LISBOA SCHOOL OF ECONOMICS & MANAGEMENT

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DISSERTATION

Robust Mean Variance

João Nuno Martins Cardoso

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João Nuno Martins Cardoso

Supervisor:

Raquel Medeiros Gaspar

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Abstract

This empirical study's objective is to evaluate the impact of robust estimation on mean variance portfolios. This was accomplished by doing a simulation on the behavior of 15 SP500 stocks. This simulation includes two scenarios: One with normally distributed samples and another with contaminated non-normal samples. Each scenario includes 200 resamples. The performance of maximum likelihood (classical) estimated portfolios and robustly estimated portfolios are compared, resulting in some conclusions: On normally distributed samples, robust portfolios are marginally less efficient than classical portfolios. However, on non-normal samples, robust portfolios present a much higher performance than classical portfolios. This increase in performance is positively correlated with the level of contamination present on the sample. In summary, assuming that financial returns do not present a normal distribution, we can state that robust estimators result in more stable mean variance portfolios.

Resumo

Este estudo empírico tem como objectivo avaliar o impacto da estimação robusta nos portefólios de média variância. Isto foi conseguido fazendo uma simulação do comportamento de 15 acções do SP500. Esta simulação inclui dois cenários: um com amostras que seguem uma distribuição normal e outro com amostras contaminadas não normais. Cada cenário inclui 200 reamostragens. O performance dos portefólios estimados usando a máxima verosimilhança (clássicos) e dos portefólios estimados de forma robusta são comparados, resultando em algumas conclusões: Em amostras normais, portefólios robustos são marginalmente menos eficientes que os portefólios clássicos. Contudo, em amostras não normais, os portefólios robustos apresentam um performance muito superior que os portefólios clássicos. Este acréscimo de performance está positivamente correlacionado com o nível de contaminação da amostra. Em suma, assumindo que os retornos financeiros têm uma distribuição não normal, podemos afirmar que os estimadores robustos resultam em portefólios de média variância mais estáveis.

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1. Introduction

The purpose of this paper is to study the effect of estimation risk on the stability of portfolios resulting from Mean Variance Theory (MVT), developed by the work of Markowitz (1952, 1959). It is focused on problems on practical applications of MVT, contrasting with the fact that in theory it produces optimal results and is the backbone of modern portfolio theory. Nonetheless, its translation into real life is not perfect, as the composition of the optimal portfolio[1] is unstable over time and too sensitive to ill estimations of parameters, resulting in it being unconsidered by many portfolio managers, topic that was explored by Michaud (1989). How can a model like this be considered the answer for portfolio allocation problems and then result in less than efficient portfolios on its practical applications? See Jobson and Korkie (1981).

To answer this question, first we should start by looking in to MVT and understand how it works. This theory takes a basket of n securities and allocates them in order to maximize expected returns for each possible risk level, on a certain investment period. Resulting in a curve in risk/expected return plan, representing the set of efficient portfolios, called *efficient frontier*. One of the advantages of MVT is its simplicity, mainly because it relies on two inputs: μ, a vector composed with n securities' expected returns and $\sum$, a $n*n$ variance-covariance matrix of the respective expected returns. In MVT the optimization problem is treated as a deterministic problem, i.e. it is assumed that both μ and $\sum$ which are be applied to the investment period are known with certainty. In real life, of course, these two inputs have to be estimated, and so they are subject to *estimation error*. The assumption that one knows for sure the future expected returns and future variance-covariance of expected returns of the different assets is difficult to stand. In practice, investors have to forecast those inputs, which inevitably results in estimation error. Estimation risk is therefore the main source of risk in MVT. As Fabozzi et al (2014) defended, the irrational relation between inputs and outputs and the great sensitivity of the portfolio allocation to changes in inputs are the main reasons for MVT not being applied by the majority of portfolio managers in practice.

1- All mentions to optimal portfolio refer to the tangent portfolio of the efficient frontier.

On this study, several questions have to be answered. First, why does MVT produce sub-optimal portfolios in practice? How can robust estimators improve the stability of the resulting portfolios? How robust estimators perform in comparison with classical Maximum Likelihood estimators under different scenarios? And finally, how different degrees of parameter deviation affect the stability of the final output of MVT?

We start by positioning this study with respect to what has already been done on this subject on chapter 2, showing several studies about MVT risk mitigation, optimization of the estimation process and portfolio optimization procedures. Additionally, in this chapter we demonstrate how this work differs from those past studies. Then on chapters 3 we present the methodology used. On chapter 4 a simulation study of 15 Standard and Poor's stocks is presented, where the performance of classical Maximum Likelihood estimators and robust estimators is compared. On chapter 5 some conclusions are drawn from the results of the simulation study.

2. Literature Review

Mean Variance Theory developed by Markowitz (1952, 1959) has been the basis of modern portfolio theory. Before him, the focus of investment selection was on evaluating securities individually in order to decide which securities yielded the higher returns. Not taking in account the risk associated with those securities. After Markowitz the focus changed to diversification and the impact of each security on a portfolio's total risk-return.

This literature review starts by taking a look at the main breakthroughs that MVT allowed, as well as the limitations of this theory on section 2.1. Subsequently, on section 2.2, are presented the major developments related with estimation risk, which is a very important topic through this paper. On the next section (2.3), are presented some important improvements and alternatives to classical MVT. To conclude, section 2.4 presents a review on robust statistics.

2.1 MVT Applications & Shortcomings

Since its creation MVT has been widely used by researchers and practitioners. In the academia it laid the ground for several equilibrium models, like CAPM, produced through the work of Lintner (1965), Mossin (1966) and Sharpe (1966). It was also important in defining and understanding the difference between systematic and diversifiable risk. For practitioners it is important in many portfolio management applications. However, despite the simplicity of the model created by Markowitz, the powerful optimization theory supporting it and the existence of software able to easily solve the optimization problem, it is still disregarded in practice due to its poor ability to adapt to estimated inputs and due to the sensibility of optimal portfolios to small changes in inputs, see Michaud (1989). If we divide MVT in two parts, first we have the process of computing the inputs and secondly the optimization of those same inputs in to portfolios. The problem is that in the first part of the theory we can't reach exact results, whereas on the second part, the process is flawless. Though, the second part only results in solid portfolio allocations if the first part produced exact inputs, which most of the times is not possible.

MVT is based on the assumption that the vector of expected returns (μ) and variance-covariance matrix (Σ) are sufficient to create the set of efficient portfolios. However this preposition does not consider an important factor. It implies that μ and Σ are known, when in practice these parameters have to be estimated with base on historical prices of those same securities. Therefore, the actual inputs of MVT are $\hat{\mu}$ and $\hat{\Sigma}$, which have some associated estimation error. This is a shortcoming of MVT, which does not take in account the fact that inputs were estimated, and include estimation error, treating them as the unbiased parameters. This is commonly referred to as "certainty equivalence", as those estimated parameters are considered equivalent to the known parameters through the whole optimization process.

2.2 Estimation Risk

Estimation risk is a fundamental factor for the inefficiency of Mean Variance efficient portfolios. Merton (1980) has shown that expected returns are extremely sensitive to estimation error, variance is slightly sensitive and covariance is the less sensitive of the three. We also know that even small changes in input parameters may result in large changes in the optimal portfolio composition, Chopra (1993). Additionally, Chopra and Ziemba (1993) concluded that "errors in means are approximately 10 times as important as errors in variances and covariances considered together". So, considering that investors have limited resources, they should focus their resources on finding the best possible estimates of expected returns, as this parameter is the one which most influences the quality of the estimated tangent portfolio.

Jobson and Korkie (1981) made a simulation study where they estimated the expected return and risk of 20 securities from a simulated dataset of 60 monthly returns. The estimated tangent portfolio had a Sharpe ratio of 0.08, compared with the Sharpe ratio of 0.34 of the actual tangent portfolio, even less than the Sharpe of 0.27 of an equally weighted portfolio of those same 20 securities, concluding that MV efficient portfolios can even be dominated by naive portfolios of the same assets due to the effect of estimation error.

Taking a different perspective, Michaud (1989) introduced the concept of “estimation error maximizers”, defending that MVT behaves as a maximizer of estimation risk. Securities that have its expected return overestimated (underestimated), its standard deviation underestimated (overestimated) or its covariance underestimated (overestimated) will be overweighted (underweighted) on the resulting portfolio. This acts as proof of the ill effect of *estimation risk* on the output of the optimal portfolio composition.

Best and Grauer (1991) made a computational simulation on the behavior of portfolios of different sizes and reached the conclusion that when changing the expected return of one asset, the composition, expected return and risk of the optimal portfolio changed drastically. Although when a non-negativity constraint was present the portfolio composition was still very sensitive to changes in one security expected return, but the expected return and risk of the portfolio tended to remain almost constant.

What all of those studies show is that estimation risk has a big impact on portfolio allocation problems. More than changing the portfolio composition, it may also affect deeply the portfolio’s expected return and risk. Hence, estimation risk has to be mitigated as much as possible, something that researchers have been doing but is still a work in progress, like we can see on the next section.

