LISBON SCHOOL OF ECONOMICS & MANAGEMENT
UNIVERSIDADE DE LISBOA

MASTER

ACTUARIAL SCIENCE

MASTER´S FINAL WORK

DISSERTATION

USE OF THE NP-APPROXIMATION TO DETERMINE THE RISK ADJUSTMENT UNDER IFRS 17 IN A NON-LIFE PORTFOLIO

TATIANA MARIA DOS SANTOS MARQUES

OCTOBER 2020

LISBON SCHOOL OF ECONOMICS & MANAGEMENT
UNIVERSIDADE DE LISBOA

MASTER
ACTUARIAL SCIENCE

MASTER´S FINAL WORK
DISSERTATION

USE OF THE NP-APPROXIMATION TO DETERMINE THE RISK ADJUSTMENT UNDER IFRS 17 IN A NON-LIFE PORTFOLIO

TATIANA MARIA DOS SANTOS MARQUES

SUPERVISION:
WALTHER NEUHAUS

OCTOBER 2020

Glossary

APRA – Australian Prudential Regulation Authority.

CAS – Casualty Actuarial Society.

CBNI – Covered but not Incurred.

CoCM – Cost of Capital Margin.

CR – Claim Rates.

CSM – Contractual Service Margin.

CTE – Conditional Tail Expectation.

DF – Development Factors.

EIOPA – European Insurance and Occupational Pensions Authority.

GAAP – General Accepted Accounting Principles.

GMM – General Measurement Model.

IAA – International Actuarial Association.

IAN – International Actuarial Note.

IASB – International Accounting Standards Board.

IBNR – Incurred but not Reported.

IFoA – Institute and Faculty of Actuaries.

IFRS – International Financial Reporting Standard.

ISAP – International Standards of Actuarial Practice.

LIC – Liability for Incurred Claims.

LRC – Liability for Remaining Coverage.

MVM – Market Value Margin.

MOCE – Margin Over Current Estimates.

MPP – Marked Point Process.

PAA – Premium Allocation Approach.

PCES – Plano de Contas para as Empresas de Seguros.

PHT – Proportional Hazards Transform.

PVFCF – Present Value of Future Cash Flows.

RBNS – Reported but not Incurred.

SCR – Solvency Capital Requirement.

SST – Swiss Solvency Test.

TRG – Transition Resource Group.

ULAE – Unallocated Loss Adjustment Expense.

VaR – Value at Risk.

ABSTRACT AND KEYWORDS

The accounting standard IFRS 17 “Insurance Contracts” will become effective as at 1 January 2023 and will require the presentation of an explicit compensation for uncertainty in the measurement of the insurance contracts liability, named the risk adjustment.

This dissertation presents a possible model to be applied to determine the risk adjustment under IFRS 17 for the non-life business. In order to do, we apply the NP-approximation, which only requires an estimate of the central second and third order moments of the distribution of the present value of future cash flows.

The determination of the risk adjustment is presented separately for the liability for incurred claims and the liability for remaining coverage. The first is further split into claims reported but not settled and claims incurred but not reported.

For the claim payment process we first present the estimate of the necessary moments assuming a compound Poisson distribution. Further on we generalise this model and present other two claims development assumptions that can be used: the multinomial and the Dirichlet.

Finally, the proposed model is applied in practice to a real non-life business portfolio and the results are presented and compared.

KEYWORDS: IFRS 17; risk adjustment; NP-approximation; liability for incurred claims; liability for remaining coverage; compound Poisson; multinomial; Dirichlet.

TABLE OF CONTENTS

TABLE OF FIGURES

Acknowledgments

First, I would like to thank my supervisor, professor Walther, for his support, guidance and invaluable knowledge that has been passed onto me. I also thank him on his constant availability to meet and discuss my questions and doubts.

I would also like to thank Nuno for his comments on the dissertation and for introducing me to actuarial science. I am also thankful to Francisco for his tips on programming in R which were very much needed on the practical application.

Finally, I am grateful to my friends and family. To my godmother Carmen, who has always been an inspiration in proving that working and studying at the same time is possible. I am thankful to my closest family: my parents, brother and grandmother who were the ones that supported me the most and encouraged me to finish the dissertation and to give my best. A special thank you to my grandfather Vitor, who even though was not physically present at this stage of my life, has had a great influence on me and always encouraged me to investigate and always want to know more, which have proven to be very useful qualities in both my academic and professional life.

1. INTRODUCTION

Starting in 2023 a new International Financial Reporting Standard (IFRS) for insurance contracts will be in force. Insurance companies reporting in IFRS will need to adapt in order to apply this new standard. In Portugal, all insurance companies will have to report their financial statements in accordance with IFRS 17.

This new standard demands that when determining the insurance contract liability, normally the largest item of a liability side of an insurance company's balance sheet, a measure for non-financial risk has to be included, the so-called risk adjustment. The risk adjustment will represent the compensation an entity requires for bearing the uncertainty in the amount and timing of the cash flows associated with the contracts.

In different already applicable frameworks, the determination of the insurance contract liability has a specific allowance for risk, a margin above the best estimate. However, the criteria and the definition of this risk adjustment is slightly different. The standard is principles-based and therefore it does not prescribe specific methods on how to determine the risk adjustment. Regardless of the method chosen by insurance companies to determine their risk adjustment, the disclosure of the confidence level is mandatory.

This dissertation presents a model that can be applied by non-life insurance companies to determine their risk adjustment for a defined confidence level, or to estimate the confidence level associated with a pre-selected risk adjustment.

Prior to the Solvency II regime, insurance companies in Norway had to report their liabilities (determine their reserves) based on a risk theoretical model proposed by Norberg and Sundt [1] and further developed and implemented by Kristiansen [2]. The security provision calibrated using this model had a similar concept to the risk adjustment. Starting from this model, we adapt and extend it in order to determine a risk adjustment compliant with the characteristics imposed in the IFRS 17 standard. The method presented below relies on the NP-approximation to determine a confidence level for the present value of future cash flows determined for a group of insurance contracts.

Chapter 2 gives a review of the literature covering the determination of the risk adjustment under IFRS 17 and other similar measures under different frameworks.

Chapter 3 covers the definition and characteristics of the risk adjustment under IFRS 17. It also mentions previous frameworks in which a similar measure had to / has to be determined.

Chapter 4 focuses on the model itself. The NP-approximation is presented and the notation and assumptions going forward are defined. Based on these assumptions, we get the general expressions for the central second and third order moments of the reported but not incurred (RBNS), incurred but not reported (IBNR) and covered but not incurred claims (CBNI), which is all the information needed to estimate the risk adjustment.

The model is first presented assuming a compound Poisson process for the claim payment process. A specific section is dedicated to parameter estimation. The assumption is then generalised so that different claim payment distributions can be used. We show as other specific cases the multinomial and the Dirichlet distributions.

A short remark on the order of presentation of the models. We started this work with the intention of adapting the model of Kristiansen [2]. We then realised that Kristiansen's original model is too limited for our purpose, as it does not include the stochastic payment process of reported claims (Kristiansen assumes that claims are fully paid, thus known, when they are reported). The compound Poisson model is a first and easy approach to the claim payment process, that we use to motivate the structure of our overall model. However, it is not the only possible model of the claim payment process. When analysing alternative models of the claim payment process – the multinomial and the Dirichlet – we discovered the general structure. We believe that our presentation, in going from a specific model to a generalised model, and thence back to other possible specifications, gives better insight into the model framework, than if we had postulated an abstract, generalised model upfront.

Chapter 5 presents the application of the model in Chapter 4 to a real non-life insurance portfolio. We have programmed the model in R and present the structure defined. The results are presented and discussed.

2. LITERATURE REVIEW

Different sources have been consulted in order to define the model in this dissertation and its compliance with the concept of risk adjustment under IFRS 17. One of the most important ones is International Accounting Standards Board's (IASB's) IFRS 17 standard [3] and accompanying basis for conclusions [4], which define the risk adjustment in a principles-based setting. The concept of the risk adjustment will be further explored in the next chapter.

Other analyses and clarifications have been provided by the IASB [5], [6] and the Transition Resource Group (TRG) [7], [8]. The IASB is the Board responsible for the development and publication of the IFRS standards, including IFRS 17. The TRG, which is composed of different stakeholders in the insurance industry, was created to discuss questions raised on implementation and to foster possible amendments and clarifications of the IFRS 17 standard.

The International Actuarial Association (IAA) has published the International Standard of Actuarial Practice, ISAP 4 [9] and an International Actuarial Note, IAN 100 [10], both specific to IFRS 17. The IAA is the worldwide association of professional actuarial organizations that publishes ISAPs and IANs in order to define model standards of practice and current approaches that can be applied by actuaries. Both the ISAPs and IANs serve only as educational documents and are not mandatory.

The IAA [10] states that "In order to determine confidence levels, it is necessary to be able to locate the value of the Fulfilment Cash Flow of a collection of insurance contracts on the probability distribution of the present value of the cashflows for the contracts." The purpose of this dissertation is indeed to obtain a liability amount corresponding to a certain confidence level on the probability distribution of the present value of the future cashflows, relying on the NP-approximation. The confidence level chosen by an insurance company will be dependent on different factors: risk aversion, risk characteristics, diversification benefits, etc.

A very complete monograph focused solely on the risk adjustment under IFRS 17 has been published by the IAA [11]. This is one of the main sources to become familiar with the concept of the risk adjustment from a qualitative and quantitative perspective.

The monograph mentions possible estimation techniques: quantile, cost of capital and Wang transform and presents some case studies for each.

Beard, Pentikäinen and Pesonen [12, pp. 108-121] cover the NP-approximation, which is used in this dissertation. They present the derivation of the formula which was originally found by Kauppi and Ojantakanen [13].

The model proposed in the following chapters is based on the work from Kristiansen [2] which was based on the work from Norberg and Sundt [1] and Norberg [14]. The models proposed by Norberg, Sundt and Kristiansen were used by the Norwegian regulator as a framework for Norwegian insurance companies to determine the minimum requirements for technical provisions in non-life insurance [15]. The framework and requirements for technical provisions are summarised in a presentation to the Baltic and Nordic Insurance Supervisory Authorities by Kristiansen [16].

Since the IASB [3] mandates the disclosure of the confidence level for the risk adjustment, one of the goals of this dissertation is to provide a simple model to determine the confidence level, even if the entity does not follow the confidence level approach to set its risk adjustment. Reback [17] presents another simple method to derive a confidence level for the risk adjustment.

Since the issue of the first exposure draft of IFRS 17 (at the time named IFRS 4 Phase II) different authors have started to dig into the concept of the risk adjustment and of different techniques to calibrate it. Different estimation techniques and their upsides and downsides are presented by the Institute and Faculty of Actuaries (IFoA) IFRS 17 for General Insurance working party [18] and England [19]. There is extensive literature in determining a probability distribution for claims reserves, in IFRS 17 named the liability for incurred claims (LIC). England [20] also mentions different methods to obtain a probability distribution for premium reserves, in IFRS 17 named the liability for remaining coverage (LRC). Once a probability distribution has been derived, the risk adjustment can be estimated using an appropriate risk measure, e.g. value at risk (VaR), conditional tail expectation (CTE) or proportional hazards transform (PHT). This approach is outlined by England, Verrall and Wüthrich [21].

The IASB [3] mentions that the risk adjustment needs to be allocated at the unit of account level but must be estimated at the level which is deemed the most appropriate by the entity. The main concepts of the standard are presented in Appendix A. The allocation of the risk adjustment to the defined level of aggregation is further explored by England [20] and the IFRS 17 contractual service margin (CSM) Working Party [22].

Different authors have presented their views on possible methods to determine the risk adjustment for non-life business. Miccolis and Heppen [23] use the Rehman-Klugman method to estimate the risk adjustment. This methodology analyses actual vs estimated values in order to obtain the volatility in the estimates and can be applied regardless of the method used to estimate ultimate losses. Moro and Krvavych [24] outline a method to determine the Solvency II risk margin confidence level. Different approximations are used in order to do so, of which the NP-approximation is one example. The approximations are applied to the same portfolios and their accuracy is compared. The Casualty Actuarial Society (CAS) Task Force on Fair Value Liabilities [25] addresses the concept of the current value of insurance liabilities with several methods proposed in order to obtain the risk adjustment.

Readers interested in approaches for life business can consult Wagner [26] for universal life products, Koetsier [27] for term life insurance, immediate annuity, insured pension and disability insurance and Coulter [28] for yearly renewable term products. Much like the work carried out for the non-life business, Chevallier, Moro, Krvavych et al. [29] apply approximations to obtain a confidence level for the risk margin in life portfolios. Similar to the purpose of this dissertation, Norberg [30] has relied on the models from Norberg and Sundt [1] and Kristiansen [2] in order to determine a risk adjustment for paid-up policies.

Prior to IFRS 17, similar concepts to the risk adjustment already existed. Gutterman [31], Risk Margin Working Group [32], Society of Actuaries [33] and Risk Margins Taskforce [34] focus on these similar concepts, risk margins and the margin over current estimates (MOCE), along possible estimation techniques. IAA [35], European Insurance and Occupational Pensions Authority (EIOPA) [36], The Institute of Actuaries of Australia [37], Collings and White [38] and Australian Prudential Regulation Authority (APRA) [39] provide an overview of the current measures similar to the risk adjustment. The concept of risk adjustment and current similar measures is covered in the following chapter.

3. Risk adjustment in IFRS 17

The risk adjustment in IFRS 17 is defined as "the compensation an entity requires for bearing the uncertainty about the amount and timing of the cash flows that arises from non-financial risk as the entity fulfils insurance contracts" IASB [3]. This amount now needs to be determined explicitly, that is, it needs to be presented separately from the best estimate.

This concept is further explained as follows:

> *The risk adjustment for non-financial risk for insurance contracts measures the compensation that the entity would require to make the entity indifferent between:*
>
> *(a) fulfilling a liability that has a range of possible outcomes arising from non-financial risk; and*
>
> *(b) fulfilling a liability that will generate fixed cash flows with the same expected present value as the insurance contracts.*

In: IASB (2017), B87 [3]

Since IFRS 17 is principles-based there are no prescribed techniques for determining the risk adjustment. The focus of the dissertation is to present a coherent model that satisfies the following requirements set out in the standard:

> *(a) risks with low frequency and high severity will result in higher risk adjustments for non-financial risk than risks with high frequency and low severity;*
>
> *(b) for similar risks, contracts with a longer duration will result in higher risk adjustments for non-financial risk than contracts with a shorter duration;*
>
> *(c) risks with a wider probability distribution will result in higher risk adjustments for non-financial risk than risks with a narrower distribution;*
>
> *(d) the less that is known about the current estimate and its trend, the higher will be the risk adjustment for non-financial risk; and*
>
> *(e) to the extent that emerging experience reduces uncertainty about the amount and timing of cash flows, risk adjustments for non-financial risk will decrease and vice versa.*

In: IASB (2017), B91 [3]

3.1 Determination of a risk adjustment prior to IFRS 17

Even though the risk adjustment as defined in the standard is a new concept, similar measures already exist in different countries, whether they are determined explicitly or implicitly.

3.1.1 Portugal

In the Portuguese General Accepted Accounting Principles (GAAP), Plano de Contas para as Empresas de Seguros (PCES), no specific allowance for risk exists in the claims reserve. This reserve is defined as the total amount needed by the insurance company to settle all outstanding claims, whether they have been reported or not, minus the amounts paid so far. The Portuguese regulator does not define the confidence level at which the claims reserve should be determined. In the market, companies tend to follow a prudent approach and determine their reserves at a level higher than the best estimate, thus introducing an implicit risk adjustment in their estimates.

This reality is common to other countries since it has always been considered good accounting practice to add a margin for uncertainty to the technical provisions. According to IASB [5], "When applying IFRS 4, many insurers include a risk margin in their measurement of insurance contract liabilities (ie an implicit or explicit allowance for risk) in addition to their estimate of the future cash flows. The extent of the risk margin is typically not disclosed."

Miccolis and Heppen also mention:

> In practice, actuaries have used a variety of technical methods and assumptions to consider such risks, and in many situations risk margins have been implicitly embedded in the assumptions or the selection or interpretation of the results of analyses or models. In the U.S. and certain other jurisdictions, the lack of sufficient adjustments for the time value of money [i.e., not discounting] has also been considered to provide an implicit risk margin. GAAP in most countries have used the amounts established in the financial statements filed with insurance regulators/supervisors, resulting in some jurisdictions including explicit risk margins in their financial statements while others have no explicit risk provisions or risk adjustments.

In: Miccolis & Heppen (2010), p. 2 [23]

3.1.2 European Union (Solvency II)

In Europe a concept similar to the risk adjustment is required since 1 January 2016, the date that the prudential regime Solvency II entered into force. For the purpose of the Solvency II balance sheet, European insurance companies determine their technical provisions at what would be considered their current value, which is composed of a best estimate and a risk margin.

However, as stated by EIOPA [36] "The approach to determining the risk margin in Solvency II is conceptually different from the risk adjustment in IFRS 17 (transfer vs entity-specific)".

The *risk adjustment* is evaluated as the compensation required by the entity for the uncertainty in the amount and timing of cash flows. The *risk margin* in Solvency II is determined as the notional amount demanded by a market participant to take over the portfolio of contracts and guarantee the full run-off of its responsibilities. Therefore, the concept of the risk adjustment is specific to the entity, while the concept of the risk margin is based on market perception.

