method to get the CDS Spread. Following O'Kane and Turnbull (2003), in this project, the approach used is the discounting model. This model focuses on calculating the present value of the periodic payments that the buyer does (Premium Leg Valuation) and the present value of the payment that the seller does in case of a credit event (Protection Leg Valuation).

With this approach, we are calculating the breakeven CDS spread. As seen above, the protection buyer pays periodic payments, and the amount of these payments is related to the PD of the reference entity. Consequently, the breakeven spread is the "fair value" to pay for protection at the beginning of a new CDS contract for a given reference entity. So, the objective is to get equal present values for the Premium leg and the Protection leg:

$$PV\ Premium\ Leg = PV\ Protection\ Leg \tag{22}$$

The CDS spread, as also discussed, is the surplus return of the underlying asset minus the return on a risk-free asset. For this approach, since structural models are used to estimate the PD of the reference entity and they do not consider directly the underlying asset features, it is not possible to link the PD of the underlying asset with the CDS spread.

2.3.1 Premium leg

The Premium Leg Valuation is then composed of two computations:

- present value of all the future premium payments weighted by the probability of survival,

$$S_0 \sum_{n=1}^{N} \Delta\left(t_{n-1,} t_n\right) Q(t, t_n) Z(t, t_n) \tag{23}$$

- the accrued amount of premium payments until the time of default (s),

$$S_0 \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \Delta\left(t_{n-1,} s\right)\left(-dQ(t, s)\right) Z(t, s) \tag{24}$$

  and the integral can be approximated by,

$$\frac{1}{2} \Delta\left(t_{n-1,} t_n\right) Z(t, t_n)\left(Q(t, t_{n+1}) - Q(t, t_n)\right) \tag{25}$$

  if we assume that a default happens during the middle of two consequents payments. This assumption makes the accrued premium equal to:

$$\frac{S_0 \, \Delta\left(t_{n-1,} t_n\right)}{2} \tag{26}$$

Then, as the premium payment is weighted by the survival probability, so this accrual amount is weighted by the discrete PD within the period, that is given by:

$$\left(Q(t, t_{n+1}) - Q(t, t_n)\right) \tag{27}$$

Summing both expressions, we have that the PV of the premium leg is equal to:

$$S_0 \frac{1}{2} \sum_{n=1}^{N} \Delta\left(t_{n-1,} t_n\right) Z(t, t_n) \left(Q(t, t_{n+1}) + Q(t, t_n)\right) \tag{28}$$

with S_0 being the CDS spread, t is the date of the contract, $\Delta\left(t_{n-1,} t_n\right)$ the time, number of days, between two consecutive payments, $Z(t, t_n)$ the discounting factor, which is calculated as $e^{-r(T-t)}$, $Q(t, t_n)$ the survival probability of the reference entity until t_n and consequently, $-Q(t, s)$ represents the default probability at time s.

2.3.2 Protection leg

To evaluate the Protection Leg, the main idea is to calculate the expected value of the amount that the protection seller will have to pay to the protection buyer in case of default. To do this, the present value of the payment is computed as the difference between the present value of the par value and the amount of the bond that is recovered, the Recovery Rate (R). This difference is also called Loss Given Default (LGD).

The Protection Leg value is equal to:

$$(1-R)\int_t^T Z(t,s)\big(-dQ(t,s)\big) \Leftrightarrow LDG\int_t^T Z(t,s)\big(-dQ(t,s)\big) \tag{29}$$

with the same variables as defined in the above formulas.

The integral can be simplified, turning into intervals the continuous time. To do this, M is defined as the discretisation of steps to integrate, so we have $K = int(M(T-t)+0.5)$, being K the periods of time used.

With this approximation, the Protection Leg value is given by:

$$LGD\sum_{k=1}^{K} Z(t,k\xi)(Q(t,(k-1)\xi) - Q(t,k\xi)) \tag{30}$$

being $\xi = (T-t)/K$.

Using these two values is possible to calculate de CDS breakeven spread, taking into account that on the formulas above we have that $t = 0$, since the breakeven spread is calculated at the beginning of a new CDS contract.

So, this equality hold:

$$S_0\frac{1}{2}\sum_{n=1}^{N}\Delta\big(t_{n-1,}t_n\big)Z(0,t_n)\,\big(Q(0,t_{n+1})+Q(0,t_n)\big) = LGD\sum_{k=1}^{K} Z(0,k\xi)(Q(0,(k-1)\xi) - Q(0,k\xi)) \tag{31}$$

and reformulating this equation, we have that the CDS breakeven spread is equal to:

$$S_0 = \frac{LGD\sum_{k=1}^{K} Z(0,k\xi)(Q(0,(k-1)\xi) - Q(0,k\xi))}{\frac{1}{2}\sum_{n=1}^{N}\Delta\big(t_{n-1,}t_n\big)Z(0,t_n)\big(Q(0,t_{n+1})+Q(0,t_n)\big)} \tag{32}$$

3. MODELLING

In order to apply the two structural models and compare results, Python was used, where various functions were defined. All functions were defined by applying the theoretical foundations of the work presented above. The workflow passed by first understanding what needed to be defined and estimated, and after that, writing the Python code.

A few difficulties were faced during different stages of the project, such as getting CDS spread market values that allowed a good analysis, understanding what could be the most efficient way of writing the code, and making checks during some parts to understand what could be causing an error.

The steps to do that were:

1. define variables
2. model Merton and First Passage models to get probabilities of default
3. model the discounted approach to get the CDS breakeven spread
4. estimate value and sigma of assets
5. apply discounted approach with both PD from the two models
6. Sensitivity analysis
7. Applying models to real data

3.1 Variables

To get analytics results from both models, some values needed to be fixed. For both models, it is needed to define the default barrier, and for both, it will be the same. This value is important to be estimated using values of the reference entity's liabilities. In order to get that, Zhan, N. , Lin, L. and Lou, T. (2013) defined that the default barrier is composed of short-term liabilities (SL) and long-term liabilities (LL). This estimation puts more weight on SL in a way that short-term debt should be repaid within less time than LL. The expression is the following:

$$H = SL + \frac{1}{2}LL \tag{33}$$

The capital structure of a firm is composed by assets, equity, and liabilities, and the value of the assets is equal to the sum of equity and liabilities. It is possible to get those values from a company's balance sheet; however, their market value is not for all. Market value of equity (MVE) can be taken from Yahoo Finance website as Market Cap. For assets, it is not the same, and so, the market value of assets and their volatility will be estimated, as will be seen in the next section. As seen on this section, the default barrier, which is the Debt of the reference entity, will need SL and LL values. These values can be observable at the balance sheet as Total current liabilities and LL, respectively. The values for MVE, Total current liabilities and LL used for applying the model were the average of quarterly results during one year and one quarter.

To get the volatility of equity, it is also possible by calculating the standard deviation of shares' returns.

The Recovery Rate used for all contracts was 40%, which is the value defined by ISDA for the contracts that can be seen on Bloomberg. For the maturity (T), in this project, were study three cases: 2, 3 and 5 years, to understand the importance of time horizon in estimating PD. In all contracts it was assumed that the frequency of payments (M) is quarterly, value that is also taken from Bloomberg contracts.

Values of market cap, equity volatility, SL and LL that were used for the five companies can be seen in Table1. For market cap, SL and LL the values are in USD.

	Market Cap	Equity volatility	SL	LL
NOK	23,356,000,000.00	0.34	11,366,666,666.67	7,038,250,000.00
TSLA	832,896,000,000.00	0.04	29,220,000,000.00	18,101,400,000.00
C	125,056,000,000.00	0.31	41,917,600,000.00	289,576,200,000.00
BA	117,076,000,000.00	0.39	97,584,000,000.00	61,127,000,000.00
CLF	6,562,000,000.00	0.57	3,311,400,000.00	8,179,200,000.00

Table 1Market Cap, Equity volatility, SL and LL for studied companies

3.2 Estimate Value and Sigma of Assets

As seen before, it is needed to estimate the value of assets and their volatility. Following Zhan, N. , Lin, L. and Lou, T. (2013), that can be done starting by differentiating the European call option on the assets, expression (4):

$$\frac{\partial E_A}{\partial V} = N(d_1) \tag{34}$$

applying the chain rule:

$$\sigma_{E_A} = \frac{\partial E_A}{\partial V}\frac{V}{E_A}\sigma_V \iff \sigma_{E_A} = N(d_1)\frac{V}{E_A}\sigma_V \iff E_A\sigma_{E_A} = N(d_1)V\sigma_V \tag{35}$$

As seen before, the value of the European option on the assets is equal to the value of the firm´s equity ($E_A = V_E$) and now with expressions (4) and (35), it is possible to estimate the value of the firm´s assets (V) and asset´s volatility (σ_V) by solving the system with those two equations.