2.3 Mean Variance Theory extensions

The inability of MVT to cope with estimated parameters and the fact that it did not acknowledge the existence of estimation error on its inputs has been criticized by several authors like Barry (1974), Bawa and Klein (1979), Brown (1976).

According to Stein (1956) for an estimator to be admissible it must dominate all other estimators on a given risk or loss function. The same author has proven the inadmissibility of sample means as estimators of expected returns. Based on this development Jobson et al(1979) proposed a James-Stein estimator to improve the efficiency of MVT, this estimator was put to the test on Jobson and Korkie (1981) and consists on a Bayesian estimator which shrinks the individual means of each security to a global mean, in a way that, the greater the volatility of each security, the greater the shrinkage, Jorion (1986) proposed a similar estimator, but shrinking the expected return estimates towards the minimum variance portfolio.

Black Litterman (1991) introduced a market based shrinkage approach which combines equilibrium expected returns based on the weighted average market return from CAPM with investor's views on each asset or group of assets and with the investor's degree of confidence on each forecast. An important assumption of this model is that the expected return of a security should be consistent with the market equilibrium unless the investor has a particular view on that security. Therefore, an investor without any views (unconstrained) should hold the market portfolio. Another possible way to correct the ill effect of estimation error is to increase the risk aversion parameter. That is exactly what Horst, Roon and Werker (2002) did, developing a risk aversion correction to take estimation risk into account when estimating mean variance tangent portfolios, this new risk aversion is higher than the standard one and the increase depends on the sample size, number of securities on the portfolio and on the curvature of the mean variance frontier. This way, considering that the estimated efficient frontier is an overestimation of the *true efficient frontier* due to the error maximization effect, increasing the risk aversion parameter nullifies some of that overestimation.

On 1998 Michaud presented a resampling methodology to mitigate estimation risk on a mean variance framework. Through statistical resampling, several risk/return estimates are created, from those estimates, tangent portfolios are created and by averaging the weights of those tangent portfolios an investor can reach a solid estimate of an optimal portfolio.

On the late 90's, a method that considers estimation error as part of the optimization process was developed. This method is called robust optimization, and was first introduced by El Ghaoui and Lebret (1997) and Ben-Tal and Nemirovski (1998). It consists on finding a solution that is satisfactory to most possible realizations of the uncertain parameters (μ, Σ). From this point, uncertainty sets are created, containing all or at least most of the possible realizations of the uncertain parameters, the greater the set, the more uncertain is the parameter (higher chance of estimation error) and more robust is the solution. The problem is then solved by finding the portfolio that maximizes utility on the worst-case scenario, from the possible values of the uncertainty set. See for instance: Lobo and Boyd (2000) and Goldfarb and Iyengar (2003). Even using any of these enhancements of MVT, and due to the stochastic behavior of the asset return process, estimation error will always exist, even if it is lower than the vanilla MVT.

2.4 Robust Statistics

What would be preferred by an investor: (i) make a solid estimation of expected returns and covariances of a set of securities or (ii) make an estimation that considers the infinitesimal probability of a sudden market crisis but isn't as reliable as the first one on a non-crisis scenario? Should investors give the same importance to extreme returns as they do to more common returns? This results in a problem of statistical robustness. Robust statistics, developed by Huber (1964) and Hampel (1968), consists on procedures to deal with situations where the underlying distribution deviates slightly from the assumed model, see Huber and Ronchetti (1981). On MVT, extreme observations with a very few chance of happening have a great effect on the estimated input parameters.

It is then important to clarify two concepts: efficiency and breakdown of estimators. Efficiency is the capability of an estimator to yield close to optimal estimates, therefore to minimize estimation errors. Breakdown is the amount of outlying observations that an estimator can withstand before it produces unstable results, see Huber and Ronchetti (1981). Usually, the more efficient an estimator is, the lower is its breakdown point, and vice versa. For example, if we compare the classical mean and median, we can see that the mean is a much more efficient estimator than the median, as it tends to produce closer to optimal estimates. However, if we multiply the highest observation on a given sample by one million, the mean of that sample would be heavily biased by it, whereas its median would remain the same. Consequently robust estimators, the ones with a considerable higher breakdown point have an implied loss of efficiency associated with gains in stability, Huber and Ronchetti (1981).

An estimator is considered robust if it can withstand extreme observations and still yield stable estimations. Classical estimators are known to result in dire results on the presence of those observations as its breakdown point is or is close to zero. One way to assess if an estimator is robust is to study its Influence Function. This function resulted from the work of Hampel (1968, 1974) as it "allows us to assess the relative influence of individual observations towards the value of an estimate or test statistic" and therefore "allows an immediate and simple heuristic assessment of the asymptotic properties of an estimate" Huber and Ronchetti (1981). Furthermore, Hampel (1986) has stated that the only necessary condition for an estimator to be robust is for it to have a bounded influence function, consequently the only condition necessary for MVT to be robust is to have its parameters estimated through robust estimators with bounded influence functions.

On this study we see how estimation error and the inclusion of "noise" on the sample, affect the quality of the estimated parameters. Studying the effect of different magnitudes and frequencies of sample contamination and how robust estimators perform in comparison to classical estimators on all those different scenarios. To do this, we use a resampling method similar to the one developed by Michaud (1998). Though, on our case, two different estimates are applied to each sample, one classical and one robust and different kind of samples are used. In addition, we are able to see the effect of "estimation error maximizers" on practice.

3. Methodology

In order to compare the performance of tangent portfolios resulting from classically and robustly estimated efficient frontiers, we simulate the returns of a set of stocks through a resampling process. This is repeated to several different scenarios. On each scenario, both types of estimations are applied to all samples, resulting in a number of different estimated parameters. From there, MVT is applied on each different scenario, for both estimation types and for all resamples, resulting in several estimated tangent portfolios which are the main tools to study the advantages and disadvantages of each estimator.

On this chapter this methodology is explained in detail. On section 3.1 we start by presenting the data which served as base for this empirical study. Section 3.2 is where the simulation methodology is presented, being then divided in subsection 3.2.1 and 3.2.2. On subsection 3.2.1 the control scenario, the one which does not contain any contamination, is explained. In opposition, subsection 3.2.2 describes the different contamination scenarios, and how the contamination process occurred. Section 3.3 shows in more detail the robust estimator which is used on this study. And finally section 3.4 demonstrates how the portfolio creation process was developed.

3.1 Data

A set of 15 Standard & Poor's stocks are the objects of the simulation, all of them having been in the index for more than 20 years. This set of stocks encompasses companies from different industries and sizes.

Table I – Stock description

Number	Name	Industry	Market Capitalization
1	AT&T, Inc.	Telecommunications	196.89B
2	The Boeing Company	Aircraft	92.62B
3	Colgate-Palmolive Co.	Consumer goods	56.2B
4	Duke Energy Corporation	Energy	48.78B
5	Ford Motor Co.	Automaker	52.24B
6	DR Horton Inc.	Construction	10.78B
7	Intel Corporation	Technology	137.9B
8	Johnson & Johnson	Pharmaceutical	261.4B
9	Oracle Corporation	Technology	153.14B
10	The Procter & Gamble Company	Consumer goods	195.02B
11	Verizon Communications Inc.	Telecommunications	178.57B
12	Coca-Cola Enterprises Inc.	Beverages	10.7B
13	The Gap, Inc.	Clothing	12.33B
14	The Hershey Company	Chocolate manufacturing	20.12B
15	Mattel, Inc.	Toy manufacturing	7.35B

Profile of 15 SP500 companies which are part of the simulation. Market capitalization in billion dollars, at the date of 28/09/2015

Since we want to mitigate idiosyncratic risk as much as possible, 15 stocks should be enough to generate diversified portfolios. See, for instance, the comments about a diversified portfolio of Reilly (1985): *"In terms of overdiversification, several studies have shown that it is possible to derive most of the benefits of diversification with a portfolio consisting of from 12 to 18 stocks. To be adequately diversified does not require 200 stocks in a portfolio" pp.(213)*.

Monthly observations from April 1995 to March 2015 were extracted for each stock. To minimize estimation error as much as possible, a sample of 240 monthly returns was used for each stock, given that according to Jobson and Korkie (1981) 60 to 100 monthly returns are not enough to eliminate estimation error. The authors also note that a sample of at least 200 monthly returns would be needed in order to minimize estimation error to a point that it would not bias our estimated tangent portfolio. Through classical maximum likelihood estimation of those monthly returns, the expected returns and variance-covariance matrix of each security was extracted.

From this point onwards, we focus our study only on the expected returns and variance-covariance matrix estimated from the initial data. Creating a vector μ composed with the annual return of each of the 15 stocks and a 15x15 matrix Σ composed with the covariances between those same returns. These values are going to be the inputs of the simulation process. Additionally, we assume μ and Σ to be next year's realized return and variance-covariance for each stock. It is important to understand that μ and Σ are related to next year as the investment horizon of the developed portfolios will be equally of one year.