Furthermore, while IFRS 17 is principles-based, Solvency II is more rules based and the calculation of the risk margin is standardised. The method applied is the cost of capital with a 6% cost of capital rate and a capital amount equal to the Solvency Capital Requirement (SCR) which is calibrated at a confidence level of 99.5%. This margin is determined net of reinsurance. The risks considered are also different since the risk margin includes more than non-financial risks, e.g. operational risk, credit risk and non-hedgeable market risk.

3.1.3 Switzerland

In Switzerland the Swiss Solvency Test (SST), a solvency regime, has been in force since 1 January 2011. It is similar in nature to Solvency II but has some differences. As stated in IAA [35], "Technical liabilities include a discounted best estimate and a Market Value Margin (MVM). The MVM represents what a rational investor would demand in excess of the best estimate of the liabilities. The MVM is estimated by calculating the Cost of Capital Margin (CoCM)." The MVM is similar in concept to the risk margin in Solvency II. Both are determined using a cost of capital approach. However, in SST capital is calibrated at a 99% CTE. The Solvency II SCR at 99.5% VaR, could be considered an approximation of 99% CTE.

3.1.4 Norway

Between 1991 and 2015 Norway had prudential regulation for non-life insurance, which involved minimum requirements for the technical provisions, including a separately identified security provision. The security provision had a similar role in NO GAAP accounts as the risk adjustment will have under IFRS 17: it was considered to belong to technical provisions and increases or releases went through the profit or loss statement. The minimum requirements were calibrated using a risk theoretic model developed by Norberg and Sundt [1] and Kristiansen [2]. The security provision was calibrated as the difference between the 99th percentile of the outstanding claim liabilities and their expected value, in such a way that the sum of the minimum requirements to the premium provision, claim provision and security provision was aiming at providing a 99% probability of adequacy of provisions in a run-off scenario. In addition, the regulation had minimum requirements to provisions for expenses and reinsurer default.

3.1.5 Canada

In Canada an explicit margin for adverse deviations is prescribed in the Canadian Actuarial Standards. Gutterman [31] states "For each assumption, actuaries have a range from which to choose a risk margin. These ranges provide for a minimal amount of conservatism while the maximum is at a level that is still not so conservative as to distort income". For the non-life business, actuaries should select margins for three assumptions: claims development, recovery from reinsurance ceded and investment return rates. Margins range from 0 to 20% depending on the assumption.

3.1.6 Australia

In Australia, the regulator requires an explicit risk margin above the central estimate for both outstanding claims liabilities and premium liabilities. According to the Prudential Standard GPS 340 Insurance Liability Valuation:

> *The risk margin is the component of the value of the insurance liabilities that relates to the inherent uncertainty that outcomes will differ from the central estimate. It is aimed at ensuring that the value of the insurance liabilities is established at an appropriate and sufficient level. (…)*

Risk margins must be determined, for each class of business and in total, on a basis that reflects the experience of the insurer. In any event, the risk margins must be valued so that the insurance liabilities of the insurer, after any diversification benefit, are not less than the greater of a value that is:

(a) determined on a basis that is intended to value the insurance liabilities of the insurer at a 75 per cent level of sufficiency; and

(b) the central estimate plus one half of a standard deviation above the mean for the insurance liabilities of the insurer.

In: APRA (2019), p. 6 [39]

4. Model adaptation

This chapter presents a model that can be used to estimate the confidence level of the risk adjustment, based on the characteristics mentioned in the previous chapter. The model allows us to use the NP-approximation in a straightforward way.

4.1. Calculating a risk adjustment with the NP-approximation

The NP-approximation can be used to obtain an approximate confidence level without having to derive a full distribution for the aggregate loss distribution, which is often difficult to determine. Insurance reserve distributions are naturally skewed as a result of a high number of small claims and a small number of more extreme claims. By taking the aggregate loss distribution's skewness into account, this approximation presents better results than the normal approximation without being too complex.

Our goal is to apply the NP-approximation to the distributions of the present value of payments of claims RBNS, claims IBNR, and claims CBNI, in order to obtain a risk adjustment for each group of claims. The first two form the basis of the liability for incurred claims, while claims CBNI form the basis of the liability for remaining coverage, as defined under IFRS 17.

We will denote by LIC the liability related to incurred claims (expired risk) and by LRC the liability related to remaining coverage (unexpired risk).

For a random variable S, the NP-approximation of its $(1-\alpha)$ percentile is:

$$F_S^{-1}(1-\alpha) \approx \mu_1(s) + z_{1-\alpha}\sqrt{\mu_2(s)} + \frac{(z_{1-\alpha}^2 - 1)}{6}\frac{\mu_3(s)}{\mu_2(s)}$$

where $z_{1-\alpha}$ denotes the $(1-\alpha)$ percentile of the standard $N(0,1)$ distribution function, μ_1 denotes the mean of S, μ_2 its variance and μ_3 its central third order moment.

Based on the NP-approximation we only need to get an estimate for the central second and third order moments of LIC and LRC in order to estimate the risk adjustment for the liability for incurred claims and the risk adjustment for the liability for remaining coverage, respectively:

$$RA_{LIC} \approx z_{1-\alpha}\sqrt{\mu_2(LIC)} + \frac{(z_{1-\alpha}^2 - 1)}{6}\frac{\mu_3(LIC)}{\mu_2(LIC)} \qquad (1)$$

$$RA_{LRC} \approx z_{1-\alpha}\sqrt{\mu_2(LRC)} + \frac{(z_{1-\alpha}^2 - 1)}{6}\frac{\mu_3(LRC)}{\mu_2(LRC)} \tag{2}$$

The derivation of the NP-approximation is detailed in Appendix B.

The approximation presented for the risk adjustment is based on an additional amount on top of the best estimate. This best estimate can be calculated in any way the actuary may deem reasonable. For a matter of consistency, we will assume throughout the dissertation that the best estimate is estimated accordingly to the model we have defined.

4.2. Notation

Let us define the notation that will be used going forward. The model allows us to separate claims RBNS from claims IBNR and claims CBNI.

Let us define the following quantities, for a given line of business:

j accident period, $j = 1,2,...$

d reporting delay after accident, $d = 0,1,2,...$

t payment delay after reporting, $t = 0,1,2,...$

J current period, valuation being required at the end of period J

N_{jd} Number of claims incurred in period j and reported in period $j + d$

X_{jdt} Overall payments of claims in incurred/reported cohort (j, d) paid in period $j + d + t$

$X_{jd} = \sum_{t\geq 0} X_{jdt}$ Overall payments of claims in incurred/reported cohort (j, d)

For $j + d + t > J, X_{jdt}$ are unobserved and, as consequence, the overall payments X_{jd} are unknown quantities.

The random, unknown future payments related to claims RBNS at the end of period J are $\mathcal{X}_{RBNS|J} = \{X_{jdt}: j = 1, \cdots, J,\ d \leq J - j,\ t > J - (j + d)\}$.

The random, unknown future payments related to claims IBNR at the end of period J are $\mathcal{X}_{IBNR|J} = \{X_{jdt}: j = 1, \cdots, J,\ d > J - j,\ t \geq 0\}$.

The random, unknown future payments related to incurred claims at the end of period J are $\mathcal{X}_{LIC|J} = \mathcal{X}_{IBNR|J} \cup \mathcal{X}_{RBNS|J} = \{X_{jdt}: j = 1, \cdots, J,\ d \geq 0,\ t > J - (j + d)\}$.

The random, unknown future payments related to claims CBNI at the end of period J are $\mathcal{X}_{LRC|J} = \{X_{jdt}: j > J,\ d, t \geq 0\}$, only for policies in force or for which there is an obligation assumed by the insurance company at the end of period J.

The notation $|J$ indicates something that is given at time J (only).

4.3. Assumptions

In the model the following stochastic assumptions are considered:

p_j Risk exposure in past accident period $j = 1, \ldots, J$

$p_{j|J}$ Unexpired risk exposure in future accident period $j > J$ at the end of period J

θ_j Random claim frequency for accident period j

$\{\pi_d: d = 0,1,2, \ldots\}$ Reporting pattern: expected proportion of ultimate claim count reported in development period d starting from the accident date, with $\sum_{d \geq 0} \pi_d = 1$

$\{\upsilon_t: t = 0,1,2, \ldots\}$ Payment pattern: expected proportion of claims paid in payment delay t starting from the reporting date, with $\sum_{t \geq 0} \upsilon_t = 1$

Conditional on Θ_j, the numbers of reported claims $\{N_{jd}: d = 0,1,2, \ldots\}$ are independent and Poisson distributed with means $p_j \Theta_j \pi_d$, i.e. $N_{jd}|\Theta_j \sim Poisson(p_j \Theta_j \pi_d)$.

The random claim frequencies Θ_j are i.i.d. with a distribution to be specified.

Conditional on the number of reported claims N_{jd}, the payments $\{X_{jdt}: t = 0,1,2, \ldots\}$ are independent with a compound Poisson distribution $X_{jdt}|N_{jd} \sim CP\left(N_{jd} \upsilon_t, G(\cdot)\right)$.

The development of reported claims in any incurred/reported cohort (j, d) is stochastically independent of everything else.

All stochastic quantities pertaining to different accident periods are independent.

All stochastic quantities pertaining to different lines of business are independent.

These assumptions are consistent with those of a marked point process (MPP) as suggested by Arjas [40] and Norberg [41]. In particular, we will utilise the result that states that conditionally, given the number of claims reported, the future development of IBNR is independent of the future development of RBNS.

The model proposed is an extension of the model previously proposed by Kristiansen [2]. In Kristiansen's model, it was assumed that all claims were settled when reported meaning that there was no payment dimension. In the model defined in this dissertation, this assumption is dropped and therefore we introduce a payment dimension. This also results in the introduction of a model for the RBNS.

4.4. Moments of discounted liabilities

Let us define the following quantities, for a given line of business:

$\mu_1(S) = E[S]$ mean of S

$\mu'_k(S) = E[S^k], k > 1$ *k*th raw moment of S

$\mu_k(S) = E\left[\left(S - E(S)\right)^k\right], k > 1$ *k*th central moment of S

We also use μ'_k and μ_k with a probability distribution (not a random variable) as argument, with the obvious meaning.

D will denote the one period discount factor defined by $D = \left(1 + \bar{r}_J\right)^{-1}$ where $\bar{r}_J$ denotes the flat yield rate that would produce the same present value of future cash flows as discounting the future payments with the spot rates from the yield curve $\left\{r_{s|J}: s = 1,2,\cdots\right\}$ applicable at reporting period J. It is important to note that a single rate can be determined separately for each of the cases presented: RBNS, IBNR and CBNI.

Having seen that only the central second and third order moments of the LIC and LRC are necessary to get an approximate value for the risk adjustment, we will derive the expressions for those moments in the following sections.

4.4.1. Moments of the Liability for Incurred Claims (LIC)

The risk adjustment for the LIC will be determined by splitting the claims RBNS from the claims IBNR. The determination of the moments is simplified by the independence assumptions.

The random present value of outstanding claim payments is then:

$$L_{RBNS|J} = \sum_{j=1}^{J} \sum_{d \leq J-j} \sum_{t > J-(j+d)} D^{j+d+t-J} X_{jdt}$$

$$L_{IBNR|J} = \sum_{j=1}^{J} \sum_{d > J-j} \sum_{t \geq 0} D^{j+d+t-J} X_{jdt}$$

and

$$LIC = L_{IBNR|J} + L_{RBNS|J}$$

For the mean, central second and third order moments we can use the assumed conditional independence, given the number of past reported claims $N_{j,\leq J-j} = \sum_{d=0}^{J-j} N_{jd}$, between future RBNS evolution and future IBNR evolution:

$$\mu_s(LIC|N_{j,\leq J-j}) = \mu_s(L_{RBNS|J}|N_{j,\leq J-j}) + \mu_s(L_{IBNR|J}|N_{j,\leq J-j}), for\ s = 1, 2, 3$$

The proof for the conditional independence is presented in Appendix C. We have used the fact that the conditional distribution of Θ_j, given $\{N_{jd}: j + d \leq J\}$, is equal to the conditional distribution of Θ_j, given $N_{j,\leq J-j}$.

Based on these results, we will need to estimate the central second and third order moments of the $L_{RBNS|J}$ and $L_{IBNR|J}$ to estimate the risk adjustment for the LIC.

From the independence assumption of claim development for different accident periods we have for the IBNR:

$$\mu_s(L_{IBNR|J}|N_{j,\leq J-j}) = \mu_s\left(\sum_{j=1}^{J} \sum_{d > J-j} \sum_{t \geq 0} D^{j+d+t-J} X_{jdt}\, |N_{j,\leq J-j}\right)$$

$$= \sum_{j=1}^{J} D^{s(j-J)} \mu_s\left(\sum_{d > J-j} \sum_{t \geq 0} D^{d+t} X_{jdt}\, |N_{j,\leq J-j}\right), for\ s = 1, 2, 3$$

Considering also the independence assumption of claim development between incurred/reported cohorts (j,d), we get for the RBNS:

$$\mu_s\left(L_{RBNS|J}|N_{j,\leq J-j}\right)=\mu_s\left(\sum_{j=1}^{J}\sum_{d\leq J-j}\sum_{t>J-(j+d)}D^{j+d+t-J}X_{jdt}\,|N_{j,\leq J-j}\right)$$

$$=\sum_{j=1}^{J}\sum_{d\leq J-j}D^{s(j+d-J)}\mu_s\left(\sum_{t>J-(j+d)}D^{t}X_{jdt}\,|N_{j,\leq J-j}\right), for\ s=1,2,3$$

4.4.2. Moments of the Liability for Reported Claims (RBNS)

From the previous section we have seen that the mean, central second and third order moments are defined based on the respective moments of $\sum_{t>J-(j+d)}D^{t}X_{jdt}$.

From the stochastic model assumed, $\{X_{jdt}: t>J-(j+d)\}$ are conditionally independent, given N_{jd}, and $X_{jdt}|N_{jd}\sim CP\left(N_{jd}\upsilon_t, G(\cdot)\right)$. In this case, the conditional distribution of $\sum_{t>J-(j+d)}D^{t}X_{jdt}$ is:

$$\sum_{t>J-(j+d)}D^{t}X_{jdt}\,|N_{jd}\sim CP\left(N_{jd}\upsilon_{>J-(j+d)},\frac{\sum_{t>J-(j+d)}\upsilon_t D^t G}{\upsilon_{>J-(j+d)}}\right)$$

The mean, central second and third order conditional moments of $\sum_{t>J-(j+d)}D^{t}X_{jdt}$ are therefore:

$$\mu_s\left(\textstyle\sum_{t>J-(j+d)}D^{t}X_{jdt}\,|N_{jd}\right)=N_{jd}\upsilon_{>J-(j+d)}\mu_s'\left(\frac{\sum_{t>J-(j+d)}\upsilon_t D^t G}{\upsilon_{>J-(j+d)}}\right)$$

$$=N_{jd}\upsilon_{>J-(j+d)}\frac{\sum_{t>J-(j+d)}\upsilon_t D^{st}\,\mu_s'(G)}{\upsilon_{>J-(j+d)}}=N_{jd}\sum_{t>J-(j+d)}\upsilon_t D^{st}\,\mu_s'(G)$$

$$=N_{jd}\gamma_{s,t>J-(j+d)}, for\ s=1,2,3$$

where we have defined $\gamma_{s,t>J-(j+d)}=\sum_{t>J-(j+d)}\upsilon_t D^{st}\,\mu_s'(G)$.

The previous result is obtained keeping in mind that the severity distribution of the sum of independent compound Poisson distributed random variables is a mixed distribution.

Therefore, we have as the mean, central second and third order conditional moments for the RBNS:

$$\mu_s\left(L_{RBNS|J}\middle|N_{jd}\right)=\sum_{j=1}^{J}\sum_{d\le J-j}D^{s(j+d-J)}\mu_s\left(\sum_{t>J-(j+d)}D^tX_{jdt}\,|N_{jd}\right)$$
$$=\sum_{j=1}^{J}\sum_{d\le J-j}D^{s(j+d-J)}N_{jd}\sum_{t>J-(j+d)}\upsilon_tD^{st}\,\mu_s'(G) \qquad (3)$$
$$=\sum_{j=1}^{J}\sum_{d\le J-j}D^{s(j+d-J)}N_{jd}\gamma_{s,t>J-(j+d)}\,,for\ s=1,2,3$$

4.4.3. Moments of the Liability for Incurred, Unreported Claims (IBNR)

We will define first the conditional moments on Θ_j. Further we will present different options to estimate Θ_j. By unconditional independence between accident years, and conditionally on Θ_j within an accident year, the conditional moments given all Θ_j are:

$$\mu_s\left(L_{IBNR|J}\middle|\Theta_1,\cdots,\Theta_J\right)=\sum_{j=1}^{J}\mu_s\left(L_{IBNR|J}(j)|\Theta_j\right)$$
$$=\sum_{j=1}^{J}\sum_{d>J-j}D^{s(j+d-J)}\mu_s\left(\sum_{t\ge0}D^tX_{jdt}|\Theta_j\right),for\ s=1,2,3$$

From the independence assumption between accident periods j the moments are given by the sum of each accident period. To simplify the derivations we will present the results for a single accident period j since the moments will be the sum of each accident period j.