The method to do that passes by getting an initial guess, rearranging the two equations, and then using a Python function to solve the next system:

$$\begin{cases} VN(d_1) - De^{-rT}N(d_2) - E_A = 0 \\ N(d_1)V\sigma_V - E_A\sigma_{E_A} = 0 \end{cases} \tag{36}$$

Replacing in the formula above, the two variables, value of assets and their sigma for the first guess approach, that are computed as below:

$$V = V_A Guess = MVE + H \tag{37}$$

$$\sigma_V = \sigma_V Guess = \sigma_{E_A} * \left(\frac{MVE}{V_A Guess}\right) \tag{38}$$

If the solver converges with a small error, it returns the estimated values; otherwise, it does not.

Since both models require these values to be estimated and this approach is in line with both models, it is used for both.

On Table2, the estimated value of assets and their volatility, for each company, can be seen. When changing the CDS contract maturity, the risk-free rate also changes, which will cause a change in the estimated values.

	Value of assets			Sigma of assets		
	2Y	3Y	5Y	2Y	3Y	5Y
NOK	37,027,847,223.02	36,443,420,537.62	35,252,474,428.39	0.21	0.22	0.23
TSLA	868,024,926,906.54	866,369,508,716.17	862,429,415,287.80	0.70	0.70	0.70
C	296,531,308,351.83	289,196,588,557.80	274,320,119,911.63	0.13	0.13	0.14
BA	234,698,821,950.89	229,419,778,961.68	218,262,971,844.59	0.19	0.20	0.21
CLF	13,277,703,368.97	12,841,239,355.44	11,862,241,141.00	0.29	0.31	0.35

Table 2 Estimated value of Assets and sigma of assets

The changes that can be seen are expected, since the higher the risk-free rate, the lower it will be the discounted value of debt, causing the value of the assets to decrease in order to balance the equation below, remembering that the market value of equity in this approach has a fixed value:

$$Equity \ = Asset \ - Debt \tag{39}$$

3.3 Merton, First Passage and Discounting models

All functions were defined in Python, recurring to some libraries as *numpy*, used to create vectors and other numerical operations; *matplotlib*, allowing the creation of plots and visualizations; *tqdm*, not mandatory but helped to track progress during tasks that took more time running; *pandas,* mainly used to calculate statistics like standard deviation and *scipy.stats,* that was used do work with the normal distribution.

It is important to note that for the discounting approach model, the function defined was the same, and only the values of PD changed based on the two methods.

All functions can be seen in Appendix B.

3.4 Data

To extract data, Yahoo Finance and Investing.com websites were used. From Yahoo Finance, the values taken were the Market Cap, Total current liabilities, and Total non-current liabilities. From Investing.com, daily share prices were extracted, allowing for the computation of the annualized equity volatility. All data used was from 02/01/2024 to 31/03/2025, quarterly. This timeframe was chosen in order to give a recent sample with enough data to be reliable. To get the market value of the CDS spread, the Bloomberg terminal was used. In order to get the spread for each company, it was needed to search for the ticket of the company on the terminal and then use the command *CDSW* that presents the window for CDS valuation. The valuation data used for the CDS spread was 01/04/2025 since it was the first day after the last data day.

To see how accurate the models are to real data, five reference entities were chosen: Nokia (NOK) Tesla (TSLA), Citigroup (C), Boeing (BA) and Cleveland-Cliffs (CLF). These companies were chosen because of their data availability, leading to more viable results, and with the objective of studying companies of different sizes, but most being large and medium-cap and with different levels of debt. Also, the objective was to reflect a diversity of areas, such as finance, technology, airline, automotive, and the manufacture of steel.

As for the risk-free rate used in all computations, this value was fixed for the entire period, allowing simple calculations. Since all data used was from the United States of America, it would make sense to use a rate also of the same area, being the currency (USD) one of the reasons, and same timeframe, and so, the yield of U.S. Treasure Notes was used. Although is not the same as an overnight deposit rate, as EONIA (Euro Overnight Index Average) or €STR (Euro Short-Term Rate), that normally are used as risk-free rate for EURO area, the U.S. Treasury Notes are accepted as a risk-free rate, since they are considered to have almost zero default risk. In this project, yields of different maturities were used, matching the maturity of the CDS contract. Data was taken from the Statista website, and these were the values used for discounting calculations:

Maturity	Yield
2Y	4.25 %
3Y	4.27%
5Y	4.38%

Table 3 Treasury notes yield for different maturities, for 2024

3.5 Sensitivity analysis

Next, it is important to study the impact that each variable has on the calculation of PD and then on the CDS breakeven spread. Although the models have differences, the sensitivity analysis produced the same general conclusions for both.

For the next analysis is important to keep in mind that as seen, PD and the CDS spread have a direct relation, which means that when PD increases/decreases, the CDS spread also increases/decreases. To perform the sensitivity analysis, each variable was increased and decreased by 10% and 20%, keeping the other variables unchanged. In the next two tables, there are the percentage changes of the CDS spread for the two structure models relative to each variable change.

	Variables change				
	-20%	-10%	0%	10%	20%
MVE	24.33%	11.34%	0.00%	-9.89%	-18.63%
equity volatility	-78.09%	-47.92%	0.00%	66.99%	153.09%
SL	-10.92%	-5.37%	0.00%	5.23%	10.25%
LL	-10.92%	-5.37%	0.00%	5.23%	10.25%
Maturity	-35.70%	-17.71%	0.00%	17.15%	33.48%
Risk free	2.45%	1.24%	0.00%	-1.18%	-2.36%
Recovery Rate	13.35%	6.67%	0.00%	-6.64%	-13.32%
Value of assets	271.12%	92.24%	0.00%	-47.83%	-72.69%
Sigma of assets	-75.88%	-45.35%	0.00%	59.02%	129.58%

Table 4 Sensitivity analysis for Merton model

	Variables change				
	-20%	-10%	0%	10%	20%
MVE	27.78%	12.68%	0.00%	-10.81%	-20.10%
equity volatility	-77.50%	-47.28%	0.00%	65.20%	148.34%
SL	-11.90%	-5.89%	0.00%	5.77%	11.44%
LL	-11.90%	-5.89%	0.00%	5.77%	11.44%
Maturity	-14.43%	-7.55%	0.00%	8.20%	17.07%
Risk free	0.55%	0.22%	0.00%	-0.13%	-0.15%
Recovery Rate	13.33%	6.66%	0.00%	-6.66%	-13.33%
Value of assets	290.96%	95.39%	0.00%	-48.29%	-73.05%
Sigma of assets	-75.25%	-44.70%	0.00%	57.44%	125.51%

Table 5 Sensitivity analysis for First Passage model

It is easy to see that the decrease in the value of equity leads to an increase in the PD and so the increase of CDS spread, being this justified by the fact that with less equity, the leverage ratio increases and so leads to more risk of default. Also, when there is more volatility, it will increase the PD value since there is a wider range of values that the equity can acquire, and so, lead to default. The impact of value of assets is stronger than the impact of MVE, since MVE is used to estimate the value of assets, that is directly used as an input on the calculation PD and consequently the CDS spread, being more sensitive on the First Passage model, which is expected, since this model more sensitive to the move of the assets during the life of the contract. The sigma of assets has an impact really similar to the equity volatility, although the first one has a smaller impact on the CDS spread than equity volatility. Both variables cause the same impact on the CDS spread since they are directly related to each other, since the equity volatility is used to estimate the sigma of assets, following Zhan, N. , Lin, L. and Lou, T. (2013).