This data serves as benchmark through the whole study, being more than once compared with the simulated data. Having this in to account, these parameters are referred to as *true expected returns* (μ) and *true covariance* (Σ). Anything else referred to as *true* is assumed to be related with these same parameters. Below we can see the *true parameters*:

$$\mu = \begin{bmatrix} 2.15 & 9.94 & 10.62 & 4.11 & 2.90 & 14.16 & 8.97 & 9.44 & 14.40 & 7.96 & 3.65 & 9.05 & 11.04 & 10.40 & 1.36 \end{bmatrix} \quad (1)$$

$$\Sigma = \begin{bmatrix}
0.0581 & 0.0013 & 0.0007 & 0.0017 & 0.0030 & 0.0025 & 0.0016 & 0.0012 & 0.0015 & 0.0007 & 0.0033 & 0.0008 & 0.0018 & 0.0009 & 0.0018 \\
0.0013 & 0.0731 & 0.0012 & 0.0006 & 0.0068 & 0.0047 & 0.0029 & 0.0012 & 0.0021 & 0.0009 & 0.0014 & 0.0037 & 0.0035 & 0.0009 & 0.0031 \\
0.0007 & 0.0012 & 0.0400 & 0.0013 & 0.0021 & 0.0014 & 0.0008 & 0.0015 & 0.0007 & 0.0021 & 0.0009 & 0.0027 & 0.0015 & 0.0008 & 0.0020 \\
0.0017 & 0.0006 & 0.0013 & 0.0555 & 0.0021 & 0.0018 & 0.0008 & 0.0017 & 0.0006 & 0.0015 & 0.0014 & 0.0016 & 0.0008 & 0.0013 & 0.0010 \\
0.0030 & 0.0068 & 0.0021 & 0.0021 & 0.2344 & 0.0089 & 0.0052 & 0.0019 & 0.0034 & 0.0020 & 0.0031 & 0.0064 & 0.0073 & 0.0012 & 0.0045 \\
0.0025 & 0.0047 & 0.0014 & 0.0018 & 0.0089 & 0.2208 & 0.0046 & 0.0016 & 0.0034 & 0.0010 & 0.0021 & 0.0054 & 0.0058 & 0.0019 & 0.0043 \\
0.0016 & 0.0029 & 0.0008 & 0.0008 & 0.0052 & 0.0046 & 0.1418 & 0.0011 & 0.0067 & -0.0001 & 0.0013 & 0.0021 & 0.0048 & 0.0005 & 0.0025 \\
0.0012 & 0.0012 & 0.0015 & 0.0017 & 0.0019 & 0.0016 & 0.0011 & 0.0339 & 0.0007 & 0.0016 & 0.0012 & 0.0019 & 0.0017 & 0.0010 & 0.0017 \\
0.0015 & 0.0021 & 0.0007 & 0.0006 & 0.0034 & 0.0034 & 0.0067 & 0.0007 & 0.1751 & -0.0007 & 0.0022 & 0.0004 & 0.0037 & -0.0003 & 0.0014 \\
0.0007 & 0.0009 & 0.0021 & 0.0015 & 0.0020 & 0.0010 & -0.0001 & 0.0016 & -0.0007 & 0.0512 & 0.0006 & 0.0026 & 0.0012 & 0.0005 & 0.0017 \\
0.0033 & 0.0014 & 0.0009 & 0.0014 & 0.0031 & 0.0021 & 0.0013 & 0.0012 & 0.0022 & 0.0006 & 0.0489 & 0.0013 & 0.0017 & 0.0006 & 0.0017 \\
0.0008 & 0.0037 & 0.0027 & 0.0016 & 0.0064 & 0.0054 & 0.0021 & 0.0019 & 0.0004 & 0.0026 & 0.0013 & 0.1334 & 0.0043 & 0.0015 & 0.0034 \\
0.0018 & 0.0035 & 0.0015 & 0.0008 & 0.0073 & 0.0058 & 0.0048 & 0.0017 & 0.0037 & 0.0012 & 0.0017 & 0.0043 & 0.1566 & 0.0007 & 0.0038 \\
0.0009 & 0.0009 & 0.0008 & 0.0013 & 0.0012 & 0.0019 & 0.0005 & 0.0010 & -0.0003 & 0.0005 & 0.0006 & 0.0015 & 0.0007 & 0.0417 & 0.0014 \\
0.0018 & 0.0031 & 0.0020 & 0.0010 & 0.0045 & 0.0043 & 0.0025 & 0.0017 & 0.0014 & 0.0017 & 0.0017 & 0.0034 & 0.0038 & 0.0014 & 0.1058
\end{bmatrix} \tag{2}$$

3.2 Monte Carlo Simulation

We use Monte Carlo simulation to set 200 resamples of 10 years of daily correlated returns for the 15 securities. This process is repeated on all different scenarios. This simulation method consists on a series of repeated random sampling developed by Metropolis (1949). We assume each stock's returns to follow a geometric Brownian motion, and therefore, this method results in samples from a multivariate normal distribution (μ, Σ). Those samples follow a trend μ and are influenced by the multivariate effect of Σ. In our case this repeated sampling creates several datasets of hypothetical historical returns. Each resample is used to estimate its own $\hat{\mu}_i$ and $\hat{\Sigma}_i$ for $i = 1, \ldots, 200$.

3.2.1 Control scenario

The first scenario is treated as the control scenario. In this case all samples were based on a geometric Brownian motion which follows a multivariate normal distribution. This results in samples of normally distributed returns on all 15 securities. On Table II we can see the descriptive statistics of each stock:

Table II – Control Scenario descriptive statistics

	Mean annual return	Annualized standard deviation	Skewness	Kurtosis	Normality test
1	-0.61%	24.07%	$7.42e^{-4}$	3.0085	Normal
2	6.02%	27.09%	$1.94e^{-4}$	3.0206	Non-normal
3	8.76%	20.01%	0.0065	3.0001	Normal
4	1.52%	23.54%	-0.0024	3.0097	Normal
5	-8.82%	48.42%	0.0011	3.0063	Normal
6	3.60%	46.98%	-0.0040	3.0021	Normal
7	1.01%	37.69%	$1.66e^{-4}$	3.0034	Normal
8	7.57%	18.40%	-0.0014	3.0017	Normal
9	5.95%	41.88%	0.0022	3.0059	Normal
10	5.67%	22.63%	0.0048	2.9935	Normal
11	1.40%	22.10%	0.0019	2.9974	Normal
12	2.25%	36.57%	0.0060	3.0128	Non-normal
13	3.17%	39.54%	$5.31e^{-4}$	3.0029	Normal
14	8.48%	20.42%	0.0105	2.9967	Non-normal
15	-3.42%	32.53%	0.0029	2.9993	Normal

Descriptive statistics of each stock on the control scenario. Both the return and standard deviation are annualizations of mean daily return and standard deviation, respectively. The normality was testes through a Jarque-Bera test with 95% confidence interval.

As we can observe the majority of the stocks follow a normal distribution. Even the ones which fail the normality test have skewness very close to 0 and kurtosis close to 3 which demonstrates that they behave in many ways like a normally distributed sample. Consequently, this first simulation does not contain many outlier observations. Therefore the main cause of estimation error are the sample size and the high standard deviation of stock returns, which by themselves can be enough to cause portfolio allocation inefficiencies. This control scenario allows, in particular, to see if without extreme observations to bias the inputs of MVT, robust estimators are less efficient than classical estimators, and if so, how significantly. It is also important to note that the estimated parameters $\hat{\mu}$ and $\hat{\Sigma}$ are going to be compared with the realized risk/return of all securities on the following year, the *true parameters* μ and Σ.

3.2.2 Contamination scenario

On the contamination scenario, various sets of 200 resamples are produced. However, in this case the Monte Carlo simulation does not follow a geometric Brownian motion. It includes a multiplicative contamination, which consists in randomly multiplying a certain percentage of the standard normally distributed matrix that is on the base of the geometric Brownian motion by a certain value, as proposed by Perret-Gentil and Victoria-Feser (2005). Through this method, it is possible to test the stability of the Mean Variance portfolios against outlier observations and small parameter fluctuations. It artificially creates extreme observations typical of financial returns and adding “noise” to the sample in order for the estimated parameters to fluctuate. This way, we are able to test if a small degree of deviation on the estimated parameters seriously affects the composition and performance of the tangent portfolios, as well as testing how both types of estimators behave on the presence of extreme observations.