In a similar derivation as shown in the previous section we get that the mean, central second and third order conditional moments of $\sum_{t\ge0}D^tX_{jdt}$ are:

$$\mu_s\left(\textstyle\sum_{t\ge0}D^tX_{jdt}\,\middle|N_{jd}\right)=N_{jd}\sum_{t\ge0}\upsilon_tD^{st}\,\mu_s'(G)=N_{jd}\gamma_s,for\ s=1,2,3$$

where we have considered $\gamma_s=\gamma_{s,t\ge0}$.

For the conditional mean we have:

$$\mu_1\left(\textstyle\sum_{t\ge0}D^tX_{jdt}\,\middle|\Theta_j\right)=\mu_1\left(\mu_1\left(.\,\middle|N_{jd},\Theta_j\right)\middle|\Theta_j\right)=\mu_1\left(N_{jd}\gamma_1\middle|\Theta_j\right)=\mu_1\left(N_{jd}\middle|\Theta_j\right)\gamma_1$$
$$=p_j\Theta_j\pi_d\gamma_1$$

Starting from the central second order conditional moment and using the previous result we get:

$$\mu_2\left(\sum_{t\geq 0} D^t X_{jdt}\,|\Theta_j\right) = \mu_1\left(\mu_2\left(.\,|N_{jd},\Theta_j\right)|\Theta_j\right) + \mu_2\left(\mu_1\left(.\,|N_{jd},\Theta_j\right)|\Theta_j\right)$$

$$= \mu_1\left(N_{jd}\gamma_2|\Theta_j\right) + \mu_2\left(N_{jd}\gamma_1|\Theta_j\right) = \mu_1\left(N_{jd}|\Theta_j\right)\gamma_2 + \mu_2\left(N_{jd}|\Theta_j\right)\gamma_1^2$$

$$= p_j\Theta_j\pi_d\gamma_2 + p_j\Theta_j\pi_d\gamma_1^2 = p_j\Theta_j\pi_d(\gamma_2 + \gamma_1^2)$$

For the third order moment we will use the following result:

$$\mu_3\left(\sum_{t\geq 0} D^t X_{jdt}\,|\Theta_j\right) = \mu_3'\left(\cdot\,|\Theta_j\right) - 3\mu_2\left(\cdot\,|\Theta_j\right)\mu_1\left(\cdot\,|\Theta_j\right) - \mu_1^3\left(\cdot\,|\Theta_j\right)$$

$$= \mu_3'\left(\cdot\,|\Theta_j\right) - 3p_j\Theta_j\pi_d(\gamma_2 + \gamma_1^2)p_j\Theta_j\pi_d\gamma_1 - \left(p_j\Theta_j\pi_d\gamma_1\right)^3$$

For the third raw moment we have:

$$\mu_3'\left(\sum_{t\geq 0} D^t X_{jdt}\,|\Theta_j\right) = \mu_1\left(\mu_3'\left(.\,|N_{jd},\Theta_j\right)|\Theta_j\right)$$

$$= \mu_1\left(\mu_3\left(\cdot\,|N_{jd},\Theta_j\right) + 3\mu_2\left(\cdot\,|N_{jd},\Theta_j\right)\mu_1\left(\cdot\,|N_{jd},\Theta_j\right) + \mu_1^3\left(\cdot\,|N_{jd},\Theta_j\right)|\Theta_j\right)$$

$$= \mu_1\left(N_{jd}\gamma_3 + 3N_{jd}\gamma_2 N_{jd}\gamma_1 + \left(N_{jd}\gamma_1\right)^3\Big|\Theta_j\right)$$

$$= \gamma_3\mu_1\left(N_{jd}|\Theta_j\right) + 3\gamma_1\gamma_2\mu_2'\left(N_{jd}|\Theta_j\right) + \gamma_1^3\mu_3'\left(N_{jd}|\Theta_j\right)$$

$$= p_j\Theta_j\pi_d\gamma_3 + 3\left(p_j\Theta_j\pi_d + \left(p_j\Theta_j\pi_d\right)^2\right)\gamma_1\gamma_2$$

$$+ \left(p_j\Theta_j\pi_d + 3\left(p_j\Theta_j\pi_d\right)^2 + \left(p_j\Theta_j\pi_d\right)^3\right)\gamma_1^3$$

where we have used the following results for the raw moments of $N_{jd}|\Theta_j$ considering that $\mu_s\left(N_{jd}|\Theta_j\right) = p_j\Theta_j\pi_d \; for \; s = 1, 2, 3$:

$$\mu_1\left(N_{jd}|\Theta_j\right) = p_j\Theta_j\pi_d$$

$$\mu_2'\left(N_{jd}|\Theta_j\right) = \mu_2\left(N_{jd}|\Theta_j\right) + \mu_1^2\left(N_{jd}|\Theta_j\right) = p_j\Theta_j\pi_d + \left(p_j\Theta_j\pi_d\right)^2$$

$$\mu_3'\left(N_{jd}|\Theta_j\right) = \mu_3\left(\cdot\,|\Theta_j\right) + 3\mu_2\left(\cdot\,|\Theta_j\right)\mu_1\left(\cdot\,|\Theta_j\right) + \mu_1^3\left(\cdot\,|\Theta_j\right)$$

$$= p_j\Theta_j\pi_d + 3\left(p_j\Theta_j\pi_d\right)^2 + \left(p_j\Theta_j\pi_d\right)^3$$

Inserting the expression for the third raw moment in $\mu_3\left(\sum_{t\geq 0} D^t X_{jdt}\,|\theta_j\right)$ we get:

$$\mu_3\left(\sum_{t\geq 0} D^t X_{jdt}\,\middle|\Theta_j\right)$$

$$= p_j\Theta_j\pi_d\gamma_3 + 3\left(p_j\Theta_j\pi_d + \left(p_j\Theta_j\pi_d\right)^2\right)\gamma_1\gamma_2$$

$$+ \left(p_j\Theta_j\pi_d + 3\left(p_j\Theta_j\pi_d\right)^2 + \left(p_j\Theta_j\pi_d\right)^3\right)\gamma_1^3$$

$$- 3p_j\Theta_j\pi_d(\gamma_2+\gamma_1^2)p_j\Theta_j\pi_d\gamma_1 - \left(p_j\Theta_j\pi_d\gamma_1\right)^3$$

$$= p_j\Theta_j\pi_d(\gamma_3 + 3\gamma_1\gamma_2 + \gamma_1^3)$$

Therefore, the mean, central second and third order conditional moments are:

$$\mu_1\left(L_{IBNR|J}(j)|\Theta_j\right) = \sum_{d>J-j} D^{(j+d-J)}\, p_j\Theta_j\pi_d\gamma_1 = A_j\Theta_j \tag{4}$$

$$\mu_2\left(L_{IBNR|J}(j)|\Theta_j\right) = \sum_{d>J-j} D^{2(j+d-J)} p_j\Theta_j\pi_d(\gamma_2+\gamma_1^2) = B_j\Theta_j \tag{5}$$

$$\mu_3\left(L_{IBNR|J}(j)|\Theta_j\right) = \sum_{d>J-j} D^{3(j+d-J)} p_j\Theta_j\pi_d(\gamma_3 + 3\gamma_1\gamma_2 + \gamma_1^3) = C_j\Theta_j \tag{6}$$

where we have defined:

$$A_j = \sum_{d>J-j} D^{(j+d-J)}\, p_j\pi_d\gamma_1 \tag{7}$$

$$B_j = \sum_{d>J-j} D^{2(j+d-J)} p_j\pi_d(\gamma_2+\gamma_1^2) \tag{8}$$

$$C_j = \sum_{d>J-j} D^{3(j+d-J)} p_j\pi_d(\gamma_3 + 3\gamma_1\gamma_2 + \gamma_1^3) \tag{9}$$

We have not considered any previous assumptions for the distribution of Θ_j in order to mention at this stage different options that can be applied on the derivation of the moments. We will outline three approaches and proceed with the final one which defines conditional moments for L_{IBNR} while maintaining a reasonable assumption for the distribution of Θ_j. The three approaches (not exhaustive) are:

a) Substitute each Θ_j by its mean;

b) Substitute each Θ_j by an estimate $\overline{\Theta}_j$ (generalization of the previous case);

c) Assume a Bayesian approach in the sense that the posterior distribution of Θ_j is adjusted by the experience on the number of reported claims $N_{j,\leq J-j}$.

Based on the third approach, the mean, central second and third order moments of the $L_{IBNR|J}$ for a certain accident period j are:

$$\mu_1\left(L_{IBNR|J}(j)\middle|N_{j,\leq J-j}\right) = \mu_1\left(A_j\Theta_j\middle|N_{j,\leq J-j}\right) = A_j\mu_1\left(\Theta_j\middle|N_{j,\leq J-j}\right) \tag{10}$$

$$\mu_2\left(L_{IBNR|J}(j)\middle|N_{j,\leq J-j}\right) = B_j\mu_1\left(\Theta_j\middle|N_{j,\leq J-j}\right) + A_j^2\mu_2\left(\Theta_j\middle|N_{j,\leq J-j}\right) \tag{11}$$

$$\begin{aligned}\mu_3&\left(L_{IBNR|J}(j)\middle|N_{j,\leq J-j}\right)\\ &= C_j\mu_1\left(\Theta_j\middle|N_{j,\leq J-j}\right) + 3B_jA_j\mu_2\left(\Theta_j\middle|N_{j,\leq J-j}\right)\\ &+ {A_j}^3\mu_3\left(\Theta_j\middle|N_{j,\leq J-j}\right)\end{aligned} \tag{12}$$

The derivation of these results is presented in Appendix D.

In order to reach a closed form solution for the conditional moments of L_{IBNR} we will assume as a specific case of the third approach the following new assumption:

The random claim frequencies Θ_j are gamma distributed, $\Theta_j \sim Gamma(\alpha, \beta)$.

From this new assumption it results that conditional on $N_{j,\leq J-j}$, the random claim frequencies Θ_j still follow a gamma distribution since we are assuming a gamma as a prior distribution and a Poisson for the likelihood function, $\Theta_j|N_{j,\leq J-j} \sim Gamma\left(\alpha + N_{j,\leq J-j}, p_j\pi_{\leq J-j} + \beta\right)$. Going forward let us define $\alpha_{j|J} = \alpha + N_{j,\leq J-j}$ and $\beta_{j|J} = p_j\pi_{\leq J-j} + \beta$.

The mean, central second and third order moments of $\Theta_j|N_{j,\leq J-j}$ are therefore:

$$\mu_1\left(\Theta_j\middle|N_{j,\leq J-j}\right) = \frac{\alpha_{j|J}}{\beta_{j|J}}$$

$$\mu_2\left(\Theta_j\middle|N_{j,\leq J-j}\right) = \frac{\alpha_{j|J}}{{\beta_{j|J}}^2}$$

$$\mu_3\left(\Theta_j\middle|N_{j,\leq J-j}\right) = 2\frac{\alpha_{j|J}}{{\beta_{j|J}}^3}$$

Substituting by the moments of $\Theta_j|N_{j,\leq J-j}$ we get:

$$\mu_1\left(L_{IBNR|J}(j)\middle|N_{j,\leq J-j}\right) = A_j \frac{\alpha_{j|J}}{\beta_{j|J}}$$

$$\mu_2\left(L_{IBNR|J}(j)\middle|N_{j,\leq J-j}\right) = B_j \frac{\alpha_{j|J}}{\beta_{j|J}} + A_j^2 \frac{\alpha_{j|J}}{{\beta_{j|J}}^2}$$

$$\mu_3\left(L_{IBNR|J}(j)\middle|N_{j,\leq J-j}\right) = C_j \frac{\alpha_{j|J}}{\beta_{j|J}} + 3B_j A_j \frac{\alpha_{j|J}}{{\beta_{j|J}}^2} + 2{A_j}^3 \frac{\alpha_{j|J}}{{\beta_{j|J}}^3}$$

4.4.4. Moments of the Liability for Remaining Coverage (LRC)

To determine the moments needed for the LRC, also defined as CBNI, the derivation is very similar to the one performed for the IBNR. The difference lies in the exposure measure considered and the estimation of $\Theta_j, j > J$.

The random present value of outstanding claim payments for the unexpired risk is:

$$L_{CBNI|J} = \sum_{j>J} \sum_{d\geq 0} \sum_{t\geq 0} D^{j+d+t-J} X_{jdt}$$

The mean, central second and third order moments conditional on Θ_j for $j > J$ are now given by:

$$\mu_1\left(L_{CBNI|J}(j)|\Theta_j\right) = \sum_{d\geq 0} D^{(j+d-J)}\, p_{j|J}\Theta_j\pi_d\gamma_1 = A_{j|J}\Theta_j \quad (12)$$

$$\mu_2\left(L_{CBNI|J}(j)|\Theta_j\right) = \sum_{d\geq 0} D^{2(j+d-J)} p_{j|J}\Theta_j\pi_d(\gamma_2 + \gamma_1^2) = B_{j|J}\Theta_j \quad (13)$$

$$\begin{aligned}\mu_3\left(L_{CBNI|J}(j)|\Theta_j\right) &= \sum_{d\geq 0} D^{3(j+d-J)} p_{j|J}\Theta_j\pi_d(\gamma_3 + 3\gamma_1\gamma_2 + \gamma_1^3)\\ &= C_{j|J}\Theta_j\end{aligned} \quad (14)$$

where we have defined:

$$A_{j|J} = \sum_{d\geq 0} D^{(j+d-J)}\, p_{j|J}\pi_d\gamma_1 \quad (15)$$

$$B_{j|J} = \sum_{d\geq 0} D^{2(j+d-J)} p_{j|J}\pi_d(\gamma_2 + \gamma_1^2) \quad (16)$$

$$C_{j|J} = \sum_{d\geq 0} D^{3(j+d-J)} p_{j|J}\pi_d(\gamma_3 + 3\gamma_1\gamma_2 + \gamma_1^3) \quad (17)$$

As mentioned, the exposure measure $p_{j|J}$ considered is an unearned / unexpired risk measure that represents the risk that the insurance company is exposed to in the following accident periods, $j > J$, for which there is already an obligation at reporting time J.

Once again, to estimate Θ_j there are different approaches that can be followed. The Bayesian approach presented for the IBNR now has no impact since there is no experience on the number of reported claims to adjust the distribution of Θ_j.

The moments for the CBNI for a given accident period j with $j > J$ are given by:

$$\mu_1\left(L_{CBNI|J}(j)\right) = A_{j|J}\mu_1(\Theta_j) \tag{18}$$

$$\mu_2\left(L_{CBNI|J}(j)\right) = B_{j|J}\mu_1(\Theta_j) + A_{j|J}^2\mu_2(\Theta_j) \tag{19}$$

$$\mu_3\left(L_{CBNI|J}(j)\right) = C_{j|J}\mu_1(\Theta_j) + 3B_{j|J}A_{j|J}\mu_2(\Theta_j) + A_{j|J}^3\mu_3(\Theta_j) \tag{20}$$

Substituting by the moments of Θ_j for $j > J$, assuming a gamma distribution, that is, $\Theta_j \sim Gamma(\alpha, \beta)$, we get:

$$\mu_1\left(L_{CBNI|J}(j)\right) = A_{j|J}\frac{\alpha}{\beta}$$

$$\mu_2\left(L_{CBNI|J}(j)\right) = B_{j|J}\frac{\alpha}{\beta} + A_{j|J}^2\frac{\alpha}{\beta^2}$$

$$\mu_3\left(L_{CBNI|J}(j)\right) = C_{j|J}\frac{\alpha}{\beta} + 3B_{j|J}A_{j|J}\frac{\alpha}{\beta^2} + 2A_{j|J}^3\frac{\alpha}{\beta^3}$$

4.5. Parameter estimation

The moments presented in the previous sections considered some parameters which need to be estimated in order to apply the model in practice. We will define possible approaches to estimate the reporting and payment pattern (π_d and υ_t), the raw moments of the severity distribution ($\mu_S'(G)$), and the moments of Θ_j. These methods will then be applied in practice in the following chapter.

Besides these parameters, there are also observable variables which are needed. The insurance company must have the number of past reported claims, $N_{j,\leq J-j}$, and exposure measures for both the IBNR and the CBNI, p_j and, $p_{j|J}$, respectively. Additionally, in order to present discounted moments, the insurance company needs to derive a discount curve according to the standard.

4.5.1. Reporting and payment pattern

The estimation of a reporting and payment pattern has been widely discussed over the years and is not the focus of this dissertation. Therefore, we will briefly present and apply basic methods that are widely used in claims reserving, namely using development factors (DF) and claim rates (CR).

The model requires a reporting pattern and a payment pattern. For the reporting pattern, the information must be organised on an accident period vs reporting delay basis, where the reporting delay is the number of periods elapsed between the accident period and the reporting period. For the payment pattern the information must follow a reporting period vs payment delay basis, where the payment delay is the number of periods elapsed between the reporting period and the payment period. The usual way of organizing claims information only by accident period and aggregate payment delay (with aggregate payment delay being the number of periods elapsed between the accident period and the payment period) does not suffice to estimate these patterns.

