LISBON SCHOOL OF ECONOMICS & MANAGEMENT
UNIVERSIDADE DE LISBOA

MASTER
ACTUARIAL SCIENCE

MASTER'S FINAL WORK
INTERNSHIP REPORT

APPLICATION OF STOCHASTIC MODELS ON THE PORTUGUESE POPULATION AND DISTORTION TO WORKERS COMPENSATION PENSIONERS EXPERIENCE.

MBELLI NJAH NKWENTI

SEPTEMBER - 2015

LISBON SCHOOL OF ECONOMICS & MANAGEMENT
UNIVERSIDADE DE LISBOA

MASTER
ACTUARIAL SCIENCE

MASTER'S FINAL WORK
INTERNSHIP REPORT

APPLICATION OF STOCHASTIC MODELS ON THE PORTUGUESE POPULATION AND DISTORTION TO WORKERS COMPENSATION PENSIONERS EXPERIENCE.

MBELLI NJAH NKWENTI

SUPERVISORS: ONOFRE ALVES SIMÕES
CARLOS EDUARDO BARRENHO DA ROSA

SEPTEMBER - 2015

Acknowledgements

I leave in the first place, a very special thanks to my family, especially my daughter, mother who is of late, my father, my brother and sisters, for believing in me and for their support and understanding that helped me get here.

I also wish to thank AXA Portugal for giving me the opportunity to be able to carry out this research, in particular special thanks goes to my Company supervisor Carlos da Rosa and the entire staffs of the Actuarial department of AXA for the invaluable guidance they provided to me throughout this work.

Finally, Professor Onofre Alves Simões for his availability and support, whose valuable contribution was critical to this work.

ABSTRACT

This study was motivated by an internship offered by AXA on the topic of pensions payable under the workers compensation (WC) line of business. There are two types of pensions: the compulsorily recoverable and the not compulsorily recoverable. A pension is compulsorily recoverable for a victim when there is less than 30% of disability and the pension amount per year is less than six times the minimal national salary.

The law defines that the mathematical provisions for compulsory recoverable pensions must be calculated by applying the following bases: mortality table TD88/90 and rate of interest 5.25% (maybe with rate of management). To manage pensions which are not compulsorily recoverable is a more complex task because technical bases are not defined by law and much more complex computations are required. In particular, companies have to predict the amount of payments discounted reflecting the mortality effect for all pensioners (this task is monitored monthly in AXA).

The purpose of this report is thus to develop a stochastic model for the future mortality of the worker's compensation pensioners of both the Portuguese market workers and AXA portfolio. Not only is past mortality modeled, also projections about future mortality are made for the general population of Portugal as well as for the two portfolios mentioned earlier.

The global model is split in two parts: a stochastic model for population mortality which allows for forecasts, combined with a point estimate from a portfolio mortality model obtained through three different relational models (Cox Proportional, Brass Linear and Workgroup PLT). The one year death probabilities for ages 0-110 for the period 2013-2113 are obtained for the general population and the portfolios. These probabilities are used to compute different life table functions as well as the not compulsorily recoverable reserves for each of the models required for the pensioners, their spouses and children under 25.

The results obtained are compared with the not compulsory recoverable reserves computed using the static mortality table (TV 73/77) that is currently being used by AXA, to see the impact on this reserve if AXA adopted the dynamic tables.

KEY WORDS: worker's compensation pensioners, compulsorily recoverable, life table functions, relational models.

RESUMO

Este estudo foi motivado por um estágio proposto pela AXA, e visa dar um contributo para a resolução do problema da correta determinação das reservas para cobrir os encargos futuros com as indemnizações no Ramo de Acidentes de Trabalho.

A questão coloca-se com particular relevância relativamente às pensões ditas 'não obrigatoriamente remíveis', pois a autoridade supervisora (ASF) deixa em parte ao critério das companhias qual o modelo de mortalidade a aplicar.

O objetivo do estágio, e que este relatório procura traduzir, foi assim o desenvolvimento de um modelo estocástico para a mortalidade dos pensionistas em análise, para o que foi necessário considerar inicialmente toda a população portuguesa, passando-se depois para a população constituída por todos os trabalhadores cobertos por apólices de Acidentes de Trabalho e, finalmente, para os trabalhadores segurados na AXA.

O modelo global é composto por um modelo estocástico para a mortalidade da população combinado com um modelo de mortalidade para o portfólio, obtido a partir de três modelos relacionais (Cox Proportional, Brass Linear and Workgroup PLT). As probabilidades de morte a um ano para as idades 0-110, ao longo do período 2013-2113, foram calculadas para a população em geral e para as duas carteiras e utilizadas na construção das correspondentes tábuas de mortalidade e funções associadas. Pôde então determinar-se o montante das reservas relativas aos pensionistas, incluindo os cônjuges e os filhos com idades inferiores a 21 anos.

Os valores obtidos para as reservas foram então comparados com os que a AXA estabeleceria, caso continuasse a usar a mesma tabela estática atualmente em vigor (TV 73/77), para se aferir sobre o impacto da eventual implementação das tábuas resultantes do estudo.

PALAVRAS-CHAVE: Acidentes de Trabalho, Pensões, Tábuas de Mortalidade, Modelos Relacionais.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER 1: INTRODUCTION

1.1 Worker's Compensation: Compensations and Annuities

This report is based on a six months internship carried out at the Actuarial Department of AXA Portugal. The purpose of the internship was to develop a stochastic model for the future mortality of pensioners in AXA worker's compensation portfolio and to study the impact on the reserves using this stochastic model instead of the static life table (TV 73/77) currently being used.

In Portugal, Workers Compensation (WC) is mandatory according to Law No.98/2009 of September 4th: all employers must insure the risk (all employees) in an insurance company; also, all self-employees must subscribe WC insurance.

This line of business includes obligations to compensate the victim and/or respective beneficiaries in case of accidents at work or if occupational diseases occur, from which a disability situation results (also including rehabilitation and reintegration). The accident on the journey between the employee's residence and the workplace and vice versa, and other specific circumstances, for example, when the accident occurred between the workplace and the place where the employee takes meals, are also covered.

The nature of the resulting disability may be temporary or permanent. The temporary disability may be partial or absolute. The permanent disability may be partial, permanent for usual work or permanent for any work.

To make it easier for the readers to understand this issue, we will divide the losses in two parts: compensations and annuities.

Compensations:

The right to reparation includes the following forms: in kind and in cash. In kind, the main benefits are medical, pharmaceutical and hospital assistance needed to restore health and work capacity. Included are also transportation and accommodation, technical help for functional disabilities, thermal treatments and dependent relatives' psychological assistance. All compensations provided by law, such as, compensation for temporary disability, death and funeral expenses, and subsidies for high disability (above 70%), house adaptation, rehabilitation and social integration are paid in cash.

The compensations for temporary disability intend to compensate the victim for the temporary loss of work capacity while under ambulatory treatment or vocational rehabilitation. In absolute temporary disability, the victim earns a daily compensation equal to 70% of daily salary during the first 12 months and 75% in the following period. For partial temporary disability, the victim has the daily compensation equal to 70% of daily salary times the degree of disability.

Annuities:

Financial compensations for permanent disability are more complex because they include not only the victim but they may include other beneficiaries (orphans, husband/wife, parents or equivalents that live together and have earnings below the social pension). In Portugal this takes a significantly different character from what is found in most European countries (Belgium, Finland and Denmark are the exceptions that we know similar to Portugal). The management of this risk is maintained in Property and Casualty team (P&C) and it is present on P&C Balance sheet. The Law defines that in absolute permanent disability for any kind of work, the victim has the right to an annual pension equal to 80% of the salary and can add 10% per dependent person until the salary limit is reached.

In absolute permanent disability for usual work, the victim has the right to an annual pension between 50% and 70% of his/her salary, depending on the functional capacity to develop another compatible work. In partial permanent disability, the victim has the right to an annual pension equal to 70% of his salary devaluated by the degree of ability. Providing additional support for third person is assigned to victims without the capacity for basic daily needs.

In case of death, the family or equivalent beneficiaries have the right to compensation:

– Husband/Wife or equivalent beneficiaries: compensation is 30% of the victim's salary until the retirement age of the beneficiary and 40% above retirement age or when disability or chronic illness is verified;

– Orphans: compensation is 20% of victim's salary if there is only one; it's 40% if there are two orphans; and 50% if there are three or more orphans (may be 80% for orphans of both parents). The orphans have the right for compensation until 25 years old as long as they are students. Orphans are entitled to a pension for life in case of disability or chronic illness;

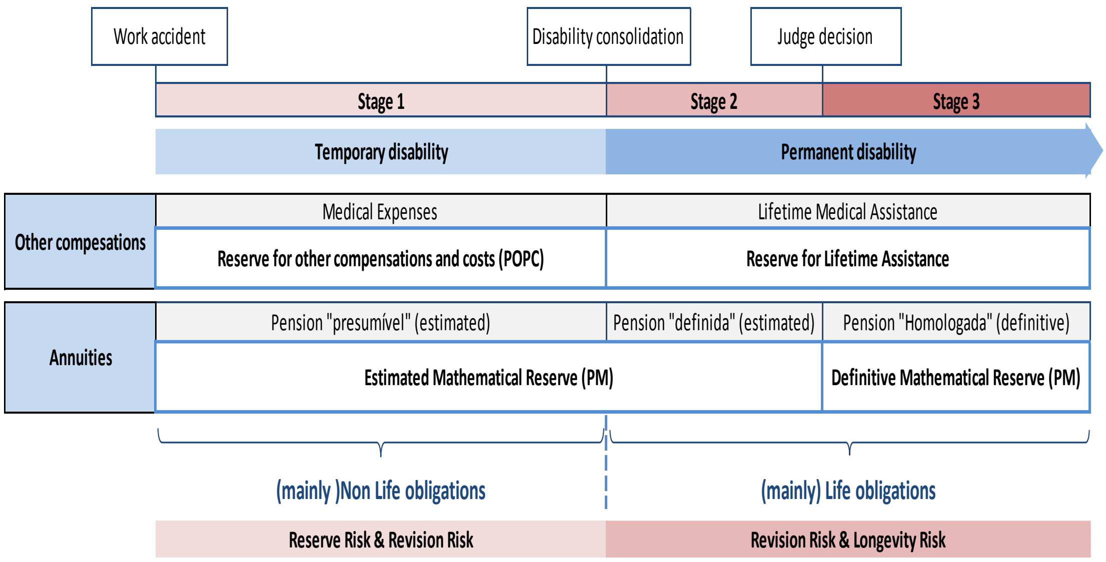
– Parents or equivalent beneficiaries: compensation is 10% of the victim's salary for each beneficiary, limited to 30% of the salary. When there isn't husband/wife or orphans, the parents or equivalents earn 15% for each until retirement age and 20% above retirement age or when disability or chronic illness is verified (however limited to 80%);

There are two types of pensions: the compulsorily recoverable and the not compulsorily recoverable. A pension is compulsorily recoverable for a victim when there is less than 30% of disability and the pension amount per year is less than six times the minimal national salary. For other beneficiaries (except orphans) only the second condition applies. On the other hand, a pension can be partially recoverable for victims if they have 30% or more of disability. The pensions for the other beneficiaries can be partially recoverable if the pension leftover is not less than six times the minimal national salary and capital redemption cannot be more than the capital that results of 30% of disability. The law defines that the mathematical provisions for compulsory recoverable and partial recoverable are calculated applying the following conditions: mortality table – TD 88/90 and rate of interest – 5.25% (maybe with rate of management).

The victims can require the revaluation of their disability once a year. In outdated law (before 2010), this situation is only possible during 10 years after the pension has been fixed.

The figure below shows the framework for WC reserve.

FIGURE 1: WORKER'S COMPENSATION FRAMEWORK: RESERVES TYPES



Source: AXA documentations

1.2 Reserving for worker's compensation

The Instituto de Seguros de Portugal (ISP) requires that Insurers have in their balance sheet mathematical reserves for permanent disability estimated from the date of occurrence of the accident. When estimated disability estimated is less 30% the annuity is paid as a lump sum (compulsory) and the capital to be paid after judge decision is calculated with legal parameters. Otherwise, insures must use best estimate parameters based on actuarial studies to evaluate it. AXA Portugal presently uses the following parameters in this case:

		Mortality table	Interest rate
Disability ≤30%	Legal parameters	TD 88/90	5.25%
Disability ≥30%	Best estimate parameters	TD 73/77	4.5%

The actualization of the annuities due to inflation is assured by a public fund (Fundo de Acidentes de Trabalho - FAT).The Fund receives from companies two types of contributions: 0.15% of sum insured and 0.85% of capital redemption of pensions' stock at December 31 (that includes the mathematical provision for third person's assistance).The capital redemption amount is calculated based on the following parameters: Mortality table – TD 88/90; Rate of interest – 5.25%; and Rate of management – 0%.The companies must predict the provision for Future "FAT" in their Balance Sheet.

Insurers must also have in their balance sheet claims reserves for other compensations (medical expenses, daily compensations and other compensation).These reserves includes a reserve for whole life medical assistance that is calculated using life techniques (best estimate parameters).

According to Brian et al. (2013), adverse reserve development in older accident years is a persistent problem in the workers' compensation industry. Predicting the final cost of workers' compensation claims is particularly difficult due to the long period of time over which claimants receive statutory indemnity and medical benefit payments. Misestimating of reserves for these claims can result in financial reporting errors, claim settlement inequities, loss of reinsurance protection due to late reporting of large claims (through "sunset" clauses) as well as a drag on current earnings. The misestimating of reserves for lifetime workers' compensation cases can stem from many issues including:

• **Insufficient historical loss development data.** Some serious lifetime injury claims can stay open for several decades, but only limited historical loss experience may be available for analysis (e.g., 10 to 20 years).

• **Significant impact of inflation on future costs.** Generally, claims adjusters establish case reserves based on today's costs without consideration of future indemnity benefit escalation and medical inflation. Compounding this issue is the relatively high workers' compensation medical escalation rate (though tempered somewhat in very recent years) compared to general or medical consumer price indices (CPIs).

• **Increases in medical utilization over time**. Case reserves often do not anticipate future intermittent medical costs such as surgeries, prosthetic replacements, and the high cost of end-of-life care. Other significant costs, such as those resulting from technology improvements, new treatments and greater use of expensive prescription narcotics also can contribute to inadequate case reserves.

• **Use of outdated or static life tables**. Even if case reserves reflect mortality considerations for lifetime claims, often the mortality assumptions do not reflect future improvements in life expectancy. Also, the averaging nature of a simplistic life expectancy approach generally underestimates gross claim costs in an inflationary environment (i.e., the impact on costs of claimants dying before and after the life expectancy is not offsetting) and changes the distribution of losses in various layers.

• **Industry case reserving practices**. Industry case reserving philosophies and practices vary widely and can lead to different incurred development patterns by company. For example, some organizations may only case reserve for a fixed number of years of payments (e.g., 5 years) or to a "settlement" value instead of an "ultimate value," leading to continual case reserve increases or "stair stepping."

We shall concentrate our work on the source of misestimating stemming from Outdated or static life tables because that was the reason for the internship.

When a pension plan guarantees the payment of annuities to the death of the beneficiary, the fact that the duration of the benefits come to be systematically greater than those implied in the mortality table used in the actuarial valuations, may create a funding problem in long term, leading to the necessity of including an additional effort on the part of contribution (s) of member (s).To cater for this, managing bodies of pension funds should choose the mortality table that best suits the profile of the population concerned. It would therefore be expected to seek somehow use dynamic tables, and not static tables TV 73/77.

The goal of this report is twofold. First we will derive a model for the mortality in the pensions portfolio of AXA, which allows for a forecast of the future mortality in this portfolio. Basically this will be done by developing two separate models and then integrating them. We will start by assessing the longevity of the Portuguese population as a whole, based on data of this group. For forecasting future mortality in the Portuguese population I will use the Poisson Lee-Carter method. Subsequently the effects of adverse selection in the AXA pensions portfolio will be implemented. These effects will be estimated by comparing past mortality in the portfolio to past population mortality. Combining the longevity projections and the effects of adverse selection future mortality in the portfolio will be forecasted. The resulting mortality model is then specific for the pensions portfolio of AXA. Secondly, this model will be used to quantify the amount of capital AXA should hold to cover their longevity risk for reserving not compulsorily recoverable pensions. The results for this internal model are then compared to those from the static table (TV 73/77) approach. On the basis of this comparison, conclusions recommendations will be made.

The structure of this thesis is as follows. Chapter 2 presents the Literature review. Chapter 3 provides a short explanation of important terms in the scope of this thesis, three models regarding the projection of future mortality for general populations of Portugal. Furthermore we also present our preferred approach regarding modeling future population mortality, the results of this and why we preferred this one. Lastly we model portfolio specific mortality. Chapter 4 first presents the use of the portfolio mortality model to quantify the amount of capital AXA should hold to cover their longevity risk for reserving not compulsorily recoverable Pensions. Second, the results for this model are then compared to those from the static table (TV 73/77) approach. Conclusions and recommendations are then presented in Chapter 5.

CHAPTER 2: LITERATURE REVIEW

Previous research on worker's compensation reserving has several discussions around the need to consider the improvement of mortality over time, with recent papers providing deeper analyses of other key assumptions and more detailed instructions on how to build a mortality model. In 1971, Ferguson points out the necessity of considering mortality in long-term pension-type workers' compensation awards. The author notes that in the calculation of tabular reserves for long term pension type awards special care must be used when an excess of loss reinsurance coverage is involved. The various reserves are calculated by breaking the gross or direct reserves (total expected payment) into its component pieces (direct reserve = net reserve + ceded reserve).The net reserve must be based on a temporary life annuity, thus taking into account both the mortality and interest discounting. The ceded reserve is based on a deferred annuity; deferred by the number of years needed to exhaust the ceding company retention.

Steeneck (1996) provides an update to Ferguson's paper, incorporating escalation of indemnity benefits and medical inflation in mortality-based forecasts. Several illustrations provide some sensitivity analysis concerning the interaction of mortality and claim cost structure. Both indemnity and medical expenses are modeled by annuities. An argument is made for the inclusion of escalation of indemnity (where applicable) and medical inflation within the annuity mathematics to provide a proper forecast of the individual gross loss and to layer that loss properly. This moves the "loss development" provision away from IBNR (incurred but not reported) reserves and into case reserves, providing greater accuracy and clarity to experience. This applies to gross, retrocessional, and net claim reserves.

Snader (1987) expands on the use of life contingency concepts in establishing reserves for claimants requiring lifetime medical care using a three phase approach-claim evaluation, medical evaluation and actuarial evaluation. This paper provides a comprehensive discussion of mortality modeling, including considerations for selecting key assumptions such as inflation, life expectancy, discounting and medical.

Gillam (1993) focuses on mortality assumptions in his discussion of the NCCI Special Call for Injured Worker Mortality Data in 1987 and 1988 and the ensuing analysis of that data. He concluded that differences in mortality, while significant, did not, at that time, imply significant redundancy or inadequacy of the tabular reserves.

Other authors discuss specific assumptions impacting a mortality-based model. For example, Blumsohn (1997) developed the comparison of a deterministic approach (using average life expectancy) versus the stochastic approach (using mortality probabilities).His paper, examines the errors resulting from using a deterministic approach to model parameters other than mortality, such as medical usage, medical inflation, cost of living adjustments (COLAs),and investment income. By assuming deterministic values for future medical usage, medical inflation, COLAS, and investment income, the calculation ignores the possibilities of higher or lower values. It is shown that these do not generally balance out, and that this deterministic parameter produces biased results. In low reinsurance layers, the commutation amount is overstated, and ín high layers it is understated. By removing deterministic assumptions from the calculation, bias is removed from the results.

In his discussion of "ultimate" loss reserves (i.e., case plus IBNR reserves estimated on an individual claim basis) in the context of runoff operations, Kahn (2002) comments on a number of important considerations, including medical escalation and longevity of claimants, that may impact model scenarios.

Sherman and Diss (2005) comment on medical cost severities, escalation rates, and mortality rates used to estimate a workers' compensation tail for the medical component of permanent disability claims. In this paper, the authors demonstrate that case reserves estimated based on the expected year of death (life expectancy approach) are significantly less than the expected value of such reserves using a life contingency cash flow approach.

Brian *et al*. (2013), provide a practical framework to construct mortality- based approach to model lifetime worker's compensation claims including a detailed discussion of the key assumptions. According to them the mortality model included nine major steps amongst which were the applications of mortality assumptions to undiscounted cash flows. Just as Sherman and Diss (2005), they also noted that the life expectancy (instead of the life contingency) approach underestimates the future liability and thus the reserves.

"Just as it is wrong to assume medical usage and inflation are fixed, so it is wrong to assume that a claimant's life-span is fixed. Assuming a deterministic life-span leads to inaccurate calculations. Likewise, assuming deterministic medical care and inflation will lead to inaccurate calculations. A deterministic life span implies that high layers of reinsurance will not be hit, when they do, in fact, have a chance of getting hit if the claimant lives long enough." (Brian *et al.,* 2013, page 16).

The above literature on worker's compensation reserving tend to center around three specific problems: selecting a mortality table to use for computation of the reserves taking into consideration the mortality improvement over time; the applicability of this mortality table to the portfolio (claimant) population; assessing the impact (of using static versus dynamic mortality tables) on the future liability and thus the reserves. The literature reviewed which discussed such problems is presented in the preceding paragraphs.