Nine different contamination scenarios are used on this study, varying in percentage of sample contamination and in magnitude of contamination. The percentages of contamination are 2.5%, 5% and 10% and the magnitude of contamination, that is, the value by which each contaminated return is multiplied, is 2.5, 5 and 10. Each percentage of contamination is tested with each magnitude, allowing for nine different contamination degrees. On Table III we present the descriptive statistics for the scenario with 5% of contaminated observation with a magnitude of 5:

Table III – Contamination Scenario descriptive statistics

	Mean annual return	Annualized standard deviation	Skewness	Kurtosis	Normality test[3]
1	-2.04%	35.66%	0.0387	19.7412	Non-normal
2	6.26%	40.13%	0.0268	19.9605	Non-normal
3	8.49%	29.67%	-0.0091	19.8181	Non-normal
4	0.41%	34.82%	-0.0264	19.7890	Non-normal
5	-9.09%	71.81%	-0.0440	20.0844	Non-normal
6	1.44%	69.59%	-0.0311	20.2157	Non-normal
7	2.5%	55.60%	-0.0398	19.7314	Non-normal
8	7.48%	27.33%	0.0029	19.7275	Non-normal
9	7.5%	61.98%	-0.0090	20.0146	Non-normal
10	4.52%	33.53%	0.0211	19.7494	Non-normal
11	0.13%	32.83%	-0.0102	19.8257	Non-normal
12	1.91%	54.21%	0.0131	20.1279	Non-normal
13	3.39%	58.78%	0.0346	20.1905	Non-normal
14	8.51%	30.25%	0.0615	19.6054	Non-normal
15	-3.88%	48.17%	0.0345	20.0864	Non-normal

Descriptive statistics of each stock on the control scenario. Both the return and standard deviation are annualizations of mean daily return and standard deviation, respectively. The normality was testes through a Jarque-Bera test with 95% confidence interval.

On opposition with the control scenario, this scenario is composed by 15 non-normally distributed securities. All stocks present fat tails, as we can see by the high kurtosis. That means that in this scenario there are many more outlier observations which may have a great impact on the estimated parameters.

3.3 Robust estimator

On all scenarios, after creating the 200 resamples, there is a need to estimate the inputs of MVT(μ, Σ). As already explained, two estimations are applied on each sample. A classical estimation based on Maximum Likelihood estimators and a robust estimation. The robust estimator chosen is Minimum Covariance Determinant, this is a multivariate M-estimator which looks into the $h\left(>\frac{n}{2}\right)$ observations (out of α) whose classical covariance matrix has the lowest possible determinant, see Rousseeuw (1984). The location estimate is then computed by taking the average of those h points and the scatter estimate is the covariance matrix of those h points. These estimates can resist $(n-h)$ outliers and so, we can assess the robustness of the estimator by looking into $\alpha=\frac{h}{n}$. In extreme cases, in order to achieve the highest resistance against outliers an α of 0.5 could be used. However, in less extreme scenarios, α should be set higher than 0.5 in order to improve finite-sample efficiency. In our case, α was set to 0.75 on all estimations. That means that the estimations produce stable results if the sample contains up to 25% outliers. Moreover, we have finite samples and so, in order to achieve a good enough efficiency, an α higher than 0.5 would be needed.

3.4 Portfolio creation process

On this study we work with three different types of portfolios: Tangent portfolios, average portfolios and, in a lesser extent, naive portfolios. All those portfolios are used to compare the performance of both estimation styles.

First and foremost, it is important to mention that on each scenario, we have two sets of portfolios, a set of classically and another of robustly estimated portfolios. After computing $\hat{\mu}_i$ and $\hat{\Sigma}_i$ for $n=1,\ldots,200$ for each estimation type, MVT results in two groups of 200 efficient frontiers. An efficient frontier is a set of efficient portfolios resulting from MVT. However, we need to find the tangent portfolio for each efficient frontier. That portfolio is the most efficient portfolio of a given efficient frontier on the presence of a risk free rate. Having a defined efficient frontier and risk free rate are the necessary tools to reach the tangent portfolio. The risk-free rate used on this study is the 12-month Euribor rate at the date of 25/2/15, 0.24%. Euribor is the rate that is behind the majority of interest rates offered by financial institutions. It can be assumed risk-free, following the assumption that the European Central Bank has no risk of default. The maturity of 12-months is also the one that makes more sense since as we know the parameters μ and $\sum$ for the next year and the investment horizon of this simulation is one year. Other assumption of this study was the non-inclusion of short selling on the portfolio creation process. This is consistent with many other studies on MVT, with the majority of portfolios on the market and it makes the study simpler and easier to understand. See for example, Chopra and Ziemba (1993) and Goldfarb and Iyengar (2003).

The second portfolio type is the “average portfolio”. That portfolio is born from the average weights of all resamples, following the process developed by Michaud (98), like mentioned on section 2.3. This portfolio will then be compared with the so called naive portfolio, the portfolio which allocates equal weights for all possible assets.

All those portfolios may differ greatly in risk and return. That is why we need a performance measure to compare any of them with one another. The instrument used is the Sharpe Ratio. Developed by William Sharpe, it is a measure of excess return per unit of risk. It is commonly used in finance and in this study is the major instrument for comparing the efficiency of each portfolio.

4. Results

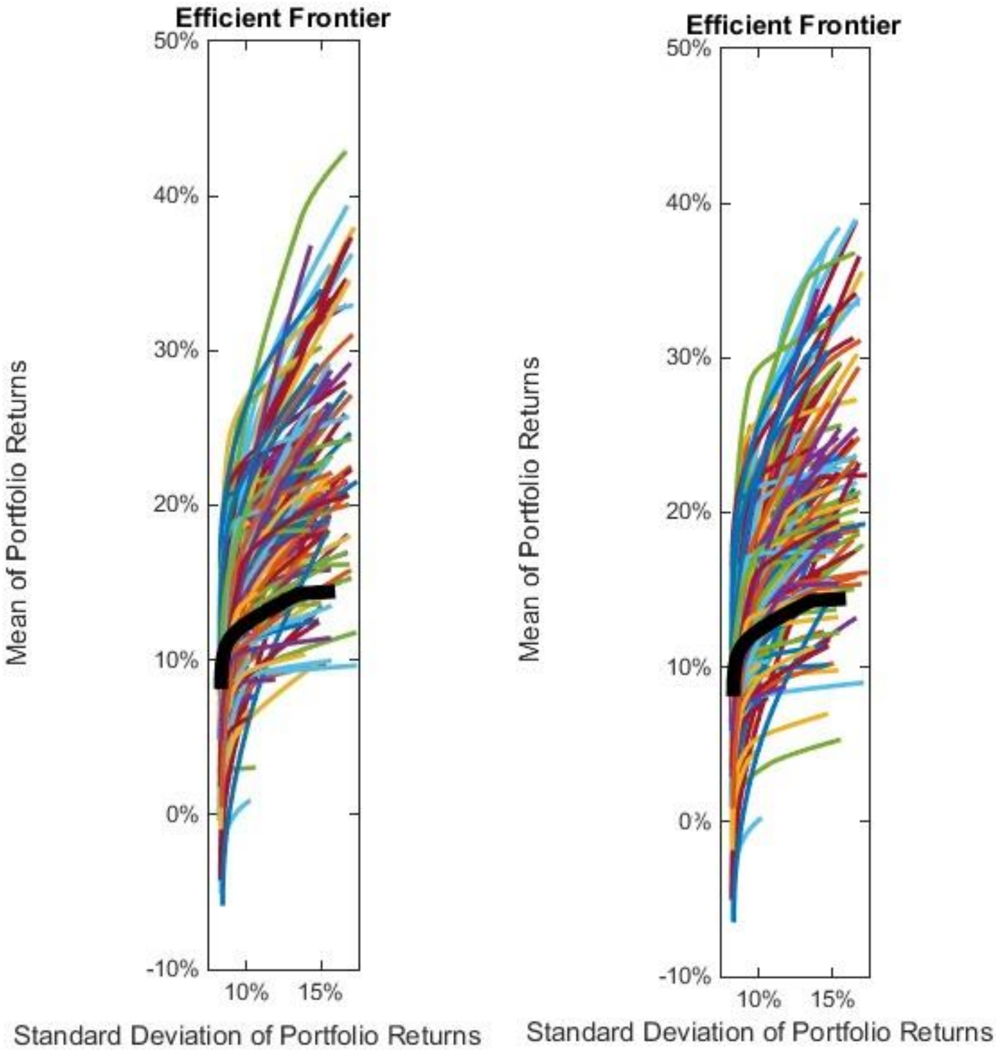
This chapter starts by presenting a broad view in to the control scenario on section 4.1. Subsequently, it presents the results for the different types of portfolios tested on that scenario, on subsection 4.1.1 we study the behavior of tangent portfolios and on subsection 4.1.2 average and naive portfolios. On section 4.2 the focus is on the contaminated scenario, with a major focus on the 5% contamination. Again, on subsection 4.2.1 we take a look at how tangent portfolios perform on contaminated samples and on subsection 4.2.2 we test how average and naive portfolios perform.