Having claims information organized this way, possible estimators for the reporting and payment pattern are given by:

$$\pi_d^*(DF) = \begin{cases} \dfrac{1}{\prod_{d'>0} DF_{d'}}, & d = 0 \\ \dfrac{\prod_{d'=1}^{d} \frac{N_{\leq J-d', \leq d'}}{N_{\leq J-d', \leq d'-1}} - \prod_{d'=1}^{d-1} \frac{N_{\leq J-d', \leq d'}}{N_{\leq J-d', \leq d'-1}}}{\prod_{d'>0} \frac{N_{\leq J-d', \leq d'}}{N_{\leq J-d', \leq d'-1}}} = \dfrac{\prod_{d'=1}^{d} DF_{d'} - \prod_{d'=1}^{d-1} DF_{d'}}{\prod_{d'>0} DF_{d'}} & \end{cases}$$

$$\pi_d^*(CR) = \frac{\frac{N_{\leq J-d,d}}{p_{\leq J-d}}}{\sum_{d'\geq 0} \frac{N_{\leq J-d',d'}}{p_{\leq J-d'}}} = \frac{CR_d}{\sum_{d'\geq 0} CR_{d'}}$$

$$v_t^*(DF) = \begin{cases} \dfrac{1}{\prod_{t'>0} DF_{t'}}, & t = 0 \\ \dfrac{\prod_{t'=1}^{t} \dfrac{\tilde{X}_{\leq J-t', \leq t'}}{\tilde{X}_{\leq J-t', \leq t'-1}} - \prod_{t'=1}^{t-1} \dfrac{\tilde{X}_{\leq J-t', \leq t'}}{\tilde{X}_{\leq J-t', \leq t'-1}}}{\prod_{t'>0} \dfrac{\tilde{X}_{\leq J-t', \leq t'}}{\tilde{X}_{\leq J-t', \leq t'-1}}} = \dfrac{\prod_{t'=1}^{t} DF_{t'} - \prod_{t'=1}^{t-1} DF_{t'}}{\prod_{t'>0} DF_{t'}} \end{cases}$$

$$v_t^*(CR) = \frac{\dfrac{\tilde{X}_{\leq J-t,t}}{\tilde{N}_{\leq J-t}}}{\sum_{t'\geq 0} \dfrac{\tilde{X}_{\leq J-t',t'}}{\tilde{N}_{\leq J-t'}}} = \frac{CR_t}{\sum_{t'\geq 0} CR_{t'}}$$

Where $\tilde{X}_{kt}$ denotes the aggregate payments at payment delay t, for claims reported in year k; and $\tilde{N}_k$ denotes the aggregate number of claim reported in year k, both aggregated across all (j, d) where $j + d = k$. This is similar to the estimation of the reporting pattern except that triangles are by reporting year and payment delay. The variables N_{jd}, p_j and X_{jdt} are observable. For a derivation of these results the reader can consult Neuhaus [42].

4.5.2. Raw moments of the severity distribution

To estimate the raw moments of the severity distribution it is important to understand the meaning of the G distribution that was previously defined. With the compound Poisson assumption, the distribution G does not represent the claim severity for each claim. Indeed, the amount of each settled claim, k, can be seen to follow itself a compound Poisson distribution with frequency parameter 1 and severity distribution G, $X_{jd}^{(k)} \sim CP\big(1, G(.)\big)$. This means that we expect one loss event and from that event we have a severity distribution G.

The actual observations are the amounts needed to settle each claim, $X_{jd}^{(k)}, j + d \leq J$. Therefore, we get the following relation between the moments of the observed variable and the raw moments of the severity distribution G:

$$\mu_1\left(X_{jd}^{(k)}\right) = 1\mu_1(G) = \mu_1(G)$$

$$\mu_2\left(X_{jd}^{(k)}\right) = 1\mu_2'(G) = \mu_2'(G)$$

$$\mu_3\left(X_{jd}^{(k)}\right) = 1\mu_3'(G) = \mu_3'(G)$$

Assume we have a sample $X^{(1)}, \cdots, X^{(n)}$ of settled claims. One option is to calculate the empirical central moments of those claims and transform them to the corresponding raw moments of G, as set out below:

$$(\mu_1)^*(G) = (\mu_1)^*\left(X^{(k)}\right) = \frac{\sum_{\forall k} X^{(k)}}{n} = \bar{X}$$

$$(\mu_2')^*(G) = (\mu_2)^*\left(X^{(k)}\right) = \frac{\sum_{\forall k}\left[X^{(k)} - \bar{X}\right]^2}{n}$$

$$(\mu_3')^*(G) = (\mu_3)^*\left(X^{(k)}\right) = \frac{\sum_{\forall k}\left[X^{(k)} - \bar{X}\right]^3}{n}$$

Another option is to fit a parametric distribution to $X^{(1)}, \cdots, X^{(n)}$ and then use the central moments from the fitted distribution to calculate the corresponding raw moments of G, in the same way as above.

4.5.3. Moments of the random claim frequency

For the moments of the IBNR and the CBNI we need to have an estimate for the random claim frequency Θ_j. We mentioned different approaches that could be applied but proceeded with the Bayesian approach. In order to apply this, we need to estimate the mean, central second and third order moments of Θ_j.

For estimating these moments also different procedures can be followed. We present three different possible methods. By using the first one, the moments can be estimated without assuming a specific distribution. For the other two approaches a specific distribution needs to be assumed. As referred previously in section 4.4.3., we proceed with a gamma distribution for Θ_j and therefore only the parameters of the distribution, α and β, need to be estimated.

1st – Method of moments

The method of moments is one of the simplest approaches that can be used to estimate the unknown moments. It matches the empirical moments with the theoretical moments in order to get an estimator for the moments. In order to apply this method, we need to follow a two-step procedure since the reporting pattern is not observed. The two steps to follow are:

1) Estimate the delay probabilities π_d (already covered in section 4.5.1.);
2) Assume the estimated delay probabilities π_d^* as fixed parameters and then estimate the moments of Θ_j.

Since $N_{j,\leq J-j}$ are observed, the equations to solve are as follows:

$$\left(\mu_1(\Theta_j)\right)^* = \left[\sum_{j=1}^{J} p_j \pi_{\leq J-j}^*\right]^{-1} \sum_{j=1}^{J} N_{j,\leq J-j}$$

$$\left(\mu_2'(\Theta_j)\right)^* = \left[\sum_{j=1}^{J} \left(p_j \pi_{\leq J-j}^*\right)^2\right]^{-1} \sum_{j=1}^{J} N_{j,\leq J-j}\left(N_{j,\leq J-j} - 1\right)$$

$$\left(\mu_3'(\Theta_j)\right)^* = \left[\sum_{j=1}^{J} \left(p_j \pi_{\leq J-j}^*\right)^3\right]^{-1} \sum_{j=1}^{J} N_{j,\leq J-j}\left(N_{j,\leq J-j} - 1\right)\left(N_{j,\leq J-j} - 2\right)$$

where we have considered the observations for all accident periods j since the Θ_j are independent and identically distributed. The derivation of the previous result is presented in Appendix E.

For the central second and third order moments we use the following relation:

$$\left(\mu_2(\Theta_j)\right)^* = \left(\mu_2'(\Theta_j)\right)^* - \left[\left(\mu_1(\Theta_j)\right)^*\right]^2$$

$$\left(\mu_3(\Theta_j)\right)^* = \left(\mu_3'(\Theta_j)\right)^* - 3\left(\mu_2(\Theta_j)\right)^*\left(\mu_1(\Theta_j)\right)^* - \left[\left(\mu_1(\Theta_j)\right)^*\right]^3$$

Thus, the method of moments allows us to estimate the first three moments of the distribution of Θ_j without any parametric assumption.

Assuming a gamma distribution for Θ_j we get the following estimators for α and β:

$$\begin{cases} \left(\mu_1(\Theta_j)\right)^* = \dfrac{\alpha^*}{\beta^*} \\ \left(\mu_2'(\Theta_j)\right)^* = \dfrac{\alpha^*(\alpha^*+1)}{(\beta^*)^2} \end{cases} \Leftrightarrow \begin{cases} \alpha^* = \dfrac{\left[\left(\mu_1(\Theta_j)\right)^*\right]^2}{\left(\mu_2'(\Theta_j)\right)^* - \left[\left(\mu_1(\Theta_j)\right)^*\right]^2} \\ \beta^* = \dfrac{\left(\mu_1(\Theta_j)\right)^*}{\left(\mu_2'(\Theta_j)\right)^* - \left[\left(\mu_1(\Theta_j)\right)^*\right]^2} \end{cases}$$

2nd – Maximum likelihood

We will follow an empirical Bayesian framework meaning that we will estimate the parameters of the prior distribution based on the observed information. For this approach we are assuming that the random claim frequencies Θ_j are gamma distributed, $\Theta_j \sim Gamma(\alpha, \beta)$. Once again, we need to follow two steps:

1) Estimate the delay probabilities π_d (already covered in section 4.5.1.);
2) Assume the estimated delay probabilities π_d^* as fixed parameters and then estimate the parameters α and β based on maximum likelihood estimation.

The likelihood function used is an unconditional one in order to have an expression to maximize dependent on the parameters of the gamma distribution, α and β:

$$L\left(N_{jd}: j + d \leq J\right) \propto \prod_{j=1}^{J} \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha + N_{j,\leq J-j})}{(\beta + p_j \pi_{\leq J-j})^{\alpha + N_{j,\leq J-j}}}$$

The derivation of the likelihood function is presented in Appendix F.

The log likelihood to be maximized in order to estimate α and β can then be solved numerically:

$$\max_{\alpha^*,\beta^*} \log L = \max_{\alpha^*,\beta^*} \sum_{j=1}^{J} \log\left(\frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma\left(\alpha + N_{j,\leq J-j}\right)}{\left(\beta + p_j \pi_{\leq J-j}\right)^{\alpha + N_{j,\leq J-j}}}\right) + c$$

$$= \max_{\alpha^*,\beta^*} \sum_{j=1}^{J} \left[\alpha \log\left(\frac{\beta}{\beta + p_j \pi_{\leq J-j}}\right) + \sum_{n=1}^{N_{j,\leq J-j}} \log\left(\frac{\alpha + n - 1}{\beta + p_j \pi_{\leq J-j}}\right)\right] + c$$

3rd – De Vylder's iterative procedure

Another method that can be followed is based on the iterative procedure proposed by De Vylder [43], [44] and also presented by Neuhaus [42] and Dubey and Gisler [45].

The best predictor for Θ_j can be written as a credibility estimator,

$$E\left(\Theta_j \middle| \mathcal{F}\right) = \mathrm{z}_j \frac{N_{j,\leq J-j}}{p_j \pi_{\leq J-j}} + \left(1 - \mathrm{z}_j\right) \frac{\alpha}{\beta}$$

where we have considered $\mathrm{z}_j = \frac{p_j \pi_{\leq J-j}}{p_j \pi_{\leq J-j} + \beta}$.

Reparametrizing the mean and variance of the gamma distribution by $\tau = \alpha/\beta$ and $\lambda = \alpha/\beta^2$, respectively, the credibility factor is then:

$$z_j = \frac{p_j \pi_{\leq J-j} \lambda}{p_j \pi_{\leq J-j} \lambda + \tau}$$

Estimating the new parameters τ and λ we will be able to get estimates for the parameters α and β. The estimation of these parameters is based on De Vylder's iterative procedure. In order to apply the iteration method we first need to estimate the delay probabilities π_d, as seen in the previous methods. Having these estimates, the iteration follows as:

1) Pick starting values $\tau^*_{(0)}$ and $\lambda^*_{(0)}$. Set the iteration number $i = 0$.
2) For $j = 1, \dots, J$, calculate the Chain Ladder estimates $\hat{\theta}_j = N_{j,\leq J-j}/p_j \pi_{\leq J-j}$.
3) For $j = 1, \dots, J$, calculate credibility factors $z_j{}^{(i)} = \frac{p_j \pi_{\leq J-j} \lambda^*_{(i)}}{p_j \pi_{\leq J-j} \lambda^*_{(i)} + \tau^*_{(i)}}$.
4) Calculate an estimate of the mean $\tau^*_{(i+1)} = \sum_{j=1}^{J} z_j^{(i)} \hat{\theta}_j \,/ \sum_{j=1}^{J} z_j^{(i)}$.
5) Calculate an estimate of the variance $\lambda^*_{(i+1)} = \frac{1}{J-1} \sum_{j=1}^{J} z_j^{(i)} \left(\hat{\theta}_j - \tau^*_{(i+1)}\right)^2$.

Repeat (3)-(5) until convergence is reached.

Finally, determine α^* and β^*:

$$\begin{cases} \dfrac{\alpha^*}{\beta^*} = \tau^* \\ \dfrac{\alpha^*}{(\beta^*)^2} = \lambda^* \end{cases} \Leftrightarrow \begin{cases} \alpha^* = \dfrac{(\tau^*)^2}{\lambda^*} \\ \beta^* = \dfrac{\tau^*}{\lambda^*} \end{cases}$$

This procedure produces unbiased estimators. The proof is presented in Appendix G. Since the estimators depend on themselves, what De Vylder calls pseudo-estimators, an iterative procedure needs to be applied.

4.6. Generalization for the moments of discounted liabilities

In the previous sections we presented and defined a model to get an estimate for the risk adjustment assuming a compound Poisson model of the claim payment process. One could generalize the claim payment model in order to be able to get an estimate for the risk adjustment based on a different set of assumptions about the claim settlement process.

The purpose of this final section is to present the generalized moments and show two other possible specific claim payment models.

The generalization is to allow any stochastic model of individual claim payments $M_{jd}^{[k]} = \left\{X_{jdt}^{[k]}: t = 0,1,\cdots\right\}$ from which we can determine the mean, central second and third order moments of $\sum_{t\geq t'} D^t X_{jdt}^{[k]}$, which we define as $m_{s,t\geq t'} = \mu_s\left(\sum_{t\geq t'} D^t X_{jdt}^{[k]}\right)$. In what follows, we assume that these moments have already been calculated.

We still assume a process similar to a MPP in the sense that the N_{jd} marks are independent and identically distributed and, given a claim, its development is a totally independent stochastic process. We also keep the assumption that $N_{jd}|\Theta_j \sim Poisson\left(p_j \Theta_j \pi_d\right)$.

For the RBNS the general moments are then given by:

$$\mu_s\left(L_{RBNS|J}\middle|N_{jd}\right) = \sum_{j=1}^{J}\sum_{d\leq J-j} D^{s(j+d-J)}\mu_s\left(\sum_{t>J-(j+d)} D^t X_{jdt}\,|N_{jd}\right)$$
$$= \sum_{j=1}^{J}\sum_{d\leq J-j} D^{s(j+d-J)} N_{jd}\mu_s\left(\sum_{t>J-(j+d)} D^t X_{jdt}^{[k]}\,|N_{jd}\right)$$
$$= \sum_{j=1}^{J}\sum_{d\leq J-j} D^{s(j+d-J)} N_{jd} m_{s,t>J-(j+d)}\,, for\ s = 1, 2, 3$$

In the previous derivation we have used the assumption that the N_{jd} marks in the (j,d) cohort are independent and identically distributed which results in $\mu_s\left(\sum_{t>J-(j+d)} D^t X_{jdt}\,|N_{jd}\right) = N_{jd}\mu_s\left(\sum_{t>J-(j+d)} D^t X_{jdt}^{[k]}\,|N_{jd}\right)$.

For the IBNR moments, conditional on Θ_j we get:

$$\mu_1\left(L_{IBNR|J}(j)|\Theta_j\right) = \sum_{d>J-j} D^{(j+d-J)} p_j \Theta_j \pi_d m_1 = A_j \Theta_j$$

$$\mu_2\left(L_{IBNR|J}(j)|\Theta_j\right) = \sum_{d>J-j} D^{2(j+d-J)} p_j \Theta_j \pi_d (m_2 + m_1^2) = B_j \Theta_j$$

$$\mu_3\left(L_{IBNR|J}(j)|\Theta_j\right) = \sum_{d>J-j} D^{3(j+d-J)} p_j \Theta_j \pi_d (m_3 + 3m_1 m_2 + m_1^3) = C_j \Theta_j$$

where we have defined $m_{s,t\geq 0} = m_s$.

Similarly to the derivation based on the compound Poisson model it is also possible to determine the moments of $L_{IBNR|J}(j)$ conditional on $N_{j,\leq J-j}$.

The moments for the LRC will follow a very similar approach to the ones for the IBNR, as already seen in section 4.4.4.

Based on the general setting defined, one can easily obtain the necessary moments by assuming specific stochastic models for the individual claim development, which is what we show in the following sub-sections.

We can also prove that the compound Poisson model shown in section 4.4. is a special case of this generalised setting. In the compound Poisson model we assumed that $X_{jdt}|N_{jd} \sim CP(N_{jd}\upsilon_t, G)$ which implies that:

$$X_{jd}^{[k]} \sim CP(1, G)$$

$$X_{jdt}^{[k]} \sim CP(\upsilon_t, G)$$

and

$$\sum_{t>J-(j+d)} D^t X_{jdt}^{[k]} \sim CP\left(\upsilon_{t>J-(j+d)}, \frac{\sum_{t>J-(j+d)} \upsilon_t D^t G}{\upsilon_{t>J-(j+d)}}\right)$$

Based on these results,

$$m_{s,t>J-(j+d)} = \mu_s\left(\sum_{t>J-(j+d)} D^t X_{jdt}^{[k]} |N_{jd}\right) = \sum_{t>J-(j+d)} \upsilon_t D^{st} \mu_s'(G) = \gamma_{s,>J-(j+d)}$$

and therefore we get the same moments for the RBNS as we obtained in the derivation with the compound Poisson model of the claim payment process.