The literature on the study of dynamic mortality is quite wide with several authors having different contributions to the subject. In the past, mortality patterns were parameterized in the form of different laws. One of them is the Gompertz-Makeham law. This assumes that the logarithm of mortality approximates a straight line when viewed over age. This law, already developed in the 19th century, seems to hold well for ages between 30 and 100 (Peters *et al.,* 2012).As mortality rates have rapidly declined in the 20th century, the need was felt to come up with not only models for present mortality, but also with models that aim to predict future mortality rates. This meant that models with only an age component needed to be expanded with a time component as well. Lee and Carter (1992) have developed a very influential and widely used model in this respect. Using central mortality rates in the United States of America, the authors presented an extrapolative model that describe the mortality of the population using a single index, designed with time-series forecasting methods. By the method of singular value decomposition, which enables the achievement of a solution of least squares, found that the result represented clearly the pattern of mortality in the study. In short, the Lee-Carter method assumes the existence of a time effect in log mortality rates, meaning that death rates in a population have a strong tendency to move up or down together over time. This indeed seems to apply to low-mortality countries (Pitacco *et al*., 2009).The Lee-Carter method does not only allow for the calculation of point estimates of future rates of mortality and life expectancies, but also for the determination of confidence intervals.

Later, Lee (2000) reviewed the model, showing applications to American, Chilean and Canadian population. Some extensions in the original model are described, in particular the breakdown by gender, as the template was initially applied to total population. The author mentions that there are several population breakdown possibilities, but the most simple and intuitive way is the separate treatment of men and women, applying the model independently to each genus. Lee (2000) thus showed that the original Lee-Carter method performed well in explaining the rise in life expectancy in the US in the period 1989-1997.

Many alterations have been proposed in the literature to either improve the statistical soundness of the model or to come to a better fit. Brouhns *et al.* (2002) for instance propose the Poisson model. This model is very similar to the Lee-Carter model, but models the number of deaths conditional on the exposure-to-risk as a Poisson random variable, whereas Lee and Carter model the central death rates as a random variable. They used maximum likelihood estimation to estimate the parameters, instead of resorting to the method of singular value decomposition originally applied in Lee-Carter (1992). Specifically, the original method is embedded in a Poisson regression model, which is perfectly suited for age–sex-specific mortality rates. This model is fitted for each sex to a set of age-specific Belgian death rates. A time-varying index of mortality is forecasted in an ARIMA framework. These forecasts are used to generate projected age-specific mortality rates, life expectancies and life annuities net single premiums. Finally, a Brass-type relational model (Brass 1974) is proposed to adapt the projections to the annuitant's population, allowing for estimating the cost of adverse selection in the Belgian whole life annuity market.

Some empirical analyses suggest that the relation logarithm of the death probability on the survival probability is approximately linear across age for fixed time. This is why Cairns *et al.* (2006) propose a model called CBD (Cairns-Blake-Dowd) based on this relation, including one age component and two time components. The model, applied to the population of England and Wales, contains two stochastic factors to represent the dynamics of mortality. The first factor affects mortality-rate dynamics at all ages in the same way, whereas the second factor affects mortality-rate dynamics at higher ages much more than at lower ages. The article then examines the pricing of longevity bonds with different terms to maturity referenced to different cohorts. We find that longevity risk over relatively short time horizons is very low, but at horizons in excess of ten years it begins to pick up very rapidly. The advantage of this model compared to the Lee-Carter method is that it does not impose perfect correlation of changes in mortality at different ages from one year to the next (Pitacco *et al.,* 2009).Plat (2009a) proposes a model which aims to combine the strong points of the Lee-Carter model and the CBD-model and which seems to provide a better fit than those models, however being slightly more complex.

Estimating the parameters has now become significantly more difficult than in the standard Lee-Carter setting, but it also makes the model more flexible. Besides including an extra time component, one could also include a cohort effect in a mortality model. Renshaw and Haberman (2006) do this for the Lee-Carter method, coming up with an Age Period Cohort (APC) model. Cairns *et al.* (2009) propose an alteration of the CBD-model to include a cohort effect. When there is a cohort effect present in mortality for a specific population, people born in a certain year or period experience significantly different changes in mortality than other people in the population. A cohort effect thus differs from a time effect, as the latter holds for the entire population. Haberman and Renshaw (2011) argue that including a cohort parameter can lead to more accurate projections, but only for countries for which there indeed exists a significant cohort effect. Typically the UK is considered as such a country. For the Portugal, there is as far as I know no conclusive evidence for the existence of a cohort effect (Coelho et al. 2010).

In the literature, a lot of research concerning mortality patterns of entire populations (as discussed above) is performed. These models can be used to predict future mortality rates and the uncertainty (particularly the longevity risk) surrounding these estimates. For insurers (and pension funds) however, it is almost equally important to know how mortality in their portfolio relates to general mortality in the country in which they are active. Therefore, also models have been developed to quantify these specific relations. It should however be noted that the number of papers written in this field is far less than the number of papers written on general mortality patterns. This would not be a problem if the data set of the insurer is of such a size that it can be seen as a specific population itself (not only in number of clients, but also in number of years for which data is available).In that case, the insurer can just apply one of the models discussed before to its own data set. Typically, however, the number of clients is considerably smaller and reliable data is only available for a small number of years. This is why an insurer will often need to revert to a model for population mortality to predict future development in mortality rates. The specific relation between mortality in the portfolio and mortality in the population can then be applied to this model (Wijk, 2012).

In the same paper in which Brouhns et al. (2002) present their Poisson model for population mortality, they also come up with a Cox Proportional Hazard model (Cox, 1972) to study mortality of Belgian annuitants relative to the general Belgian population. This model is based on the linear relationship on the logarithmic scale, which was already observed by Brouhns and Denuit (2001). To quantify this relationship, they resorted to a relational model inspired by the work of Brass (1974), which sought to relate the mortality under Belgian population to the population of pension fund pensioners of this country through proper function. The use of said relational models moreover arises quite frequently in the literature. For example, Delwarde *et* *al.* (2004) make a collection of various relational models, including a model of Cox proportional hazards and the relational model Brass (Brass, 1974) which applied to several data sets.

Pitacco *et al. (*2009) on the other hand suggest that differences in mortality between different socio-economic groups have widened over time. In the same paper, Pitacco et al. (2009) propose a new model for portfolio mortality which assumes the rate of decline in mortality to be the same for both the general and the insured population. Also, they assume the relation between population mortality and portfolio mortality to be constant over time. They consider mortality data in Belgium, distinguished by gender and by type of annuity (individual or group).They find coefficients of determination R2 between 97.2% and 99.8% for males and females respectively. Note that in all these papers the analysis is limited to the ages 65-98.Confidence intervals for the regression is constructed as for the Cox Proportional Hazard model.

Another important model is the one developed by Plat (2009).It makes no assumption about the relation between population and portfolio mortality being constant over time or not. Plat (2009) wants to estimate portfolio mortality factors for ages *x = x1, ..., xm* and years t = *t1, ..., tn.* For every year *t*, the author employs a regression model to approximate different vectors of portfolio mortality factors in year *t.* This regression model is fitted using Generalized (or Weighted) Least Squares based on the observed number of deaths. Plat (2009) applies this model to data from a large portfolio of about 100,000 insured males above the age of 65, and to a medium-sized portfolio of about 45,000 insured males aged above 65.For these portfolios he finds an AR(1) as the most appropriate to model the relationship between population and portfolio mortality.

Unlike in the previously mentioned relational models, Plat (2009) does find ways to combine the stochastic characteristics of both the population and portfolio mortality model. In order to simulate mortality rates for both the population and the portfolio, he needs to know more about the correlations. He uses the technique of Seemingly Unrelated Regression (SUR), which imposes the need to use the same historical observation periods for both population and portfolio mortality. The technique of SUR does not require the processes to be similar, hence the name. The combined processes can then be fitted by first estimating the parameters equation by equation, by means of Ordinary Least Squares. In this way, both the uncertainty in the population mortality model and the uncertainty in the portfolio mortality model are represented in the combined model.

In the same report in which Workgroup PLT (2010) presents their model for (future) population mortality, they also demonstrate their model for portfolio specific mortality. This model is based on the mortality as experienced by the pension insurers that have provided data. These companies account for approximately eighty percent of the pension insurance market in the Netherlands. PLT assumes that the relation between portfolio mortality and population mortality does not change over time. So future portfolio mortality can be projected based on a forecast of the population mortality and the existing relation. The actual estimation is only done for the ages 29.5 up to 94.5. For lower ages, they assume the factors to be constant at the level of age 29.5. Also, they assume that the effects of adverse selection have disappeared at age 104.5.Between ages 94.5 and 104.5, PLT assumes a linear relation, which they extrapolate until value 1 is reached. For men, PLT finds the relation for ages 29.5-94.5 to be linear. For women however they find a quadratic fit. PLT is able to provide life expectancies in 2058 (the final year of their projection) for people with pension insurance at one of the companies that have provided data. They find a (period) life expectancy at birth of 87.93 years for men and of 88.90 years for women. Remaining life expectancies at age 65 in 2058 are 23.77 years for men and 25.39 years for women.

To close this chapter, the approach of assessing the impact on the future liability and thus the reserves of worker’s compensation using dynamic mortality tables rather than static ones (Brian et al.2013).The authors express the difference in life expectancy of using different mortality tables. They concluded that a mortality-based approach is a valuable alternative to traditional property/casualty methods for estimating the future liability for mature claims with stable future annual payments, such as lifetime workers’ compensation claims. Actuaries can use such an approach to estimate liabilities directly or to enhance traditional reserving for mature, stable, lifetime claims by corroborating tail factors used in loss development methods. Either way, consideration of a mortality calculation can enhance reserve projections, which is particularly important in the context of negotiating claim settlements, commutations and loss portfolio transfers, reserving for run-off books of business, and reinsurance reporting, as well as the collection or allocation of funds for insolvent companies, state guaranty funds and the run-off of state second injury funds. The use of a mortality-based approach will provide valuable insights into the variability of the liabilities through sensitivity testing of the key assumptions (such as variability of mortality) and provide information that may be used to better manage costs.

CHAPTER 3: PROJECTIONS OF MORTALITY

3.1 Stochastic Mortality Rates

There is a vast literature on stochastic modeling of mortality rates. Often used models are for example those of Lee and Carter (1992), Brouhns et al (2002), Renshaw and Haberman (2006), Cairns et al (2006a), as already stated in the literature review. These models are generally tested on a long history of mortality rates for large country populations, such as the United Kingdom or the United States. However, the ultimate goal is to quantify the risks for specific insurance portfolios.

In practice, however there is often not enough insurance portfolio specific mortality data to fit such stochastic mortality models reliably, because:

- The historical period for which observed mortality rates for the insurance portfolio are available is usually shorter, often in a range of only 5 to 15 years.
- The number of people in an insurance portfolio is much less than the country population.

So fitting the before mentioned stochastic mortality models to the limited mortality data of insurers, measured in insured amounts, will in many cases not lead to reliable results. In practice, this issue is often solved by applying a (deterministic) portfolio experience factor to projected (stochastic) mortality rates of the whole country population. We will thus model population mortality and portfolio specific mortality separately by first modelling the population mortality stochastically and later applying a deterministic portfolio factor to these projected stochastic population rates.

The following paragraphs present explanation of some useful terms, the data used, models and results from the estimation of the mortality of the Portuguese population and that for the Portfolio of Workers Compensation in Portugal and AXA.

3.2 *EXPLANATION OF TERMS*

In this section we will briefly explain some terms commonly used in mortality studies and which will be used throughout this report as well.

- **Exposure-to-risk**: the exposure-to-risk $E_{x,t}$, denotes the number of person years lived during year t by people aged *x* at the start of the year. Assuming that people who die during a year have on average been alive during half of the year, the exposure-to-risk can be approximated by the number of survivors plus half the number of deaths in this group.
- **Central death rate**: the central death rate $m_{x,t}$ is defined as the total number of people aged x who have died during year *t* **(**$D_{x,t}$**)** divided by the exposure-to-risk of age group *x* during year *t*. In formula: $m_{x,t}$= $D_{x,t}$**/** $E_{x,t}$.
- **Death probability***:* the probability $q_{x,t}$ that an individual aged *x* at the start of year *t* will die before having reached year *t + 1*.This quantity is closely related to the central death rate $m_{x,t}$.Throughout this report, I use the relation $q_{x,t}$ = 1−exp(−$m_{x,t}$) from Wijk (2012). $q_{x,t}$ is a one-year death probability. One can however also define an *s*-

year death probability (probability of dying within s years) after having reached age x in year t by ${}_{s}q_{x,t}$.From now on, the term death probability will refer to a one-year death probability.

- **Survival probability**: the survival probability $p_{x,t}$ (the probability that a person aged x will survive year t), is defined by $p_{x,t}$ **= 1 −** $q_{x,t}$.Like for the death probabilities, one can also define the probability of surviving an additional s years after having reached year t by ${}_{s}p_{x,t}$ **= 1 −** ${}_{s}q_{x,t}$.In applications s will typically be an integer, but it need not be.
- **Force of mortality**: the force of mortality $\mu_{x,t}$ is defined by $\mu_{x,t}$= $\lim_{s \downarrow 0}$ ${}_{s}q_{x,t}/s$.It is also referred to as the instantaneous rate of mortality at the age x in the year t. A typical assumption in the literature (for example Wijk 2012) is that the force of mortality is piecewise constant. We will also adopt this assumption, which is an essential one when the analysis is done forage groups with widths of one year. This assumption implies that the force of mortality becomes equal to the central death rate $m_{x,t}$.
- **Period life expectancy**: the (remaining) life expectancy calculated for a person in year t, based on mortality rates which hold for year t. For instance if this person is aged 40 in year t, the survival probability to reach the age 50 in year $t + 10$ will entirely be based on the survival probabilities for people aged 41, 42, etc.in year *t.* For a person aged x in year *t,* the remaining period life expectancy $e_{x,t}$ is defined by:
  $e_{x,t} = \sum_{\tau=1}^{\omega-x} {}_{\tau}p_{x,t} + 1/2$

  where ω denotes the maximum age an individual can reach. The first term calculates the number of complete years lived, the half is to compensate for the fact that, on average, a person dies half a year after the last and half a year before his next birthday. Note that the maximum age to be obtained is of course unknown, so typically an assumption is made. Throughout this report ω equals 110.

- **Cohort life expectancy**: as above, but then cohort mortality rates are used. This means that the survival probability to reach the age 50 for the person discussed above is based on the probability to reach age 41 in year t, age 42 in year $t+1$, age 43 in year $t+2$ *etcetera.* These future death probabilities are not deterministic at time *t,* but stochastic. Therefore the cohort life expectancy can only be calculated if assumptions about the future death probabilities are made, for instance, that they are according to a best estimate. When one does not assume that mortality rates are constant over time, the cohort life expectancy differs from the period life expectancy. If mortality rates decline over time, cohort life expectancy is higher than period life expectancy.

3.3 PROJECTION OF PORTUGUESE POPULATION MORTALITY

3.3.1 DATA

Mortality data that we have used for the population longevity model comes from Human Mortality Database (HMD), which makes use of numbers provided by University of California Berkeley (Institute for Demographic Research).We have used total data for men and women. The sample covers the period starting in 1940 ($t = t1$) and ending in 2012 ($t = tn$), which is the latest year for which data from HMD is available. Death rates are provided for one-year age intervals and one-year period intervals. The first age group ($x = x1$) is that of persons aged 0, the last age group ($x = xm$) is that of persons aged 110. HMD provides data up to age group 110+.

The following figures report for the Portuguese population the pattern of logarithm of death rates according to age and time. Several behaviors are shown respectively for male, female and total population.

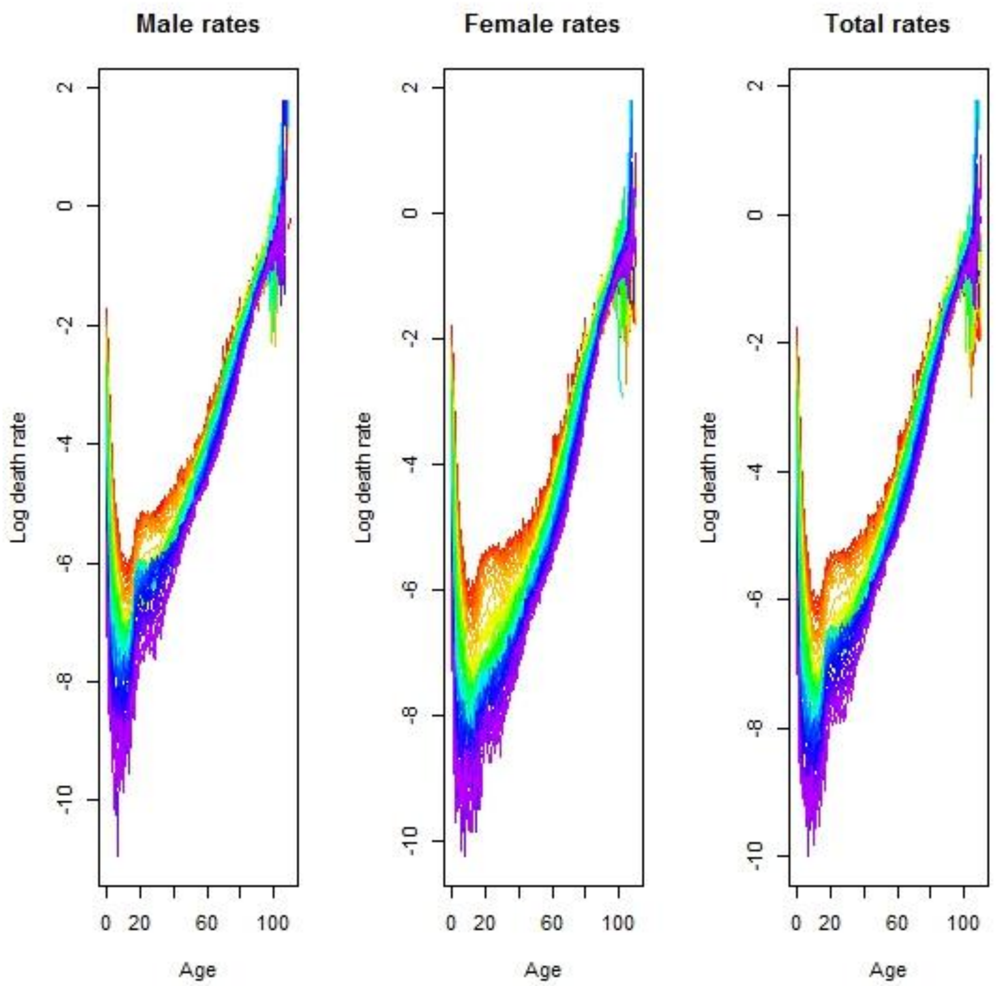


Figure 2: log central mortality rates against age (years 1940 to 2012)
Source: HMD (2012)

For all countries and for males, females and the total population, the values of q_x follow exactly the same pattern as a function of age*, x.* Figure 1 shows the Portuguese mortality rates for males, females and the total population. Note that we have plotted these on a logarithmic scale in order to highlight the main features. Also, although the information plotted consists of values of q_x for *x = 0, 1, ..., 110,* we have plotted a continuous line as this gives a clearer representation. We note the following points from Figure 2 (see Dickson et al. 2011):

- The value of q_0 is relatively high. Mortality rates immediately following birth, perinatal mortality, are high due to complications arising from the later stages of pregnancy and from the birth process itself. The value of q_x does not reach this level again until about age 80.
- As expected the average mortality grows when age increases.
- The rate of mortality is much lower after the first year, less than 10% of its level in the first year, and declines until around age 10.
- Furthermore it is clearly visible the young mortality hump in the age-range (15,20) due to accidental deaths.
- Mortality rates increase from age 10, with the accident hump creating a relatively large increase between ages 10 and 20, a more modest increase from ages 20 to 40, and then steady increases from age 4

Secondly, an initial exploration for trends in the data was conducted by plotting the logarithm of empirical mortality rates against calendar year for each age separately.

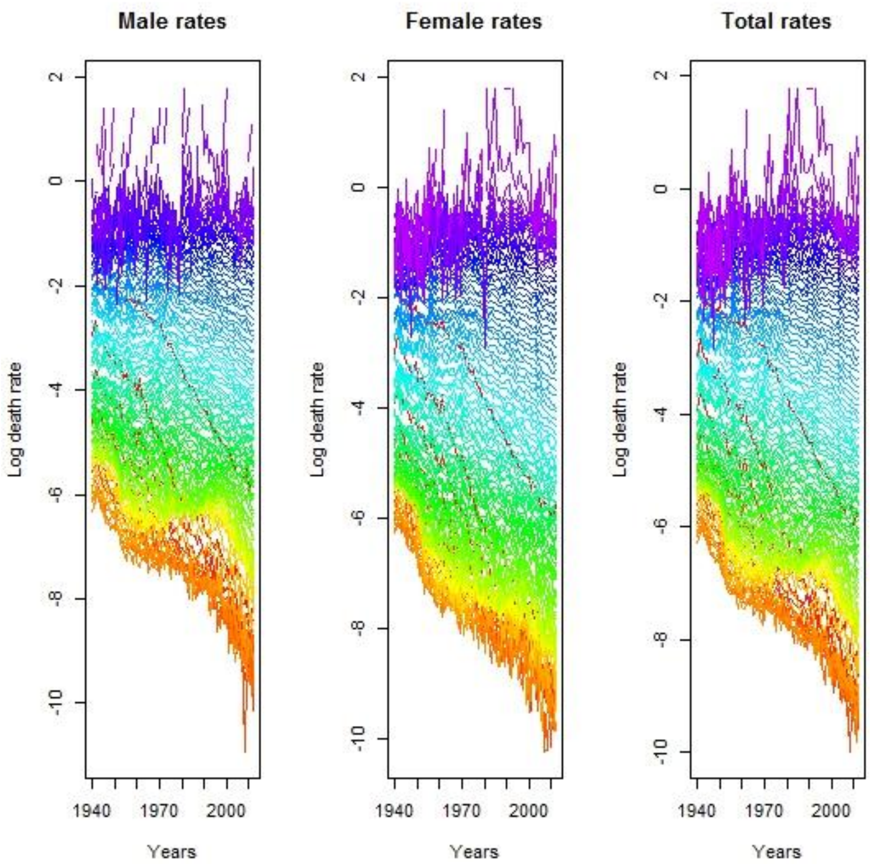


Figure 3: log central mortality rates against time (ages 0 to 110)
Source: HMD (2012)

Figure 3 confirms there is a pronounced increase in the rate of improvement in mortality, stemming from the 1940s, in all age bands. This is a feature that was noted also by Wilmoth (2000) for the USA and Lee (2000).For both males and females and for the total population, there is a pronounced decrease in the rate at which mortality has been improving over the past quarter century, compared with the preceding quarter century.