Another important variable is the short and long-term debt that defines the default barrier. Along with the rise of the default barrier also the CDS spread increases since the distance between the value of assets and debt is smaller and the other way around. For both, the impact on the CDS spread is the same since the changes are not so different, reflected on the default barrier. Also, as expected by the definition of both models, it can be seen that the First Passage model is more sensitive to changes in the default barrier.

There is a direct relationship between the time to maturity and the CDS spread, since when the maturity is longer more time there is to happen a credit event to occur. When analysing the Recovery Rate, it is seen that when it increases, it makes the CDS spread decrease its value. If the RR is higher, it means that the protection buyer will receive more from the defaultable asset if he holds the asset, and so, we will receive less from the CDS, since LGD will be less. With that payment at default being less, the protection buyer will not pay so much for the protection, leading to a decrease in the CDS spread.

For both models, it is also possible to check that the variables that most impact the change of the CDS spread, along with other external effects, that are not considered on this project, are the equity volatility, value and sigma of assets. This can be justified by the fact that PD´s are estimated based on the move of the assets during the life of the contract.

4. RESULTS

In this section, the results of applying the models defined to real data will be presented.

Even before applying the models is known that they will not retrieve the exact same value as the market value seen in Bloomberg. This can be due to the use of because average values, or because the timeframe is not the best, or by the reason that market values take more external variables that will influence the result. However, this analysis will be done in more detail after the presentation of the results.

On Table6 and Table7, results of PD values, in percentage, from the two structural models for different CDS contract maturities can be seen and the corresponding values of the CDS spread, in bps, for each PD value.

	Merton			First Passage			Bloomberg		
	2Y	3Y	5Y	2Y	3Y	5Y	2Y	3Y	5Y
NOK	0.09	0.60	3.08	0.18	1.26	6.63	1.62	2.34	3.77
TSLA	0.29	1.86	9.02	0.51	3.22	14.55	1.79	3.25	7.38
C	0.19	0.98	4.15	0.42	2.35	11.11	1.57	2.61	5.73
BA	0.84	2.96	9.18	1.78	6.44	20.62	3.53	5.09	8.12
CLF	7.65	15.82	31.02	15.32	31.19	58.13	6.19	12.28	29.17

Table 6 PD values from Merton and First Passage models and market values from Bloomberg

	Merton			First Passage			Bloomberg		
	2Y	3Y	5Y	2Y	3Y	5Y	2Y	3Y	5Y
NOK	2.58	11.56	35.71	5.35	24.37	77.76	52.64	56.93	65.12
TSLA	8.30	35.95	104.96	14.97	62.36	172.34	48.52	60.93	87.29
C	5.45	19.03	50.15	12.38	45.93	137.40	42.49	48.88	67.24
BA	24.54	58.26	113.77	52.41	128.34	268.81	96.49	96.52	96.57
CLF	234.19	341.59	473.71	484.80	734.26	1138.25	171.09	241.87	392.60

Table 7 CDS spread values from discounting model using PD values from Merton and First Passage model and market values from Bloomberg

4.1 Merton VS First Passage

As expected by the definition of the two structural models, the First Passage model estimates higher probabilities of default, since by definition, default can occur in any time between the beginning until the maturity of the contract. Consequently, it will also generate higher CDS spread, since by definition, the higher the PD the higher the spread.

The difference between both models' values can be seen as the risk premium to pay driven by the possibility of an early default. With this analysis we can confirm that the First Passage model is more conservative towards risk, giving the possibility of default before the maturity. The reasoning is more or less the same as the price to pay for an option that can only be exercised at the maturity, a European option, or an option that can be exercised during the maturity of the contract, an American option. As can be seen in Hull (2006), the price of an American option is higher than the European one, due to the willingness of being exercised at any time until maturity. The same happens with Merton and First Passage model, being in this case the PD the price and the European option the Merton model and the American option the First passage model.

When looking at the graphs (see Appendix1) of the PD term structure on Appendix 1, it is visible that there are different curves between each company. The graphs that can be seen, plot the accumulated PD during the time of the CDS contract for each company, each structure model, and each one of the three different maturities studied. In all the graphs, the expected occurs: the higher the distance to default (DD), the smaller PD, being distance to default the time that is left until the maturity of the contract. It was expected an upward-sloping term structure of PD curves, which can be seen. This slope is justified by the accumulative PD that increases as time passes by and companies are more exposed to uncertainty for longer time horizons.

To better understand these results, it is also important to consider the ratio between assets and liabilities, since it has already been discussed, the relation between these two variables is one of the principal drivers of both models.

The financial ratio that gives us that information is a leverage ratio that is called Debt-to-Assets ratio, that passes by divide liabilities by assets, in order to understand how much of the assets is financed by debt. When a company is more financed by debt it increases its risk, as debt, when comparing to equity, is a higher riskier form of financing. This is true since companies are required to pay debtholders them interest, regardless of being profitable or not. With that, it is understandable that a higher Debt-to-Assets ratio reflect a riskier company when analysing leverage, because it tells us that there is a high percentage of assets financed by debt.

On the table below, Debt-to-Assets ratio for each company can be seen:

	2Y	3Y	5Y
NOK	0.497056	0.505027	0.522089
TSLA	0.054516	0.05462	0.05487
C	1.117905	1.146258	1.20842
BA	0.676233	0.691793	0.727155
CLF	0.865406	0.89482	0.96867

Table 8 Debt-to-Assets ratio for each company

For C, BA and CLF, the increasing of the PD on the first half of the 5 years is higher than for the other two companies. This conclusion is taken by the convexity of the PD term structure of NOK and TSLA while for C, BA and CLF, the curves are concave.

We have seen that the DD is the value inside the normal function, which is equal to d_2. When we are at the beginning of the contract, the denominator is small, which will generate a high negative value inside the normal distribution. So, by the form of the normal distribution, if this value is really behind the mean, it will give a small error and so a small PD, and that also increases slowly. On the other side, on the numerator of d_2, we have the logarithm of the inverted leverage ratio, that passes by divide the assets by the debt, in this specific case, being the debt equal to the default barrier. So, for a high V/H values, which is the same as having a small Debt-to-assets ratio value, it will give a high negative value inside the normal distribution, and so, a small PD. That is the reason why NOK and TSLA PD graphs start to increase with a small slope, while C, BA and CLF, which have a high Debt-to-Assets ratio, see their PD increase faster, making the PD curves have a concave form.

The difference seen in each graph between the three different maturities is coming from the different risk-free rate used for discount computations. As seen in the sensitivity analysis, as the risk-free rate increases, the CDS Spread decreases, which is the same as having the PD value to decrease. That relationship can be seen by the plots of PD in Appendix A.

The reasoning is the same for the First Passage model, although has already discussed, by adding the possibility of early default, it will generate higher PD's, as seen on the plots.

4.2 Analytical VS Market

Analytical values from both structural methods in some cases underestimate risk of default, leading to smaller PD and consequently, smaller CDS spread than the market values. This is true for any Merton PD value for 2Y and 3Y maturities, with exception of CLF, that is the smallest company being studied and at the same time the one with higher equity volatility, which will cause higher PD estimated, as can be seen by the sensitivity analysis. The same reasoning happens for the First Passage model, only with exception of BA and CLF.

By *Table 8*, we can see that BA and CLF are two of the three companies with the most leverage, resulting in higher PD values for both. However, there is TSLA that, although it does present a smaller Debt-to-Assets ratio, is one of the companies with higher analytical PD values. This result is justified by the fact that TSLA is the company with higher estimated sigma of assets, that by the construction of the models, will impact the PD values, leading to higher values and as seen on the sensitivity analysis, sigma of assets has more impact on the PD and CDS spread than the STL and LTL.

Although a certain increase in PD generates a proportional increase in the value of the CDS spread in both analytical models, this is not the case with market values. For example, in the case of BA, the PD increases by 3 percent between a maturity of 3Y and 5Y, while the spread has only increased by 0.05 bps, which is very small compared to the change in PD, and that does not happen with any analytical value. For example, for NOK, on the Merton model, an increase of 2.48 percent of PD from a contract of 3Y to one with 5Y maturity, generates an increase of 24.15 bps.

It is important to notice that the rating of riskier companies is in line with the rating that the market does, by the PD and market spread, both rating CLF, TSLA and BA as the three most risky companies.