The moments of the IBNR obtained from the derivation based on the compound Poisson model are the following:

$$\mu_1\left(L_{IBNR|J}(j)|\Theta_j\right) = \sum_{d>J-j} D^{(j+d-J)}\, p_j \Theta_j \pi_d \gamma_1 = A_j \Theta_j$$

$$\mu_2\left(L_{IBNR|J}(j)|\Theta_j\right) = \sum_{d>J-j} D^{2(j+d-J)} p_j \Theta_j \pi_d (\gamma_2 + \gamma_1^2) = B_j \Theta_j$$

$$\mu_3\left(L_{IBNR|J}(j)|\Theta_j\right) = \sum_{d>J-j} D^{3(j+d-J)} p_j \Theta_j \pi_d (\gamma_3 + 3\gamma_1\gamma_2 + \gamma_1^3) = C_j\Theta_j$$

The moments of $\sum_{t\geq 0} D^t X_{jdt}^{[k]}$ are:

$$m_s = \mu_s\left(\sum_{t\geq 0} D^t X_{jdt}^{[k]} \,|N_{jd}\right) = \sum_{t\geq 0} \upsilon_t D^{st} \mu_s'(G) = \gamma_s$$

and therefore we also prove that for the IBNR the compound Poisson model is a special case.

4.6.1. Multinomial distribution

First, we will define as N_{jdt} the number of claims incurred in period j, reported in period $j+d$ and settled in period $j+d+t$ and assume that:

$$\left(N_{jd0}, N_{jd1}, \ldots\right)|N_{jd} \sim Multinomial\left(N_{jd}, \upsilon_0, \upsilon_1, \cdots\right)$$

In this specific setting a dependence within the cohorts (j,d) exists since the number of reported claims is fixed at N_{jd} and only the settlement delay may vary. This can be seen as a two-step process in the sense that there is a random event generating N_{jd} claims each with a possible different settlement time $T_{jd}^{[k]}$. The settlement time for the *k*th claim, $T_{jd}^{[k]}$ is a random variable with $\Pr\{T=t\} = \upsilon_t$.

For the claim development it results that:

$$X_{jdt}|N_{jdt} \sim G^{N_{jdt}*}(\cdot)$$

meaning that there are N_{jdt} claims settled at time t, with severities independent and identically distributed, $X_{jdt}^{[k]} \sim G(.)$

We also assume that $X_{jdt}^{[k]}$ and $T_{jd}^{[k]}$ are independent.

Based on the previous assumptions we get the following result for the raw moments of $\sum_{t>t'} D^t X_{jdt}^{[k]}$:

$$\mu_s'\left(\sum_{t>t'} D^t X_{jdt}^{[k]}\right) = \sum_{t>t'} D^{st} \Pr\left\{T_{jd}^{[k]} = t\right\} \mu_s'\left(X_{jd}^{[k]}\right) = \sum_{t>t'} D^{st} \nu_t \mu_s'\left(X_{jd}^{[k]}\right)$$

In the moments presented we have decided to discount the claims by the number of periods elapsed until the settlement date. Even though not entirely realistic it is a needed approximation in order to simplify the application of the discount factor. The assumption is realistic, however, in lines of insurance where the major portion of claim payments is disbursed at the time of settlement (typically liability insurance).

Having obtained the raw moments, the central moments needed, $m_{s,t>t'}$, are obtained based on the following relations: $m_{1,t>t'} = \mu_1$, $m_{2,t>t'} = \mu_2' - (\mu_1)^2$ and $m_{3,t>t'} = \mu_3' - 3\mu_2'\mu_1 + 2(\mu_1)^3$.

4.6.2. Dirichlet distribution

For this specific model we assume each mark $M_{jd}^{[k]} = \left\{X_{jdt}^{[k]}: t = 0,1,\cdots\right\}$ is distributed as $\left(B_0^{[k]}, B_1^{[k]}, \dots, B_T^{[k]}\right) \cdot X^{[k]}$, where $\left(B_0^{[k]}, B_1^{[k]}, \dots, B_T^{[k]}\right) \sim Dirichlet(\delta_0, \delta_1, \dots, \delta_T)$ and $X^{[k]} \sim G$ and $\left(B_0^{[k]}, B_1^{[k]}, \dots, B_T^{[k]}\right)$ and $X^{[k]}$ are stochastically independent. We can look at this setting as having a stochastic claim severity $X^{[k]}$ paid in accordance with a stochastic payment pattern $\left(B_0^{[k]}, B_1^{[k]}, \dots, B_T^{[k]}\right)$.

Based on the previous assumptions we get the following result for the raw moments of $\sum_{t>t'} X_{jdt}^{[k]}$:

$$\mu_s'\left(\sum_{t>t'} X_{jdt}^{[k]}\right) = \mu_s'\left(B_{t>t'}^{[k]} \cdot X^{[k]}\right) = \frac{\delta_{t>t'}(\delta_{t>t'} + 1)\cdots(\delta_{t>t'} + s - 1)}{\delta(\delta + 1)\cdots(\delta + s - 1)} \mu_s'\left(X^{[k]}\right), for\ s = 1,2,3$$

The proof for the result of the raw moments of $B_{t>t'}^{[k]}$ is presented in Appendix H.

The previous moments do not incorporate discount. In order to do so we can consider:

$$\mu_s'\left(\sum_{t>t'} D^t X_{jdt}^{[k]}\right) \approx \frac{\delta_{t>t'}(\delta_{t>t'} + 1)\cdots(\delta_{t>t'} + s - 1)}{\delta(\delta + 1)\cdots(\delta + s - 1)} \mu_s'\left(X_{jd}^{[k]}\right) \frac{\sum_{t>t'} \upsilon_t D^{st}}{\upsilon_{>t'}}, for\ s = 1,2,3$$

where $\upsilon_t = \delta_t/\delta$ can be seen as the expected payment pattern implied by the Dirichlet distribution.

Once again, the central moments, $m_{s,t>t'}$, are obtained based on the following relations: $m_{1,t>t'} = \mu_1$, $m_{2,t>t'} = \mu_2' - (\mu_1)^2$ and $m_{3,t>t'} = \mu_3' - 3\mu_2'\mu_1 + 2(\mu_1)^3$.

5. Application to a non-life portfolio

5.1. Structure of the program

We have programmed the model presented in the previous chapter in R. The code can be viewed in the following link.

The program is structured into six scripts:

- “Functions - general”, “Functions - parameters”, “Functions - cash flows” and “Functions – results” where all the functions needed for the practical application are defined. Only the script “Functions - general” is necessary to apply the model. The remaining scripts define functions to estimate the inputs needed for the general functions and for the presentation of results. Future users of the program can estimate these inputs by themselves according to the methods they deem more appropriate.
- “Data” where the data obtained is organised and structured to be used as an input in the calculation of the risk adjustment.
- “Results” where the entire code is run and the results are obtained.

The structure of the general functions is presented using flowcharts in Appendix I.

5.2. Data needed

We have applied the model proposed to two risk groups (general liability) from a real-life dataset of compensation claims. The data has been modified to avoid its traceability.

The data consisted of a list of claims evaluated at valuation years from 2007 to 2017. For each claim we had information on accident year, reporting year, amount paid in that period, whether the claim was already settled and if so, the settlement year. We also obtained an exposure measure for each accident year. Given this information we were able to estimate all the necessary inputs. The reporting date considered was the 31 December 2017. For the discount rate we have considered EIOPA’s risk free rate plus volatility adjustment as at 31 December 2017. We had no information on exposure for the CBNI and therefore we have constructed that input based on past exposure.

In the end, the needed inputs to determine the risk adjustment are:

- Exposure measure for each accident period j, p_j and $p_{j|J}$;
- Reporting and payment pattern, π_d and υ_t;
- Discount factor, D;
- Raw moments of the severity distribution, $\mu_s'(G)$;
- Number of reported claims, N_{jd};
- Moments of the random claim frequency Θ_j, $\mu_s(\Theta_j)$.

5.3. Results

Below we present the results obtained for both risk groups for LIC and LRC. We have estimated the reporting and payment pattern based on development factors and the moments of Θ_j were determined by applying De Vylder's iterative procedure.

The following chart shows the results obtained for the present value of future cash flows (the mean of the model defined) for both risk groups as well as the risk adjustment at an 85% confidence level for the CP model. For the IBNS, the RA is not the sum of the RBNS and IBNR because of the diversification benefits.

TABLE I

PVFCF AND RA FOR RBNS, IBNR AND CBNI FOR BOTH RISK GROUPS AT A 85% CONFIDENCE LEVEL FOR THE CP

		RBNS	IBNR	IBNS	CBNI
Risk group 5	PVFCF	130,585,694	114,454,774	245,040,468	33,390,160
	$z_{0.85}\sqrt{\mu_2}$ ("standard deviation adjustment")	26,102,123	25,095,140	36,208,934	13,450,142
	$\frac{(z_{0.85}^2-1)}{6}\frac{\mu_3}{\mu_2}$ ("skewness adjustment")	110,775	113,743	112,200	112,088
Risk group 6	PVFCF	66,596,035	147,842,123	214,438,159	19,929,450
	$z_{0.85}\sqrt{\mu_2}$	12,901,068	20,024,384	23,820,443	9,975,075
	$\frac{(z_{0.85}^2-1)}{6}\frac{\mu_3}{\mu_2}$	101,520	97,309	98,544	126,282

One of the goals of this practical application lies on the comparison of the risk adjustment to the mean, which is presented in the graphs below. The graphs show the relative size of risk adjustment over the present value of future cash flows for different confidence levels ranging from 50% to 99.5%.

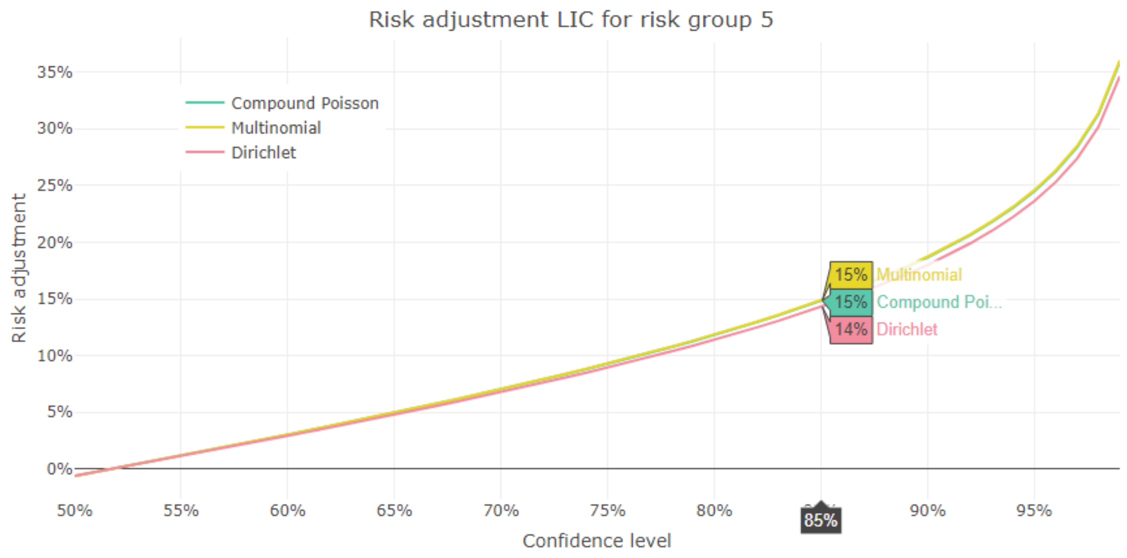

Figure 1 Relative size of risk adjustment over PVFCF for the LIC of risk group 5

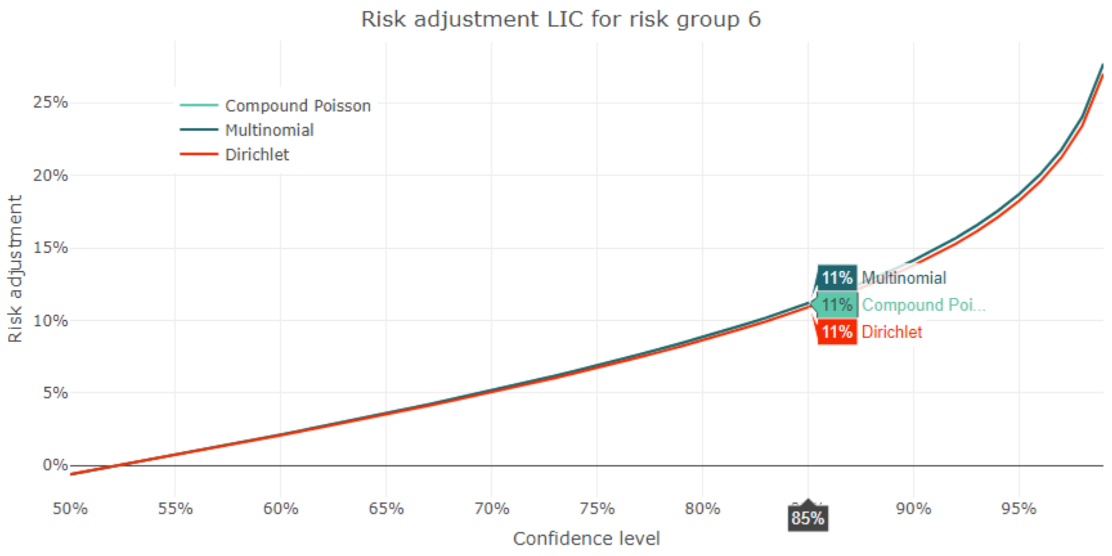

Figure 2 Relative size of risk adjustment over PVFCF for the LIC of risk group 6

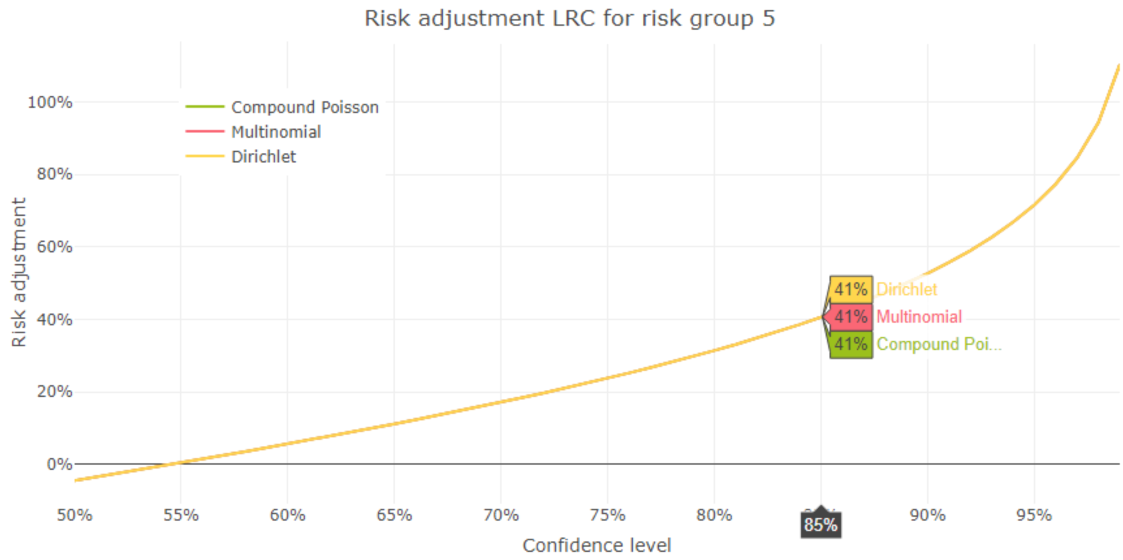

Figure 3 Relative size of risk adjustment over PVFCF for the LRC of risk group 5

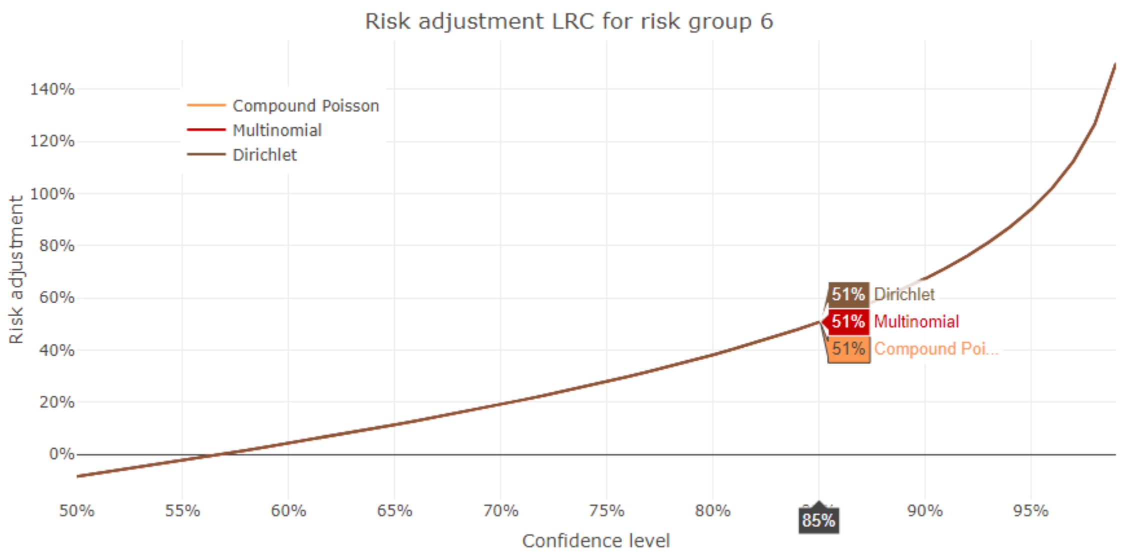

Figure 4 Relative size of risk adjustment over PVFCF for the LRC of risk group 6

As can be seen, the results are very similar regardless of the assumptions chosen for the claim payment model, that is, compound Poisson, multinomial or Dirichlet. The differences between these assumptions increases with the confidence level. For the LIC the Dirichlet results in a lower confidence level. In this practical application we have considered $\delta = 1$ for the Dirichlet distribution. This parameter affects the volatility of the model and as it approaches zero, we draw closer to the multinomial. Overall, the compound Poisson presents less volatility than the multinomial.