3.3.2 MODELS

Given an appropriate model, forecasts of the single parameter could be then used to generate forecasts of the level and age distribution of mortality for the next few decades, Lee and Carter (1992).There are several candidates for the model.

We have decided to use three different models to predict future mortality of the general Portuguese population. The models are: the Original Lee-Carter method (Lee and Carter, 1992) with Gaussian errors, the Poisson Lee-Carter method (Brouhns et al., 2002) and the age-period-cohort (APC) variant of the Lee-Carter method including a cohort effect (Renshaw and Haberman 2006).Over the years the Lee-Carter method has evolved following proposals by other scientists. The reason that we have chosen for the Lee-Carter method and its improvements is there are relatively simple model, but not less accurate compared to other models, Wijk (2012).

Model 1: The Lee-Carter Model (Lee and Carter, 1992)

The original Lee–Carter method was used to aggregate (sexes combined) US data. Carter and Lee (1992) implemented their model for males and females separately, showing that the two series are best treated as declining independently. Wilmoth (1993) applied Lee–Carter methods to forecast Japanese mortality and also experimented with variants of this model. Lee and Nault (1993) applied Lee–Carter methods to model Canadian mortality and Brouhns and Denuit (2001) did the same for Belgian statistics, Coelho et al. (2010) and Pateiro (2013) applied it to the Portuguese population. It should be noted that the Lee–Carter method does not attempt to incorporate assumptions about advances in medical science or specific environmental changes; no information other than previous history is taken into account. This means that this approach is unable to forecast sudden improvements in mortality due to the discovery of new medical treatments or revolutionary cures. Similarly, future deteriorations caused by epidemics, the apparition of new diseases or the aggravation of pollution cannot enter the model, Brouhns et al. (2002).

Model 2: The Poisson Lee-Carter Model (Brouhns et al., 2002)

The Poisson-Lee-Carter model has some advantages over the classical version of the model that make it especially attractive. First, the model explicitly recognizes the integer nature of $D_{x,t}$ unlike the Lee-Carter method. Second, the model drops the assumption of homoscedasticity of the error term and recognizes the greater variability of $\mu_{x,t}$ at older ages. Third, the possibility of using maximum likelihood methods to estimate the parameters instead of using the least squares method implemented by singular value decomposition makes the estimation more efficient. Finally, contrary to the classical Lee-Carter approach there is thus no need of a second-stage estimation of the time-index level of mortality (k_t) since the error applies directly on the number of deaths in the Poisson regression approach, see Coelho et al. (2010).

The Lee–Carter methodology is a mere extrapolation of past trends. All purely extrapolative forecasts assume that the future will be in some sense like the past. Some authors (see, Gutterman and Vanderhoof (1999)) severely criticized this approach because it seems to ignore underlying mechanisms. As pointed out by Wilmoth (2000), such a critique is valid only in so far as such mechanisms are understood with sufficient precision to offer a legitimate alternative method of prediction. The understanding of the complex interactions of social and biological factors that determine mortality levels being still imprecise, the extrapolative approach to prediction is particularly compelling in the case of human mortality.

Model 3: The Poisson Lee-Carter Model with cohort effects (Renshaw and Haberman, 2006)
In a more recent development, the basic setting has been further extended to include an additional bilinear term, containing a second period effect (as in Renshaw and Haberman, 2003b) or a cohort effect (as in Renshaw and Haberman, 2006). In particular, the latter approach sheds new light on the early 20th century England and Wales mortality patterns.
Thus, the basic Lee-Carter model can be transformed into a more general framework in order to analyse the relationship between age and time and their joint impact on the mortality rates.

3.3.3 FITTING THE MODELS

Model 1: The Lee-Carter Model (Lee and Carter, 1992)
The Lee-Carter method (Lee and Carter, 1992) combines a demographic model, describing the historical change in mortality, a method for fitting the model and a time series model for the time component which is used for forecasting. The classical two-factor Lee-Carter model is

$$\ln\left(m_{x,t}\right) = \alpha_x + \beta_x k_t + \varepsilon_{x,t} \qquad (1)$$

where $m_{x,t}$ denotes the central mortality rate at age *x* in year *t.* When model (1) is fitted by ordinary least-squares (OLS), interpretation of the parameters is quite simple.

α_x: the fitted values of α_x exactly equals the average of $\ln\left(m_{x,t}\right)$ over time t so that exponential is the general shape of the mortality Schedule.

β_x: represents the age-specific patterns of mortality change. It indicates the sensitivity of the logarithm of the force of mortality at age *x* to variations in the time index k_t.

k_t: represents the time trend.The actual forces of mortality change according to an overall mortality index k_t modulated by an age response β_x .The shape of the β_x profile tells which rates decline rapidly and which decline slowly over time in response of a change in k_t .

The error term $\varepsilon_{x,t}$, with mean 0 and variance σ_ε^2, reflects particular age-specific historical influence not captured in the model.

The equation underpinning the Lee-Carter model is known to be over parameterized. To ensure model identification, Lee and Carter (1992) add the following constraints to the parameters**:**

$$\sum_{x\,min}^{x\,max} \beta_x=1, \qquad \sum_{x\,min}^{x\,max} k_t=0,$$

to obtain unique parameter estimates. As a result of these constraints, the parameter α_x is calculated simply by averaging the $\ln(m_{x,t})$ over time.

A) OLS estimation

The main statistical tool of Lee and Carter (1992) is least-squares estimation via singular value decomposition of the matrix of $\ln(m_{x,t})$ .The model (1) is fitted to a matrix of age-specific observed forces of mortality using singular value decomposition (SVD).Specifically, the $\alpha_x{'}s, \beta_x{'}s \; and \; k_t{'}s$ are such that they minimize

$$\sum_{x,t} (\ln(m_{x,t}) - \alpha_x - \beta_x k_t)^{2} \qquad (2)$$

It is worth mentioning that model (1) is not a simple regression model, since there are no observed covariates in the right-hand side. The minimization of (2) consists in taking for α_x the row average of the $\ln(m_{x,t})$'s, and to get the $\beta_x{'}s \; and \; k_t{'}s$ from the first term of an SVD of the matrix $\ln(m_{x,t}) - \alpha_x$.This yields a single time-varying index of mortality k_t .

Before proceeding directly to modeling the parameter k_t as a time series process, the $k_t{'}s$ are adjusted (taking α_x and β_x estimates as given) to reproduce the observed number of deaths $\sum_x D_{x,t}$ i.e. the $k_t{'}s$ solve the equation

$$\sum_x D_{x,t} = E_{x,t} \exp(\alpha_x + \beta_x k_t) \qquad (4)$$

So, the $k_t{'}s$ are reestimated so that the resulting death rates (with the previously estimated $\alpha_x{'}s$ and $\beta_x{'}s$) applied to the actual risk exposure, produce the total number of deaths actually observed in the data for the year *t* in question.

There are several advantages to make this second-stage estimate of the parameters k_t .In particular, it avoids sizable discrepancies between predicted and actual deaths (occurring because the first step is based on logarithms of death rates).Other advantages are discussed by Lee (2000).

B) Modelling the Index of Mortality

Having developed and fitted the demographic model, we are now ready to move to the problem of forecasting. An important aspect of Lee–Carter methodology is that the time factor k_t is intrinsically viewed as a stochastic process. To forecast, Lee and Carter assume that α_x and β_x remain constant over time and forecast future values of k_t using a standard ARIMA univariate time series model.

The first step is to find an appropriate ARIMA time series model for the mortality index k_t .

Box–Jenkins methodology (identification–estimation–diagnosis) is used to generate the appropriate ARIMA time series model for the mortality indexes of the various models estimated (see Box and Jenkins, 1970).These forecasts in turn yield projected age-specific mortality rates and life expectancies.

After carrying out the standard model identification procedures we can find that an ARIMA (1,1,0) with drift best describes the mortality index (see Appendix 3 for details).The ARIMA model is given by:

$$\boldsymbol{k_t - \emptyset_1 k_{t-1} = c_t} + \mathcal{E}_t + \mathcal{E}_{t-1}$$

The constant term $\boldsymbol{c_t}$ indicates the average annual change of k_t, and it is this change that drives the forecasts of the long-run change in mortality. The $\mathcal{E}_t$ is the independent disturbance (random error).

Model 2: The Poisson Lee-Carter Model (Brouhns et al., 2002)

According to Alho (2000), the Lee-Carter model described in equation (1) above is not well suited to the situation of interest. As already mentioned, the main drawback of the OLS estimation via SVD is that the errors are assumed to be homoscedastic. This is related to the fact that for inference we are actually assuming that the errors are normally distributed, which is quite unrealistic. The logarithm of the observed force of mortality is much more variable at older ages than at younger ages because of the much smaller absolute number of deaths at older ages. Because the number of deaths is a counting random variable, according to Brillinger (1986), the Poisson assumption appears to be plausible, Brounhs et al. (2002).

The Poisson Lee-Carter Model assumes that the age-specific forces of mortality are constant within bands of age and time. More formally, given any integer age *x* and calendar year *t*, he assume that

$$\mu_{x+\epsilon,t+\tau} = \mu_{x,t} \quad \text{for } 0\leq\mathcal{E},\ \boldsymbol{\tau}<1$$

Under this constant force of mortality assumption, $\mu_{x,t}$ may be estimated as the quotient between the number of deaths and the number of exposed to the risk of dying or $E_{x,t}$.Brouhns et al. (2002) developed a maximum likelihood estimation solution of the Lee-Carter model based on the assumption that $D_{x,t}$, the number of deaths recorded at age x during calendar year t , follows a Poisson distribution, i.e.,

$$D_{x,t} \sim \text{Poisson}(\mu_{x,t}E_{x,t}) \qquad (5)$$

with

$$\mu_{x,t} = \exp(\alpha_x + \beta_x k_t) \qquad (6)$$

where the parameters are still subject to the constraints in equation (1). The meaning of α_x, β_x and k_t parameters is essentially the same as the classical Lee-Carter model.

A) Maximum likelihood estimation

The model preserves the log-bilinear structure for $m_{x,t}$ but replaces the classical assumptions on the error term $\varepsilon_{x,t}$ by a Poisson law for $D_{x,t}$.In spite of this, parameters α_x, β_x and k_t maintain,in essence, their original interpretation.Instead of resorting to SVD procedures, parameter estimates maximize the following log-likelihood function:

$$L(\alpha_x, \beta_x, k_t)=\sum_{x\ min}^{xmax}\sum_{t\ min}^{tmax}\{d_{x,t}(\alpha_x+\beta_x k_t)-E_{x,t}\exp(\alpha_x+\beta_x k_t)\}+c \qquad (7)$$

where *c* is a constant. The presence of the log-bilinear term $\beta_x k_t$ in (6) prevents the estimation of model parameters using standard statistical packages (e.g. OLS) that include Poisson regression. Because of this, we resort to an iterative algorithm for estimating log-bilinear models developed by Goodman (1979) based on a Newton-Raphson algorithm. Finally, a reparametrization of the model is necessary in order to guarantee that the parameter estimates α_x, β_x, k_t generated by the ML procedure verify the model identification constraints.

To forecast, as in the Lee–Carter method, we use the above time series methods to make long-run forecasts of age–sex-specific mortality rates.

B) Modelling the Index of Mortality

We do not modify the time series part of the Lee–Carter methodology. Estimates of $\alpha_x and\ \beta_{x,}$ are used with forecasted k_t to generate other lifetable functions.After carrying out the standard model identification procedures we can find that an ARIMA(0,2,2) without drift best describes the mortality index (see Appendix 3 for details).The ARIMA model is given by:

$$\boldsymbol{k_t - k_{t-1} - k_{t-2}} = \mathcal{E}_t + \boldsymbol{\theta_1}\, \mathcal{E}_{t-1} + \boldsymbol{\theta_2}\, \mathcal{E}_{t-2}$$

Model 3: The Poisson Lee-Carter Model with cohort effects (Renshaw and Haberman, 2006)

In the current application, we follow the APC modelling framework and fitting methodology proposed by Renshaw and Haberman (2006) that specifies the force of mortality by a generalized structure written as

$$\mu_{x,t} \quad = \exp(\alpha_x + \beta_x^{(0)}(i_{t-x}) + \beta_x^{(1)} k_t) \qquad (8)$$

where α_x maps the main age profile of mortality and i_{t-x} and k_t represent the cohort and period effects, respectively; parameters $\beta_x^{(0)}$ and $\beta_x^{(1)}$ measure the corresponding interactions with age.

A) Maximum likelihood estimation

The parameter estimates maximize the following log-likelihood function:

$$L(\alpha_x, \beta_x, i_{t-x}, k_t) = \sum_{x\,min}^{xmax} \sum_{t\,min}^{tmax} \Big\{ d_{x,t}(\alpha_x + \beta_x^{(0)}(i_{t-x}) \ + \beta_x^{(1)} k_t) - E_{x,t} \exp(\alpha_x + \beta_x^{(0)}(i_{t-x}) \ + \beta_x^{(1)} k_t) \Big\} + c \qquad (9)$$

where c is a constant. To ensure model identification, we add the following constraints to the parameters<u>**:**</u>

$$\sum_{x\,min}^{x\,max} \beta_x^{(0)} = \sum_{x\,min}^{x\,max} \beta_x^{(1)} = 1, \qquad \text{and} \qquad \sum_{x\,min}^{x\,max} k_t = 0$$

As before, the presence of the log-bilinear term $\beta_x^{(0)}(i_{t-x})$ and $\beta_x^{(1)} k_t$ in (8) prevents the estimation of model parameters using standard statistical packages that include Poisson regression. Because of this, we resort to an iterative algorithm for estimating log-bilinear models developed by Goodman (1979) based on a Newton-Raphson algorithm. Finally, a reparametrization of the model is necessary in order to guarantee that the parameter estimates for α_x, $\beta_x^{(0)}$, $\beta_x^{(1)}$, k_t generated by the ML procedure verify the model identification constraints.

As mentioned, the fitting methodology implemented in this application is based on an iterative algorithm that minimizes the deviance function. That is, we make use of a cyclical updating process of the parameter estimates until the minimum difference between the likelihood of the fitted model and the likelihood of the saturated model (i.e.one parameter for each observation) is achieved. Thus, the updating mechanism for a given parameter θ is provided by the Newton-Raphson minimization method applied to the deviance function as can be seen on Appendix 1.

Using the complete data set convergence could not be achieved and thus the deviance function could not be minimized. Thus in this application we make use of the data using a restricted age range from 18-85 so that convergence could be achieved, the deviance function minimized and the model parameters obtained.

B) Modelling the Index of Mortality

To forecast, as in the Lee–Carter method we use the same time series methods to make long-run forecasts of trend parameters i_{t-x} *and* k_t.

After carrying out the standard model identification procedures we can find that an ARIMA(1,1,0) without drift best describes the mortality index (see Appendix 3 for details). The ARIMA model is given by:

$$\boldsymbol{k_t} - \emptyset_1 \boldsymbol{k_{t-1}} = \boldsymbol{c_t} + \mathcal{E}_t + \mathcal{E}_{t-1}.$$

3.3.4. RESULTS

In the previous section we have described the three Lee-Carter methods we have used in the estimation. In this section, the results of these approaches for modeling future mortality in the Portuguese population are presented and discussed.

An extract of the estimated α_x , β_x, $\beta_x^{(0)}$, and $\beta_x^{(1)}$ (for the total population) of the different models is given in table 1 below. The full numerical values are presented in Appendix 2.

AGE	**MODEL 1**		**MODEL 2**		**MODEL 3**		
	α_x	β_x	α_x	β_x	α_x	$\beta_x^{(0)}$	$\beta_x^{(1)}$
18	-6.76545	0.013416	-6.72009	0.012826	-6.43238	0.028755	0.017501
19	-6.6869	0.013062	-6.64259	0.01244	-6.4682	0.029366	0.019462
20	-6.63557	0.012926	-6.59936	0.012756	-6.52836	0.031109	0.022206
…	…	…	…	…	…	…	…
83	-2.08696	0.004504	-2.08089	0.004582	-2.06065	0.006017	0.007938
84	-1.96552	0.004929	-1.96247	0.00477	-1.9185	0.006123	0.007589
85	-1.84533	0.005592	-1.84438	0.005018	-1.7872	0.006087	0.007081

Table 1: estimates α_x , β_x, $\beta_x^{(0)}$, and $\beta_x^{(1)}$

An extract of the estimated i_{t-x} and k_t (for the total population) of the different models is given in table 2 below. The full numerical values are presented in Appendix 2.

Year	Model 1	Model 2	Model 3	
	k_t	k_t	k_t	i_{t-x}
1940	74.578	65.958	0	53.577
1941	80.4121	73.121	3.2435	53.478
1942	78.69425	69.453	0.6468	53.380
…	…	…	…	…
2011	-76.894	-99.5736	-156.936	0
2012	-79.155	-98.1178	-158.53	0

Table 2: Estimates of parameters i_{t-x} *and* k_t.

The resulting values for the parameters of the ARIMA models are given in Table 2, for the κt 's obtained via the classical Lee–Carter method, for the Poisson case and for the model with cohort effects. The detailed results are presented in Appendix 3.

Model	c_t	$\emptyset_1$	θ_1	θ_2	$\mathcal{E}_t$
Model 1	-2.1612	0.3116	0	0	0.1184
Model 2	0	0	-1.5482	0.6883	0.0929
Model 3	-1.7135	-0.3562	0	0	0.1118

Table 3: Estimates of parameters of ARIMA models

Figures 2, 3, 4 and 5 plot the estimated α_x, $\beta_x^{(0)}$, $\beta_x^{(1)}$, k_t and i_{t-x} (for the total population) of the diferente models.This clearly illustrates the fact that similar trends are observed. Appendix 2, contain the detailed numerical values.

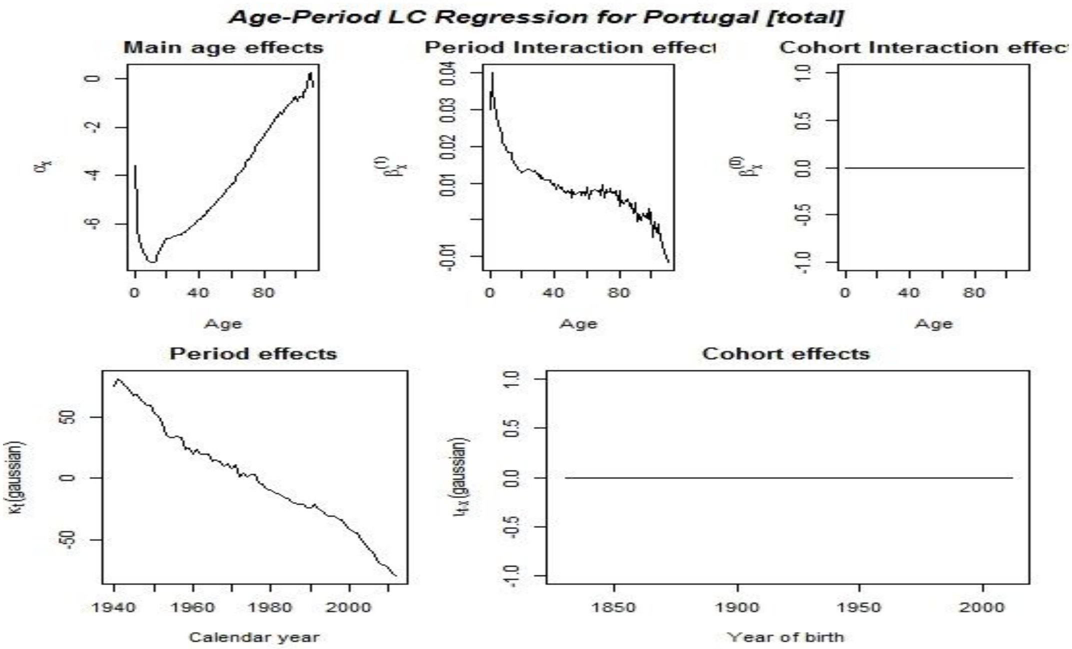


Figure 4: Original Lee-Carter with Gaussian errors.