4.3 Error Analysis

After having the results of both models, it is important to study what is the approximation error of analytical results to market ones, in order to check the performance and accuracy of both structural models to real market data for the different companies.

Following Bruce, Peter, Bruce, Andrew (2019), to perform this analysis, three metrics were calculated: prediction error (PE), mean absolute error (MAE) and root mean squared error (RMSE). In order to perform those analyses, three formulas were applied.

For the prediction error:

$$PE = \frac{mv - a}{mv} \tag{40}$$

for the mean absolute error:

$$MAE = \frac{1}{n}\sum_{i=1}^{n} |mv_i - a_i| \tag{41}$$

and for the root mean squared error:

$$RMSE = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(mv_i - a_i)^2} \tag{42}$$

where mv is the observable market value from Bloomberg, a is the analytical result from each model and n is the number of observations.

The prediction error allows us to check if the models are good for practical decision making, since it divides the difference between the observable market value and the analytical value by the observable market value. The MAE metric computes an average of all errors, which can distribute the impact of large errors. However, although it is a simple metric, it allows to get an intuitive overview of how much the analytical values deviate from the market values. The RMSE calculate the square root of the average of squared differences between market values and analytical values, putting more weight on large errors than MAE, since the differences are squared. These three metrics were chosen since they are well recognized in error analysis when working with a wide range of models.

	Merton			First Passage		
	2Y	3Y	5Y	2Y	3Y	5Y
NOK	94.44%	74.36%	18.30%	88.89%	46.15%	-75.86%
TSLA	83.80%	42.77%	-22.22%	71.51%	0.92%	-97.15%
C	87.90%	62.45%	27.57%	73.25%	9.96%	-93.89%
BA	76.20%	41.85%	-13.05%	49.58%	-26.52%	-153.94%
CLF	-23.59%	-28.83%	-6.34%	-147.50%	-153.99%	-99.28%

Table 9 Prediction error values for Probabilities of default

	Merton			First Passage		
	2Y	3Y	5Y	2Y	3Y	5Y
NOK	95.10%	79.69%	45.16%	89.84%	57.19%	-19.42%
TSLA	82.89%	41.00%	-20.24%	69.15%	-2.35%	-97.43%
C	87.17%	61.07%	25.42%	70.86%	6.04%	-104.34%
BA	74.57%	39.64%	-17.81%	45.68%	-32.97%	-178.36%
CLF	-36.88%	-41.23%	-20.66%	-183.36%	-203.58%	-189.93%

Table 10 Prediction error values for CDS spread

The results from the Prediction error reinforce the conclusions made on the two previous chapters: Merton model can estimate values that are more in line with market values for contracts with 5Y maturity and for First Passage model it happens for 3Y contracts maturity, since there are the periods for which the error are smaller within each model, for both PD and CDS spreads.

The only exception is the same as seen before, CLF. For this company, the more accurate PD result for the First Passage model is for the 5Y contract, instead of for the 3Y as for all the other companies when using the First Passage model, being this value practically double the market value. What can also be different from other companies is that, for the other four companies, the PD error is basically reflected on the CDS spread error, meaning that the maturity for PD, for each model, that has the highest PE, also has the highest PE of CDS spread, and vice versa. However, that is not happening for the First Passage model for CLF. It is visible that, although the 5Y maturity contract generates the most accurate value when compared to the Bloomberg value for PD, for the CDS spread, it is the value for the 2Y maturity that is closer to the market one. This can be a specific case where it is notable that market spreads include more information and are not so linearly related to the analytical PD, having as input more external information than the analytical methods used.

This analysis also shows that estimation errors are not linear, meaning that analytical models can make a good estimate for a certain maturity, but when chasing it, it can get a high deviation from the market value, as can be seen for TSLA. First Passage model estimates a PD value for a 3Y maturity contract with only 0.92 percent deviation from market value; however, for the other two maturities, we see a huge increase in the estimation error.

PD

MAE	Merton	First Passage
2Y	1.22	2.11
3Y	1.49	3.09
5Y	0.97	8.12

Table 11 MAE values for probabilities of default

RMSE	Merton	First Passage
2Y	1.78	4.28
3Y	2.22	8.49
5Y	1.43	14.72

Table 12 RMSE values for probabilities of default

CDS spread

MAE	Merton	First Passage
2Y	52.47	113.98
3Y	47.64	112.23
5Y	32.50	217.15

Table 14 MAE values for CDS spread

RMSE	Merton	First Passage
2Y	54.14	144.65
3Y	54.74	221.15
5Y	40.85	345.83

Table 13 RMSE values for CDS spread

Since the First Passage model is more conservative towards risk, it could be expected that the results of this model would be more in line with the market values, since even not incorporating external market conditions, getting higher values than the Merton model it could give results more in line with the market ones. However, that is not what happened, as can be easily seen by the MAE and RMSE tables. Both tables show that for any maturity, the Merton model has a better performance than the First Passage model, since the error value is lower in any case for the first model.

When comparing the RMSE with the MAE, it is also notable that the model with higher deviations from the observable market values is the First Passage model. This happens since, has discussed, RMSE takes more weight for higher deviations, and it is possible to check that for all maturities, the biggest differences between analytical values and market values are coming from the First Passage model, for both PD and CDS spread.

5. CONCLUSIONS

The main objective of this project was to understand how CDS can be evaluated by applying structural models and to understand how changes to the models affect the ending result, in this case, both PD and the CDS spread. This study allowed the use of quantitative finance, derivatives, mathematical finance, and programming knowledge.

After conducting this study, there are some conclusions that can be made. One of the main conclusions taken is the significant differences between the results of the two models, that was driven mainly by the sensitivity of the models to the capital structure of the companies and the way default can occur within the models. The results were in line with the expectation, since the First Passage model allows default to occur any time until maturity, reflecting better the risk of earliest default events, it estimates higher values than the Merton model, a result that was true for every company and every maturity.

The sensitivity analysis allowed us to see the inputs that affect the CDS spread the most, which is the same as analysing the sensitivity to the PD, since for the CDS spread, the same approach was used. With the analysis made, it was visible that there are four main drivers of the different results, in descending order of impact: value of assets, equity volatility, sigma of assets, and maturity. These results are also in line with the way the models estimate the PD, since the main impact is driven by the way assets are higher or lower than the default barrier and how they move during the time until maturity.

It was also checked that the ratio between assets and debt also impacts the way PD changes during times, higher leveraged companies having PD curves more aggressive and faster.

When compared to observable values, models showed a limited capacity to replicate market values, specifically the First Passage model, which overestimates PD for the higher contract maturity. So, it can be concluded that, although having some limitations, like just allowing default only at maturity, the Merton model generated results closer to the market ones, especially for longer maturities. This suggests that, even being a simpler model with some unrealistic features, it can be good for a first approximation, being better also for maturities like 5 years.

As already discussed, although First Passage allows early default, which already takes a limitation of the Merton model, both models continue to have some limitations. An important one is the assumed constant volatility, an assumption made already on the Black - Scholes formula that it is known to not be true. For the discounting model to get the CDS spread, the default is assumed to occur at the middle of the two premium payments, which is also understandable, as it is not true for market situations.

Also, since the work was done with structural models that base the results on the capital structure of companies, PD values do not reflect external factors like market liquidity, counterparty risk, macroeconomic conditions, regulation changes, and demand and supply.

Another important limitation is the values used as input, since some of them were averages used to simplify calculations, and some were also estimations, which will impact the final results.

For future studies, it could be useful to use reduced-form models that are sensitive to market conditions; structural models extended with stochastic volatility, allowing to take the limitation of constant volatility; incorporate some macroeconomic features into the models, which can allow to have a dynamic default barrier to align with some macro variables.