For the specific risk groups used, the risk adjustment as a proportion of the mean is lower for the LIC when compared to the LRC, which is easily understood since there is more uncertainty for CBNI claims than for RBNS and IBNR claims combined.

The proportion of the risk adjustment for the LIC compared to the mean ranges from $[0\%, 35\%]$ depending on the confidence level chosen. For the LRC this percentage ranges from $[0\%, 140\%]$. The percentages obtained vary according to the characteristics of the line of business under analysis but can serve as a good indication of how much risk adjustment could be considered for a certain confidence level and a similar line of business.

6. Conclusion

The purpose of this dissertation was to present a model that could be applied by insurance companies to estimate their risk adjustment under IFRS 17 for the non-life business. We have defined a general model that allows us to estimate the central second and third order moments needed to calculate a risk adjustment, using the NP-approximation.

The framework presented is based on a set of assumptions and as for any model there is a trade-off between complexity and accuracy. Our purpose was to present a model that could be fairly easy to understand and apply and coherent in its assumptions. Different assumptions can be explored starting from the general setting and more complex approaches can be pursued in the estimation of the parameters. In the end, by looking at the results obtained one can see a range for the relative size of the risk adjustment compared to the PVFCF which can be applied and used as reference.

It is intuitive to see that the main characteristics of the risk adjustment as defined in the standard are satisfied in the model presented. Indeed, risks with low frequency and high severity as well as risks with a wider probability distribution will result in a higher risk adjustment given the higher central second and third order moments. Contracts with a longer duration result in a higher risk adjustment given that there are more years to be considered in the estimation. Also, the more experience and information there is about the risk and its current estimate and trend the lower the risk adjustment since in principle the estimation will be more accurate and less uncertain.

The results presented can also be useful even if an insurance company does not intend to apply an approach to estimate the risk adjustment based on the confidence level. An insurance company applying a different approach can use this as reference for the approximate confidence level to which its risk adjustment corresponds, which is a mandatory disclosure.

One of the concerns that can be raised by a practicing actuary is the historical data needed in the model. To be able to apply this model it is not necessary to have such a complete dataset as the one mentioned in 5.2. For example, if the number of reported claims is not available this measure can be approximated by the expected number of claims, using a reporting pattern and an a priori claim frequency. If the claim reporting process is reasonably stable and correctly modelled, this approach will give very similar results.

If one is willing to employ more rather than less data, for the RBNS liability one could base the risk adjustment on the actual number of open claims instead of relying on the total number of reported claims. One could also try to model dependency between the severity distribution and the time to settle a claim and/or the reporting delay, since in fact there is some dependency between these variables.

Another issue that can be raised regards the consideration of Unallocated Loss Adjustment Expenses (ULAE) reserve in the LIC. In the model presented we have not considered any amount of reserves for ULAE although it should be considered in the LIC. Nevertheless, its impact is not expected to be significant especially for the determination of the risk adjustment.

As further research the accuracy of the model can be better tested. For example, by using simulation more accurate results could be obtained for the assumptions made since the percentiles would be retrieved from the simulated distributions instead of the NP-approximation.

Lastly, we have assumed as one of the main assumptions independency between lines of business and accident periods which may result in higher diversification benefits than the ones verified. As further research in this area one could try to incorporate the dependency and correlation between lines of business and accident periods into the model as well as allocating the determined risk adjustment to the unit of account level (the level of aggregation prescribed by the standard).

References

[1] R. Norberg and B. Sundt, “Draft of a system for solvency control in non-life insurance,” *ASTIN Bulletin,* vol. 15, no. 2, pp. 149-169, 1985.

[2] A. Kristiansen, “Modeller og metoder for beregning av det forsikringstekniske ansvaret og dets komponenter,” Master thesis from the University of Oslo, unpublished, 1991.

[3] International Accounting Standards Board, IFRS 17 Insurance Contracts, 2017.

[4] International Accounting Standards Board, Basis for Conclusions on IFRS 17 Insurance Contracts, 2017.

[5] International Accounting Standards Board, “Effects Analysis | IFRS 17 Insurance Contracts,” 2017. [Online]. Available: https://cdn.ifrs.org/-/media/project/insurance-contracts/ifrs-standard/ifrs-17-effects-analysis.pdf.

[6] International Accounting Standards Board, “Feedback Statement | IFRS 17 Insurance Contracts,” 2017. [Online]. Available: https://cdn.ifrs.org/-/media/project/insurance-contracts/ifrs-standard/ifrs-17-feedback-statement.pdf.

[7] Transition Resource Group for IFRS 17 Insurance Contracts, “Determining the risk adjustment for non-financial risk in a group of entities,” May 2018. [Online]. Available: https://cdn.ifrs.org/-/media/feature/meetings/2018/may/trg-for-ifrs-17/ap02-risk-adjustment-in-a-group-of-entities.pdf.

[8] Transition Resource Group for IFRS 17 Insurance Contracts, “Reporting on other questions submitted,” April 2019. [Online]. Available: https://cdn.ifrs.org/-/media/feature/meetings/2019/april/trg-for-ifrs-17/ap2-reporting-on-other-questions-submitted.pdf.

[9] International Actuarial Association, “International Standard of Actuarial Practice 4,” November 2019. [Online]. Available: https://www.actuaries.org/IAA/Documents/CTTEES_ASC/Final_ISAPs_Posted/IAA_ISAP4_ApprovedFinal_21Nov2019.pdf.

[10] International Actuarial Association, “International Actuarial Note 100 - Application of IFRS 17 Insurance Contracts (Exposure Draft),” January 2019. [Online]. Available: https://www.actuaries.org/iaa/IAA/Publications/IANs/IAA/Publications/international_actuarial_notes.aspx.

[11] International Actuarial Association, Risk Adjustments for Insurance Contracts under IFRS 17, Ontario: International Actuarial Association, Ottawa, 2018.

[12] R. E. Beard, T. Pentikäinen and E. Pesonen, Risk Theory - The Stochastic Basis of Insurance, London / New York: Chapman and Hall, 1984.

[13] L. Kauppi and P. Ojantakanen, “Approximations of the Generalized Poisson Function,” *ASTIN Bulletin Vol.5 No.2,* pp. 213-226.

[14] R. Norberg, “A Contribution to Modelling of IBNR Claims,” *Scandinavian Actuarial Journal,* vol. 1986, pp. 155-203, 1986.

[15] The Banking, Insurance and Securities Commission of Norway Finance and Insurance Department, “Notes on Risk–Theoretic Methods Applied for Estimating Minimum Requirements for Technical Provisions in Non–life Insurance The Norwegian Case,” 2000.

[16] A. Kristiansen, “Requirements for Technical Provisions in Non-life Insurance - The Norwegian Case,” in *Baltic-Nordic Seminar in Oslo – 21 and 22 January 1999,* 1999.

[17] L. Reback, “IFRS 17 Risk Adjustment Confidence Level Disclosure,” *Financial Reporter,* 2017.

[18] IFoA IFRS 17 for General Insurance working party, “IFRS 17 Managing the Risk Adjustment in Practice,” 6 June 2018. [Online]. Available: https://www.actuaries.org.uk/system/files/field/document/A3%20-%20McMasterCronje.pdf.

[19] P. England and M. Facey, “IFRS 17 Risk Adjustments, and Risk Margins using the Cost-of-Capital approach: Estimating Future Capital Requirements,” 2017. [Online]. Available:

https://www.actuaries.org.uk/system/files/field/document/F6_Peter%20England%20and%20Matt%20Facey_0.pdf.

[20] P. England, “IFRS 17 Risk Adjustments: Reserving or Capital Modelling?,” in *Presented to Institute and Faculty of Actuaries, 25 April 2019,* Available: https://www.actuaries.org.uk/system/files/field/document/G4_Peter%20England_0.pdf, 2019.

[21] P. England, R. Verrall and M.V.Wüthrich, “On the Lifetime and One-Year Views of Reserve Risk, with Application to IFRS 17 and Solvency II Risk Margins,” Available at SSRN: https://ssrn.com/abstract=3141239, 2018.

[22] IFRS 17 CSM Working Party, “IFRS 17: Determining the risk adjustment at the group of contracts level,” [Online]. Available: https://www.linkedin.com/pulse/ifrs-17-determining-risk-adjustment-group-contracts-level-yousuf.

[23] R. S. Miccolis and D. E. Heppen, “A Practical Approach to Risk Margins in the Measurement of Insurance Liabilities for Property and Casualty (General Insurance) under Developing International Financial Reporting Standards,” in *29th International Congress of Actuaries, March 2010,* Cape Town, South Africa, 2010.

[24] E. D. Moro and Y. Krvavych, “Probability of Sufficiency of Solvency II Reserve Risk Margins: Practical Approximations,” Available at SSRN: https://ssrn.com/abstract=2652088, 2016.

[25] CAS Task Force on Fair Value Liabilities, “White Paper on Fair Valuing Property/Casualty Insurance Liabilities,” 2000. [Online]. Available: https://www.casact.org/research/tffvl/whitepaperfinal.pdf.

[26] H. S. a. R. K. Darryl Wagner, “IFRS 17 Risk Adjustment - Insights from a Practical Example,” *The Financial Reporter,* 2018.

[27] L.-R. Koetsier, “Optimising choices with respect to the risk adjustment in IFRS 17,” Master thesis from the Faculty of Science, Radboud University Nijmegen, 2018.

[28] B. Coulter, "Risk adjustments for life insurers: Using a GI approach in a life insurance context," November 2016. [Online]. Available: https://actuaries.org.nz/wp-content/uploads/2016/07/6-paper-Risk-adjustments.pdf.

[29] F. Chevallier, E. D. Moro, Y. Krvavych and I. Rudenko, "Probability of sufficiency of the risk margin for life companies under IFRS 17," in *International Congress of Actuaries*, Berlin, 2018.

[30] E. Norberg, *Risk Adjustment Gjensidige Pensjonsforsikring,* Oslo: Internal note, 2018.

[31] S. Gutterman, "Chapter 17 - Risk and Uncertainty," in *IAA Risk Book*, International Actuarial Association, 2017.

[32] Risk Margin Working Group, "Measurement of Liabilities for Insurance Contracts: Current Estimates and Risk Margins," 2009. [Online]. Available: https://www.actuaries.org/LIBRARY/Papers/IAA_Measurement_of_Liabilities_2009-public.pdf.

[33] Society of Actuaries, "Analysis of Methods for Determining Margins for Uncertainty under a Principle-Based Framework for Life Insurance and Annuity Products," 2009. [Online]. Available: https://www.soa.org/globalassets/assets/files/research/projects/research-analysis-life-annuity.pdf.

[34] Institute of Actuaries of Australia Risk Margins Taskforce, "A Framework for Assessing Risk Margins," in *16th General Insurance Seminar*, Coolum, 2008.

[35] International Actuarial Association, Stochastic Modeling Theory and Reality from an Actuarial Perspective, Ontario: International Actuarial Association, Ottawa, 2010.

[36] European Insurance and Occupational Pensions Authority, "EIOPA's analysis of IFRS 17 Insurance Contracts," 18 October 2018. [Online]. Available: https://register.eiopa.europa.eu/Publications/Reports/EIOPA-18-717_EIOPA_Analysis_IFRS_17_18%2010%202018.pdf.

[37] The Institute of Actuaries of Australia, “Research and Data Analysis Relevant to the Development of Standards and Guidelines on Liability Valuation for General Insurance,” 20 November 2001. [Online]. Available: https://www.actuaries.asn.au/Library/Reports/2001/RelevantDevSGLiabilityValGI.pdf.

[38] S. Collings and G. White, “APRA Risk Margin Analysis,” in *Institute of Actuaries of Australia XIII th General Insurance Seminar*, 2001.

[39] Australian Prudential Regulation Authority, “Prudential Standard GPS 340 Insurance Liability Valuation,” July 2019. [Online]. Available: https://www.legislation.gov.au/Details/F2018L00738.

[40] E. Arjas, “The Claims Reserving Problem in Non-life Insurance: Some Structural Ideas,” *ASTIN Bulletin Volume 19, No. 2*, pp. 139-152, 1989.

[41] R. Norberg, “Prediction of Outstanding Liabilities II Model Variations and Extensions,” *ASTIN Bulletin Volume 29, No. 1*, pp. 5-25, 1999.

[42] W. Neuhaus, Outstanding Claims in General Insurance, Lisbon/Oslo: Unpublished lecture notes on loss reserving, 2014.

[43] F. De Vylder, “Parameter Estimation in Credibility Theory,” *ASTIN Bulletin 10 Vol. 1*, pp. 99-112, (1978).

[44] F. De Vylder, “Practical Credibility Theory with Emphasis on Optimal Parameter Estimation,” *ASTIN Bulletin 12 Vol. 2*, pp. 115-132, 1981.

[45] A. Dubey and A. Gisler, “On Parameter Estimators in Credibility,” *MItteilungen der Vereinigung schweizerischer Versicherungsmathematiker*, no. 2, pp. 187-212, 1981.

[46] C. M. Ramsay, “A Note on the Normal Power Approximation,” *ASTIN Bulletin Vol. 21 No. 1*, pp. 147-150, 1991.

APPENDICES

A. Main concepts in IFRS 17

Under IFRS 17 the insurance contract liability that determines the liability for insurance contracts and claims the insurance company has will be split into a liability for incurred claims and a liability for remaining coverage. In the non-life business two measurement models can be applied to each unit of account to measure these liabilities: the general measurement model (GMM) or the premium allocation approach (PAA).

Only when applying the GMM will there be a need to determine the risk adjustment for both the LIC and the LRC. Under the PAA the LRC is determined as an unearned premium reserve, therefore only the risk adjustment for the LIC will be necessary. Nevertheless, when performing the onerous contract test insurance companies must have an estimate of the risk adjustment for the LRC regardless of the measurement models.

The PAA is a simplified model and its application is always optional, on the condition that any of the two eligibility conditions are met: the coverage period of the contracts is one year or less or there are no material differences in the measurement of the LRC between the GMM or the PAA.

The estimation of the risk adjustment can be performed at whatever level the actuary deems as the most reasonable. However, in order to determine the CSM of the group of contracts, the risk adjustment needs to be allocated at the unit of account level. The unit of account level is the final grouping that results from the level of aggregation rules defined in the standard. The insurance contracts must be grouped based on three steps. First, insurance companies must define portfolios which comprise similar risk managed together. Secondly, for each portfolio, contracts must be split in three groups according to their profitability: onerous, profitable with no significant possibility of becoming onerous and the remaining profitable contracts. Lastly, an insurance company must not include in the same group contracts issued more than one year apart (which is informally known as the annual cohort requirement).

B. Derivation of the NP-approximation

We will use the collective model where N is called the number of claims. X_i is the payment or also referred to as loss. S is called the aggregate payment or loss random variable.

We define μ_S, σ_S and γ_S as the mean, standard deviation and skewness of S. The cumulative distribution function of S we denote by F.

The NP-approximation is based on the Edgeworth series. The interested reader can find its complete derivation in Beard, Pentikäinen and Pesonen [12, pp. 108-121].

The basic idea is that the standardised aggregate loss random variable, $Z = \frac{S-\mu_S}{\sigma_S}$, can be approximated by a quadratic polynomial,

$$Z \approx Y + \frac{\gamma_S}{6}(Y^2 - 1)$$

with $Y \sim N(0,1)$, as stated by Ramsay [46].

Based on the Edgeworth series and considering only the first two terms, the distribution function of Y is approximated by:

$$F(y) \approx N(y) - \frac{\gamma_S}{6} N^{(3)}(y)$$

The third order derivative of the standard normal distribution function is

$$N^{(3)}(y) = (y^2 - 1)\phi(y)$$

with $\phi(y)$ denoting the standard normal density function.