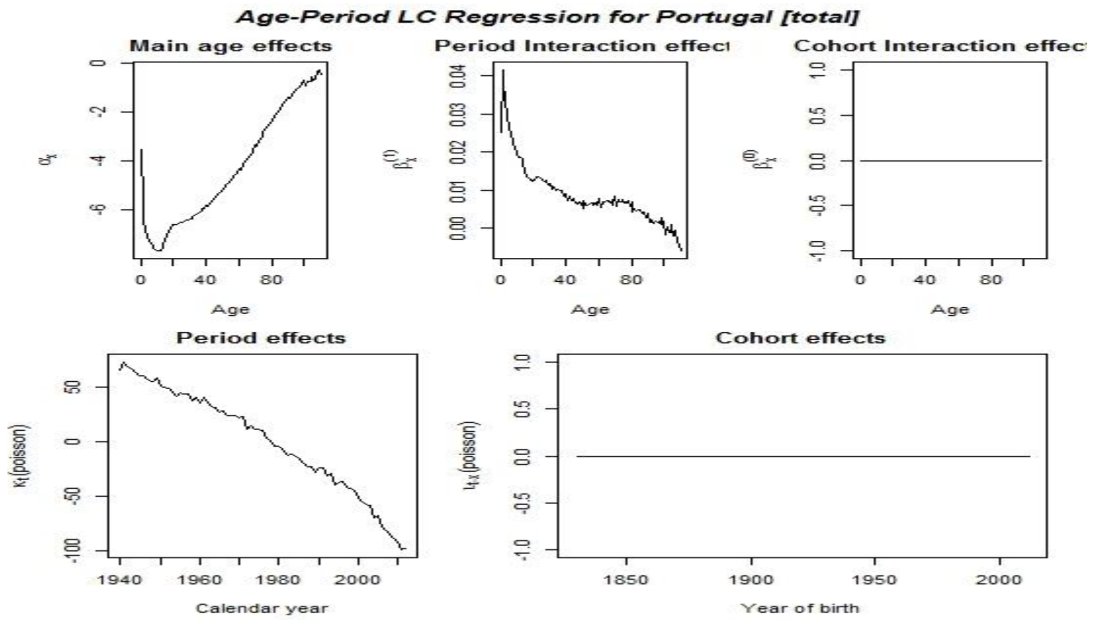


Figure 5: Lee-Carter with Poisson Errors

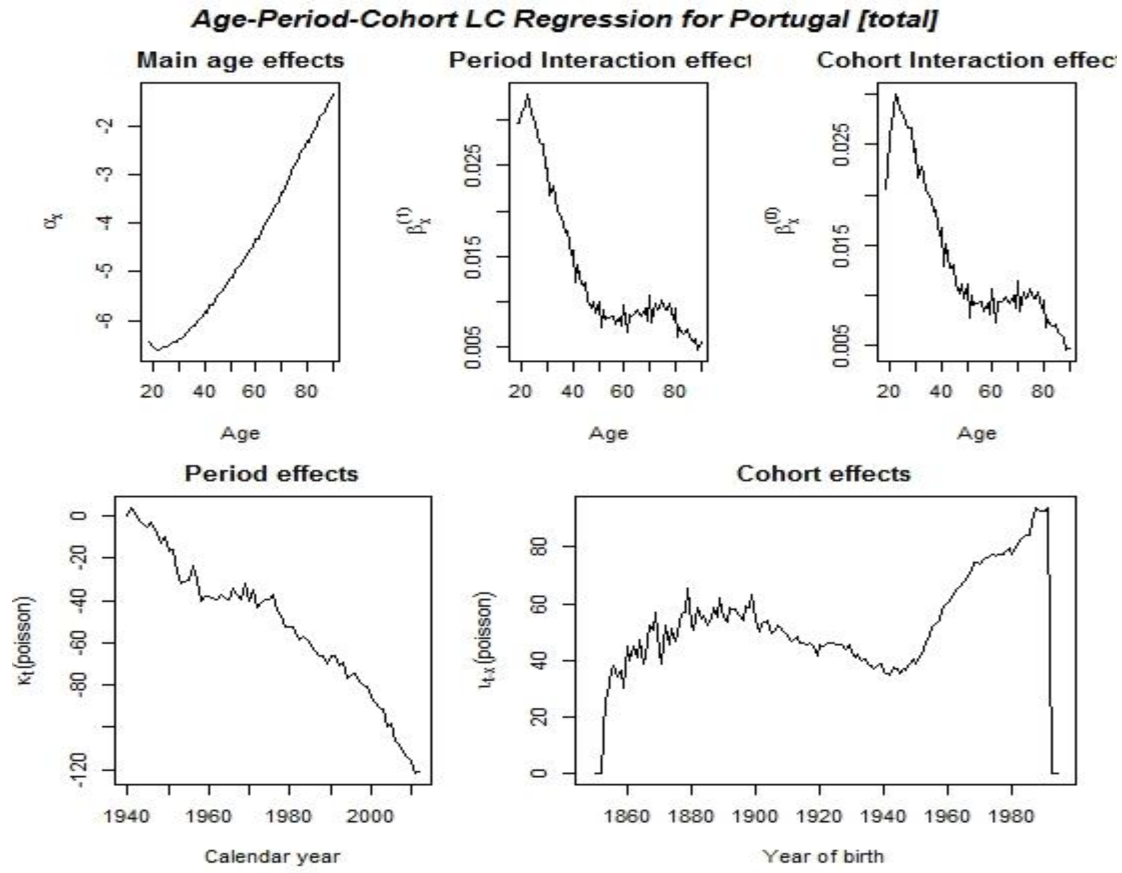


Figure 6: Lee-Carter with Cohort effects

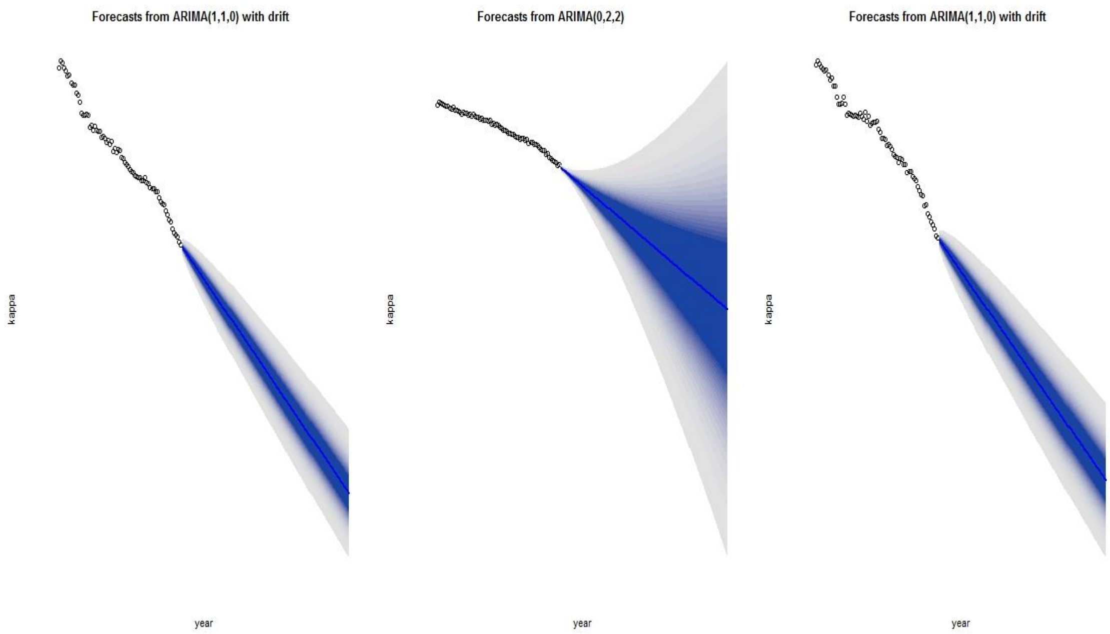


Figure 7: Forecasting of k_t by ARIMA

The figures indeed present a pattern for α_x which is consistent with previous results in the literature, see for instance De Waegenaere et al. (2010). At age 0 mortality is quite high due to infant mortality, after which it is decreasing until the age of 10. Afterwards it is approximately linearly increasing, except for the 'accident hump' noticeable for young adults. However when cohort effects are taken into consideration the decrease is from 0 to 20 years thereafter there is a linear increase.

The pattern of $\beta_x^{(0)}$ and $\beta_x^{(1)}$ shows that young children have profited most (high $\beta_x^{(0)}$) from the decrease in mortality over time. Again the pattern shows close resemblance with previous results in the literature, see for instance De Waegenaere et al. (2010a). Note also that variation in the value is higher for lower ages, showing that the mortality rates over time have varied more for the young.

Estimates for k_t are initially obtained for years 1940-2012 and displayed in the figures. As expected, k_t has a decreasing trend with the increment of time.

The reconstituted sex-specific forces of mortality are then used to generate sex-specific life expectancies as shown in the figure below:

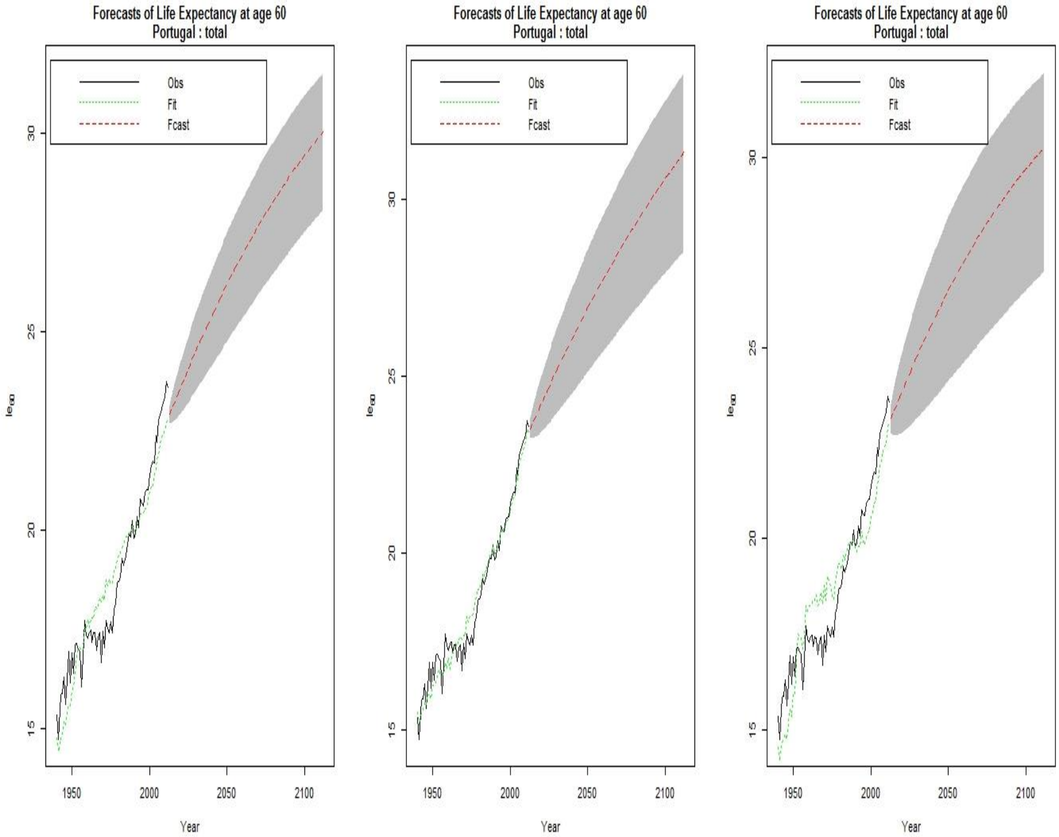


Figure 8: From left to right: observed, estimated and forecasted life expectancy of models 1,2 and 3.

Figure 8 shows that the Poisson Lee-Carter (model 2) has the best fit. We thus will use it as the base model to model the population mortality of the Portuguese market. The mortality rates obtained will then be used as reference rate to model the mortality rate of Workers compensation for the Portuguese market as well as the Portfolio of AXA in the next sections.

3.4 PROJECTION OF PORTFOLIO SPECIFIC MORTALITY FOR PORTUGUESE MARKET OF PENSIONERS AND AXA PORTFOLIO OF PENSIONERS

In the literature, a lot of research concerning mortality patterns of entire populations (as discussed in Chapter 2) is available. These models can be used to predict future mortality rates and the uncertainty (longevity risk) surrounding these estimates. For insurers (and pension funds) however, it is almost equally important to know how mortality in their portfolio relates to general mortality in the country in which they are active. Therefore, also models have been developed to quantify these specific relations. It should however be noted that the number of papers written in this field is far less than the number of papers written on general mortality patterns. This would not be a problem if the data set of the insurer is of such a size that it can be seen as a specific population itself (not only in number of clients, but also in number of years for which data is available).In that case, the insurer can just apply one of the models discussed in the previous section to its own data set. Typically however the number of clients is considerably smaller and reliable data is only available for a small number of years. This is why an insurer will often need to revert to a model for population mortality to predict future development in mortality rates. The specific relation between mortality in the portfolio and mortality in the population can then be applied to this model (Wijk,2012).In practice the issue of modelling portfolio mortality is solved by applying a deterministic portfolio experience factor to projected stochastic mortality rates of the whole population.

In this section we will discuss the way we model the mortality as experienced in the pension's portfolio of both the Portuguese market and AXA Portugal .The first subsection is about the data we have used, the second is about the models we have adopted for our analysis, third estimation procedure and results and the last comments.

3.4.1 DATA

The data was provided by the Actuarial Department of AXA. The information for the Portuguese market as a whole spans the period from 2006 up to and including 2013 and that for AXA spans the period from 2006 up to and including 2014.Unfortunately the data from 2013 and 2014 is not of much use, since we have no access to population mortality data for these years and we can therefore not compare the mortality in population and portfolio for those years. This is why our analysis is only based on the years 2006 up to and including 2012.

For every year 2006-2012, we were provided with the number of insureds per age("N.º pessoas expostas ao risco (EX)") and gender. Also, for every year we received the number of deaths per age (Mortalidade real) and gender. Using these numbers we could find the observed death probabilities for each age (central mortality rates and initial mortality rates).For example for AXA portfolio the initial and central mortality rate respectively are given by $q.AXA = \frac{Death.AXA}{Exp.AXA}$ and $m.AXA = -\log(1 - q.AXA)$.

As the sample is not that big (number of deaths for AXA portfolio aggregated over all age groups and the total period: 759; number of deaths for pension portfolio of Portugal aggregated over all age groups and the total period: 2431) as compared to 699,988 for the whole Portuguese population, it was thus not possible to study the specific portfolio mortality for each year separately, hence we studied for the total time period. This same approach is applied by Wijk(2012), Brouhns et al. (2002) and Denuit(2007) in modelling portfolio mortality.

An extract of the data is given below. The complete data set is given on Appendix 4.

AGE	EXP. AXA	DEATH AXA	EXP. WC	DEATH WC	EXP.POP.	DEATH POP.
0	4.5	0	11.5	0	700000	2239
1	22	0	46	0	697761	212
2	29	0	93	0	697549	133
…	…	…	…	…	…	…
109	1.1	1	5	1	52	28
110	1	1	9	1	25	25
TOTAL	43,711.5	759	207,378	2,431.577	56,174,158	699,988

Table 4: Aggregate Exposure to Risk and number of Deaths from 2006 to 2012 for Market WC, AXA portfolio and General Population.

3.4.2 MODELS

The idea is to build a function f(μ_x) and to relate the mortality in a population under study (the Worker's compensation portfolio, in our case) to that in a reference population whose mortality rates are μ_x^{ref} (the whole Portuguese population, in our case).

In this work we will again estimate three relational models: a relational model based on cox proportional hazards (Cox, 1972) used by Delwarde et al. (2007), the Brass Linear Model(Brass, 1974) used by Brouhns et al. (2002) and Workgroup PLT (2010) used by the Dutch Association of Insurers.

Model 1: Proportional Hazard Model of Cox (Cox, 1972)

Parametric models allow fine comparison of mortality, but are obviously subject to misspecification: the parametric form can be false, discrediting the conclusions reached. The proportional hazards model proposed by Cox (1972) overcomes a rigid parametric formulation.

He postulates that the portfolio mortality rates which the actuary is interested in are proportional to those of the reference population, with the factor of proportionality not being dependent on age or time. Specifically the rates $m_{x,t}^{portfolio}$ are related to the reference rate $m_{x,t}^{population}$ by the relation

$$m_{x,t}^{portfolio} = \theta\, m_{x,t}^{population} \qquad (10)$$

where $m_{x,t}^{portfolio}$ reflects the one-year central mortality rate of the insureds, the $m_{x,t}^{population}$ are the central rates of mortality of the general population and θ is a portfolio mortality factor independent of age and time,also: $x = 0,1, \dots\dots\dots\dots\dots, 110$ and $t = 2006,2007, \dots\dots\dots, 2012$.

This is the simplest relational model linking the mortality group of interest represented by $m_{x,t}^{portfolio}$ to that of the reference population represented by $m_{x,t}^{population}$,using the equation 10.Equation 10 assumes that that the relation between portfolio mortality and population mortality does not change over time and age. So, future portfolio mortality can be projected based on a forecast of the population mortality and the relation described in equation 10.

Model 2: Workgroup PLT (2010)

Workgroup PLT (2010) imposes a simple model, characterized by the relation

$$q_{x,t}^{portfolio} = e_x\, q_{x,t}^{population} \qquad (11)$$

where $q_{x,t}^{portfolio}$ reflects the one-year death probabilities of the insureds,the $q_{x,t}^{population}$ are the death probabilities of the general population and e_x is a portfolio mortality factor dependent of age.PLT assumes that the relation between portfolio mortality and population mortality does not change over time. So future portfolio mortality can be projected based on a forecast of the population mortality and the relation described in (11).PLT obtain the actual estimates only for the ages 29.5 up to 94.5.For lower ages, they assume the factors to be constant at the level of age 29.5.Also, they assume that the effects of adverse selection have disappeared at age 104.5 (e_x = 1 for this age and higher). Between ages 94.5 and 104.5, PLT assumes a linear relation, which they extrapolate until value 1 is reached.

Model 3: Brass Linear Model (Brass,1974)

The validity of model (1) has however been questioned by many authors. In the same paper in which Brouhns et al. (2002) present their Poisson model for population mortality, they also come up with a Brass linear model to study mortality of Belgian annuitants relative to the general Belgian population. This model is based on the linear relationship on the logarithmic scale, which was already observed by Brouhns and Denuit (2001).The specification of the model is as follows

$$log m_{x,t}^{portfolio} = \theta_1 + \theta_2 log\, m_{x,t}^{population} + \varepsilon_{x,t} \qquad (12)$$

where $m_{x,t}^{portfolio}$ denotes central death rates for the portfolio considered, $m_{x,t}^{portfolio}$ denotes central death rates for the general population and $\varepsilon_{x,t}$ is the iid error term with mean 0 and variance σ_{ε}^2.If this relation remains valid over time (this issue will be discussed later in this section),I can relate the future mortality rates in the portfolio to the future mortality rates of the population via

$$m_{x,t}^{portfolio} = \exp(\theta_1)\, (m_{x,t}^{population})^{\theta_2} \qquad (13)$$

3.4.3 FITTING THE MODELS (RELATIONAL AND COMBINED MODELS)

In this subsection we present how to combine the models for population mortality and mortality of people in the pension's portfolio. First the parameters of the relational models presented in section 3.4 above were estimated using linear regression. The regression thus gives us the point estimates of θ , e_x , θ_1 and θ_2 together with their standard errors.As in WIJK (2012),the actual regression is however done for the ages 19 up to 86.For lower ages , we assume the portfolio mortality to be same as the population mortality and thus use a factor of one. Between ages 87 and 110,we assume the parameter estimate will be the same as that estimated between 19 to 86.This higher ages were not included in our regression because the data size(Exposed to Risk and Number of deaths) were very small. All this will be taken into consideration when using the future mortality rates of the population to forecast that of the portfolio.

The future portfolio mortality rates ($m_{x,t}^{portfolio}$ and $q_{x,t}^{portfolio}$) depend on α_x , β_x and k_t from the Poisson Lee- Carter Model and on the parameters θ , e_x , θ_1 and θ_2 from the relational models.These parameters are all stochastic and there is no information available on the correlation between the Lee-Carter parameters and the relational model parameters. Then there is no way to come to a fully equipped stochastic model for future mortality in the portfolio. Brouhns et al. (2002) therefore propose to only implement the point estimates of the parameters for the portfolio mortality model into the stochastic population mortality model. I will adopt this method. It is important to note that in this way the only randomness considered in the total mortality model comes from the Lee-Carter model. The relation between population and portfolio mortality is now seen as deterministic, which leads to believe that in this way total uncertainty is (slightly) underestimated.

The forecasted mortality rates of the Portfolio were thus obtained by applying the deterministic parameters of the relational models to the dynamic mortality rates of the population for the estimated future years and ages 19-110,in the ages 0- 18 it was assumed the population rates equal the portfolio rates.

3.4.4 Results

The above section presents how to obtain the parameters relating the past mortality experience of the portfolio of pensioners and the mortality experience of the population. It also explains how to combine the future experience of the population with the parameter estimated to get the future mortality experience of the portfolio. In this section we present the results of both the relational models as well as the combined models for the portfolio of worker's compensation market and portfolio of AXA pensioners.

1) PORTFOLIO OF WORKER'S COMPENSATION MARKET.

A) RELATIONAL MODELS

We have to run the three relational models (Proportional Cox, Workgroup PLT and Brass Linear) for the periods 2006 – 2012 and ages 0-110.The table below shows the results of the different models.