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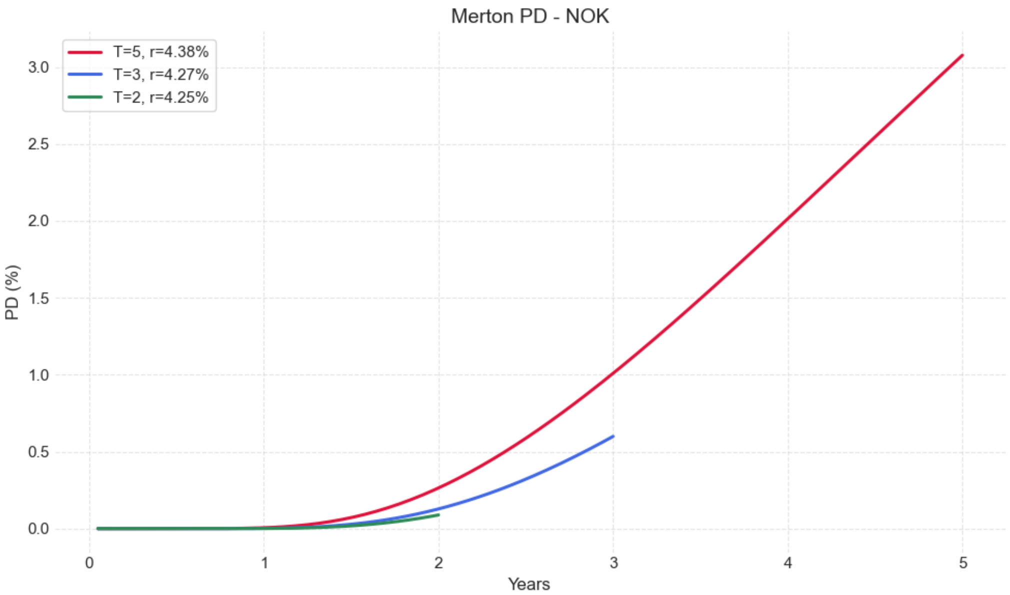
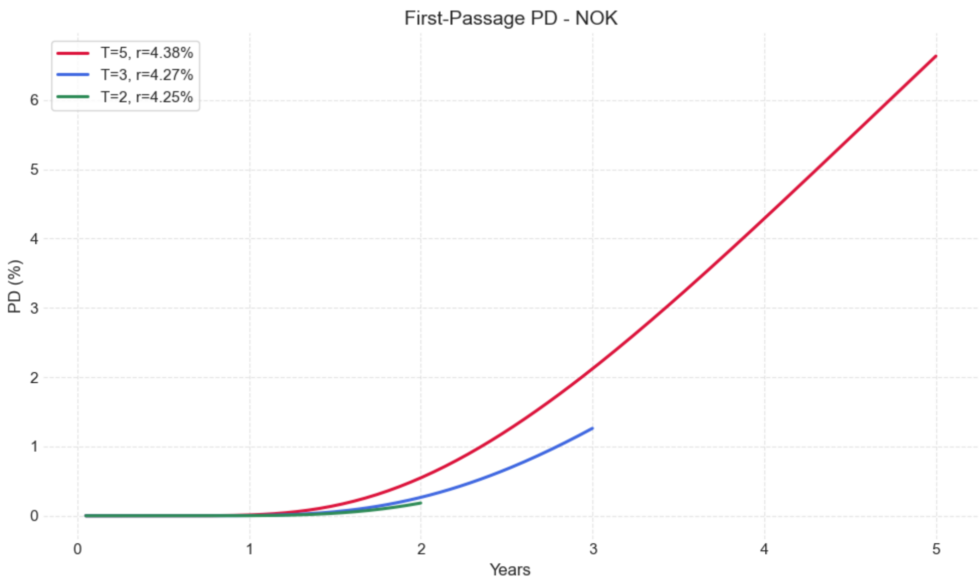
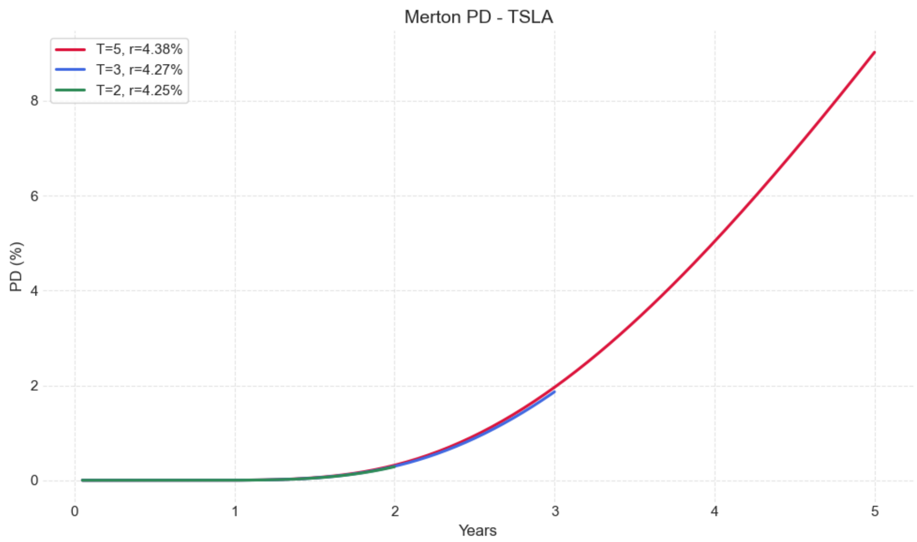
APPENDIX A

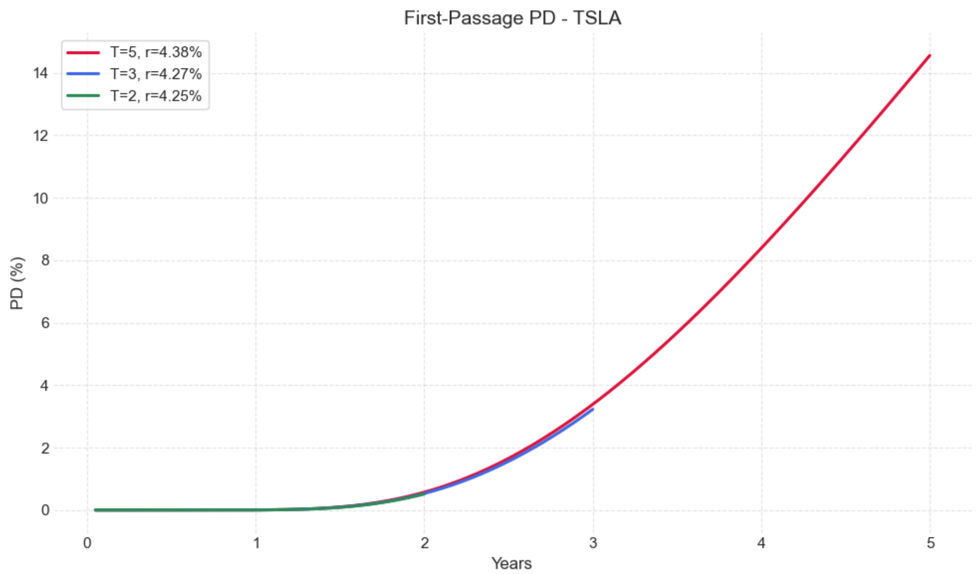
PD term structure, for both models, plotted in graphs for the different companies studied and different contracts maturities.