For the distribution function of Y we end up getting

$$F(y) \approx N(y) - \frac{\gamma_S}{6}(y^2 - 1)\phi(y)$$

and

$$F(y + \Delta y) \approx F(y) + F'(y)\Delta y \approx N(y) - \frac{\gamma_S}{6}(y^2 - 1)\phi(y) + \phi(y)\Delta y$$

$$= N(y) + \left[\Delta y - \frac{\gamma_S}{6}(y^2 - 1)\right]\phi(y)$$

By defining

$$\Delta y = \frac{\gamma_S}{6}(y^2 - 1)$$

we get

$$F\left(y + \frac{\gamma_S}{6}(y^2 - 1)\right) \approx N(y)$$

Solving for y

$$y + \frac{\gamma_S}{6}(y^2 - 1) = \frac{s - \mu_S}{\sigma_S}$$

results in

$$y = -\frac{3}{\gamma_S} + \sqrt{\frac{9}{\gamma_S^2} + 1 + \frac{6}{\gamma_S}\frac{s - \mu_S}{\sigma_S}}$$

The NP-approximation is then defined as:

$$F(s) \approx N\left[-\frac{3}{\gamma_S} + \sqrt{\frac{9}{\gamma_S^2} + 1 + \frac{6}{\gamma_S}\frac{s - \mu_S}{\sigma_S}}\right]$$

where N denotes the cumulative $N(0,1)$ distribution function.

It gives an approximate distribution for the aggregate loss random variable, S which is defined based only on the mean, variance and skewness of S. These three measures are completely defined by knowing the first raw moment and the central second and third order moments of S.

This approximation is properly applicable if $Z > 1$, meaning that $S > \mu_S + \sigma_S$, that is, we need to be on the right tail of the distribution for the approximation to be properly applied. The approximation yields better results for distributions with skewness between 0 and 1.

C. Proof that RBNS and IBNR are independent conditional on the number of claims

We want to prove that the RBNS is conditionally independent from the IBNR when we know the past number of reported claims.

Firstly, we assume that claim development of different accident periods are independent.

For one accident period, we drop the subscript j and define:

RBNS: n = reported claim number, x = reported claim amount,

IBNR: $\bar{n}$ = unreported claims, $\bar{x}$ = unreported claim amount.

The density function in a generalised way is:

$$p(\theta, n, \bar{n}, x, \bar{x}) = u(\theta) \underbrace{p(n|\theta)p(\bar{n}|\theta)}_{\substack{cond.indep.\\ given\ \theta}} \underbrace{g^{n*}(x)g^{\bar{n}*}(\bar{x})}_{\substack{severities\ indep.\\ of\ everything}}$$

Where g^{n*} is the density function of the nth convolution of G.

First calculate:

$$p(n, \bar{n}, x, \bar{x}) = \int u(\theta)p(n|\theta)p(\bar{n}|\theta)g^{n*}(x)g^{\bar{n}*}(\bar{x})du(\theta)$$

Similarly:

$$p(n, x) = \int u(\theta)p(n|\theta)g^{n*}(x)du(\theta)$$

Then we find that the distribution of $(\bar{n}, \bar{x})$ (i.e. IBNR) is independent of x (reported claim amount):

$$\begin{aligned} p(\bar{n}, \bar{x}|n, x) &= \frac{\int u(\theta)p(n|\theta)p(\bar{n}|\theta)\cancel{g^{n*}(x)}g^{\bar{n}*}(\bar{x})du(\theta)}{\int u(\theta)p(n|\theta)\cancel{g^{n*}(x)}du(\theta)} \\ &= \frac{\int u(\theta)p(n|\theta)p(\bar{n}|\theta)du(\theta)}{\int u(\theta)p(n|\theta)du(\theta)} \cdot g^{\bar{n}*}(\bar{x}) = \frac{p(n, \bar{n})}{p(n)} \cdot g^{\bar{n}*}(\bar{x}) \\ &= p(\bar{n}|n) \cdot g^{\bar{n}*}(\bar{x}) \end{aligned}$$

So the distribution of future IBNR depends on the past observations only through the claim number.

D. Derivation of the conditional moments of the Liability for Incurred, Unreported Claims (IBNR)

The derivation of the mean, central second and third order conditional moments of $L_{IBNR|J}(j)$ shown in expressions $(10), (11)$, and (12) is presented as follows:

$$\mu_1\left(L_{IBNR|J}(j)\middle|N_{j,\leq J-j}\right) = \mu_1\left(\mu_1\left(.\middle|\Theta_j, N_{j,\leq J-j}\right)\middle|N_{j,\leq J-j}\right) = \mu_1\left(A_j\Theta_j\middle|N_{j,\leq J-j}\right)$$
$$= A_j\mu_1\left(\Theta_j\middle|N_{j,\leq J-j}\right)$$

$$\mu_2\left(L_{IBNR|J}(j)\middle|N_{j,\leq J-j}\right) = \mu_1\left(\mu_2\left(.\middle|\Theta_j, N_{j,\leq J-j}\right)\middle|N_{j,\leq J-j}\right) + \mu_2\left(\mu_1\left(.\middle|\Theta_j, N_{j,\leq J-j}\right)\middle|N_{j,\leq J-j}\right)$$
$$= \mu_1\left(B_j\Theta_j\middle|N_{j,\leq J-j}\right) + \mu_2\left(A_j\Theta_j\middle|N_{j,\leq J-j}\right)$$
$$= B_j\mu_1\left(\Theta_j\middle|N_{j,\leq J-j}\right) + A_j^2\mu_2\left(\Theta_j\middle|N_{j,\leq J-j}\right)$$

$$\mu_3\left(L_{IBNR|J}(j)\middle|N_{j,\leq J-j}\right) = \mu_3'\left(.\middle|N_{j,\leq J-j}\right) - 3\mu_2\left(.\middle|N_{j,\leq J-j}\right)\mu_1\left(.\middle|N_{j,\leq J-j}\right) - \mu_1^3\left(.\middle|N_{j,\leq J-j}\right)$$
$$= \mu_1\left(\mu_3\left(.\middle|\Theta_j, N_{j,\leq J-j}\right) + 3\mu_2\left(.\middle|\Theta_j, N_{j,\leq J-j}\right)\mu_1\left(.\middle|\Theta_j, N_{j,\leq J-j}\right) + \mu_1^3\left(.\middle|\Theta_j, N_{j,\leq J-j}\right)\middle|N_{j,\leq J-j}\right)$$
$$- 3\left(B_j\mu_1\left(\Theta_j\middle|N_{j,\leq J-j}\right) + A_j^2\mu_2\left(\Theta_j\middle|N_{j,\leq J-j}\right)\right)A_j\mu_1\left(\Theta_j\middle|N_{j,\leq J-j}\right) - \left(A_j\mu_1\left(\Theta_j\middle|N_{j,\leq J-j}\right)\right)^3$$
$$= \mu_1\left(C_j\Theta_j\middle|N_{j,\leq J-j}\right) + 3\mu_1\left(B_j\Theta_jA_j\Theta_j\middle|N_{j,\leq J-j}\right) + \mu_1\left(\left(A_j\Theta_j\right)^3\middle|N_{j,\leq J-j}\right)$$
$$- 3\left(B_j\mu_1\left(\Theta_j\middle|N_{j,\leq J-j}\right) + A_j^2\mu_2\left(\Theta_j\middle|N_{j,\leq J-j}\right)\right)A_j\mu_1\left(\Theta_j\middle|N_{j,\leq J-j}\right) - \left(A_j\mu_1\left(\Theta_j\middle|N_{j,\leq J-j}\right)\right)^3$$
$$= C_j\mu_1\left(\Theta_j\middle|N_{j,\leq J-j}\right) + 3B_jA_j\left(\mu_2\left(\Theta_j\middle|N_{j,\leq J-j}\right) + \mu_1^2\left(\Theta_j\middle|N_{j,\leq J-j}\right)\right)$$
$$+ {A_j}^3\left(\mu_3\left(\Theta_j\middle|N_{j,\leq J-j}\right) + 3\mu_2\left(\Theta_j\middle|N_{j,\leq J-j}\right)\mu_1\left(\Theta_j\middle|N_{j,\leq J-j}\right) + \mu_1^3\left(\Theta_j\middle|N_{j,\leq J-j}\right)\right)$$
$$- 3\left(B_j\mu_1\left(\Theta_j\middle|N_{j,\leq J-j}\right) + A_j^2\mu_2\left(\Theta_j\middle|N_{j,\leq J-j}\right)\right)A_j\mu_1\left(\Theta_j\middle|N_{j,\leq J-j}\right) - \left(A_j\mu_1\left(\Theta_j\middle|N_{j,\leq J-j}\right)\right)^3$$
$$= C_j\mu_1\left(\Theta_j\middle|N_{j,\leq J-j}\right) + 3B_jA_j\mu_2\left(\Theta_j\middle|N_{j,\leq J-j}\right) + {A_j}^3\mu_3\left(\Theta_j\middle|N_{j,\leq J-j}\right)$$

E. Estimators for the moments of the random claim frequency using the method of moments

To get estimators for the moments of Θ_j using the method of moments, we will use factorial moments. Assume that $N \sim Poisson(\lambda)$, then the factorial moments are given by:

$$E\left[\prod_{i=0}^{s-1}(N-i)\right] = \sum_{n=0}^{\infty}\left[\prod_{i=0}^{s-1}(n-i)\right]\frac{\lambda^n}{n!}e^{-\lambda} = \sum_{n=s}^{\infty}\frac{\lambda^n}{(n-s)!}e^{-\lambda} = \sum_{n=0}^{\infty}\frac{\lambda^{n+s}}{n!}e^{-\lambda} = \lambda^s$$

Therefore, we get the following conditional moments:

$$E\left(N_{j,\leq J-j}|\Theta_j\right) = p_j\Theta_j\pi_{\leq J-j}$$

$$E\left(N_{j,\leq J-j}\left(N_{j,\leq J-j}-1\right)|\Theta_j\right) = \left(p_j\Theta_j\pi_{\leq J-j}\right)^2$$

$$E\left(N_{j,\leq J-j}\left(N_{j,\leq J-j}-1\right)\left(N_{j,\leq J-j}-2\right)|\Theta_j\right) = \left(p_j\Theta_j\pi_{\leq J-j}\right)^3$$

Which give us the following regression equations:

$$E\left(N_{j,\leq J-j}\right) = E\left(E\left(N_{j,\leq J-j}\middle|\Theta_j\right)\right) = E\left(p_j\Theta_j\pi_{\leq J-j}\right) = p_j\pi_{\leq J-j}\mu_1\left(\Theta_j\right)$$

$$E\left(N_{j,\leq J-j}\left(N_{j,\leq J-j}-1\right)\right) = E\left(E\left(N_{j,\leq J-j}\left(N_{j,\leq J-j}-1\right)\middle|\Theta_j\right)\right) = E\left(\left(p_j\Theta_j\pi_{\leq J-j}\right)^2\right)$$
$$= \left(p_j\pi_{\leq J-j}\right)^2\mu_2'\left(\Theta_j\right)$$

$$E\left(N_{j,\leq J-j}\left(N_{j,\leq J-j}-1\right)\left(N_{j,\leq J-j}-2\right)\right) = E\left(E\left(N_{j,\leq J-j}\left(N_{j,\leq J-j}-1\right)\left(N_{j,\leq J-j}-2\right)\middle|\Theta_j\right)\right)$$
$$= E\left(\left(p_j\Theta_j\pi_{\leq J-j}\right)^3\right) = \left(p_j\pi_{\leq J-j}\right)^3\mu_3'\left(\Theta_j\right)$$

Since $N_{j,\leq J-j}$ are observed we get:

$$\left(\mu_1\left(\Theta_j\right)\right)^* = \left[\sum_{j=1}^{J}p_j\pi_{\leq J-j}\right]^{-1}\sum_{j=1}^{J}N_{j,\leq J-j}$$

$$\left(\mu_2'\left(\Theta_j\right)\right)^* = \left[\sum_{j=1}^{J}\left(p_j\pi_{\leq J-j}\right)^2\right]^{-1}\sum_{j=1}^{J}N_{j,\leq J-j}\left(N_{j,\leq J-j}-1\right)$$

$$\left(\mu_3'\left(\Theta_j\right)\right)^* = \left[\sum_{j=1}^{J}\left(p_j\pi_{\leq J-j}\right)^3\right]^{-1}\sum_{j=1}^{J}N_{j,\leq J-j}\left(N_{j,\leq J-j}-1\right)\left(N_{j,\leq J-j}-2\right)$$

where we have considered the observations for all accident periods j since the Θ_j are independent and identically distributed.

F. Derivation of the likelihood function to estimate the distribution of the random claim frequency

Some of the most usual models present extreme assumptions on claim frequencies Θ_j:

- $\Theta_j = \theta$ identical claim frequencies (assumption on Bornhuetter-Ferguson);
- Θ_j established in isolation (assumption on Chain Ladder).

More realistically, claim frequencies differ from period to period but we have an idea of the claim frequency that is expected. We are more in the middle of the two extreme assumptions.

The best predictor for N_{jd} $(j + d > J)$ in terms of mean squared error is $\overline{N_{jd}} = E\left(N_{jd}\middle|\mathcal{F}\right)$, with $\mathcal{F}$ the observed data. Using the conditional independence and the distribution assumption on $N_{jd}|\mathcal{F}$: $E\left(N_{jd}\middle|\mathcal{F}\right) = p_j E\left(\Theta_j\middle|\mathcal{F}\right)\pi_d$.

Assuming that $\Theta_j \sim Gamma(\alpha, \beta)$, the conditional distribution is still a gamma since the family of gamma distributions forms a family of conjugate priors to the Poisson distribution. The posterior distribution is gamma distributed with parameters: $\alpha_{j|J} = \alpha + N_{j,\leq J-j}$ and $\beta_{j|J} = p_j \pi_{\leq J-j} + \beta$.

From this distribution and after some rearrangements we get the following credibility expression:

$$E\left(\Theta_j\middle|\mathcal{F}\right) = \frac{\alpha + N_{j,\leq J-j}}{p_j \pi_{\leq J-j} + \beta} = \frac{N_{j,\leq J-j}}{p_j \pi_{\leq J-j} + \beta} + \frac{\alpha}{p_j \pi_{\leq J-j} + \beta}$$
$$= \frac{p_j \pi_{\leq J-j}}{p_j \pi_{\leq J-j} + \beta} \frac{N_{j,\leq J-j}}{p_j \pi_{\leq J-j}} + \frac{\beta}{p_j \pi_{\leq J-j} + \beta} \frac{\alpha}{\beta} = \mathrm{z}_j \frac{N_{j,\leq J-j}}{p_j \pi_{\leq J-j}} + \left(1 - \mathrm{z}_j\right) \frac{\alpha}{\beta}$$

where we have considered $\mathrm{z}_j = \frac{p_j \pi_{\leq J-j}}{p_j \pi_{\leq J-j} + \beta}$.

In this setting, the estimate of Θ_j can be seen as a weighted average of two extreme scenarios: a Chain Ladder type estimate and the a priori mean.

We follow an empirical Bayesian framework since the prior distribution is not known in advance. Even though we have assumed a gamma distribution, we do not know the parameters of that distribution and therefore we need to estimate them and we will do so given the observations.

The likelihood function used is an unconditional one in order to have an expression to maximize dependent on the parameters of the gamma distribution, α and β:

$$L\left(\sum_{j=1}^{J}\sum_{d=0}^{J-j} N_{jd}\right) = \prod_{j=1}^{J} L\left(\sum_{d=0}^{J-j} N_{jd}\right) = \prod_{j=1}^{J}\int_{0}^{+\infty} L\left(\sum_{d=0}^{J-j} N_{jd}\,|\theta_j\right) f(\theta_j)d\theta_j$$

$$= \prod_{j=1}^{J}\int_{0}^{+\infty}\left[\prod_{d=0}^{J-j} L(N_{jd}|\theta_j)\right] f(\theta_j)d\theta_j$$

$$= \prod_{j=1}^{J}\int_{0}^{+\infty}\left[\prod_{d=0}^{J-j} \frac{(p_j\theta_j\pi_d)^{N_{jd}} e^{-p_j\theta_j\pi_d}}{N_{jd}!}\right] \frac{\beta^{\alpha}\theta_j^{\alpha-1}e^{-\beta\theta_j}}{\Gamma(\alpha)} d\theta_j$$

$$= \prod_{j=1}^{J}\int_{0}^{+\infty}\left[\prod_{d=0}^{J-j} \theta_j^{\alpha+N_{jd}-1} e^{-\theta_j(\beta+p_j\pi_d)}\right] \frac{\beta^{\alpha}}{\Gamma(\alpha)} d\theta_j \left(\prod_{d=0}^{J-j}\frac{(p_j\pi_d)^{N_{jd}}}{N_{jd}!}\right)$$

$$\propto \prod_{j=1}^{J}\int_{0}^{+\infty} \theta_j^{\alpha+N_{j,\leq J-j}-1} e^{-\theta_j(\beta+p_j\pi_{\leq J-j})} \frac{\beta^{\alpha}}{\Gamma(\alpha)} d\theta_j$$

$$= \prod_{j=1}^{J}\frac{\beta^{\alpha}}{\Gamma(\alpha)}\frac{\Gamma(\alpha+N_{j,\leq J-j})}{(\beta+p_j\pi_{\leq J-j})^{\alpha+N_{j,\leq J-j}}}\int_{0}^{+\infty}\frac{(\beta+p_j\pi_{\leq J-j})^{\alpha+N_{j,\leq J-j}}\ \theta_j^{\alpha+N_{j,\leq J-j}-1}e^{-\theta_j(\beta+p_j\pi_{\leq J-j})}}{\Gamma(\alpha+N_{j,\leq J-j})} d\theta_j$$

$$= \prod_{j=1}^{J}\frac{\beta^{\alpha}}{\Gamma(\alpha)}\frac{\Gamma(\alpha+N_{j,\leq J-j})}{(\beta+p_j\pi_{\leq J-j})^{\alpha+N_{j,\leq J-j}}}$$