Models	Parameter estimated	Estimated values	Standard errors	R^2
Proportional Cox	θ	0.96607	0.01367	98.71%
Workgroup	e_x	0.96898	0.01352	98.68%
Brass Linear	θ_1 θ_2	-0.78493 0.79194	0.13696 0.02407	94.25%

Table 5: Results of relational model for WC market

The highest coefficient of determination R^2 (98.71%) is obtained using the proportional cox model. For the other models R^2 is 98.68% and 94.25%.

B) COMBINED MODELS

Now that we have developed a model for the future population mortality in the Portuguese and a model for the portfolio mortality relative to the mortality of the Worker's Compensation population, we can combine these to predict the future mortality in WC portfolio. In this setting of the combined model, the future portfolio mortality rates ($m_{x,t}^{portfolio}$ and $q_{x,t}^{portfolio}$) depend on α_x , β_x $and\, k_t$ from the Poisson Lee- Carter Model and the parameters θ , e_x , θ_1 and θ_2 from the relational models as already mentioned before.

As already stated in subsection 3.3.4, this will do by implementing a point estimate of the model for portfolio mortality into the stochastic model for population mortality. Also a decision about what to do with the ages that are not included in the regression (ages 0-18 and 86-110) on which the estimates are based is required. For 0-18 years, the future mortality of the portfolio is the same as that for the general population while for ages 86-110 years the factors estimated for ages 19-85 to forecast the future mortality of these ages. This is done by replacing the parameters (θ , e_x , θ_1 and θ_2) in either the Cox, Brass Linear or Workgroup model as well as the forecasted mortality rates estimated for the general population.

An extract of the results of the forecasted one year death probabilities for the combined model using the Proportional Cox Model.Notice how in the table the probability of death at older ages from (106 years) is decreasing instead of increasing e.g. for the year 2013, the probability of death at 106 years is 0.581668 and at 107 years the figure is 0.518834. This is probably because of the limited amount of data avaible at these ages leading to such errors.

Age	Probabilities ($m_{x,t}^{WC}$)				
0	2013	2014	…	2111	2112
1	0.002366	0.002233	…	8.606782e-06	8.1274081e-06
2	5.38511e-05	4.89920e-05	…	5.086265e-09	4.6273215e-09
…	…	…	…	…	…
106	0.581668	0.584199	…	0.890077	0.893949
107	0.518834	0.519224	…	0.558426	0.558845

Table 6: Forecasted one year death probabilities for the combined model using the Proportional Cox Model.

In Table 6, is presented a problem that affects, without further consideration the Lee-Carter family. We can notice that in the table the probability of death at older ages from (106 years) is decreasing instead of increasing e.g. for the year 2013, the probability of death at 106 years is 0.581668 and at 107 years the figure is 0.518834. This is probably because of the limited amount of data avaible at these ages leading to such errors. One way to solve this problem is use the Lee and Carter (1992) log-bilinear model and its extension by Brouhns et al. (2002) based on heteroskedastic Poisson error structures, together with a new variant of the model proposed by Bravo (2010) in which the Poisson-Lee-Carter framework includes a limit life table to which future mortality improvements converge.

2) AXA PORTFOLIO

A) RELATIONAL MODELS

As before we ran the three relational models (Proportional Cox, Workgroup PLT and Brass Linear) for the periods 2006 – 2012 and ages 0-110 using the AXA portfolio. The table below shows the results of the different models.

Models	Parameter estimated	Estimated values	Standard errors	R^2
Proportional Cox	θ	1.03593	0.03373	93.37%
Workgroup	e_x	1.03806	0.03391	93.33%
Brass Linear	θ_1	-1.5939	0.2617	67.76%
	θ_2	0.5417	0.0460	

Table 7: Results of relational model for AXA Portfolio

The highest coefficient of determination R^2 (93.37%) is obtained using the proportional cox model. For the other models, 93.33% and 67.76%.

B) COMBINED MODELS

Here we do the same as we did for WC portfolio. The only difference is that we replace the WC portfolio by the AXA portfolio.

An extract of the results of the forecasted one year death probabilities for the combined model using the Brass linear model. Notice how in the table the probability of death at older ages from (106 years) is decreasing instead of increasing e.g. for the year 2013, the probability of death at 106 years is 0.15432 and at 107 years the figure is 0.145054.

Age	Probabilities ($m_{x,t}^{AXA}$)				
0	2013	2014	…	2111	2112
1	0.002365	0.002233	…	8.606782e-06	8.1274081e-06
2	5.38511e-05	4.899209e-05	…	5.086265e-09	4.62732159e-09
…	…	…	…	…	…
106	0.15432	0.154684	…	0.194314	0.194771
107	0.145054	0.145113	…	0.150949	0.15101

Table 8: Forecasted one year death probabilities for the combined model using the Brass Linear Model.

3) LIFE EXPECTANCY COMPUTED FOR POPULATION AND PORTFOLIO.

By implementing the coefficients of the portfolio mortality models into the Lee-Carter model, we can find period life expectancies for different ages. In general the life expectancies computed using the general population were higher than these computed using the models on the portfolio. An extract of the results is presented below.

AGE	POP	BRASS LINEAR	COX	WORKGROUP	DIFF B/W POP AND BRASS	DIFF B/W POP AND COX	DIFF B/W POP AND WORKGROUP
0	88.40	81.97	87.93	88.12	6.43	0.470	0.2837
1	87.45	80.99	86.99	87.17	6.46	0.467	0.2839
2	86.34	79.85	85.87	86.05	6.48	0.466	0.2841
…	…	…	…	…	…	…	…
98	1.71	2.17	1.56	1.687	0.460	0.1521	0.02646
99	1.18	1.38	1.111	1.111	0.200	0.0758	0.0119
100	0.5	0.5	0.2	0.5	0	0	0

Table 9: Life expectancy for population and portfolio.

As can be seen from the table above, the life expectancy computed using the population for an individual aged 0 was 88.404 years while that computed using the Brass linear model for the same age was 81.973 years. Thus the life expectancy of the population is higher than that of the portfolio by 6.43 years. For the details of difference between the life expectancy computed for the population and using the portfolio see Appendix 5.

Also as can be seen from the table above, the life expectancy computed using the population for an individual aged 0 was 88.404years while that computed using the Cox Model for the same age was 87.93 years. Thus the life expectancy of the population is higher than that of the portfolio by 0.47 years. For the details of difference between the life expectancy computed for the population and using the portfolio see appendix 5.

For all the different portfolio models the life expectancy computed for WC market portfolio and AXA were the same for all ages. For details of the life expectancies computed for WC market portfolio and AXA see Appendix 5.

CHAPTER 4: LONGEVITY RISK ASSESSMENT OF NON RECOVER-ABLE PENSIONS

Briefly, the longevity risk is the risk of an individual surviving beyond the originally anticipated, coming so when the chances of death are systematically lower than expected. In terms of calculating the present value of the liabilities associated with benefit pension plans defined, dynamic mortality tables to incorporate future enhancements of longevity, help to capture some of this risk. Still, they do not eliminate it completely, because the future trend of mortality is random, implying that there may be systematic deviations from the predictions obtained, regardless of the model used (Pitacco, 2002), materializing in an increase of the current value responsibilities for funding.

Thus, Longevity Risk is associated with insurance obligations (such as annuities) in which a company guarantees to make a series of payments until the death of the beneficiary. A decrease in mortality rates leads to an increase in the technical provisions. This risk will be tackled only for not compulsorily recoverable pensions as this was the reason for my internship.

Longevity Risk has only impact in pensions not compulsorily recoverable and Life Assistance, Rosa (2012).In general, due to disability, the victims of accident do not have the same mortality behavior as the Portuguese population. For this purpose, the tables (TV73/77) currently used by AXA are onerous and inadequate because they donot retain the mortality behavior of the portfolio.

Companies are required to manage pensions which are not compulsorily recoverable. This means that the company will support a series of payments until the death of the pensioner. Companies have to predict the amount of payments discounted reflecting the mortality effect for all pensioners (this task is monitored monthly).However, a decrease on mortality rates leads to an increase in technical provisions. This is one of the Life and Savings (L&S) risks, known as Longevity Risk. In recent years, people have got better health care; science and technology have evolved in cases of cancer or other diseases, and so on. Hence an increase in the life expectancy is expected. Companies have to be prepared for this scenario, Rosa (2012).

The Lifetime Assistance is a provision that companies create to assist more complicated victims’ cases. For example: victims who are in wheelchairs; victims that use advanced prosthesis to address causes of the accident; victims who need regular surgeries to keep and/or not to deteriorate their quality of life; and so on. These are some of the regular needs which follow the victims until their death. In the company under study, this provision is only calculated for compensations that are paid 15 years after the occurrence of the claim. The severity of annual payments and the longevity of these complex cases are the two risks implicit in lifetime assistance.

Annuities Management and Lifetime Assistance are supported by a mortality table reference and using a given interest rate. This reference table and interest rates are the baseline for longevity risk assessment after reserving.

Presently WC reserves in AXA are based on the static mortality table (TV73/77) and interest rate equal to 4.5%, representing the base line for longevity risk. However, this base line may be outdated due to lifetime expectancies improvement as well as variations in interest rates. We will only focus in the first problem; lifetime expectancies improvement. In our studies we have thus considered these life expectancies improvement by considering a dynamic mortality table for the entire population and additional relational models for the portfolios of WC and AXA. We will in this chapter thus analyse the impact on the reserve of non recoverable pension (thus longevity risk) resulting from different mortality tables.

The difference between reserves calculated using the base line and using the new mortality are presented in table 10 under results at the end of this chapter. This amount represents the expected insufficient reserves. However this amount only makes sense in a long term view of longevity risk. Companies should have assets to prevent this scenario occurrence.

4.1 Components of the reserve

Let Y_x be a random variable representing the present value of unit monthly payments that will be made in advance for a pensioner aged x and let $\mathrm{E}[Y_x]$ be its expected present value. According to Rosa (2012), this expected present value of Y_x depends on the type of beneficiary (Victims, orphans, husband /wife or parents).

VICTIMS: $\mathrm{E}[Y_x] = \ddot{a}_x^{(12)} \approx a_x + \frac{13}{24}$ [EPV of a whole life annuity in advance] (14)

ORPHANS: $\mathrm{E}[Y_x] = \ddot{a}_{x:\overline{25-x}|}^{(12)} = \ddot{a}_x^{(12)} - {}_{25-x}E_x \times \ddot{a}_{25}^{(12)}$ [EPV of a temporarily life annuity in advance] (15)

HUSBAND/WIFE: $\mathrm{E}[Y_x] = \ddot{a}_{x:\overline{65-x}|}^{(12)} + \frac{4}{3} X\ {}_{65-x}E_x \times \ddot{a}_{65}^{(12)}$ [EPV of a whole life annuity in advance] (16)

PARENTS: $\mathrm{E}[Y_x] = \ddot{a}_{x:\overline{65-x}|}^{(12)} + \frac{4}{3} X\ {}_{65-x}E_x \times \ddot{a}_{65}^{(12)}$ [EPV of a whole life annuity in advance] (17)

Where:

$a_{x} = \sum_{k=1}^{\infty} v^k\ {}_kP_x$ [EPV of a whole life annuity in arrear] (18)

$v = (1+i)^{-1}$ [Present value of 1 due one year hence] (19)

${}_kP_x = Pr[T_x > t] = S_x(t)$ [Prob. a life aged x survived for at least t years] (20)

${}_kq_x = Pr[T_x \leq t] = 1 - S_x(t)$ [Prob. a life aged x does not survive beyond age $x+t$] (21)

${}_nE_x = v^n\ {}_nP_x$ [Expected present value of the pure endowment] (22)

(x) denotes a life aged x.

T_x is a continuous random variable used to model the future lifetime of (x).

$S_x(t)$ represents the probability that (x) survives for at least t years, and $S_x(t)$is known as the survival function.

i represents the effective rate of interest per annum.

$a_{\ x}$ represent an annuity, first payment at the end of a year, to continue during the life of (*x*).

$\ddot{a}_x^{(12)}$ represent an annuity of (*x*) payable by 12 instalments of *1/12* each throughout the year, the first payment being at the start of the first month.

$\ddot{a}_{x:\overline{n-x}|}^{(12)}$ represent temporarily life annuity in advance for *n-x* years on (*x*) payable by 12 instalments of 1/12 each throughout the year.

${}_{n-x}E_x$ represent the value of an endowment on *(x)* payable at the end of *n-x* years if (*x*) is then alive.

Note that in (16), we don't consider the possibility of the husband or the wife getting married again. In this case, the husband or the wife loses the right to pension but the company has to pay three times the annual pension amount in one time. We observed from the Portuguese Association of Insurers (known as APS) benchmark study reported by each company to Portuguese Insurance Institute (ISP), that since 2006,this possibility has almost not been used. The worst scenario for companies is to consider the rate of remarriage equal to zero and it represents a cost (see Rosa, 2012).

Let R^j be the reserves for an annuities portfolio at time j, then,

$R^j = \sum_{k=1}^{K} R_k^j$

Where K is the number of pensioners and R_k^j is the reserve for beneficiary *k* at time *j* (depending of annual amount and the expected present value$\mathrm{E}[Y_x]$.
It is expected that some pensioners die during accounting year $(j, j+1]$ and release reserve at time j+1.When it does not occur the reserve will be recalculated at time j +1 with the pensioner one year older.

Let P^{j+1} be the payments that will occur during accounting year $(j, j+1]$,

$$P^{j+1} = \sum_k^K \frac{P_k}{12} X \, [\, \ddot{a}_{12\ i} \text{ X } j_w + \ddot{a}_{6\ i} \ X \ (1 - j_w\ )\,]$$

where P_k is the annual amount paid to pensioner k,

$$j_k = \begin{cases} 1, \text{pensioner } w, \text{doesn't die during accounting year (j, j + 1)} \\ 0, \text{pensioner } w \text{ dies during accounting year (j, j + 1)} \end{cases},$$

and the factor $\ddot{a}_{\overline{n}|i}$ corresponds to the present value of the n certain monthly payments of one monetary unit in advance (not depending on human life) and $i´$ is the nominal annual rate of interest convertible in 12 times per year.

4.2 Fitting the components

The reserves for portfolio of annuities at time j, R^j is computed based on the components defined in Section 6.1 above using a Macro in Excel program. The Macro runs as follows:

- We input on one excel sheet the dynamic mortality rates obtained for the general population or these obtained for the portfolio of WC / AXA.
- On another sheet we define other parameters such as the one year death probabilities, the life expectancy for each age from 0 – 100, including the components of the reserve we defined above.
- For each of the models (Population, Brass Linear, Cox, Workgroup PLT), we run the Macro keeping the components of the reserve as output elements.
- This output elements are entered on a different sheet.
- We then use these elements to compute the difference between the reserve computed using the static table and the reserve now computed using the dynamic table.

4.3 Results

The impact using the various models is presented in the table below:

Method	Reference	Impact
Lee-Carter (BASE)	Total Population	9.8%
Cox	AXA Experience	8.7%
Brass Linear	AXA Experience	5.2%
Workgroup	AXA Experience	9.1%
Cox	WC Experience	8.7%
Brass Linear	WC Experience	6.2%
Workgroup	WC Experience	7.7%

Table 10: impact on reserve using dynamic mortality table

With reference to the static table, the dynamic mortality table computed with the general population had a 9.8% impact on the reserves. This implied that if the company were to consider the dynamic table and not the static one it should increase it reserves by 9.8% to be able to meet up with its obligations in the future. The impact on the reserves using the mortality table and the various portfolio models on either the WC portfolio or AXA Portfolio are presented in Table 10 above.

CHAPTER 5: CONCLUSIONS AND RECOMMENDATIONS

As stated before in section 1.2 in chapter 1,the misestimating of reserves for lifetime worker's compensation cases can stem from a variety of issues including: insufficient historical loss development data, significant impact of inflation on future cost, increases in medical utilization over time, use of outdated or static life tables and industry case reserving practices. Our studies however concentrated on the impact resulting from the use of static tables as this was the reason for the internship.

The goal of this report thus was to develop a stochastic model for future mortality in the Portuguese WC market portfolio and AXA pensions portfolio, and to investigate the impact on the reserves using this stochastic model instead of the static life table (TV 73/77) currently being used.

As the data set of mortality in the portfolio was not large enough to come up with a stochastic model solely based on the data from the WC or AXA Portfolio, we had to take a different approach. First we had to obtain values for the probabilities of death with reference to the Portuguese population and then adjust the results to the pensioner's population of WC and AXA through relational models. It was found that, in general, the quality of fit is satisfactory. These probabilities were then used to compute the not compulsorily recoverable reserves required for the pensioners, their spouses and children under 25.The results obtained are compared with the noncompulsory recoverable reserves computed using the static mortality table (TV 73/77) that is currently being used by AXA to see the impact on this reserve if AXA adopted the dynamic tables.

Three different models were used to predict the future mortality of the general Portuguese population: the Original Lee-Carter with Gaussian errors, the Poisson Lee-Carter and the APC variant of the Lee-Carter method including cohort effects. The model parameters α_x , β_x, $\beta_x^{(0)}$, $\beta_x^{(1)}$ $and\ k_t$ are fitted by either OLS or Maximum likelihood method.By assuming that the other parameters are constant over time,the time parameter k_t is forecasted for 100 years (2013-2112) using a standard ARIMA univariate time series model. Estimates of α_x , β_x, $\beta_x^{(0)}$, $\beta_x^{(1)}$ are used with forecasted values of k_t to generate other life table functions including the sex-specific forces of mortality and the life expectancy.Based on the fit of the life expectancy for the different models, it was observed that the Poisson Lee-Carter had the best fit to the estimated data. The forecasted mortality rates obtained using this method was then used as reference rate to model the mortality rate of Workers compensation for the Portuguese market as well as the Portfolio of AXA.

The forecasted mortality rates of the Portfolio were obtained by applying the deterministic parameters of the relational models to the dynamic mortality rates (reference rates) of the population for the estimated future years in ages 19-110 and in the ages 0- 18 it was assumed the population rates equal the portfolio rates. Three different relational models were used to obtain the deterministic parameters: Workgroup PLT, Brass Linear Model and Proportional Cox Model. The parameters of the relational models (θ,e_x,θ_1 and θ_2) and their standard errors were estimated using linear regression.

Using a Macro in excel, by inputting the dynamic mortality rates obtained for the general population or that for the portfolios we obtained the life expectancy for different ages. As already stated in subsection 3 of section 3.3.5, the life expectancy using the general population was higher than those computed using the different relational models on the portfolio. However, for all the different portfolio relational models the life expectancy computed for WC market portfolio and AXA were the same for all ages.

The second part of my internship was devoted to the calculation of the impact on the reserves using this stochastic model instead of the static life table (TV 73/77) currently being used. We calculated this impact on the reserves for the three different portfolios (population, WC and AXA).For the WC and AXA portfolios, we looked at the impact resulting from the three different relational models of Workgroup PLT, Brass Linear and Proportional Cox.

The impact caused by the use of dynamic mortality tables, in principle, reflects more accurately the population's and Portfolio's mortality profile in question and the respective evolution. As already shown in section 6.3, the impact on the reserves using the dynamic mortality tables was generally higher in all cases than that computed using the TV tables. This was as expected because the TV tables are static in nature and does not incorporate the increase in longevity over the past decades.

In the ideal situation we would have modeled population mortality and portfolio mortality simultaneously. To make this possible, a larger data set would be needed. Furthermore we need to assume that the relation between population mortality and portfolio mortality remains constant over time. AXA should continue the gathering of data on mortality in the portfolio. In that way it might in the future be possible to investigate how the relation between portfolio mortality and population mortality has changed over time and to incorporate this into the portfolio mortality model. At this point in time, no conclusion can be drawn in this respect, as the data set is simply too small. Experts do not agree whether the general relation between mortality of a population and a subpopulation has changed or remained constant in the past (wijk, 2012).This is the main shortcoming of the method we used.

Regarding future research, some improvements can be made to the research performed here's discussed in an earlier paragraph, AXA needs to collect more data in order to come to more advanced models and more reliable conclusions regarding the mortality in the portfolio. The underlying model for population mortality can be improved as well. Throughout this report we have made some assumptions regarding this model. I have for instance based the prediction of future mortality rates in the population entirely on the Poisson Lee-Carter method, whereas there are also other accurate models known in the literature such as Cairns-Blake-Dowd (CBD) model used by Cairns *et al* (2006).Thus more accurate models could be used in future research.

Furthermore by applying the deterministic parameters of the relational models to the dynamic mortality rates (reference rates) of the population to obtain the forecasted mortality rates of the Portfolio, we assume the relationship between the portfolio and population is constant over time and thus deterministic. For future research this could however be modelled stochastically as Plat (2009).This however is more complicated and will require a significant amount of time and possibly the use of faster computer software.

Another proposal for improvement would be to consider implementing a stochastic term structure for the interest rates .As it is now, I have neglected interest rate risk, assuming that the rates will not change over time. Of course this is not really in line with practice, so either the use of this model should be combined with the appropriate hedging of interest rate risk in the market, or it should include a stochastic term structure of interest rates if it would actually be implemented in practice. Furthermore individual mortality risk is not included. Even though the portfolio is large and the effects of the individual mortality risk being implemented are likely to be not big, results will be (slightly) more accurate when this form of risk is considered as well. However just as considering stochastic relationship between population and portfolio is time consuming, this will also be very time consuming.

In general, this study contributes to deepen the understanding of the potential impact on the general Portuguese population, Portuguese market WC portfolio and AXA pension portfolio of using dynamic rather than static mortality table to compute the reserves.

References

Alho, J.M. (2002). Discussion of Lee (2000). North American Actuarial Journal 4 (1), 91-93.