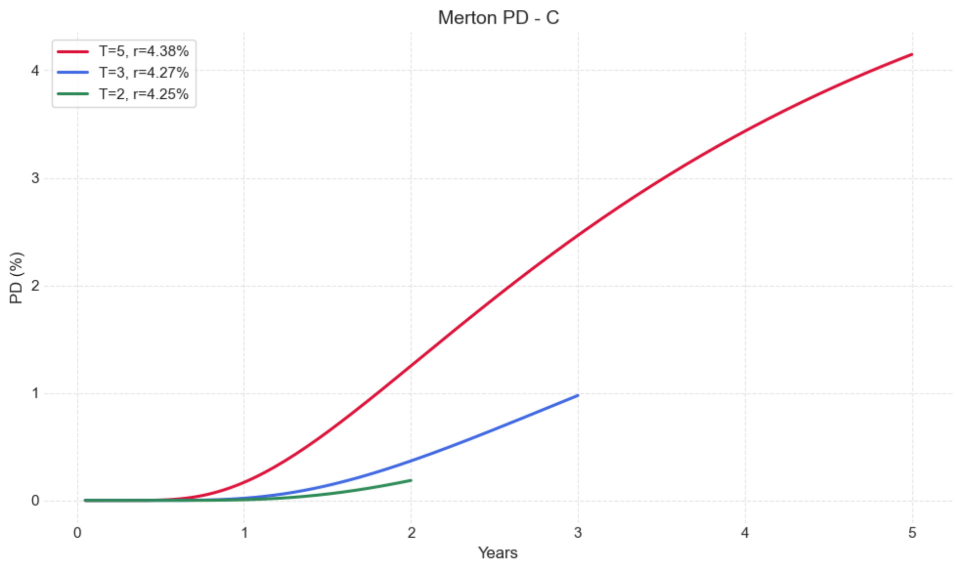




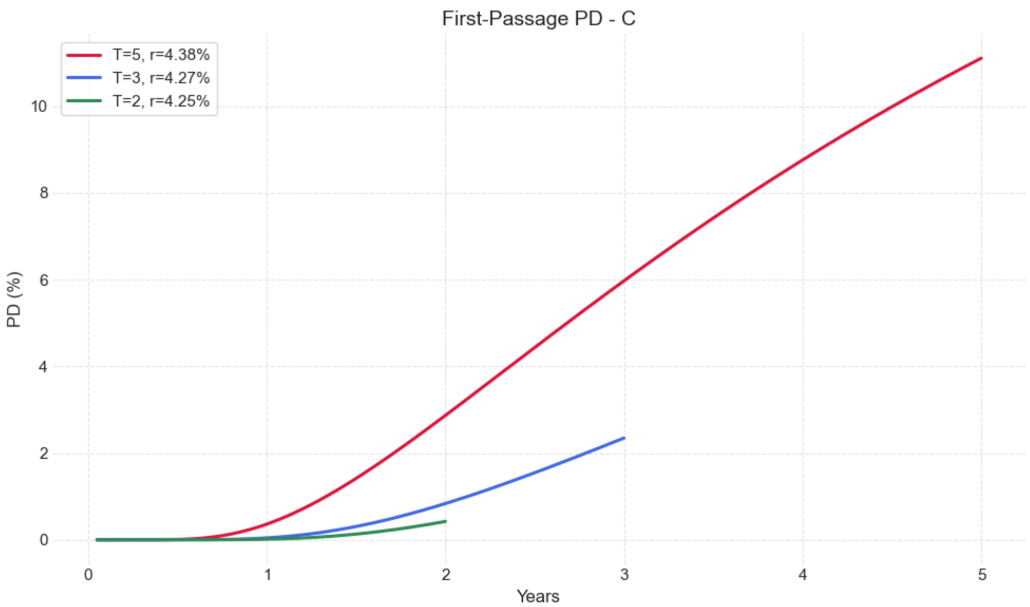
Merton PD - TSLA
T=5, r=4.38%
T=3, r=4.27%
T=2, r=4.25%
PD (%)
Years



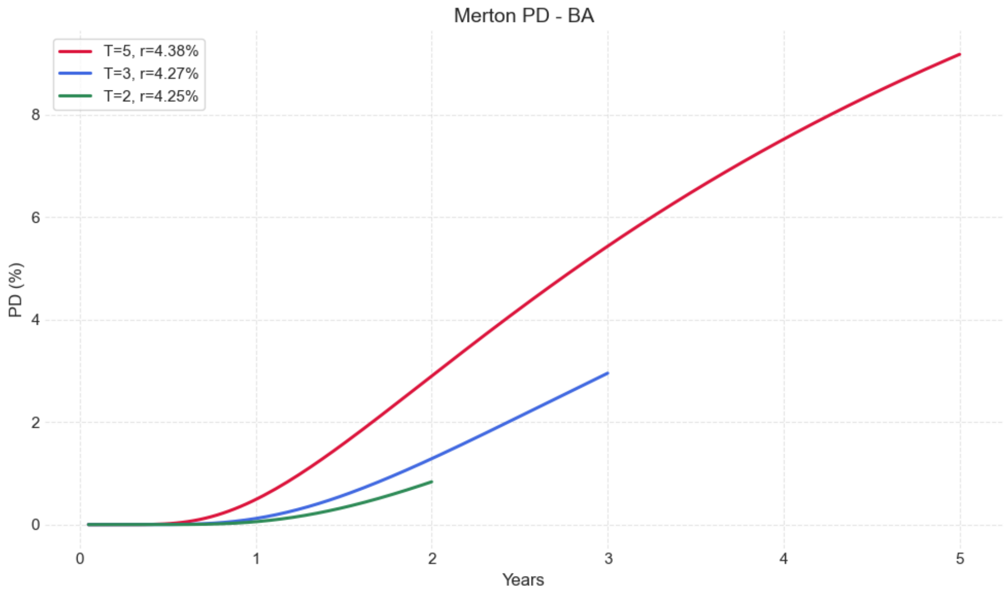
First-Passage PD - TSLA
T=5, r=4.38%
T=3, r=4.27%
T=2, r=4.25%
PD (%)
Years

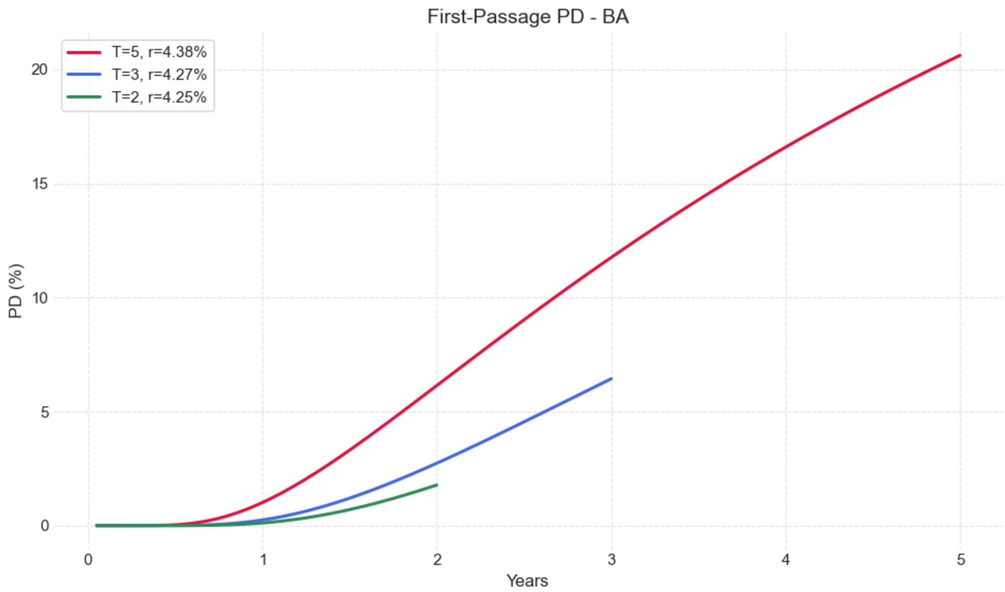


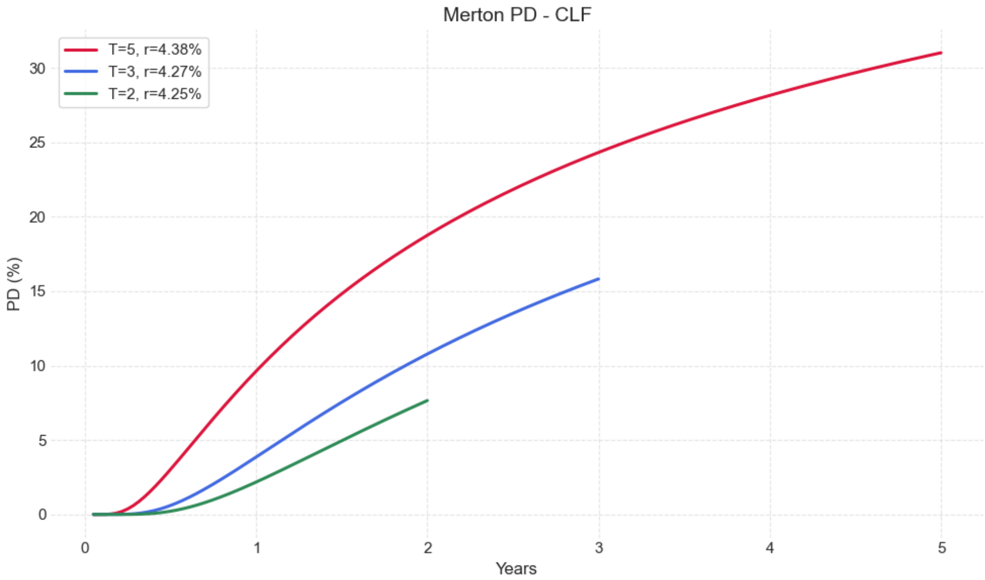
Merton PD - C
T=5, r=4.38%
T=3, r=4.27%
T=2, r=4.25%
PD (%)
Years