G. Proof De Vylder's iterative procedure produces unbiased estimators

De Vylder's iterative procedure gives the following estimators for the mean and variance, respectively:

$$\tau^* = \sum_{j=1}^{J} z_j \hat{\theta}_j \,/ \sum_{j=1}^{J} z_j$$

$$\lambda^* = \frac{1}{J-1} \sum_{j=1}^{J} z_j \left(\hat{\theta}_j - \tau^*\right)^2$$

The estimators presented for the mean and the variance are unbiased, that is, $E[\tau^*] = \tau$ and $E[\lambda^*] = \lambda$:

$$E[\tau^*] = E\left[\frac{\sum_{j=1}^{J} z_j \hat{\theta}_j}{\sum_{j=1}^{J} z_j}\right] = \frac{\sum_{j=1}^{J} z_j E\left[\hat{\theta}_j\right]}{\sum_{j=1}^{J} z_j} = \frac{\sum_{j=1}^{J} z_j \tau}{\sum_{j=1}^{J} z_j} = \tau$$

since

$$E\left[\hat{\theta}_j\right] = E\left[E\left[\frac{N_{j,\le J-j}}{p_j \pi_{\le J-j}} |\theta_j\right]\right] = E\left[\frac{p_j \theta_j \pi_{\le J-j}}{p_j \pi_{\le J-j}}\right] = E\left[\theta_j\right] = \tau$$

$$\begin{aligned}
E[\lambda^*] &= E\left[\frac{1}{J-1} \sum_{j=1}^{J} z_j \left(\hat{\theta}_j - \tau^*\right)^2\right] = \frac{1}{J-1} \sum_{j=1}^{J} z_j E\left[\left(\hat{\theta}_j - \tau^*\right)^2\right] \\
&= \frac{1}{J-1} \sum_{j=1}^{J} z_j \left(\mu_2\left(\hat{\theta}_j - \tau^*\right) + E^2\left[\hat{\theta}_j - \tau^*\right]\right) = \frac{1}{J-1} \sum_{j=1}^{J} z_j \mu_2\left(\hat{\theta}_j - \tau^*\right) \\
&= \frac{1}{J-1} \sum_{j=1}^{J} z_j \left(\mu_2\left(\hat{\theta}_j\right) + \mu_2(\tau^*) - 2Cov\left(\hat{\theta}_j, \tau^*\right)\right) \\
&= \frac{1}{J-1} \sum_{j=1}^{J} z_j \left(\frac{\lambda}{z_j} + \frac{\lambda}{\sum_{j=1}^{J} z_j} - 2\frac{\lambda}{\sum_{j=1}^{J} z_j}\right) \\
&= \frac{1}{J-1} \left(\sum_{j=1}^{J} z_j \frac{\lambda}{z_j} + \frac{\sum_{j=1}^{J} z_j \lambda}{\sum_{j=1}^{J} z_j} - 2\frac{\sum_{j=1}^{J} z_j \lambda}{\sum_{j=1}^{J} z_j}\right) = \frac{1}{J-1} \lambda(J+1-2) = \lambda
\end{aligned}$$

In the previous proof we have used the following results:

$$\begin{aligned}\mu_2(\hat{\theta}_j) &= \mu_1\left(\mu_2(\hat{\theta}_j|\theta_j)\right) + \mu_2\left(\mu_1(\hat{\theta}_j|\theta_j)\right) \\ &= \mu_1\left(\mu_2\left(\frac{N_{j,\le J-j}}{p_j\pi_{\le J-j}}\middle|\theta_j\right)\right) + \mu_2\left(\mu_1\left(\frac{N_{j,\le J-j}}{p_j\pi_{\le J-j}}\middle|\theta_j\right)\right) \\ &= \mu_1\left(\frac{p_j\theta_j\pi_{\le J-j}}{\left(p_j\pi_{\le J-j}\right)^2}\right) + \mu_2\left(\frac{p_j\theta_j\pi_{\le J-j}}{p_j\pi_{\le J-j}}\right) = \frac{\mu_1(\theta_j)}{p_j\pi_{\le J-j}} + \mu_2(\theta_j) \\ &= \frac{\tau}{p_j\pi_{\le J-j}} + \lambda = \frac{\lambda}{z_j}\end{aligned}$$

$$\mu_2(\tau^*) = \mu_2\left(\frac{\sum_{j=1}^J z_j\hat{\theta}_j}{\sum_{j=1}^J z_j}\right) = \frac{\sum_{j=1}^J z_j^2\mu_2(\hat{\theta}_j)}{\left(\sum_{j=1}^J z_j\right)^2} = \frac{\sum_{j=1}^J z_j^2\frac{\lambda}{z_j}}{\left(\sum_{j=1}^J z_j\right)^2} = \frac{\lambda}{\sum_{j=1}^J z_j}$$

$$Cov(\hat{\theta}_j, \tau^*) = Cov\left(\hat{\theta}_j, \frac{\sum_{j=1}^J z_j\hat{\theta}_j}{\sum_{j=1}^J z_j}\right) = \frac{\sum_{j=1}^J z_j\,\mu_2(\hat{\theta}_j)}{\sum_{j=1}^J z_j} = \frac{\sum_{j=1}^J z_j\frac{\lambda}{z_j}}{\sum_{j=1}^J z_j} = \frac{\lambda}{\sum_{j=1}^J z_j}$$

H. The raw moments of outstanding Dirichlet distribution

It is a known result that the sum of independent variables with a gamma distribution is still gamma distributed, that is, if $X_1, \cdots, X_n$ are independent and $X_i \sim Gamma(\delta_i, \gamma)$ then $X = \sum_{i=1}^{n} X_i \sim Gamma(\sum_{i=1}^{n} \delta_i, \gamma)$.

Furthermore, if $X_1 \sim Gamma(\delta_1, \gamma)$, $X_2 \sim Gamma(\delta_2, \gamma)$ and independent then $Z = X_1 + X_2 \sim Gamma(\delta_1 + \delta_2, \gamma)$ and $W = \frac{X_1}{X_1 + X_2} \sim Beta(\delta_1, \delta_2)$.

The variable whose moments we want to determine, $B_{t>t'}^{[k]}$, can be seen as the quotient of two gamma distributed random variables, that is:

$$B_{t>t'}^{[k]} = \frac{X_{t>t'}}{X_1 + X_2 + \cdots} = \frac{X_{t>t'}}{Z}$$

where $X_{t>t'} \sim Gamma(\delta_{t>t'}, \gamma)$ and $Z \sim Gamma(\delta, \gamma)$.

Seeing that $Z - X_{t>t'} \sim Gamma(\delta_{t \leq t'}, \gamma)$ we rearrange the previous expression:

$$B_{t>t'}^{[k]} = \frac{X_{t>t'}}{X_{t>t'} + (Z - X_{t>t'})}$$

and conclude from the second result that $B_{t>t'}^{[k]} \sim Beta(\delta_{t>t'}, \delta_{t \leq t'})$.

The beta distribution $B_{t>t'}^{[k]} \sim Beta(\delta_{t>t'}, \delta_{t \leq t'})$ has support in the interval $[0,1]$ and can be seen as a generalization of the uniform distribution which allows the random variable to have a behavior different than a constant value over its support. Its density function is:

$$f\left(B_{t>t'}^{[k]}\right) = \frac{\Gamma(\delta_{t>t'} + \delta_{t \leq t'})}{\Gamma(\delta_{t>t'})\Gamma(\delta_{t \leq t'})} {B_{t>t'}^{[k]}}^{\delta_{t>t'} - 1} \times \left(1 - B_{t>t'}^{[k]}\right)^{\delta_{t \leq t'} - 1}$$

for $B_{t>t'}^{[k]} \in [0,1]$.

The raw moments of $B_{t>t'}^{[k]}$ are then given by:

$$E\left[\left(B_{t>t'}^{[k]}\right)^{s}\right]=\int_{0}^{1}\left(B_{t>t'}^{[k]}\right)^{s}\frac{\Gamma(\delta_{t>t'}+\delta_{t\le t'})}{\Gamma(\delta_{t>t'})\Gamma(\delta_{t\le t'})}B_{t>t'}^{[k]}{}^{\delta_{t>t'}-1}\times\left(1-B_{t>t'}^{[k]}\right)^{\delta_{t\le t'}-1}dB_{t>t'}^{[k]}$$

$$=\frac{\Gamma(\delta_{t>t'}+\delta_{t\le t'})}{\Gamma(\delta_{t>t'})\Gamma(\delta_{t\le t'})}\frac{\Gamma(\delta_{t>t'}+s)\Gamma(\delta_{t\le t'})}{\Gamma(\delta_{t>t'}+s+\delta_{t\le t'})}$$

$$\times\int_{0}^{1}\frac{\Gamma(\delta_{t>t'}+s+\delta_{t\le t'})}{\Gamma(\delta_{t>t'}+s)\Gamma(\delta_{t\le t'})}B_{t>t'}^{[k]}{}^{\delta_{t>t'}+s-1}\times\left(1-B_{t>t'}^{[k]}\right)^{\delta_{t\le t'}-1}dB_{t>t'}^{[k]}$$

$$=\frac{\Gamma(\delta)\Gamma(\delta_{t>t'}+s)}{\Gamma(\delta_{t>t'})\Gamma(\delta+s)}=\frac{(\delta-1)!\,(\delta_{t>t'}+s-1)!}{(\delta_{t>t'}-1)!\,(\delta+s-1)!}$$

$$=\frac{\delta_{t>t'}(\delta_{t>t'}+1)\ldots(\delta_{t>t'}+s-1)}{\delta(\delta+1)\ldots(\delta+s-1)}$$

I. Structure of the general functions defined in the R program

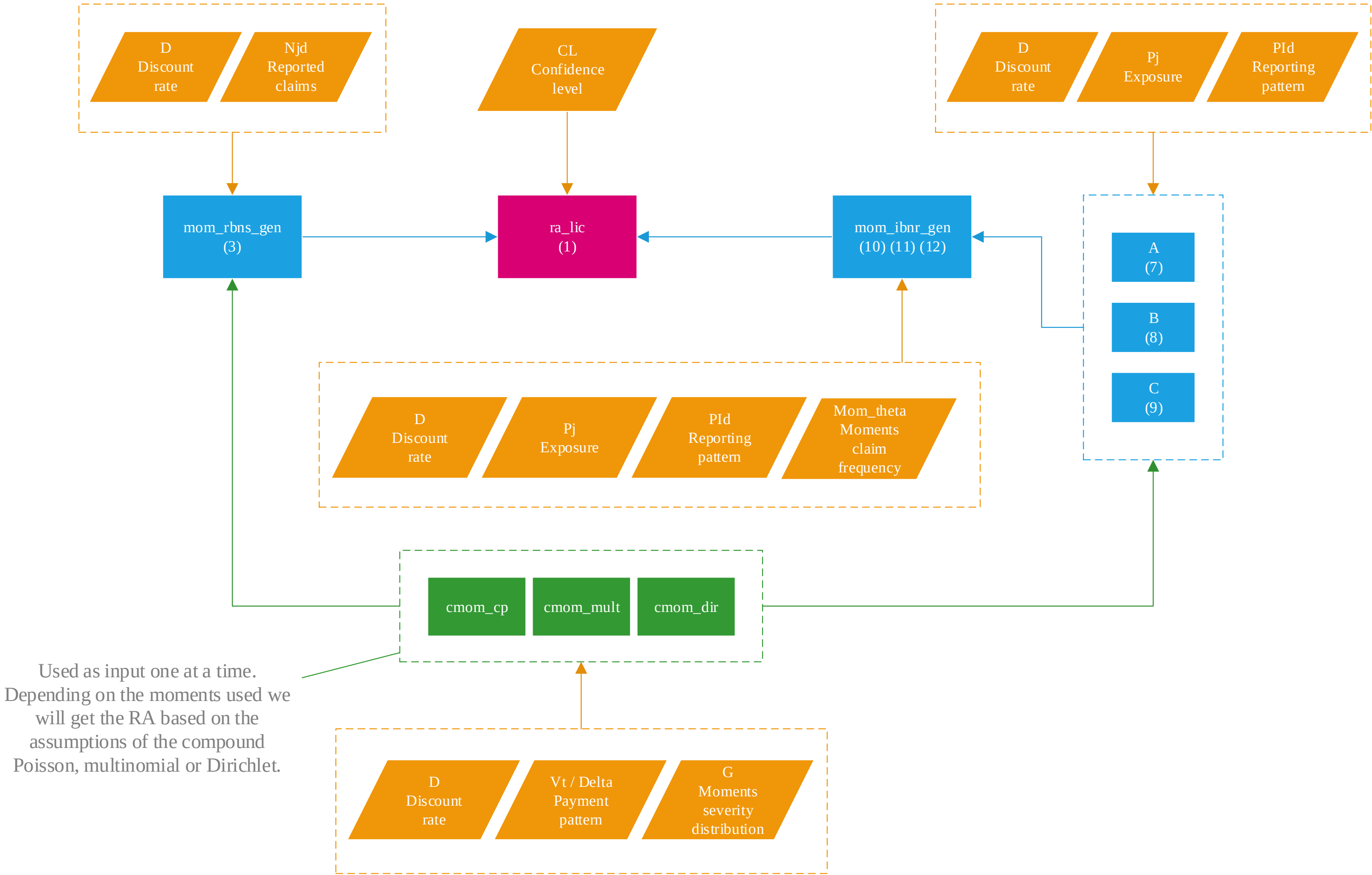

Figure 5 Structure of function to determine the risk adjustment for the LIC

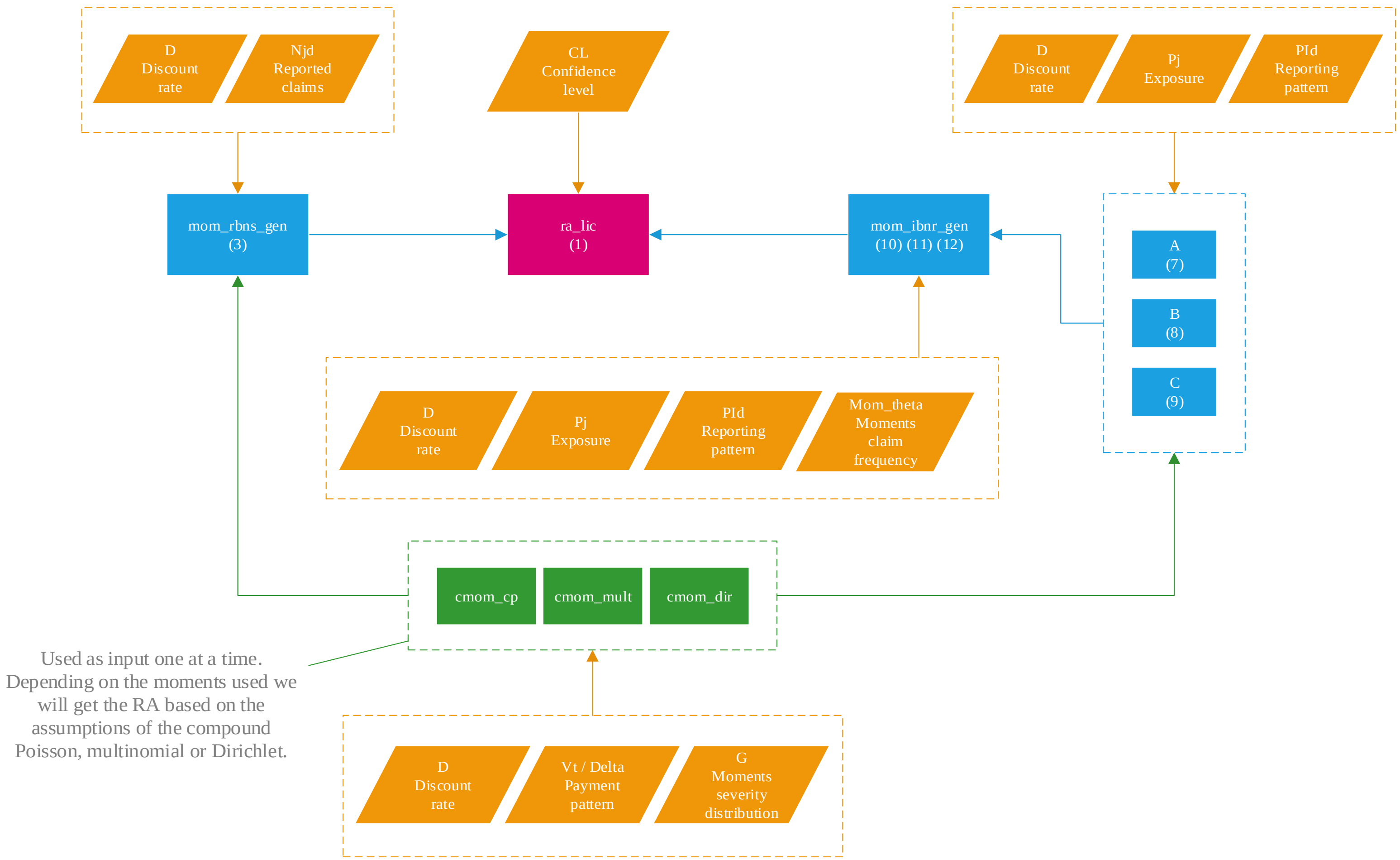

Figure 6 Structure of function to determine the risk adjustment for the LRC