Blumsohn, Gary. (1997). Levels of Determinism in Workers Compensation Reinsurance Commutations. Casualty Actuarial Society Forum Spring, pages 53-114.

Brass, W. (1974). Mortality models and their uses in demography. Transactions of the Faculty of Actuaries 33, 123-132.

Brian A. Jones, Craig J. Scukas, Kathryn S. Frerman, Melissa S. Holt and Vicki A. Fendley (2013). A Mortality-Based Approach to Reserving for Lifetime Workers' Compensation Claims. Casualty Actuarial Society E-Forum, Fall 2013.

Brillinger, D.R., 1986. The natural variability of vital rates and associated statistics. Biometrics 42, 693–734.

Brouhns, N. and Denuit, M. (2001). Risque de longévité et rentes viagères. III. Elaboration de tables de mortalité prospectives pour la population assurée belge et évaluation du cout de l'antiselection. Belgian Actuarial Bulletin, 2, 64-72.

Natacha Brouhns, Michel Denuit and Jeroen K. Vermunt (2002). A Poisson log-bilinear regression approach to the construction of projected life-tables. Insurance: Mathematics and Economics, 31, 373-393.

Cairns, A.J.G., Blake, D. and Dowd, K. (2006). A two-factor model for stochastic mortality with parameter uncertainty. The Journal of Risk and Insurance 73 (4), 687-718.

Cairns, A.J.G., Blake, D. and Dowd, K. (2006a). A Two-Factor Model for Stochastic Mortality with Parameter Uncertainty: Theory and Calibration. The Journal of Risk and Insurance, 73, 687-718.

Coelho, E., Magalhães, M.G. and Bravo, J.M. (2010). Mortality and Longevity Projections for the Oldest-Old in Portugal. Work session on demographic projections, Bucharest, 10-12 October 2007, 117-132.

Cox, D.R. (1972). Regression models and lifetables. Journal of the Royal Statistical Society – Series B (Methodology) 34 (2), 187-220.

Stevens R., De Waegenaere A. and Melenberg B. (2010). Calculating capital requirements for longevity risk in life insurance products. Using an internal model in line with Solvency II. Available from Netspar website.

Stevens R., De Waegenaere A. and Melenberg B. (2010a). Longevity risk in pension annuities with exchange options: The effect of product design. Insurance: Mathematics and Economics, 46,222-234.

Delwarde, A., Denuit, M. and Eiler, P. (2004). Smoothing the Lee-Carter and Poisson log-bilinear models for mortality forecasting: A penalized log-likelihood approach. Statistical Modelling, 7(1), 29-48.

Delwarde, A., Kachakhidze, D., Olie, L. and Denuit, M. (2004). Modèles linéaires et additifs généralisés, maximum de vraisemblance local et méthodes relationnels en assurance sur la vie. Bulletin Français d'Actuariat 6 (12), 77-102.

Denuit, M. (2007). Comonotonic approximations to quantiles of life annuity conditional expected present value. Insurance: Mathematics and Economics, 42, 831-838.

Dickson D., Hardy M. and Waters H. (2011) Actuarial mathematics for life contingent risks, 2nd ed., Cambridge, Cambridge University Press.

Ferguson, Ronald E. (1971). Actuarial Note on Workmen's Compensation Loss Reserves. Proceedings of the Casualty Actuarial Society, Volume LVIII, pages 51-57.

Gillam, William R. (1993). Injured Worker Mortality. Proceedings of the Casualty Actuarial Society, Volume LXXX, pages 34-54.

Goodman, L.A. (1979). Simple models for the analysis of association in cross-classifications having ordered categories. Journal of the American Statistical Association 74 (367), 537-552.

Gutterman, S. and Vanderhoof, I.T. (1999). Forecasting changes in mortality: a search for a law of causes and effects. North American Actuarial Journal 2, 135-138.

Haberman, S. and Renshaw, A.E. (2011). A comparative study of parametric mortality projection models. Insurance: Mathematics and Economics, 48(1), 35-55.

Instituto de Seguros de Portugal (1996). Norma Regulamentar nº 21/1996 de 5 de dezembro http://www.isp.pt.

Instituto de Seguros de Portugal (2008). Relatório do Sector Segurador e dos Fundos de Pensões 2008. http://www.isp.pt.

Instituto de Seguros de Portugal (2011). Relatório do Sector Segurador e dos Fundos de Pensões 2011. http://www.isp.pt.

Kahn, James (2002). Reserving for Runoff Operations - A Real Life Claims Specific Methodology for Reserving a Workers' Compensation Runoff Entity. Casualty Actuarial Society Forum, Fall, pages 139-210.

Lee, Ronald D. and Carter, L. (1992). Modeling and Forecasting U.S. Mortality: Differentials in Life Expectancy by Sex. International Journal of Forecasting, 8(3), 393-412.

Lee, Ronald D. (2000). The Lee-Carter method of forecasting mortality, with various extensions and applications. North American Actuarial Journal, 4, 80-93.

Li, N. and Lee, R. (1993). Coherent Mortality Forecasts for a Group of Populations: An Extension of the Lee-Carter Method. Demography, 42, 575-594.

Makeham, W.M. (1860). On the law of mortality and the construction of annuity tables. The Assurance Magazine and Journal of the Institute of Actuaries 8 (6), 301-310.

Pateiro D.A.D. (2013). Development of dynamic life tables for the Portuguese market of pension funds and assessment of the longevity risk. ISEG, Master's Thesis.

Peters, F., Nusselder, W. and Mackenbach, J. (2012). Assessing the model risk of longevity. The projection model of the Actuarial Society in comparison with two other Dutch projections. Netspar NEA paper, yet to be published.

Pitacco, E., Denuit, M., Haberman, S., and Olivieri, A. (2009). Modelling Longevity Dynamics for Pension and Annuity Business. Oxford University Press.

Plat, R. (2009). Stochastic Portfolio Specific Mortality and the Quantification of Mortality Basis Risk. Insurance: Mathematics and Economics, 45(1), 123-132.

Plat, R. (2009a). On stochastic mortality modeling. Insurance: Mathematics and Economics, 45(3), 393-404.

Renshaw, A.E. and Haberman, S. (2003). Lee-Carter mortality forecasting: A parallel generalized linear modelling approach for England and Wales mortality projections. Insurance: Mathematics and Economics, 52(1): 119-137.

Renshaw, A.E. and Haberman, S. (2003a). Lee-Carter mortality forecasting with age specific enhancement. Insurance: Mathematics and Economics, 33:255-272.

Renshaw, A.E. and Haberman, S. (2006). A cohort-based extension to the Lee-Carter model for mortality reduction factors. Insurance: Mathematics and Economics, 38:556-570.

Rosa, C.E.B. (2012). An internal model for workers compensation. ISEG, Master's Thesis.

Sherman, Richard E. and Diss, Gordon F. (2005). Estimating the Workers Compensation Tail. Proceedings of the Casualty Actuarial Society, Volume XCII, pages 579-678.

Snader, Richard H. (1987). Reserving Long Term Medical Claims. Proceedings of the Casualty Actuarial Society, Volume LXXIV, pages 322-353.

Steeneck, Lee R. (1996). Actuarial Note on Workmen's Compensation Loss Reserves-25 Years Later," Casualty Actuarial Society Forum, pages 245-272

Wijk V.M. (2012). Mortality in a pensions portfolio and the application to internal model for solvency II. Network for studies on Pensions, Aging and Retirement.

Wilmoth, J.R. (1993). Computational methods for fitting and extrapolating the Lee-Carter model of mortality change. Technical report. Department of Demography. University of California, Berkeley.

Wilmoth, J.R. (1995). Are Mortality Rates Falling at Extremely High Ages? An Investigation Based on a Model Proposed by Coale and Kisker. Population Studies, 49(2), 281-295.

Wilmoth, J.R. (2000). Demography of longevity: past, present and future trends. Experimental Gerontology, 35, 1111-1129.

Workgroup PLT (2010). Generatietafels Pensioenen 2010. Verbond van Verzekeraars. Material only intended for pension insurers.

Data

Human Mortality Database. University of California, Berkeley (EUA), and Max Planck Institute for Demographic Research (Alemanha). Disponível em www.mortality.org/.

Portuguese market Workers compensation and AXA pension portfolio data was acquired within the company.

APPENDICES

APPENDIX 1: ITERATIVE NEWTON-RAPHSON FITTING ALGORITHM

The fitting methodology implemented in the Poisson Lee-Carter is based on an iterative algorithm that minimizes the deviance function. That is, we make use of a cyclical updating process of the parameter estimates until the minimum difference between the likelihood of the fitted model and the likelihood of the saturated model (i.e.one parameter for each observation) is achieved. Thus, the updating mechanism for a given parameter is provided by the Newton-Raphson minimization method applied to the deviance function, which can be expressed as

$$u(\hat{\theta}) = \hat{\theta} - \frac{\frac{\partial D}{\partial \theta}}{\frac{\partial^2 D}{\partial \theta^2}} \qquad (1)$$

The deviance function with Poisson error structure given by:

$$D(y_{x,t}\hat{y}_{x,t}) = \sum_{x,t} dev(x,t) = \sum_{x,t} 2\, \omega_{x,t} \left\{ y_{x,t} log \frac{y_{x,t}}{\hat{y}_{x,t}} - (y_{x,t} - \hat{y}_{x,t}) \right\}$$

where $dev(x,t)$ are the deviance residuals that depend on a set of prior weights $\omega_{x,t}$ where $\omega_{x,t}$= 1 is assigned to each non-empty data cell, with $\omega_{x,t}$ = 0 for empty cells.

Looking at the deviance function above with Poisson error structure, we can observe that:

$$\frac{\partial D}{\partial \theta} = \sum \frac{\partial\, dev}{\partial \theta} = \sum 2\omega \left\{ -y \frac{\hat{y}'}{\hat{y}} + \hat{y}' \right\}$$

$$= \sum 2\omega \frac{\hat{y}'}{\hat{y}} (\hat{y} - y) = \sum 2\omega\alpha(\hat{y} - y) \ , \qquad (2)$$

Where

$$\hat{y}' = \frac{\partial \hat{y}}{\partial \theta} \rightarrow \begin{cases} \dfrac{\partial \hat{y}}{\partial \alpha_x} = \hat{y} \\ \dfrac{\partial \hat{y}}{\partial \beta_x} = k_t \hat{y} = \alpha \hat{y} \\ \dfrac{\partial \hat{y}}{\partial k_t} = \beta_x \hat{y} \end{cases} \quad such\ that \begin{cases} \alpha = 1 \\ \alpha = k_t \\ \alpha = \beta_x \end{cases}$$

Making use of the above simplified notations, we can express the second partial derivative of the deviance function as follows:

$$\frac{\partial^2 D}{\partial \theta^2} = \sum 2\omega\alpha\, \hat{y}' = \sum 2\omega\, \alpha^2 \hat{y} \qquad (3)$$

Substituting the expressions (2) and (3) into (1) yields the following general fitting routine:

$$u(\hat{\theta}) = \hat{\theta} - \frac{\sum 2\omega\, \alpha(\hat{y} - y)}{\sum 2\omega\, \alpha^2 \hat{y}} = \hat{\theta} + \frac{\sum 2\omega\, \alpha(\hat{y} - y)}{\sum 2\omega\, \alpha^2 \hat{y}}$$

We note that similar updating rule can be determined in the case of the model with Gaussian distributed errors (see Renshaw and Haberman, 2006).Without going into further details, we note that the ilc package implements the updating algorithms corresponding to the Lee- Carter models with both Gaussian and Poisson error structures. For the purpose of the current study, in the following parts we focus on the detailed estimation methodology of the Poisson Lee – Carter modelling framework.

Updating cycle of the passion LC fitting

1. **Get appropriate initial values:**

$$\alpha_x = \frac{1}{n}\sum_t \log \hat{m}_{x,t}$$
$$\beta_x = \frac{1}{n}; \hat{k}_t = 0.$$

→ calculate fitted values $\hat{y}(\hat{\alpha}_x \hat{\beta}_x \hat{k}_t)$ → calculate deviance $D(y_{x,t}\hat{y}_{x,t})$.

2. **Update parameter $\hat{\alpha}_x$:**

$$\alpha_x = \hat{\alpha}_x - \frac{\sum_t 2\omega(y-\hat{y})}{\sum_t 2\omega(\hat{y})}$$

→ calculate fitted values $\hat{y}(\hat{\alpha}_x \hat{\beta}_x \hat{k}_t)$ → calculate deviance $D(y_{x,t}\hat{y}_{x,t})$.

3. **Update parameter $\hat{k}_t$:**

$$\hat{k}_t = \hat{k}_t + \frac{\sum_t 2\omega(y-\hat{y})}{\sum_t 2\hat{\beta}_x^2\omega(\hat{y})}$$

- adjust the updated parameter such that $\hat{k}_t = \hat{k}_t - \bar{\hat{k}}_t$;

→ calculate fitted values $\hat{y}(\hat{\alpha}_x \hat{\beta}_x \hat{k}_t)$ → calculate deviance $D(y_{x,t}\hat{y}_{x,t})$.

4. **Update parameter: $\hat{\beta}_x$:**

$$\hat{\beta}_x = \hat{\beta}_x + \frac{\sum_t 2\omega(y-\hat{y})}{\sum_t 2\hat{k}_x^2\omega(\hat{y})}$$

→ calculate fitted values $\hat{y}(\hat{\alpha}_x \hat{\beta}_x \hat{k}_t)$ → calculate deviance $D_u(y_{x,t}\hat{y}_{x,t})$.

5. **Check deviance convergence:**

$$\Delta D = D - D_u,$$

where D_u is the updated deviance at step 4.

-if $\Delta D > 1 \ X \ 10^{-6} \ \rightarrow goto\ step\ 2.$

-Stop iterative process once $\Delta D \approx 0$ and take the fitted parameters as the ML estimates to the observed data.

-Alternatively, stop if D<0 for a consecutive 5 updating cycles and consider using other starting values or declare the iterations non-convergent.

6. **Once convergence is achieved, re-scale the interaction parameters: $\boldsymbol{\hat{\beta}_x}$ and $\boldsymbol{\hat{k}_t}$:**

$$\hat{\beta}_x = \frac{\hat{\beta}_x}{\sum_x \hat{\beta}_x}$$

$$\hat{k}_t = \hat{k}_t \; X \; \sum_x \hat{\beta}_x$$

in order to satisfy the usual LC model constraints $\sum_t k_t = 0 \; and \; \sum_x \beta_x$

APPENDIX 2: Values of α_x, β_x, k_t and i_{t-x}

	LEE CARTER		POISSON LEE CARTER		POISSON LEE CARTER WITH COHORT				LEE CARTER		POISSON LEE CARTER		POISSON LEE CARTER WITH COHORT		
x	ax	bx	ax	bx	ax	bx	bx0	x	ax	bx	ax	bx	ax	bx	bx0
0	-3.614	0.030	-3.522	0.025	.	.	.	61	-4.340	0.006	-4.334	0.005	-4.340	0.007	0.008
1	-5.532	0.040	-5.663	0.041	.	.	.	62	-4.190	0.008	-4.189	0.007	-4.190	0.009	0.011
2	-6.192	0.036	-6.384	0.038	.	.	.	63	-4.101	0.008	-4.098	0.007	-4.101	0.008	0.011
3	-6.648	0.032	-6.807	0.033	.	.	.	64	-4.015	0.008	-4.012	0.007	-4.015	0.009	0.011
4	-6.916	0.029	-7.035	0.030	.	.	.	65	-3.900	0.008	-3.898	0.007	-3.900	0.009	0.011
5	-7.144	0.026	-7.241	0.026	.	.	.	66	-3.821	0.008	-3.819	0.007	-3.821	0.009	0.011
6	-7.302	0.025	-7.373	0.025	.	.	.	67	-3.734	0.007	-3.731	0.007	-3.734	0.008	0.011
7	-7.362	0.023	-7.428	0.023	.	.	.	68	-3.607	0.008	-3.605	0.007	-3.607	0.009	0.012
8	-7.493	0.021	-7.568	0.022	.	.	.	69	-3.561	0.007	-3.552	0.006	-3.561	0.008	0.010
9	-7.570	0.021	-7.629	0.021	.	.	.	70	-3.353	0.010	-3.351	0.008	-3.353	0.010	0.014
10	-7.613	0.020	-7.656	0.020	.	.	.	71	-3.363	0.006	-3.354	0.006	-3.363	0.007	0.010
11	-7.610	0.018	-7.648	0.019	.	.	.	72	-3.179	0.008	-3.177	0.008	-3.179	0.009	0.012
12	-7.619	0.018	-7.661	0.019	.	.	.	73	-3.088	0.008	-3.085	0.007	-3.088	0.009	0.012
13	-7.561	0.018	-7.583	0.018	.	.	.	74	-2.969	0.008	-2.965	0.007	-2.969	0.009	0.012
14	-7.401	0.017	-7.415	0.017	.	.	.	75	-2.831	0.009	-2.830	0.008	-2.831	0.010	0.013
15	-7.228	0.015	-7.230	0.015	.	.	.	76	-2.751	0.007	-2.746	0.007	-2.751	0.009	0.012
16	-7.048	0.015	-7.016	0.014	.	.	.	77	-2.642	0.007	-2.638	0.007	-2.642	0.009	0.012
17	-6.911	0.014	-6.872	0.013	.	.	.	78	-2.497	0.008	-2.496	0.007	-2.497	0.009	0.013
18	-6.765	0.013	-6.720	0.013	-6.432	0.029	0.018	79	-2.456	0.006	-2.445	0.006	-2.456	0.008	0.011
19	-6.687	0.013	-6.643	0.012	-6.468	0.029	0.019	80	-2.318	0.008	-2.317	0.007	-2.318	0.009	0.012
20	-6.636	0.013	-6.599	0.013	-6.528	0.031	0.022	81	-2.336	0.004	-2.322	0.004	-2.336	0.006	0.008
21	-6.611	0.013	-6.571	0.013	-6.611	0.032	0.025	82	-2.167	0.005	-2.164	0.005	-2.167	0.007	0.009
22	-6.603	0.013	-6.562	0.014	-6.603	0.034	0.026	83	-2.087	0.005	-2.081	0.005	-2.061	0.006	0.008
23	-6.586	0.014	-6.548	0.014	-6.586	0.033	0.026	84	-1.966	0.005	-1.962	0.005	-1.918	0.006	0.008
24	-6.544	0.014	-6.512	0.013	-6.544	0.032	0.026	85	-1.845	0.006	-1.844	0.005	-1.787	0.006	0.007
25	-6.543	0.013	-6.510	0.013	-6.543	0.031	0.025	86	-1.775	0.004	-1.771	0.004	.	.	.
26	-6.514	0.013	-6.479	0.013	-6.514	0.030	0.025	87	-1.697	0.004	-1.691	0.004	.	.	.
27	-6.485	0.013	-6.451	0.012	-6.485	0.029	0.024	88	-1.579	0.004	-1.576	0.004	.	.	.
28	-6.453	0.013	-6.420	0.013	-6.453	0.029	0.025	89	-1.549	0.002	-1.538	0.003	.	.	.
29	-6.450	0.012	-6.418	0.011	-6.450	0.026	0.023	90	-1.385	0.005	-1.384	0.004	.	.	.
30	-6.377	0.012	-6.348	0.012	-6.377	0.026	0.023	91	-1.456	0.000	-1.432	0.002	.	.	.
31	-6.376	0.011	-6.349	0.010	-6.376	0.023	0.021	92	-1.312	0.001	-1.303	0.002	.	.	.
32	-6.309	0.012	-6.283	0.011	-6.309	0.024	0.022	93	-1.266	0.000	-1.259	0.002	.	.	.
33	-6.248	0.012	-6.221	0.011	-6.248	0.023	0.021	94	-1.199	0.000	-1.198	0.001	.	.	.
34	-6.211	0.011	-6.187	0.010	-6.211	0.021	0.020	95	-1.074	0.002	-1.076	0.002	.	.	.
35	-6.145	0.011	-6.122	0.010	-6.145	0.021	0.020	96	-1.010	0.001	-1.014	0.002	.	.	.
36	-6.098	0.011	-6.076	0.010	-6.098	0.020	0.019	97	-0.953	0.000	-0.964	0.001	.	.	.
37	-6.049	0.010	-6.030	0.010	-6.049	0.019	0.018	98	-0.810	0.003	-0.812	0.003	.	.	.
38	-5.969	0.011	-5.951	0.010	-5.969	0.019	0.019	99	-0.847	0.000	-0.862	0.001	.	.	.
39	-5.926	0.009	-5.913	0.009	-5.926	0.016	0.016	100	-0.702	0.002	-0.703	0.002	.	.	.
40	-5.826	0.010	-5.811	0.009	-5.826	0.017	0.017	101	-0.904	-0.005	-0.909	-0.002	.	.	.
41	-5.830	0.008	-5.823	0.007	-5.830	0.013	0.013	102	-0.734	-0.001	-0.729	0.000	.	.	.
42	-5.696	0.009	-5.684	0.009	-5.696	0.015	0.016	103	-0.737	-0.003	-0.733	-0.001	.	.	.
43	-5.664	0.008	-5.658	0.007	-5.664	0.013	0.013	104	-0.750	-0.004	-0.738	-0.001	.	.	.
44	-5.589	0.008	-5.581	0.007	-5.589	0.013	0.013	105	-0.522	-0.001	-0.545	0.001	.	.	.
45	-5.501	0.009	-5.493	0.008	-5.501	0.013	0.014	106	-0.523	-0.006	-0.699	-0.002	.	.	.
46	-5.459	0.008	-5.456	0.007	-5.459	0.011	0.012	107	-0.268	-0.006	-0.655	0.000	.	.	.
47	-5.390	0.007	-5.388	0.006	-5.390	0.010	0.011	108	0.164	-0.009	-0.300	-0.003	.	.	.
48	-5.276	0.008	-5.272	0.007	-5.276	0.011	0.012	109	0.277	-0.010	-0.254	-0.004	.	.	.
49	-5.237	0.007	-5.236	0.006	-5.237	0.009	0.010	110	-0.331	-0.012	-0.465	-0.006	.	.	.
50	-5.121	0.008	-5.119	0.007	-5.121	0.011	0.012								
51	-5.123	0.006	-5.121	0.005	-5.123	0.007	0.009								
52	-4.993	0.008	-4.993	0.007	-4.993	0.009	0.011								
53	-4.923	0.007	-4.923	0.006	-4.923	0.008	0.010								
54	-4.862	0.007	-4.860	0.006	-4.862	0.008	0.010								
55	-4.776	0.008	-4.775	0.007	-4.776	0.009	0.011								
56	-4.701	0.008	-4.700	0.007	-4.701	0.009	0.011								
57	-4.639	0.007	-4.638	0.006	-4.639	0.008	0.009								
58	-4.535	0.008	-4.534	0.007	-4.535	0.008	0.010								
59	-4.486	0.007	-4.483	0.006	-4.486	0.007	0.009								
60	-4.323	0.009	-4.322	0.008	-4.323	0.010	0.012								