4.1 Control scenario

We build 200 classically estimated efficient frontiers and 200 robustly estimated efficient frontiers. The resulting efficient frontiers can be observed below, with the *true efficient frontier* represented by the thicker black curve:

Figure 1 – True Efficient Frontier and 200 estimated efficient frontiers

(a) classical (b) robust



Visual representation of estimated efficient frontiers and True efficient frontier, represented by the thicker black curve. On (a) 200 classically estimated efficient frontiers and on (b) 200 robustly estimated efficient frontiers

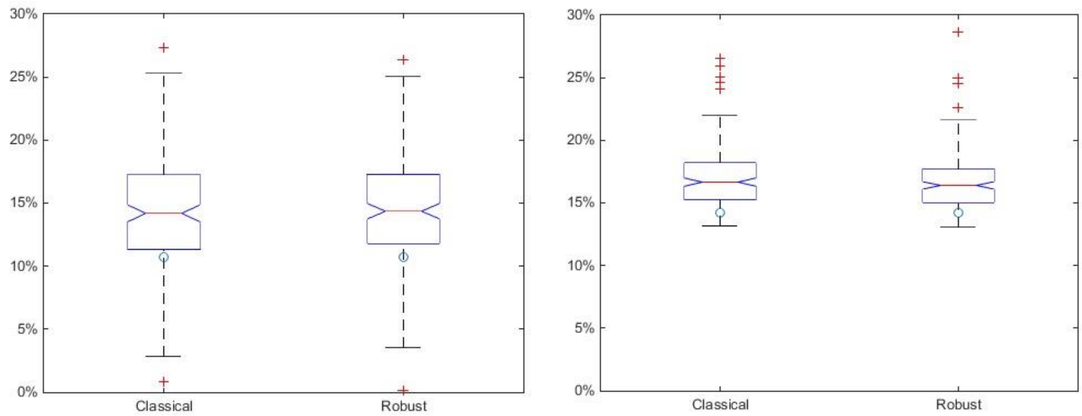
Figure 1 shows the great variety of estimated efficient frontiers that MVT created, both by classical and robust estimation. As we could observe on Table II of chapter 3, the resample process created a huge dispersion of returns on all securities. This, of course, led to a great dispersion of efficient frontiers. We can also notice that a great part of the frontiers are above the *true efficient frontier*, which may be an evidence of the "estimation error maximizer" effect. By overweighing securities with overestimated returns, there is a much greater chance to create efficient frontiers above the *true efficient frontier*.

4.1.1 Tangent Portfolios

The next step is to look at the composition of the classically estimated and robustly estimated tangent portfolios on each different sample. On Figure 2 we can see the distribution of the estimated tangent portfolios' risk and return, the red line represents the median, the point with the exact number of higher and lower observations, the edges of the box represent the

Figure 2 – Tangent Portfolios

(a) return (b) risk

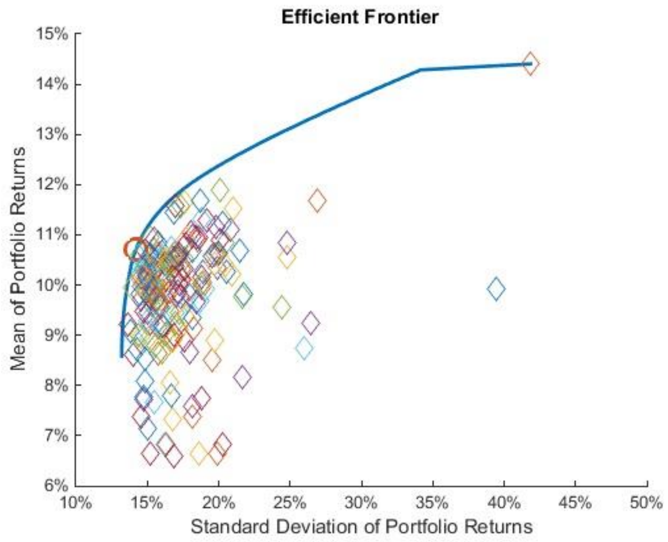


Dispersion of estimated tangent portfolios' return and risk, both for classical estimation and robust estimation. True optimal return and risk represented by the blue circles.

25% and 75% percentiles and the upper and lower black lines represent the highest and lowest observations, respectively. The red crosses represent observations considered outliers. Additionally, the *true tangent risk* and return are represented by blue circles. As we can see, classical and robust portfolios present roughly the same dispersion of observations, both on risk and return. Moreover, approximately 75% of estimated tangent returns are above the *true tangent return* of 10.71% and range from around 2% to almost 25%. The estimated tangent risk level is higher than the *true risk* level of 14.2% on around 90% of the observations, though it has a much smaller range of observations, going from 13% to 22%. The great dispersion of tangent returns is associated with the already discussed sampling variability associated with the high standard deviations of the 15 securities. The fact that estimated tangent returns happen to be much higher than the *true tangent return* is a reflection of the "estimation error maximizer" effect of MVT which causes MVT to allocate a greater part of the budget on assets with returns positively affected by estimation error and less on assets negatively affected with estimation error. Therefore, these assets with higher estimation error have a higher weight on the estimated portfolio than on the *true tangent portfolio*. This results in estimated tangent portfolios with higher return and slightly higher risk than the *true tangent portfolio*'s risk and return. Concluding, as estimation error on returns has a higher effect on tangent portfolio composition, those securities with overestimated returns will have more weight on the tangent portfolio and consequently its tangent return will be overestimated and to a lesser extent its risk will also be overestimated.

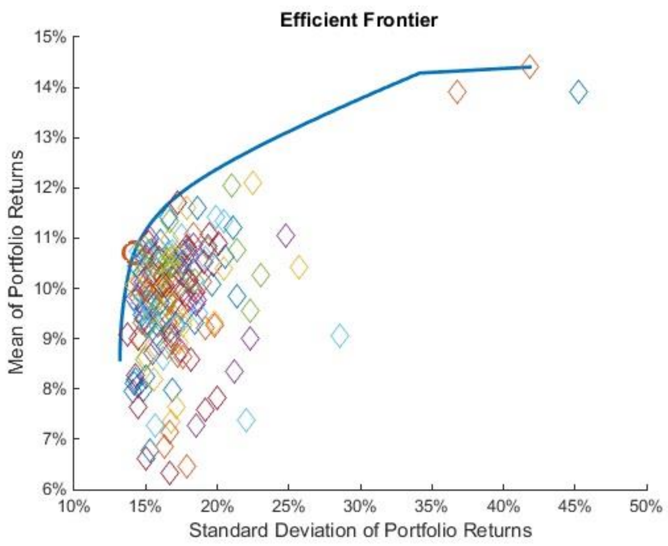
Up to this point, we have been looking at the behavior of the tangent portfolios on their own simulated samples. Now we study their performance in "reality". On this empirical study "reality" is represented by the parameters (μ, Σ) as in equations (1) and (2). By collecting the weights of the estimated tangent portfolios and applying them on the *true efficient frontier*, we are able to know the performance of those estimated portfolios for the 1 year investment horizon:

Figure 3 – Classical risk/return scatter



Scatter of the risk/return of 200 classically estimated tangent portfolios, on the *true efficient frontier* plus the true tangent portfolio, represented by an orange circle.

Figure 4 - Robust risk/return scatter



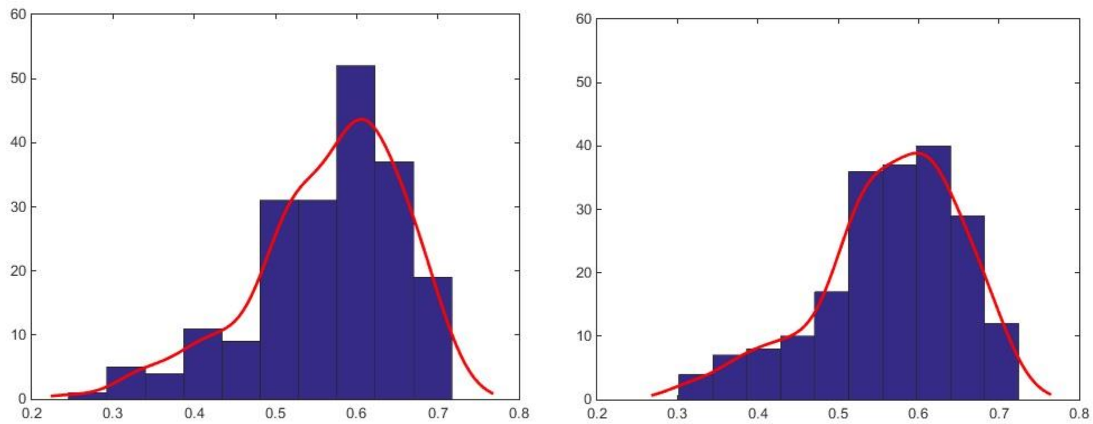
Scatter of the risk/return of 200 robustly estimated tangent portfolios, on the *true efficient frontier* plus the true tangent portfolio, represented by an orange circle.

Figures 3 and 4 show the true position of all tangent portfolios with respect to the efficient frontier, using both classical and robust estimation. On both cases the *true tangent portfolio* is represented by an orange circle on the efficient frontier. Looking at the scatter we can notice that all estimated tangent portfolios result in less than optimal portfolios on practice. This happens due to the fact that the estimated parameters ($\hat{\mu}_j^i \, \hat{\Sigma}_j^i$), $i = (classical, robust), \quad j = (1, \ldots, 200)$) which served has base for its computation had an associated *estimation error.* Combining this with the "estimation error maximizer" effect of MVT referred to on section 2.2, lead to tangent portfolios which overweighted securities with overestimated returns and underestimated standard deviations, and vice versa. As estimation error is positively correlated with standard deviation and in this study all 15 securities present high standard deviations is easy to understand the cause of the large range of risk/returns and consequent inefficiency presented by these estimated portfolios.