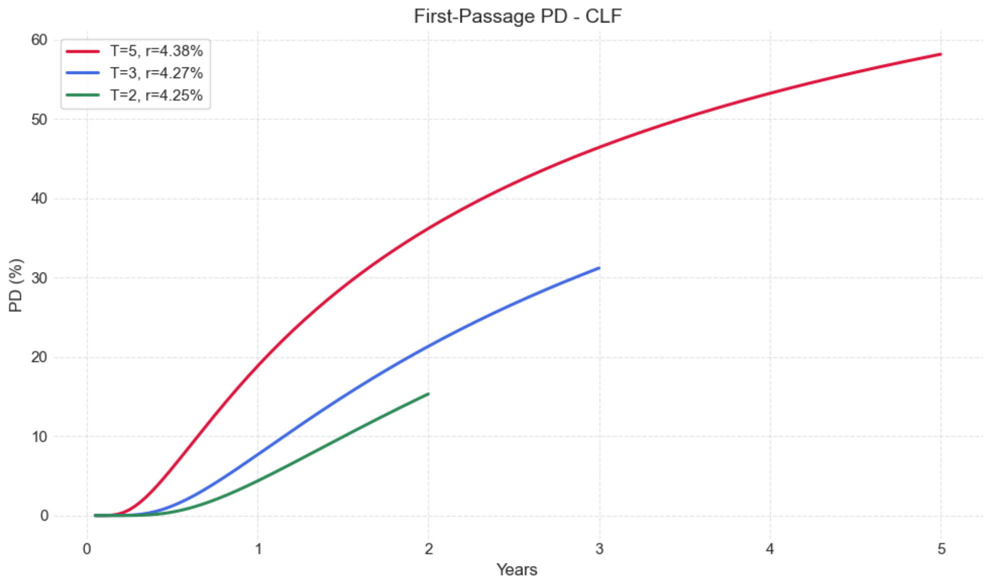


First-Passage PD - C
T=5, r=4.38%
T=3, r=4.27%
T=2, r=4.25%
PD (%)
Years


Merton PD - BA
T=5, r=4.38%
T=3, r=4.27%
T=2, r=4.25%
PD (%)
Years
0
2
4
6
8
0
1
2
3
4
5


First-Passage PD - BA
T=5, r=4.38%
T=3, r=4.27%
T=2, r=4.25%
PD (%)
Years
0
5
10
15
20
0
1
2
3
4
5


Merton PD - CLF
T=5, r=4.38%
T=3, r=4.27%
T=2, r=4.25%
PD (%)
Years


First-Passage PD - CLF
T=5, r=4.38%
T=3, r=4.27%
T=2, r=4.25%
PD (%)
Years

APPENDIX B

In this appendix, it is presented the python code used on this project.

Code to install Python libraries:

```
pip install matplotlib
pip install pandas
pip install tqdm
pip install yfinance
```

```python
import numpy as np
from scipy.stats import norm
import matplotlib.pyplot as plt
import warnings
import pandas as pd
from tqdm import tqdm
```

Discount factor, as $Z(t, t_n)$:

```python
def discount_factor(r, T_to_discount):
    if T_to_discount < 0: return 1.0
    if T_to_discount == 0: return 1.0
    return np.exp(-r * T_to_discount)
```

Code to get the survival probability by Merton model:

```
def merton_survival_prob(V0, H, r, sigma_V, T):
    if sigma_V <= 0 and T > 1e-9: return np.nan
    if T < 0: return np.nan
    #T=0
    if T < 1e-9: return 1.0 if V0 > F else 0.0
    #sigma_V=0 (e T>0)
    if sigma_V < 1e-9:
        final_value = V0 * np.exp(r * T)
        return 1.0 if final_value > H else 0.0
    with np.errstate(divide='ignore', invalid='ignore'):
        sqrt_T = np.sqrt(T)
        log_V0_H = np.log(V0 / H)
        drift_term = (r + 0.5 * sigma_V**2) * T
        vol_term = sigma_V * sqrt_T
        if abs(vol_term) < 1e-16: return 1.0 if log_V0_H + drift_term > 0 else 0.0
        d1 = (log_V0_H + drift_term) / vol_term
        d2 = d1 - vol_term
        survival_prob = norm.cdf(d2)
        if np.isnan(survival_prob):
            if d2 > 30: return 1.0
            if d2 < -30: return 0.0
            return np.nan
    return survival_prob
```

Code to get the PD by Merton model, which is get by doing $1 - survival\ probability$:

```
def merton_pd(V0, H, r, sigma_V, T):
    sp = merton_survival_prob(V0, H, r, sigma_V, T)
    if np.isnan(sp):
        return np.nan
    else:
        return np.clip(1.0 - sp, 0.0, 1.0)
```

Implementing First-Passage model to get survival probability:

```
def calculate_fpt_survival_prob(V0, H, T, r, sigma_V):
    epsilon = 1e-12
    pd_fpt = np.nan
    if V0 <= 0 or H <= 0 or T < 0 or sigma_V < 0:
        return np.nan
    if T < epsilon:
        return 1.0
    if H >= V0:
        pd_fpt = 1.0
    elif sigma_V < epsilon:
        V_T = V0 * np.exp(r * T)
        pd_fpt = 1.0 if V_T <= H else 0.0
    else:
        try:
            nu = r - 0.5 * sigma_V**2
            sqrt_T_safe = np.sqrt(T)
            sigma_V_sqrt_T = sigma_V * sqrt_T_safe
            log_H_V0 = np.log(H / V0)
            if abs(sigma_V_sqrt_T) < epsilon:
                 V_T_drift = V0 * np.exp(nu * T)
                 pd_fpt = 1.0 if V_T_drift <= H else 0.0
            else:
                h1 = (log_H_V0 + nu * T) / sigma_V_sqrt_T
                h2 = (log_H_V0 - nu * T) / sigma_V_sqrt_T
                term1 = norm.cdf(h1)
                exponent_base = -2 * nu
                exponent_den = sigma_V**2
                exponent = exponent_base / exponent_den
                base = V0 / H
                log_magnitude = exponent * np.log(base)
                if log_magnitude > 700: term2_exp = float('inf')
                elif log_magnitude < -700: term2_exp = 0.0
                else: term2_exp = np.exp(log_magnitude)
                term2 = term2_exp * norm.cdf(h2)
                if not np.isfinite(term1) or not np.isfinite(term2):
                    pd_fpt = np.nan
                else:
                    pd_fpt = term1 + term2
                    pd_fpt = np.clip(pd_fpt, 0.0, 1.0)
        except (OverflowError, ValueError, ZeroDivisionError):
              pd_fpt = np.nan #
    if np.isnan(pd_fpt):
        return np.nan
    else:
        survival_prob = 1.0 - pd_fpt
        return survival_prob
```

Code to get the PD by First-Passage model, doing $1 - survival\ probability$:

```
def fpt_pd(V0, H, r, sigma_V, T):
    sp = calculate_fpt_survival_prob(V0, H, T, r, sigma_V)
    if np.isnan(sp):
        return np.nan
    else:
        return np.clip(1.0 - sp, 0.0, 1.0)
```

These next 3 parts of code are similar to the approach for the First Passage model, only changing the “sp” variable that is defined with the function “calculate_fpt_survival_prob”.