LEE CARTER		POISSON LEE CARTER	POISSON LEE CARTER WITH COHORT					LEE CARTER		POISSON LEE CARTER	POISSON LEE CARTER WITH COHORT				
t	kt	kt	kt	x	itx	x	itx	t	kt	kt	kt	x	itx	x	itx
1940	74.58	65.96	0.00	1855	0	1928	54.57	2000	-41.60	-50.93	-111.93	1915	50.91	1988	155.17
1941	80.41	73.12	3.24	1856	0	1929	56.54	2001	-43.60	-55.02	-115.23	1916	51.08	1989	155.24
1942	78.69	69.45	0.64	1857	0	1930	54.99	2002	-44.35	-57.35	-118.66	1917	51.91	1990	156.99
1943	74.75	67.13	-2.48	1858	32.35	1931	53.46	2003	-49.86	-58.77	-120.39	1918	50.95	1991	159.29
1944	72.16	64.18	-4.26	1859	26.13	1932	54.86	2004	-53.07	-70.19	-128.82	1919	48.12	1992	0
1945	67.16	60.99	-6.17	1860	38.68	1933	53.89	2005	-57.58	-67.89	-128.51	1920	52.35	1993	0
1946	68.08	61.11	-4.45	1861	33.71	1934	54.50	2006	-59.15	-78.68	-136.86	1921	52.09	1994	0
1947	61.71	56.16	-9.78	1862	37.66	1935	53.40	2007	-65.56	-81.24	-139.83	1922	53.11		
1948	59.37	55.03	-14.51	1863	33.35	1936	53.57	2008	-69.10	-84.93	-143.57	1923	54.00		
1949	59.42	58.41	-12.51	1864	38.15	1937	53.17	2009	-70.50	-88.43	-147.13	1924	54.65		
1950	52.83	51.01	-19.55	1865	30.17	1938	54.44	2010	-72.35	-91.59	-150.30	1925	54.33		
1951	50.82	50.30	-19.24	1866	32.59	1939	55.54	2011	-76.89	-99.57	-156.94	1926	54.62		
1952	44.50	49.33	-29.88	1867	42.20	1940	53.58	2012	-79.16	-98.12	-158.53	1927	55.85		
1953	34.97	46.36	-36.13	1868	40.68	1941	53.48								
1954	33.54	41.72	-35.81	1869	46.56	1942	53.38								
1955	32.99	44.53	-35.39	1870	34.47	1943	56.78								
1956	33.95	44.27	-29.72	1871	31.46	1944	56.58								
1957	33.20	44.40	-35.95	1872	43.06	1945	55.43								
1958	23.18	37.89	-46.35	1873	37.19	1946	57.73								
1959	24.66	40.75	-44.44	1874	42.71	1947	58.00								
1960	20.03	35.42	-44.96	1875	38.92	1948	60.56								
1961	23.61	41.27	-46.09	1876	41.86	1949	63.33								
1962	20.03	35.42	-47.06	1877	47.65	1950	61.85								
1963	19.16	32.62	-46.20	1878	48.33	1951	64.90								
1964	19.82	31.50	-47.39	1879	56.21	1952	68.03								
1965	14.08	27.37	-48.35	1880	45.86	1953	72.16								
1966	14.90	28.52	-44.04	1881	43.78	1954	74.27								
1967	13.45	24.41	-47.19	1882	50.87	1955	78.35								
1968	9.96	23.59	-49.90	1883	47.61	1956	80.18								
1969	11.53	24.52	-43.17	1884	48.99	1957	82.15								
1970	8.30	22.38	-51.75	1885	46.39	1958	86.56								
1971	10.94	22.90	-46.83	1886	48.92	1959	89.52								
1972	1.69	11.68	-55.85	1887	52.49	1960	90.43								
1973	4.34	14.81	-53.80	1888	49.85	1961	92.92								
1974	1.07	11.39	-53.02	1889	56.19	1962	96.40								
1975	3.34	11.66	-52.97	1890	50.71	1963	98.52								
1976	3.01	10.37	-51.55	1891	49.51	1964	100.38								
1977	-3.74	3.58	-59.08	1892	54.47	1965	102.37								
1978	-4.36	1.21	-61.84	1893	54.06	1966	105.05								
1979	-7.91	-3.84	-67.13	1894	54.93	1967	108.17								
1980	-9.69	-4.56	-67.61	1895	52.87	1968	111.93								
1981	-11.08	-6.72	-68.16	1896	52.05	1969	112.92								
1982	-13.60	-12.27	-74.16	1897	56.59	1970	113.41								
1983	-15.14	-11.65	-73.48	1898	56.46	1971	116.45								
1984	-16.19	-13.69	-75.16	1899	60.59	1972	117.70								
1985	-19.23	-15.66	-78.07	1900	53.12	1973	119.55								
1986	-19.92	-19.41	-81.81	1901	49.95	1974	121.08								
1987	-21.13	-23.00	-84.35	1902	52.62	1975	121.65								
1988	-21.11	-23.08	-84.97	1903	53.57	1976	123.06								
1989	-23.64	-28.52	-89.27	1904	53.91	1977	123.97								
1990	-23.59	-24.31	-86.28	1905	50.34	1978	126.36								
1991	-20.98	-24.34	-86.98	1906	52.12	1979	128.53								
1992	-24.87	-30.99	-91.62	1907	53.38	1980	127.27								
1993	-26.46	-28.91	-91.13	1908	53.01	1981	130.94								
1994	-29.83	-39.28	-99.20	1909	52.44	1982	135.27								
1995	-30.24	-37.22	-97.68	1910	51.42	1983	137.09								
1996	-30.69	-36.70	-98.29	1911	49.79	1984	139.67								
1997	-32.69	-41.56	-102.23	1912	50.92	1985	141.01								
1998	-33.20	-43.29	-104.52	1913	52.13	1986	147.13								
1999	-38.14	-44.98	-106.48	1914	50.40	1987	154.26								

APPENDIX 3: DETAILS OF BEST ARIMA MODEL OF k_t

1) **LEE-CARTER**

> dkt=ts(diff(mod1$kt))

> mod1$kt

Time Series:

Start = 1940

End = 2012

Frequency = 1

> fit.arima<- auto.arima(mod1$kt,d=NA,D=NA, start.p=0, start.q=0,stationary=F,

+ ic=c('bic'),approximation=F, trace=T, stepwise=F, test=c("kpss"),lambda=NULL)

ARIMA(0,1,0) : 395.1812,ARIMA(0,1,0) with drift : 364.3593,ARIMA(0,1,1) : 398.3444,

ARIMA(0,1,1) with drift : 363.3718, ARIMA(0,1,2) : 389.1066,ARIMA(0,1,2) with drift : 366.4862

ARIMA(0,1,3) : 393.356, ARIMA(0,1,3) with drift : 368.2294, ARIMA(0,1,4) : 393.1885

ARIMA(0,1,4) with drift : 367.5125,ARIMA(0,1,5) : Inf,ARIMA(0,1,5) with drift : Inf

ARIMA(1,1,0) : 397.3616,ARIMA(1,1,0) with drift : 362.0476,ARIMA(1,1,1) : Inf

ARIMA(1,1,1) with drift : 365.9306,ARIMA(1,1,2) : 393.1857,ARIMA(1,1,2) with drift : 369.7864

ARIMA(1,1,3) : 397.432, ARIMA(1,1,3) with drift : 371.6032,ARIMA(1,1,4) : Inf,ARIMA(1,1,4) with drift : 362.8203

ARIMA(2,1,0) : 385.4566,ARIMA(2,1,0) with drift : 365.7811,ARIMA(2,1,1) : Inf,

ARIMA(2,1,1) with drift : 370.0576,ARIMA(2,1,2) : 377.9872,ARIMA(2,1,2) with drift : 363.525,

ARIMA(2,1,3) : 382.2248,ARIMA(2,1,3) with drift : Inf,ARIMA(3,1,0) : 385.7829,

ARIMA(3,1,0) with drift : 370.0569, ARIMA(3,1,1) : Inf,ARIMA(3,1,1) with drift : 374.1278

ARIMA(3,1,2) : Inf,ARIMA(3,1,2) with drift : Inf,ARIMA(4,1,0) : 389.9457, ARIMA(4,1,0) with drift : 372.3211

ARIMA(4,1,1) : Inf,ARIMA(4,1,1) with drift : 372.9079,ARIMA(5,1,0) : 376.168,ARIMA(5,1,0) with drift : 367.5025

> summary(fit.arima)

Series: mod1$kt

ARIMA(1,1,0) with drift

Coefficients:

```
        ar1    drift
    -0.3116  -2.1612
s.e. 0.1184   0.2569
```

sigma^2 estimated as 8.111: log likelihood=-174.61

AIC=355.22 AICc=355.57 BIC=362.05

2) POISSON LEE-CARTER

```
> dkt=ts(diff(mod2$kt))

> mod2$kt

Time Series:

Start = 1940

End = 2012

Frequency = 1

> fit.arima<- auto.arima(mod2$kt,d=NA,D=NA, start.p=0, start.q=0,stationary=F,

+ ic=c('bic'),approximation=F, trace=T, stepwise=F, test=c("kpss"),lambda=NULL)

 ARIMA(0,2,0)   : 467.2347,ARIMA(0,2,1)  : 401.7537,ARIMA(0,2,2) : 381.2286, ARIMA(0,2,3)   : 385.0868

 ARIMA(0,2,4) : 388.9104,ARIMA(0,2,5)    : 391.2,ARIMA(1,2,0)    : 418.1157, ARIMA(1,2,1)   : 383.0906

 ARIMA(1,2,2)     : 384.6727, ARIMA(1,2,3)   : 388.6895, ARIMA(1,2,4)   : 392.0277, ARIMA(2,2,0) : 400.8891

 ARIMA(2,2,1) : 385.3563,ARIMA(2,2,2)  : 388.4378, ARIMA(2,2,3) : 392.6725, ARIMA(3,2,0)  : 402.1582

 ARIMA(3,2,1)   : 389.6186, ARIMA(3,2,2  : 392.5282, ARIMA(4,2,0)    : 395.9773,ARIMA(4,2,1)     : 389.9606

 ARIMA(5,2,0)    : 397.3721

> summary(fit.arima)

Series: mod2$kt

ARIMA(0,2,2)

Coefficients:

         ma1     ma2

      -1.5482  0.6883

s.e. 0.0929  0.1082

sigma^2 estimated as 10.05:  log likelihood=-184.22

AIC=374.44   AICc=374.8   BIC=381.23
```

3) POISSON LEE-CARTER WITH COHORT EFFECT

```
> dkt=ts(diff(mod3$kt))

> mod3$kt

Time Series:

Start = 1940

End = 2012

Frequency = 1


> fit.arima<- auto.arima(mod3$kt,d=NA,D=NA, start.p=0, start.q=0,stationary=F,
```

+ ic=c('bic'),approximation=F, trace=T, stepwise=F, test=c("kpss"),lambda=NULL)

ARIMA(0,1,0) : 417.8834,ARIMA(0,1,0) with drift : 404.4476, ARIMA(0,1,1) : 421.3118,

ARIMA(0,1,1) with drift : 399.9199, ARIMA(0,1,2) : 420.7906,ARIMA(0,1,2) with drift : 403.6702,

ARIMA(0,1,3) : 425.0076, ARIMA(0,1,3) with drift : 407.4443,ARIMA(0,1,4) : 428.3452,

ARIMA(0,1,4) with drift : 410.8913, , ARIMA(0,1,5) : 423.3543,ARIMA(0,1,5) with drift : 407.2451,

ARIMA(1,1,0) : 420.8936,ARIMA(1,1,0) with drift : 399.277,ARIMA(1,1,1) : 424.3392,

ARIMA(1,1,1) with drift : 403.5084, ARIMA(1,1,2) : 424.9951, ARIMA(1,1,2) with drift : 407.785,

ARIMA(1,1,3) : 429.2697,ARIMA(1,1,3) with drift : 411.6593,ARIMA(1,1,4) : Inf,

ARIMA(1,1,4) with drift : 414.3706, ARIMA(2,1,0) : 421.3635,ARIMA(2,1,0) with drif : 403.5054,

ARIMA(2,1,1) : Inf, ARIMA(2,1,1) with drift : 406.0974,ARIMA(2,1,2) : Inf *,ARIMA(2,1,2) with drift : 412.0561,ARIMA(2,1,3) : Inf *,ARIMA(2,1,3) with drift : Inf,ARIMA(3,1,0) : 422.2006,

ARIMA(3,1,0) with drift : 407.7467, ARIMA(3,1,1) : 417.2216, ARIMA(3,1,1) with drift : 409.5323

ARIMA(3,1,2) : 419.675,ARIMA(3,1,2) with drift : Inf,ARIMA(4,1,0) : 426.4772

ARIMA(4,1,0) with drift : 409.5262,ARIMA(4,1,1) : 421.4978,ARIMA(4,1,1) with drift : 410.9313

ARIMA(5,1,0) : 417.8855, ARIMA(5,1,0) with drift : 408.3854

> summary(fit.arima)

Series: mod3$kt

ARIMA(1,1,0) with drift

Coefficients:

```
       ar1    drift
   -0.3562  -1.7135
s.e. 0.1118   0.3228
```

sigma^2 estimated as 13.7: log likelihood=-193.22

AIC=392.45 AICc=392.8 BIC=399.28

APPENDIX 4: DETAIL RESULT OF AGGREGATE EXPOSURED TO RISK AND NUMBER DEATHS FROM 2006 FOR MARKET WC, AXA PORTFOLIO AND GENERAL

Age	Exp.AXA	Death.AX	Exp.WC	Death.WC	Exp.POP	Death.POI	Age	Exp.AXA	Death.AX	Exp.WC	Death.WC	Exp.POP	Death.POI
0	4.5	0	11.5	0	700000	2239	61	868.5	15	4278.5	53.01186	631062	4816
1	22	0	46	0	697761	212	62	835.5	6	4027	41.01417	626245	5112
2	29	0	93	0	697549	133	63	789.5	7	3761.5	34.00464	621132	5496
3	34.5	0	148.5	0	697414	103	64	727	9	3613	40.01688	615637	5862
4	48	1	218.5	1	697311	99	65	701.5	7	3411.5	38.02837	609776	6518
5	52	0	292.5	1	697212	85	66	640	7	3187	50.01342	603258	6823
6	70.5	0	367	0	697125	74	67	634	12	3125.5	44.00647	596435	7476
7	93	0	472	0	697053	78	68	608.5	13	2889.5	47.01545	588957	8139
8	108	0	553	0	696975	92	69	622.5	11	2807	54.01393	580818	8707
9	120	0	671	0	696885	88	70	634	13	2743	45.02559	572113	9575
10	137.5	1	774.5	3	696796	79	71	607.5	16	2682	60.0237	562537	10427
11	154	0	869.5	0	696717	87	72	593.5	15	2598.5	63.02088	552110	11415
12	172.5	1	1017.5	0.005618	696630	97	73	589	14	2527.5	40.01587	540695	12526
13	189	0	1117	0	696533	97	74	577.5	18	2339	61.03125	528169	13624
14	206	1	1270.5	1	696436	116	75	536	14	2179.5	71.03161	514544	14805
15	223	0	1368.5	0	696317	149	76	480.5	24	2028	63.05199	499741	16179
16	255.5	1	1577.5	0	696168	178	77	424	20	1867.5	63.04803	483561	17734
17	250.5	2	1754	2	695993	210	78	385	13	1727	75.02687	465825	19288
18	239.5	2	1779	1	695783	273	79	358	17	1554.5	69.07281	446537	21144
19	252	0	1752.5	1	695507	299	80	317.5	24	1393.5	68.05104	425391	22807
20	237.5	2	1655.5	0.00304	695208	333	81	281	13	1245	90.08696	402586	24573
21	242.5	1	1622.5	2.002967	694876	342	82	260	18	1108	73.06575	378012	26071
22	247.5	2	1648	3	694533	361	83	251	22	964	74.07386	351941	27508
23	253	2	1523	0	694172	360	84	226	18	850.5	66.12162	324433	28306
24	228	2	1477.5	5	693813	367	85	196	18	716	63.08871	296127	28801
25	233.5	1	1199.5	1	693447	372	86	173	20	650	64.09677	267325	29345
26	237.5	1	1130.5	2.004396	693075	366	87	148.5	21	540	55.12088	237981	28420
27	263	1	1232	3	692708	396	88	127.5	11	456.5	38.09524	209560	27591
28	271	1	1437	0	692312	384	89	116.5	12	418	46.18033	181969	27142
29	308	3	1484.5	2.003401	691927	417	90	93	11	335.5	32.21622	154829	24948
30	351	1	1652.5	4	691513	474	91	83	9	286.5	34.15094	129880	23015
31	368.5	7	1806	6	691039	496	92	64.5	17	242	30.18557	106864	20408
32	393	1	1981.5	3	690544	524	93	49.5	13	191	26.16842	86457	18273
33	444	3	2159.5	4	690020	585	94	38.5	6	163	35.26667	68184	15630
34	466	3	2333.5	2.002255	689436	645	95	37	6	132	28.25926	52554	13053
35	496.5	0	2495	4	688792	717	96	27	9	113.5	23.1	39501	10617
36	529.5	1	2689.5	4.002141	688075	760	97	28.5	0	83.5	11.08696	28885	8368
37	567.5	2	2791.5	4.005753	687312	811	98	20	5	77.5	20.24	20516	6381
38	614.5	5	2959	7.001717	686502	899	99	18.5	5	54	14.3	14135	4700
39	606	5	3082	9	685603	1004	100	9.5	6	50	7.666667	9434	3341
40	676.5	2	3235.5	5	684596	1103	101	7	0	30	6.769231	6093	2287
41	675	2	3419.5	10.00303	683495	1186	102	6.5	1	26	6.4	3805	1508
42	763.5	4	3743	14.00154	682310	1316	103	6	1	16	3.285714	2295	958
43	772.5	4	3673	8.002859	680994	1405	104	4	2	15.5	1	1338	586
44	803.5	4	3828.5	10.00404	679588	1518	105	3	1	15.5	3.5	752	343
45	896	1	4099.5	12.0014	678070	1619	106	2.5	2	15	1	409	194
46	914	10	4217.45	21.0026	676450	1781	107	3	2	6	2	214	105
47	940	6	4247.5	16.00133	674671	1970	108	2.5	0	7	0.4	108	57
48	947.5	5	4318	14.0051	672699	2176	109	1.5	1	5	1	52	28
49	1015	10	4486.5	27.00385	670523	2255	110	1	1	9	1	25	25
50	1035.5	6	4533	18.00118	668268	2402		43711.5	759	207378	2431.577	56174158	699988
51	1067.5	12	4536	26.00988	665863	2626							
52	1073.5	6	4701.5	23.00676	663237	2767							
53	1030.5	19	4636.5	31.00248	660470	3044							
54	1035	8	4684.5	20.00369	657425	3176							
55	990.5	19	4605.5	31.00261	654249	3323							
56	945.5	11	4487.5	27.00494	650927	3419							
57	977.5	19	4571.5	24.00654	647508	3820							
58	953.5	8	4502.5	29.00383	643687	4032							
59	998	10	4415	17.02208	639654	4061							
60	967	14	4360.5	35.00537	635595	4533							

APPENDIX 5: Life expectancy for Portuguese population, portfolio of WC market and AXA portfolio.