Even if Figure 3 and 4 give us an idea about how estimated tangent portfolios behave on the following year, visual inspection is not enough to reach solid conclusions. Therefore, there is a need to measure the performance of each portfolio through the Sharpe ratio. The *true tangent portfolio* has a Sharpe Ratio of 0.7373. That portfolio is the point of the efficient frontier that maximizes the Sharpe Ratio for the defined risk free rate. All the estimated portfolios are under that same efficient frontier. Thus, it can be instantly affirmed that all estimated portfolios have a lower Sharpe Ratio than the *true tangent portfolio*. This statement was then confirmed by computing each portfolio Sharpe Ratio, see Figure 5. In order to compare both estimation methods with each other and with the *true tangent portfolio* an average of each of the 200 portfolios Sharpe Ratio was taken.

Figure 5 – Sharpe Ratio distribution

(a) classical (b) robust



Distribution of the Sharpe Ratio of 200 tangent portfolios on the true efficient frontier. (a) is referent to classically estimated portfolios and (b) to robustly estimated portfolios

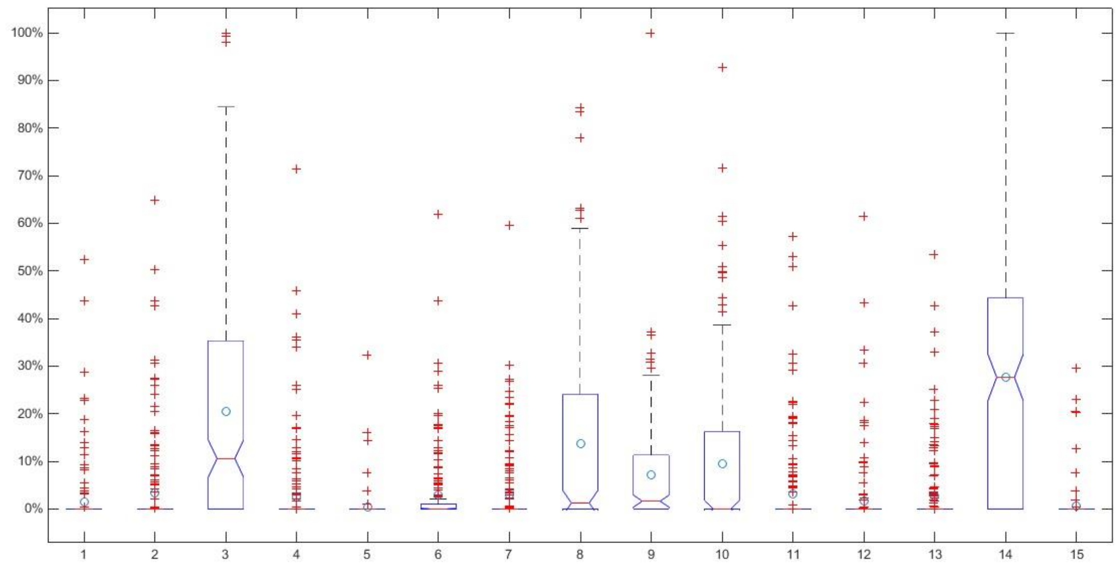
The results show that on average a classically estimated tangent portfolio has a Sharpe of 0.5654 and a robustly estimated tangent portfolio has a Sharpe of 0.5643. The classical estimation leads to a marginally higher performance measure. This confirms the theory of robust statistics, which defines robust estimators as less efficient estimators on a well distributed sample, that is, a sample without outlying observations as is the case on this first scenario, see Table II. Additionally, classically estimated portfolios have a standard deviation of its Sharpe ratio of 0.0852 against 0.0929 of robustly estimated portfolios. This adds to the point that robustly estimated portfolios are less efficient than classically estimated ones. What can be concluded from these results is that when building a portfolio through MVT, on a sample without any outlying observations, classical estimation is marginally more efficient than robust estimation.

4.1.2 Average and naive portfolio

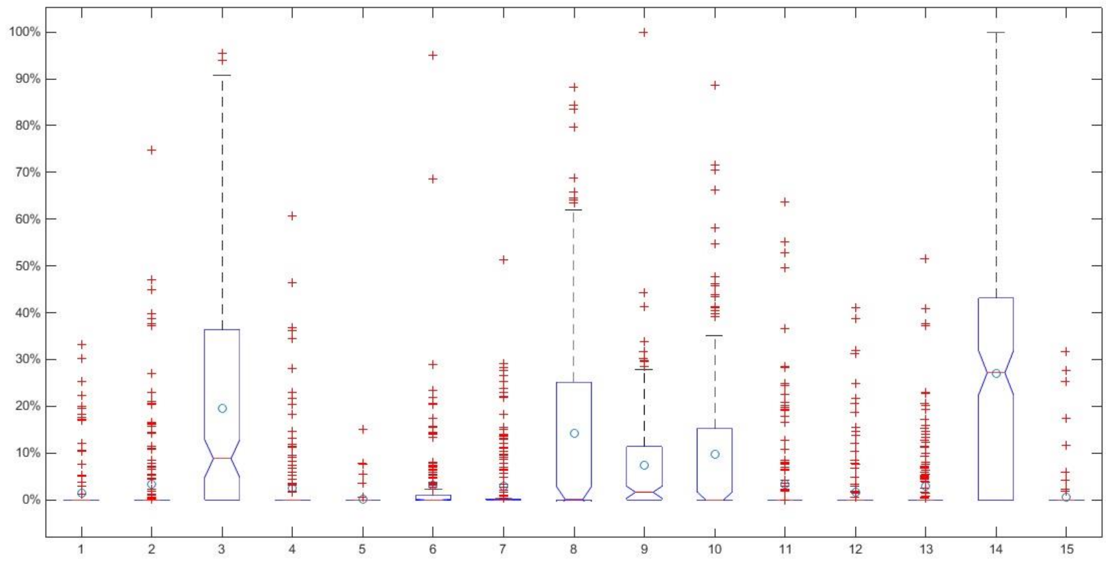
On this study we have the opportunity of creating portfolios through the resampling method developed in Michaud (1998). Doing an average of the weight each security has on the 200 tangent portfolios, both for the classical and robust case, results in two average weighted portfolios, one for the classical estimation and another for the robust estimation, see Figure 6.

Figure 6 – Dispersion of tangent portfolios' allocation

(a) classical



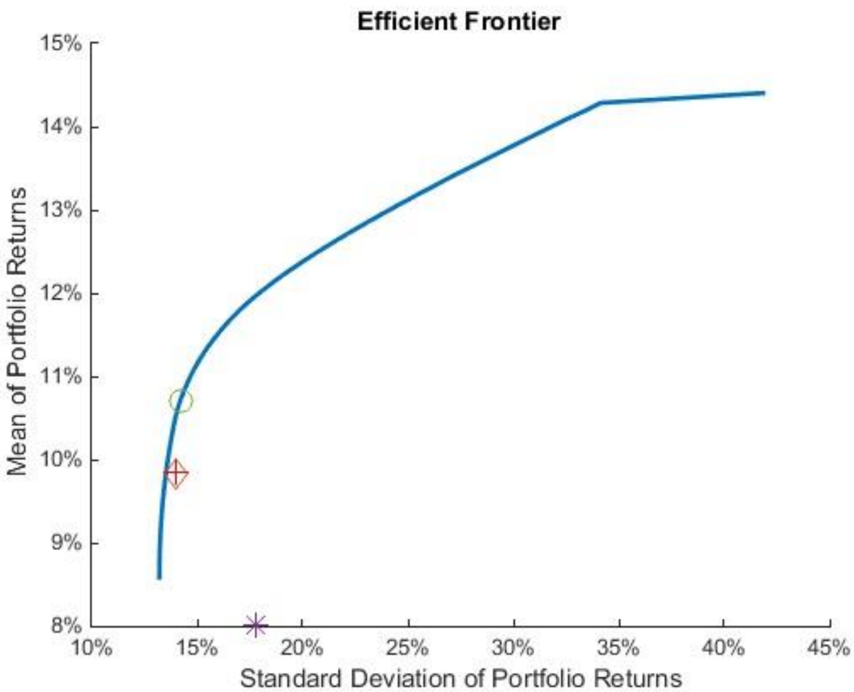
(b) robust



Dispersion of the tangent portfolio weights of each asset on the 200 resamples, both for classical estimation (a) and robust estimation (b). The blue circle represents the average weight which will be used below.

If we apply those two portfolios to the *true efficient frontier* we can see how those average weighted portfolios behave on practice and compare it with the behavior of a naively created portfolio. That behavior can be observed below on Figure 7:

Figure 7 – Average and Naive Portfolios



Average weighted, naive and true tangent portfolios on the *true efficient frontier.* The green circle represents the true tangent portfolio. The red cross represents the average robust portfolio. The orange diamond represents the average classical portfolio and the blue dot represents the naive portfolio.