Code used to calculate Premium leg, Protection leg and CDS spread:

```
def calculate_premium_leg_pv_M(t, tn, T_cds, V0, H, r, sigma_V):
  if not isinstance(tn, np.ndarray): tn = np.array(tn)
  if len(tn) == 0: return 0.0
  if np.any(tn <= t) or T_cds <= t: return np.nan
  premium_leg_pv = 0.0
  last_payment_time = t
  for payment_date in tn:
      process_stub = False; actual_payment_date = payment_date
      if payment_date > T_cds:
          if last_payment_time < T_cds: process_stub = True; actual_payment_date = T_cds
          else: break
      time_to_event = actual_payment_date - t
      delta_t = actual_payment_date - last_payment_time
      df = discount_factor(r, time_to_event) #
      sp = merton_survival_prob(V0, H, r, sigma_V, time_to_event)
      if np.isnan(sp) or np.isnan(df): return np.nan
      premium_leg_pv += delta_t * df * sp
      last_payment_time = actual_payment_date
      if process_stub: break
  return max(premium_leg_pv, 1e-16)


def calculate_protection_leg_pv_M(t, T_cds, R, M, V0, H, r, sigma_V):
  if T_cds <= t or M <= 0: return np.nan
  LGD = 1.0 - R
  total_intervals = int(M * (T_cds - t) + 0.5)
  if total_intervals <= 0: return 0.0
  epsilon = (T_cds - t) / total_intervals
  protection_leg_pv = 0.0
  sp_prev = 1.0
  for k in range(1, total_intervals + 1):
      T_k = k * epsilon
      sp_curr = merton_survival_prob(V0, H, r, sigma_V, T_k)
      if np.isnan(sp_curr): return np.nan
      marginal_pd = max(0.0, sp_prev - sp_curr)
      df = discount_factor(r, T_k)
      if np.isnan(df): return np.nan
      protection_leg_pv += LGD * df * marginal_pd
      sp_prev = sp_curr
  return protection_leg_pv
```

```
def calculate_fair_cds_spread(premium_leg_pv, protection_leg_pv):
    if np.isnan(premium_leg_pv) or np.isnan(protection_leg_pv): return np.nan
    spread = protection_leg_pv / premium_leg_pv
    if not np.isfinite(spread): return np.inf
    return spread
```

Calculate the default barrier:

```
def calculate_H(short_term_debt,long_term_debt):
    if short_term_debt < 0 or long_term_debt < 0:
        raise ValueError("Values should not be negative")
    return short_term_debt + 0.5 * long_term_debt
```

Code used to estimate the value of assets and their sigma, as explained in Chapter 3.2:

```
def black_scholes_merton_components(VA, sigma_VA, H, T, r):
    if VA <= 0 or sigma_VA <= 0 or T <= 0 or H <= 0:
        return -np.inf, -np.inf
    sigma_sqrt_T = sigma_VA * np.sqrt(T)
    if sigma_sqrt_T < 1e-10:
        if VA > H * np.exp(-r * T):
            d1 = np.inf
            d2 = np.inf
            Nd1 = 1.0
            Nd2 = 1.0
            E_model = VA - H * np.exp(-r * T)
            N_d1_sigma_VA_VA = sigma_VA * VA
        else:
            d1 = -np.inf
            d2 = -np.inf
            Nd1 = 0.0
            Nd2 = 0.0
            E_model = 0.0
            N_d1_sigma_VA_VA = 0.0
    else:
        d1 = (np.log(VA / H) + (r + 0.5 * sigma_VA**2) * T) / sigma_sqrt_T
        d2 = d1 - sigma_sqrt_T
        Nd1 = norm.cdf(d1)
        Nd2 = norm.cdf(d2)
        E_model = VA * Nd1 - H * np.exp(-r * T) * Nd2
        N_d1_sigma_VA_VA = Nd1 * sigma_VA * VA
    return E_model, N_d1_sigma_VA_VA
```

```python
def merton_system_equations(unknowns, E_observed, sigma_E_observed, H, T, r):
    VA, sigma_VA = unknowns
    if VA <= 0 or sigma_VA <= 0:
        return [1e10, 1e10]
    try:
        E_model, N_d1_sigma_VA_VA = black_scholes_merton_components(VA, sigma_VA, H, T, r)
        error1 = E_model - E_observed
        error2 = N_d1_sigma_VA_VA - sigma_E_observed * E_observed
        if not (np.isfinite(error1) and np.isfinite(error2)):
            return [1e10, 1e10]
        return [error1, error2]
    except (ValueError, OverflowError, ZeroDivisionError):
        return [1e10, 1e10]
```

```python
def estimate_merton_parameters(E_observed, sigma_E_observed, short_term_debt, long_term_debt, T, r):
    if E_observed <= 0 or sigma_E_observed <= 0 or T <= 0:
        print("Error: E_observed, sigma_E_observed and T must be positive")
        return np.nan, np.nan
    try:
        H = calculate_H(short_term_debt, long_term_debt)
        if H <= 0:
            print("Error: H must be positive.")
            return np.nan, np.nan
    except ValueError as e:
        print(f"Error: in H: {e}")
        return np.nan, np.nan
    VA_guess = E_observed + H
    sigma_VA_guess = sigma_E_observed * (E_observed / VA_guess)
    VA_guess = max(VA_guess, 1e-6)
    sigma_VA_guess = max(sigma_VA_guess, 1e-6)
    initial_guesses = [VA_guess, sigma_VA_guess]
    sol = root(merton_system_equations, initial_guesses,
               args=(E_observed, sigma_E_observed, H, T, r),
               method='lm')
    if sol.success:
        VA_estimated, sigma_VA_estimated = sol.x
        if VA_estimated > 0 and sigma_VA_estimated > 0:
            final_residuals = merton_system_equations(sol.x, E_observed, sigma_E_observed, H, T, r)
            if np.allclose(final_residuals, [0, 0], atol=1e-5):
                return VA_estimated, sigma_VA_estimated
            else:
                print(f"Solver converged, final residuals close to zero: {final_residuals}.")
                return VA_estimated, sigma_VA_estimated
        else:
            print(f"Solver converged to negative values: VA={VA_estimated}, sigma_VA={sigma_VA_estimated}")
            return np.nan, np.nan
    else:
        print(f"Solver did not converge: {sol.message}")
        return np.nan, np.nan
```

Application of all functions defined above, having as input the values calculated before with the other functions:

```
equity_market_value =  #observed value
equity_volatility = #observed value
short_term_debt = #observed value
long_term_debt =  #observed value
T = #2.0; 3.0; 5.0
r = # 0.045; 0.0427; 0.0438
R = 0.4
M = 4
```

CDS spread for Merton method:

```
VA_est, sigma_VA_est = estimate_merton_parameters(
    E_observed=equity_market_value,
    sigma_E_observed=equity_volatility,
    short_term_debt=short_term_debt,
    long_term_debt=long_term_debt,
    T=T,
    r=r
)

print("Estimated Asset Value (VA):", VA_est)
print("Estimated Asset Volatility (sigma_VA):", sigma_VA_est)

H = calculate_H(short_term_debt, long_term_debt)
PD = merton_pd(VA_est, H, r, sigma_VA_est, T)
print(F)
print(PD)

t = 0.0
T_cds = T
tn = np.arange(t + 1/M, T_cds + 1e-6, 1/M)

premium_leg = calculate_premium_leg_pv_M(t, tn, T_cds, VA_est, H, r, sigma_VA_est)
protection_leg = calculate_protection_leg_pv_M(t, T_cds, R, M, VA_est, H, r, sigma_VA_est)
print(premium_leg )
print(protection_leg )
cds_spread = protection_leg / premium_leg *10_000
```

CDS spread for First-Passage method:

```
VA_est, sigma_VA_est = estimate_merton_parameters(
    E_observed=equity_market_value,
    sigma_E_observed=equity_volatility,
    short_term_debt=short_term_debt,
    long_term_debt=long_term_debt,
    T=T,
    r=r
)

H = calculate_H(short_term_debt, long_term_debt)
PD_F = fpt_pd(VA_est, H, r, sigma_VA_est,T)

print(H)
print(PD_F)

t = 0.0
T_cds = T
tn = np.arange(t + 1/M, T_cds + 1e-6, 1/M)

premium_leg_F = calculate_premium_leg_pv_FPT(t, tn, T_cds,VA_est,H, r, sigma_VA_est)
protection_leg_F = calculate_protection_leg_pv_FPT(t, T_cds, R, M, VA_est, H, r, sigma_VA_est)

print(premium_leg_F )
print(protection_leg_F )
cds_spread_F = protection_leg_F / premium_leg_F * 10_000
```