		BRASS LINEAR		COX		WORKGROUP		POP-BRAS	POP-COX	POP-WORK			BRASS LINEAR		COX		WORKGROUP		POP-BRAS	POP-COX	POP-WOR
x	POPULATI	WC	AXA	WC	AXA	WC	AXA			WC	x	POPULATI	WC	AXA	WC	AXA	WC	AXA			WC
0	88.40434	81.97333	81.97333	87.93432	87.93432	88.12059	88.12059	6.431009	0.47002	0.283745	61	24.23304	24.27725	24.27725	23.72448	23.72448	23.94155	23.94155	-0.04421	0.508565	0.291486
1	87.45393	80.99046	80.99046	86.98657	86.98657	87.16994	87.16994	6.46347	0.467369	0.283996	62	23.31734	23.55539	23.55539	22.81067	22.81067	23.02965	23.02965	-0.23805	0.506665	0.287694
2	86.33741	79.85382	79.85382	85.87087	85.87087	86.0533	86.0533	6.48359	0.466541	0.284112	63	22.39217	22.82034	22.82034	21.88708	21.88708	22.10797	22.10797	-0.42817	0.505088	0.284195
3	85.2488	78.73848	78.73848	84.78041	84.78041	84.96309	84.96309	6.510322	0.468389	0.285711	64	21.4777	22.09352	22.09352	20.97427	20.97427	21.19721	21.19721	-0.61582	0.503423	0.280483
4	84.15967	77.62269	77.62269	83.68942	83.68942	83.87235	83.87235	6.536981	0.470247	0.287323	65	20.57181	21.37234	21.37234	20.07008	20.07008	20.29519	20.29519	-0.80053	0.501732	0.276623
5	83.07025	76.50667	76.50667	82.59813	82.59813	82.7813	82.7813	6.563581	0.472117	0.288948	66	19.67499	20.65699	20.65699	19.17497	19.17497	19.40238	19.40238	-0.98199	0.500024	0.272612
6	81.98046	75.39035	75.39035	81.50646	81.50646	81.68987	81.68987	6.590115	0.473999	0.290586	67	18.79093	19.95069	19.95069	18.29268	18.29268	18.52255	18.52255	-1.15977	0.498248	0.268377
7	80.88996	74.27341	74.27341	80.41407	80.41407	80.59773	80.59773	6.616551	0.47589	0.292236	68	17.92533	19.25823	19.25823	17.42902	17.42902	17.66153	17.66153	-1.3329	0.496309	0.263796
8	79.79922	73.15629	73.15629	79.32142	79.32142	79.50532	79.50532	6.642926	0.477793	0.293901	69	17.06379	18.566	18.566	16.56926	16.56926	16.80457	16.80457	-1.50221	0.494528	0.259224
9	78.70745	72.03828	72.03828	78.22775	78.22775	78.41187	78.41187	6.66917	0.479705	0.295576	70	16.23532	17.89901	17.89901	15.74298	15.74298	15.9813	15.9813	-1.6637	0.492337	0.254012
10	77.61493	70.91963	70.91963	77.1333	77.1333	77.31767	77.31767	6.695303	0.481627	0.297264	71	15.38927	17.21349	17.21349	14.89847	14.89847	15.13995	15.13995	-1.82422	0.490803	0.249323
11	76.52214	69.80078	69.80078	76.03858	76.03858	76.22318	76.22318	6.721363	0.483562	0.298967	72	14.6006	16.57144	16.57144	14.11218	14.11218	14.35709	14.35709	-1.97084	0.488424	0.243519
12	75.4291	68.68175	68.68175	74.94359	74.94359	75.12841	75.12841	6.747347	0.48551	0.300684	73	13.80031	15.9166	15.9166	13.31363	13.31363	13.56219	13.56219	-2.11629	0.486679	0.238125
13	74.33507	67.56188	67.56188	73.8476	73.8476	74.03266	74.03266	6.77319	0.487467	0.302413	74	13.02759	15.27936	15.27936	12.54281	12.54281	12.79526	12.79526	-2.25177	0.484785	0.232331
14	73.24089	66.44193	66.44193	72.75145	72.75145	72.93673	72.93673	6.798963	0.489439	0.304157	75	12.27721	14.65493	14.65493	11.79433	11.79433	12.05093	12.05093	-2.37772	0.482882	0.226278
15	72.14777	65.32299	65.32299	71.65634	71.65634	71.84185	71.84185	6.824777	0.491434	0.305923	76	11.53255	14.03091	14.03091	11.05114	11.05114	11.31222	11.31222	-2.49836	0.481405	0.220331
16	71.05728	64.20649	64.20649	70.56381	70.56381	70.74956	70.74956	6.850786	0.493463	0.307716	77	10.82744	13.43074	13.43074	10.34782	10.34782	10.61358	10.61358	-2.6033	0.47962	0.21386
17	69.97028	63.09319	63.09319	69.47474	69.47474	69.66074	69.66074	6.877085	0.495533	0.309542	78	10.14531	12.8421	12.8421	9.667372	9.667372	9.938092	9.938092	-2.69679	0.477939	0.207218
18	68.88609	61.98248	61.98248	68.38845	68.38845	68.5747	68.5747	6.903619	0.497642	0.311399	79	9.475529	12.25767	12.25767	8.998813	8.998813	9.274887	9.274887	-2.78214	0.476716	0.200641
19	67.80496	60.87454	60.87454	67.30517	67.30517	67.49168	67.49168	6.930425	0.499791	0.313288	80	8.867074	11.70747	11.70747	8.392569	8.392569	8.673797	8.673797	-2.8404	0.474505	0.193278
20	66.72579	59.91463	59.91463	66.22468	66.22468	66.41151	66.41151	6.811154	0.501103	0.314281	81	8.235902	11.13941	11.13941	7.761919	7.761919	8.049261	8.049261	-2.9035	0.473983	0.186641
21	65.64613	58.95446	58.95446	65.14372	65.14372	65.33086	65.33086	6.691676	0.502413	0.315271	82	7.694258	10.61954	10.61954	7.222953	7.222953	7.515404	7.515404	-2.92528	0.471305	0.178854
22	64.56658	57.99615	57.99615	64.06288	64.06288	64.25034	64.25034	6.570432	0.503706	0.31624	83	7.152699	10.09494	10.09494	6.683023	6.683023	6.981297	6.981297	-2.94224	0.469677	0.171403
23	63.48434	57.02923	57.02923	62.97928	62.97928	63.16706	63.16706	6.455112	0.505059	0.317279	84	6.653533	9.588389	9.588389	6.186275	6.186275	6.489903	6.489903	-2.93486	0.467258	0.16363
24	62.40196	56.06338	56.06338	61.89556	61.89556	62.08366	62.08366	6.338574	0.506401	0.318303	85	6.170958	9.085004	9.085004	5.705778	5.705778	6.015004	6.015004	-2.91405	0.465181	0.155954
25	61.32013	55.1011	55.1011	60.81242	60.81242	61.00084	61.00084	6.219034	0.507713	0.319291	86	5.700165	8.581781	8.581781	5.236339	5.236339	5.551669	5.551669	-2.88162	0.463826	0.148495
26	60.23868	54.14155	54.14155	59.72968	59.72968	59.91843	59.91843	6.097125	0.509	0.320248	87	5.276926	8.095657	8.095657	4.816086	4.816086	5.136217	5.136217	-2.81873	0.46084	0.140709
27	59.15809	53.18634	53.18634	58.64784	58.64784	58.83694	58.83694	5.97175	0.510249	0.321158	88	4.881248	7.614685	7.614685	4.424272	4.424272	4.748416	4.748416	-2.73344	0.456977	0.132833
28	58.07811	52.23433	52.23433	57.56664	57.56664	57.75608	57.75608	5.843775	0.511467	0.32203	89	4.487552	7.125467	7.125467	4.03335	4.03335	4.362291	4.362291	-2.63792	0.454202	0.12526
29	56.99759	51.28155	51.28155	56.4849	56.4849	56.67469	56.67469	5.716037	0.512686	0.322901	90	4.156486	6.653565	6.653565	3.710313	3.710313	4.039499	4.039499	-2.49708	0.446173	0.116987
30	55.91965	50.33827	50.33827	55.40583	55.40583	55.59598	55.59598	5.581381	0.513822	0.32367	91	3.792978	6.154298	6.154298	3.350776	3.350776	3.683587	3.683587	-2.36132	0.442201	0.109391
31	54.84241	49.39793	49.39793	54.32748	54.32748	54.51801	54.51801	5.44448	0.514927	0.3244	92	3.515184	5.675455	5.675455	3.088388	3.088388	3.414699	3.414699	-2.16027	0.426796	0.100485
32	53.76899	48.46984	48.46984	53.25307	53.25307	53.444	53.444	5.299153	0.515917	0.324989	93	3.227791	5.172162	5.172162	2.818001	2.818001	3.136449	3.136449	-1.94437	0.40979	0.091342
33	52.69344	47.5358	47.5358	52.17649	52.17649	52.36781	52.36781	5.157643	0.516954	0.32563	94	2.965077	4.653949	4.653949	2.582337	2.582337	2.884012	2.884012	-1.68887	0.38274	0.081065
34	51.62044	46.60963	46.60963	51.10252	51.10252	51.29426	51.29426	5.010806	0.517912	0.326172	95	2.710414	4.108634	4.108634	2.367492	2.367492	2.641113	2.641113	-1.39822	0.342921	0.069301
35	50.55048	45.6922	45.6922	50.0317	50.0317	50.22388	50.22388	4.858278	0.51878	0.326602	96	2.408797	3.513053	3.513053	2.110107	2.110107	2.351983	2.351983	-1.10426	0.298691	0.056814
36	49.4817	44.77798	44.77798	48.96209	48.96209	49.15472	49.15472	4.703716	0.519609	0.326979	97	2.083843	2.871398	2.871398	1.846306	1.846306	2.0412	2.0412	-0.78755	0.237537	0.042644
37	48.4148	43.86864	43.86864	47.89442	47.89442	48.08752	48.08752	4.546164	0.520382	0.327282	98	1.713678	2.171251	2.171251	1.561505	1.561505	1.687217	1.687217	-0.45757	0.152173	0.026462
38	47.35052	42.96573	42.96573	46.82944	46.82944	47.02303	47.02303	4.384787	0.52108	0.327488	99	1.187046	1.380532	1.380532	1.11116	1.11116	1.175135	1.175135	-0.19349	0.075886	0.011911
39	46.28722	42.06493	42.06493	45.76548	45.76548	45.95957	45.95957	4.222295	0.521748	0.32765	100	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0	0	0
40	45.23153	41.18153	41.18153	44.70931	44.70931	44.90397	44.90397	4.050005	0.52222	0.327564											
41	44.17715	40.30035	40.30035	43.6545	43.6545	43.84973	43.84973	3.876802	0.522657	0.327427											
42	43.13208	39.43814	39.43814	42.60922	42.60922	42.80509	42.80509	3.69394	0.522861	0.326995											
43	42.08544	38.57167	38.57167	41.56233	41.56233	41.75884	41.75884	3.513767	0.523107	0.326601											
44	41.04658	37.71957	37.71957	40.52341	40.52341	40.72061	40.72061	3.327004	0.523166	0.325964											
45	40.01141	36.87322	36.87322	39.48827	39.48827	39.6862	39.6862	3.138198	0.523141	0.32521											
46	38.9784	36.02945	36.02945	38.45532	38.45532	38.65401	38.65401	2.948949	0.523074	0.324387											
47	37.95514	35.20149	35.20149	37.43236	37.43236	37.63187	37.63187	2.753654	0.522784	0.323272											
48	36.93981	34.38514	34.38514	36.41749	36.41749	36.61789	36.61789	2.554674	0.522321	0.321923											
49	35.92713	33.57127	33.57127	35.40532	35.40532	35.60663	35.60663	2.355857	0.521815	0.320497											
50	34.92212	32.76755	32.76755	34.40097	34.40097	34.60327	34.60327	2.154566	0.521152	0.318853											
51	33.9163	31.96059	31.96059	33.39576	33.39576	33.59906	33.59906	1.955703	0.520536	0.317236											
52	32.9297	31.17973	31.17973	32.4102	32.4102	32.61463	32.61463	1.749974	0.519503	0.315078											
53	31.93664	30.38747	30.38747	31.41798	31.41798	31.62353	31.62353	1.549177	0.518659	0.313114											
54	30.95594	29.61012	29.61012	30.43836	30.43836	30.64512	30.64512	1.345819	0.517574	0.310815											
55	29.9774	28.83329	28.83329	29.46091	29.46091	29.66893	29.66893	1.144108	0.516489	0.308472											
56	29.00296	28.05942	28.05942	28.4876	28.4876	28.69693	28.69693	0.943539	0.515365	0.306033											
57	28.03344	27.28938	27.28938	27.51925	27.51925	27.72996	27.72996	0.744066	0.514193	0.303482											
58	27.07757	26.53368	26.53368	26.56478	26.56478	26.77698	26.77698	0.543887	0.512788	0.300588											
59	26.12322	25.7771	25.7771	25.6118	25.6118	25.82554	25.82554	0.346126	0.511426	0.297682											
60	25.18308	25.03433	25.03433	24.67323	24.67323	24.88864	24.88864	0.148745	0.509843	0.294436											

APPENDIX 6: Example of the result of computation of the different components of workers compensation using the Brass linear model on the WC portfolio of Pensioners.

x	Pensionistas $a_x^{(12)}$	Orfãos $a_{x:25-x}^{(12)}$	Ascendentes $a_{x:65-x}^{(12)}$	Ascendentes $_{65-x\|}a_x^{(12)}$	Ascendentes $_x^{(12)}$ + 4/3* $_{65-x}$	Viuvas (sem remaridação) $a_{x:65-x}^{(12)}$	Viuvas (sem remaridação) $_{65-x\|}a_x^{(12)}$	Viuvas (sem remaridação) $_x^{(12)}$ + 4/3* $_{65-x}$	ëx
0	21.717	15.140	21.051	0.667		21.051	0.667		81.97333
1	21.709	14.829	21.014	0.695		21.014	0.695		80.99046
2	21.655	14.472	20.931	0.724		20.931	0.724		79.85382
3	21.597	14.099	20.843	0.754		20.843	0.754		78.73848
4	21.537	13.708	20.752	0.785		20.752	0.785		77.62269
5	21.473	13.301	20.656	0.818		20.656	0.818		76.50667
6	21.407	12.875	20.555	0.852		20.555	0.852		75.39035
7	21.337	12.430	20.450	0.887		20.450	0.887		74.27341
8	21.263	11.965	20.340	0.923		20.340	0.923		73.15629
9	21.186	11.479	20.224	0.962		20.224	0.962		72.03828
10	21.104	10.972	20.103	1.001		20.103	1.001		70.91963
11	21.019	10.441	19.976	1.042		19.976	1.042		69.80078
12	20.929	9.887	19.843	1.085		19.843	1.085		68.68175
13	20.834	9.307	19.704	1.130		19.704	1.130		67.56188
14	20.735	8.702	19.558	1.177	21.127	19.558	1.177	21.127	66.44193
15	20.630	8.069	19.405	1.225	21.039	19.405	1.225	21.039	65.32299
16	20.522	7.408	19.247	1.275	20.947	19.247	1.275	20.947	64.20649
17	20.409	6.718	19.081	1.328	20.852	19.081	1.328	20.852	63.09319
18	20.291	5.997	18.909	1.383	20.752	18.909	1.383	20.752	61.98248
19	20.168	5.244	18.729	1.440	20.648	18.729	1.440	20.648	60.87454
20	20.090	4.468	18.587	1.503	20.591	18.587	1.503	20.591	59.91463
21	20.008	3.656	18.439	1.568	20.531	18.439	1.568	20.531	58.95446
22	19.923	2.805	18.285	1.637	20.468	18.285	1.637	20.468	57.99615
23	19.830	1.913	18.122	1.709	20.400	18.122	1.709	20.400	57.02923
24	19.735	0.979	17.951	1.784	20.329	17.951	1.784	20.329	56.06338
25	19.636	19.636	17.774	1.862	20.256	17.774	1.862	20.256	55.1011
26	19.533	19.533	17.589	1.944	20.181	17.589	1.944	20.181	54.14155
27	19.428	19.428	17.398	2.030	20.105	17.398	2.030	20.105	53.18634
28	19.319	19.319	17.200	2.119	20.026	17.200	2.119	20.026	52.23433
29	19.205	19.205	16.992	2.213	19.943	16.992	2.213	19.943	51.28155
30	19.090	19.090	16.778	2.312	19.860	16.778	2.312	19.860	50.33827
31	18.970	18.970	16.555	2.415	19.775	16.555	2.415	19.775	49.39793
32	18.850	18.850	16.327	2.523	19.691	16.327	2.523	19.691	48.46984
33	18.722	18.722	16.086	2.636	19.601	16.086	2.636	19.601	47.5358
34	18.592	18.592	15.837	2.755	19.511	15.837	2.755	19.511	46.60963
35	18.460	18.460	15.580	2.880	19.420	15.580	2.880	19.420	45.6922
36	18.323	18.323	15.312	3.011	19.327	15.312	3.011	19.327	44.77798
37	18.183	18.183	15.034	3.149	19.232	15.034	3.149	19.232	43.86864
38	18.039	18.039	14.745	3.293	19.136	14.745	3.293	19.136	42.96573
39	17.889	17.889	14.444	3.445	19.038	14.444	3.445	19.038	42.06493
40	17.741	17.741	14.136	3.606	18.943	14.136	3.606	18.943	41.18153
41	17.588	17.588	13.814	3.774	18.846	13.814	3.774	18.846	40.30035
42	17.436	17.436	13.483	3.953	18.754	13.483	3.953	18.754	39.43814
43	17.277	17.277	13.136	4.140	18.657	13.136	4.140	18.657	38.57167
44	17.117	17.117	12.778	4.339	18.563	12.778	4.339	18.563	37.71957
45	16.953	16.953	12.405	4.548	18.469	12.405	4.548	18.469	36.87322
46	16.784	16.784	12.015	4.769	18.373	12.015	4.769	18.373	36.02945
47	16.615	16.615	11.612	5.003	18.283	11.612	5.003	18.283	35.20149
48	16.445	16.445	11.194	5.251	18.196	11.194	5.251	18.196	34.38514
49	16.270	16.270	10.757	5.513	18.108	10.757	5.513	18.108	33.57127
50	16.092	16.092	10.301	5.791	18.023	10.301	5.791	18.023	32.76755
51	15.906	15.906	9.822	6.084	17.934	9.822	6.084	17.934	31.96059
52	15.726	15.726	9.328	6.398	17.859	9.328	6.398	17.859	31.17973
53	15.533	15.533	8.805	6.728	17.776	8.805	6.728	17.776	30.38747
54	15.340	15.340	8.260	7.080	17.700	8.260	7.080	17.700	29.61012
55	15.140	15.140	7.687	7.453	17.624	7.687	7.453	17.624	28.83329
56	14.934	14.934	7.085	7.849	17.550	7.085	7.849	17.550	28.05942
57	14.721	14.721	6.453	8.269	17.478	6.453	8.269	17.478	27.28938
58	14.509	14.509	5.790	8.719	17.415	5.790	8.719	17.415	26.53368
59	14.287	14.287	5.090	9.197	17.353	5.090	9.197	17.353	25.7771
60	14.065	14.065	4.355	9.710	17.302	4.355	9.710	17.302	25.03433

x	Pensionistas $a_x^{(12)}$	Orfãos $a_{x:25-x}^{(12)}$	Ascendentes $a_{x:65-x}^{(12)}$	Ascendentes $_{65-x\|}a_x^{(12)}$	Ascendentes $_x^{(12)}$ + 4/3* $_{65-x}$	Viuvas (sem remaridação) $a_{x:65-x}^{(12)}$	Viuvas (sem remaridação) $_{65-x\|}a_x^{(12)}$	Viuvas (sem remaridação) $_x^{(12)}$ + 4/3* $_{65-x}$	ëx
61	13.827	13.827	3.577	10.250	17.243	3.577	10.250	17.243	24.27725
62	13.599	13.599	2.759	10.840	17.212	2.759	10.840	17.212	23.55539
63	13.355	13.355	1.892	11.464	17.176	1.892	11.464	17.176	22.82034
64	13.107	13.107	0.973	12.133	17.151	0.973	12.133	17.151	22.09352
65	12.853	12.853	0.000	12.853	12.853	0.000	12.853	12.853	21.37234
66	12.592	12.592			12.592			12.592	20.65699
67	12.328	12.328			12.328			12.328	19.95069
68	12.062	12.062			12.062			12.062	19.25823
69	11.787	11.787			11.787			11.787	18.566
70	11.518	11.518			11.518			11.518	17.89901
71	11.227	11.227			11.227			11.227	17.21349
72	10.954	10.954			10.954			10.954	16.57144
73	10.663	10.663			10.663			10.663	15.9166
74	10.373	10.373			10.373			10.373	15.27936
75	10.082	10.082			10.082			10.082	14.65493
76	9.781	9.781			9.781			9.781	14.03091
77	9.487	9.487			9.487			9.487	13.43074
78	9.191	9.191			9.191			9.191	12.8421
79	8.889	8.889			8.889			8.889	12.25767
80	8.601	8.601			8.601			8.601	11.70747
81	8.291	8.291			8.291			8.291	11.13941
82	8.008	8.008			8.008			8.008	10.61954
83	7.713	7.713			7.713			7.713	10.09494
84	7.423	7.423			7.423			7.423	9.588389
85	7.128	7.128			7.128			7.128	9.085004
86	6.824	6.824			6.824			6.824	8.581781
87	6.525	6.525			6.525			6.525	8.095657
88	6.222	6.222			6.222			6.222	7.614685
89	5.905	5.905			5.905			5.905	7.125467
90	5.593	5.593			5.593			5.593	6.653565
91	5.251	5.251			5.251			5.251	6.154298
92	4.916	4.916			4.916			4.916	5.675455
93	4.552	4.552			4.552			4.552	5.172162
94	4.165	4.165			4.165			4.165	4.653949
95	3.743	3.743			3.743			3.743	4.108634
96	3.262	3.262			3.262			3.262	3.513053
97	2.722	2.722			2.722			2.722	2.871398
98	2.109	2.109			2.109			2.109	2.171251
99	1.384	1.384			1.384			1.384	1.380532
100	0.542	0.542			0.542			0.542	0.5