The orange diamond represents the average weighted classical portfolio and the red cross inside it represents the average weighted robust portfolio. Additionally, the green circle on top of the *true efficient frontier* is the *true tangent portfolio* and the blue dot on the bottom is the naive portfolio, with an equally weighted allocation between the 15 securities. As we can see, both classical and robust portfolios are below the efficient frontier, though having a higher return and lower risk than the naive portfolio. Also, they present roughly the same risk and return, which shows that robust estimators are as good as classical estimators in an environment where they were expected to be less efficient. Looking at each portfolio's Sharpe ratio, we can confirm that both estimators result in equally efficient portfolios as they have exactly the same Sharpe ratio of 0.6848, not too far from the Sharpe ratio of the *true tangent portfolio* of 0.7373. Another fact worth considering is that a Sharpe ratio of 0.6848 is way higher than the average Sharpe calculated beforehand for the 200 classically estimated portfolios (0.5403) and 200 robustly estimated portfolios (0.5354). This happens because by doing the average of the weights of the 200 portfolios, we are mitigating the error maximization effect caused by estimation error. If in some portfolios certain assets' weights were overestimated, on other portfolios the same assets' weights might have been underestimated, assuming that there is the same chance of happening positive and negative estimation errors. So, by doing the average of those weights we are nullifying the overestimations with the underestimations and reaching closer to the *true tangent portfolio* performance.

4.2 Contamination Scenario

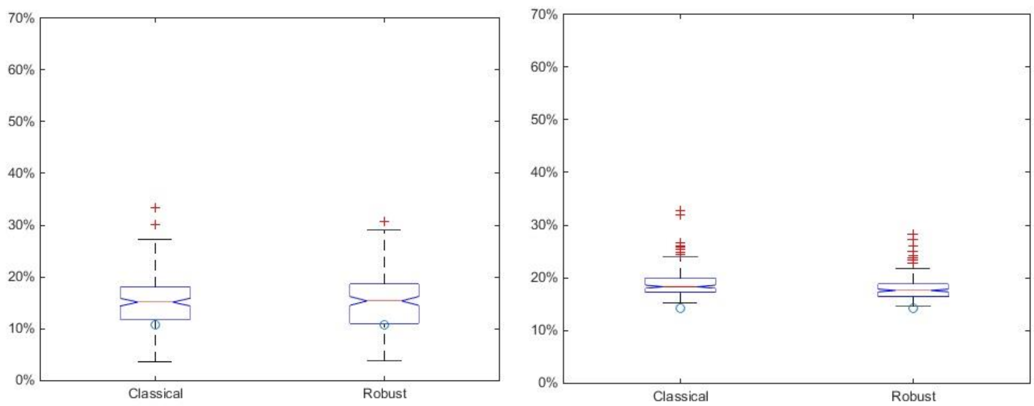
On a second stage, the resampling process was repeated for the contamination scenario. Again, two methods of evaluating the performance of both estimators are developed. Firstly we put each of the 400 estimated tangent portfolios (200 classical and 200 robust) on the *true efficient frontier* and study its Sharpe ratio. Secondly we do an average of the weights of the 200 classically and 200 robustly estimated tangent portfolios, apply those portfolios on the *true efficient frontier* and then look at its Sharpe ratio. In this part the focus is on the contaminations of 5% of observations by the three magnitudes of contamination of 2.5, 5 and 10. On annex the same results for the 2.5% and 10% contaminations can be observed.

4.2.1 Tangent portfolios

Let's start by looking at the distribution of risk/return of both classical and robust estimated tangent portfolios for the three magnitudes of contamination on Figures 8, 9 and 10:

Figure 8 - Dispersion of estimated tangent portfolios (5% contamination 2.5 magnitude)

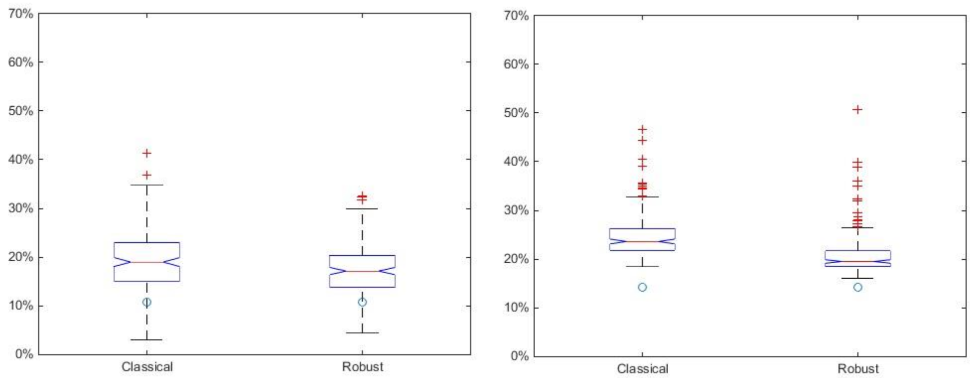
(a) return **(b) risk**



Dispersion of tangent portfolios' return (a) and risk (b) on the 5% contamination, 2.5 magnitude scenario. The blue circles represent the true tangent portfolio return and risk, respectively

Figure 9 - Dispersion of estimated tangent portfolios (5% contamination 5 magnitude)

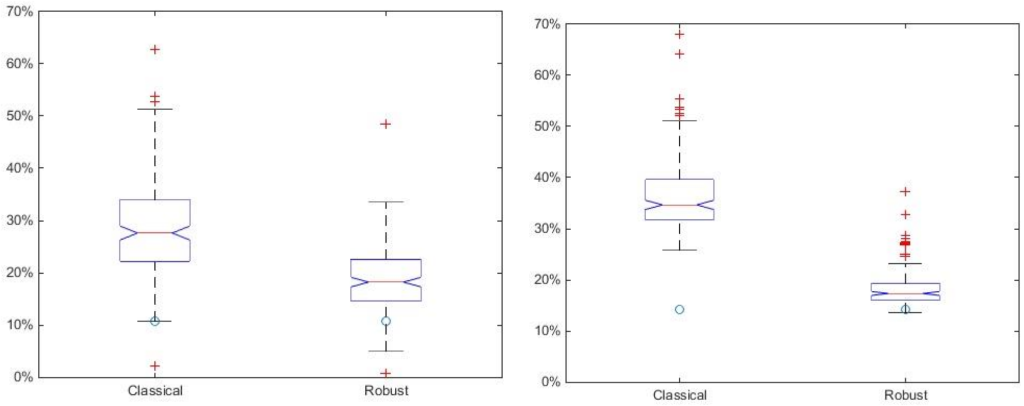
(a) return **(b) risk**



Dispersion of tangent portfolios' return (a) and risk (b) on the 5% contamination, 5 magnitude scenario. The blue circles represent the true tangent portfolio return and risk, respectively

Figure 10 - Dispersion of estimated tangent portfolios (5% contamination 10 magnitude)

(a) return **(b) risk**



Dispersion of tangent portfolios' return (a) and risk (b) on the 5% contamination, 10 magnitude scenario.
The blue circles represent the true tangent portfolio return and risk, respectively

The red line represents the median, which is the observation that has the same number of higher and lower observations, the top and bottom blue lines represent the 75% and 25% percentiles, respectively. The black line on top represents the higher observation and the bottom black line the lower observation. Additionally the red crosses are observations treated as outliers and the blue circle represents the expected return and risk of the *true tangent portfolio*. Identically to the base scenario the estimated expected returns of the three contaminations are around 75% to 80% of the times higher than the *true expected return* and the risk almost always higher than the *true risk*. On the 2.5 magnitude contamination there are no noticeable differences between the classical and the robust estimator, which tells us that a magnitude of 2.5 is not enough to produce extreme enough observations on these samples. However, as the magnitude of contamination increases we can identify a pattern of growth and dispersion of the classically estimated portfolios, both on expected return and risk. In opposition robustly estimated portfolios maintain roughly the same distribution of risk/return on all magnitudes. This is a proof of the ability of robust estimators to produce stable results even in situations where the sample is not a good source of information for the parameters needed due to the presence of outlying observations or in situations when some securities parameters fluctuate.

Now it is important to apply the weights of each portfolio to the *true efficient frontier* in order to observe how they will perform on the next year. On the following table we can see the average of the Sharpe ratios of the classically and robustly estimated portfolios with 2.5%, 5% and 10% contamination under the three different magnitudes.

Table IV – Sharpe Ratio of tangent portfolios on the true efficient frontier

Sharpe Ratio – 2.5% contamination

Magnitude	Classical		Robust	
	Mean	Standard deviation	Mean	Standard deviation
2.5	0.5566	0.0963	0.5499	0.1058
5	0.5342	0.0919	0.5494	0.0851
10	0.5055	0.1055	0.5338	0.0966

Sharpe Ratio – 5% contamination

Magnitude	Classical		Robust	
	Mean	Standard deviation	Mean	Standard deviation
2.5	0.5587	0.0881	0.5525	0.0864
5	0.5217	0.1152	0.5370	0.1047
10	0.4823	0.1061	0.5262	0.1078

Sharpe Ratio – 10% contamination

Magnitude	Classical		Robust	
	Mean	Standard deviation	Mean	Standard deviation
2.5	0.5395	0.0965	0.5337	0.1025
5	0.4958	0.1025	0.5034	0.1042
10	0.4520	0.1058	0.4823	0.1068

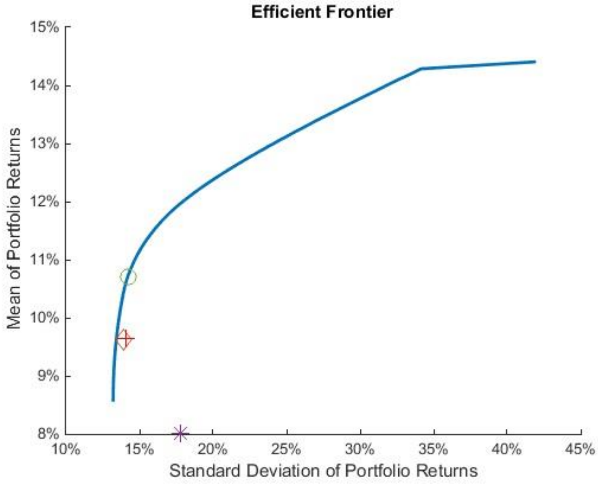
Mean and standard deviation of the Sharpe ratio of estimated tangent portfolios on the *true efficient frontier* for classical estimation and robust estimation and for all contamination scenarios

To start, is important to note that again all the estimated portfolios are below the *true efficient frontier* and so, none of them presents a Sharpe ratio equal or higher than the ratio of the *true tangent portfolio* (0.7373). Looking at Table IV, we can see that for the lower magnitude classically estimated portfolios present a higher Sharpe ratio than robustly estimated portfolios. That difference is marginal and confirms that this lower magnitude is not enough to create extreme enough observations in order for robust estimators to perform better than classical estimators. However, as the magnitude increases we can see that the average Sharpe ratio of classically estimated portfolios decreases much more than the Sharpe of robustly estimated portfolios which proves that on the 5 and 10 magnitudes of contamination robust estimation is a much more viable tool to create mean variance portfolios on an environment without a well distributed sample. On the 2.5% contamination the pattern of decline of Sharpe ratio as the magnitude increases is common with the 5% contamination. Again, on the 2.5 magnitude classically estimated portfolios present a slightly higher Sharpe ratio but on the two higher magnitudes robustly estimated portfolios have a higher Sharpe ratio, which remains stable as the magnitude increases. On the 10% contamination we can observe that once more on the 2.5 magnitude classically estimated portfolios have a marginally higher Sharpe ratio but on the other two magnitudes robustly estimated portfolios have a higher Sharpe. However, on this higher percentage of contamination we can see that the robustly estimated portfolios' average Sharpe ratio is not stable. As the magnitude of contamination increases, the average Sharpe ratio decreases considerably, perhaps due to the inability of the robust estimator to deal with the total amount of outlying observations present on that extreme scenario of contamination.

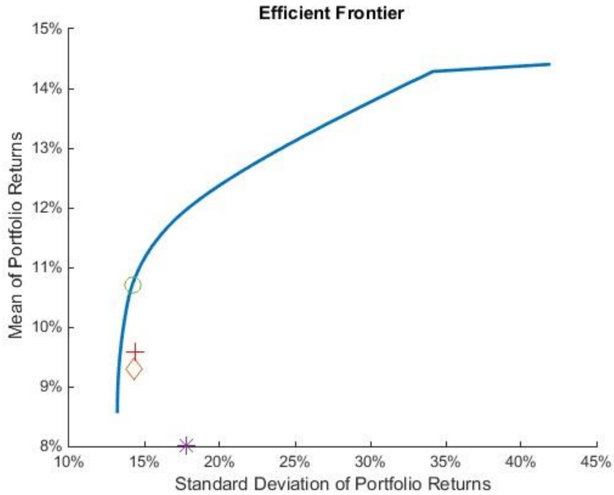
4.2.2 Average and naive portfolio

Repeating the same process of averaging the weights of the various estimated portfolios on the 5% contaminated samples and for all three magnitudes, we produce one average weighted classical portfolio and one average weighted robust portfolio for each magnitude. Once more, on Figure 11 we have the *true efficient frontier*, the *true tangent portfolio* represented by a green circle, the average weighted classical portfolio represented by an orange diamond, the average weighted robust portfolio represented by an orange cross and the naive portfolio represented by a blue dot.

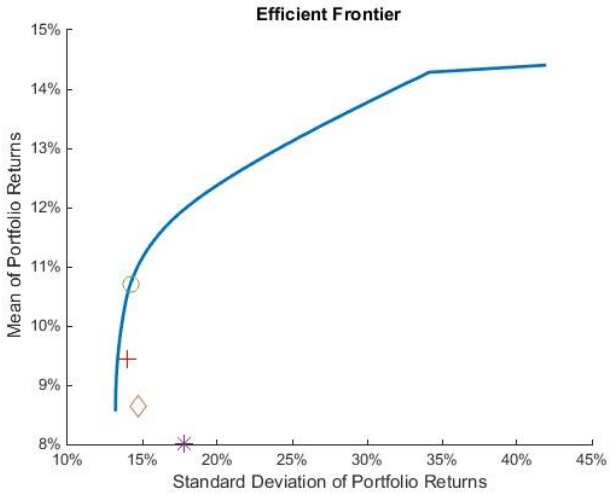
Figure 11 - Average and Naive Portfolios



(a)



(b)



(c)

Average weighted, naive and true tangent portfolios on the *true efficient frontier* for the 5% contamination scenario. The green circle represents the true tangent portfolio. The red cross represents the average robust portfolio. The orange diamond represents the average classical portfolio and the blue dot represents the naive portfolio. On (a) is represented the 2.5 magnitude, on (b) the 5 magnitude and on (c) the 10 magnitude.

On the 2.5 magnitude the two average weighted portfolios seem to have roughly the same risk/return, though the classically estimated portfolio has a Sharpe ratio of 0.673 compared with a Sharpe ratio of 0.6693 of the robustly estimated portfolio. This fact is consistent with the previous considerations taken, namely that a magnitude of 2.5 is not enough to create significant outlying observations which would create "noise" on the sample. On the opposite side, the 5 and 10 magnitudes displays a decrease in performance by the average weighted classical portfolio. As the magnitude increases the classical portfolio's Sharpe ratio decreases considerably, as can be observed on Table V:

Table V – Sharpe Ratio of average portfolios

Sharpe Ratio

Magnitude	2.5% contamination		5% contamination		10% contamination	
	Classical	Robust	Classical	Robust	Classical	Robust
2.5	0.679	0.6726	0.673	0.6693	0.6585	0.6533
5	0.6499	0.6743	0.6329	0.6475	0.5995	0.6061
10	0.6017	0.6483	0.5713	0.6553	0.5382	0.5728

Sharpe ratio of average weighted portfolios on the *true efficient frontier* for all contamination scenarios.

Once again, averaging the weights of the different estimated portfolios provides portfolios with a much higher performance than the average Sharpe of each estimated portfolio. Additionally the average weighted robust portfolio remains stable with an increase in magnitude, as opposed to the average weighted classical portfolio. Another proof of the stability of robust methods of estimation to outlying observations and distancing of the sample from the realized risk/return. On the 2.5% and 10% contaminations the pattern remains. On the 2.5% contamination, average weighted classical portfolios decrease as the magnitude increases, when average weighted robust portfolios remain stable. On 10% contamination both classes of portfolios decrease as the magnitude increases, though the decline on average weighted classical portfolios Sharpe ratio is higher than the decline on average weighted robust portfolios.

5. Conclusion

Mean Variance Theory is based on the assumption that expected returns and covariances between securities are the only necessary inputs in to create efficient portfolios. These inputs are not known, have to be estimated and are therefore subject to *estimation error*. This puts a great importance on the estimation process. This study tests the ability that two possible estimators (classical maximum likelihood estimators and robust estimators) have to withstand outlying observations which may have a higher than desired effect on the parameter estimation, and to withstand small parameter deviations and still remain stable. With the meaning of stable being: to produce close to efficient portfolios on majority of the possible scenarios. It can be stated that on scenarios of uncertainty, where the sample may not be a perfect source of information for the estimation of the parameters needed for MVT, robust estimators produce closer to optimal results than classical estimators. Only on the scenarios where the sample is a close to perfect reflection of the *true parameters* do classical estimators produce better results than robust estimators. However, the results that classical estimators produce on these situations are only marginally superior, whereas on the extreme contamination scenarios, where the sample doesn't exactly represent the *true parameters* and contains many outlying observations, robust estimators have a much higher performance. Which makes robust estimation a good tool of risk mitigation, as it produces close to efficient results even on the worst case scenario. In comparison, classical estimation produces slightly more efficient results but only on the best case scenario. Future research might study the performance of different types of robust estimators against on another, try to study which contamination percentage and magnitude would be a better reflection of the uncertainty observable on financial data. Other possible developments would be to repeat the simulation process with another distribution or apply a different weight to positive and negative contaminations.

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