LISBON SCHOOL OF ECONOMICS & MANAGEMENT
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MASTER OF SCIENCE IN MATHEMATICAL FINANCE

MASTER'S FINAL WORK

DISSERTATION

RISK NEUTRAL DENSITY FUNCTIONS AND THE COVID-19 PANDEMIC

JOÃO PEDRO DA ROCHA COSTA MAIA

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SUPERVISOR:

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Resumo

Este trabalho retira conclusões sobre as expectativas do mercado através da análise de funções Risk Neutral Density (RND) por via das opções do Standard and Poor's 500. Após uma revisão da literatura sobre este assunto, é proposta uma abordagem paramétrica para estimar as funções RND, é usada a abordagem de two lognormal mixed densities. As medidas descritivas sobre as funções RND permitem concluir um padrão que parece estar de acordo com grande parte do estudo sobre o assunto. Quando surgem períodos de stress, há um aumento à esquerda do coeficiente de assimetria, uma diminuição do coeficiente de curtose e a volatilidade é geralmente mais elevada. Chega-se à conclusão de que os mercados tiveram dificuldades em compreender completamente o impacto económico da pandemia. Neste período de choque, os investidores não foram capazes de adaptar as suas perspetivas sobre a futura trajetória da política monetária. Os resultados mostram que mal anteciparam as decisões dos agentes da política monetária ou de outros responsáveis, mudando a perceção à medida que as autoridades alteravam a sua postura. Várias são as aplicações propostas neste trabalho para a utilização das funções RND.

Abstract

This paper takes conclusions on markets expectations by analysing risk-neutral density (RND) functions on options of the Standard and Poor's 500. After a review of the literature on this subject a parametric approach is proposed to estimate RND functions, a two lognormal technique. The descriptive statics on the RND functions allow to conclude a pattern that seems to be in line with much of the work on the subject. When distressed periods arise there is an increase in left skewness, a decrease in kurtosis and volatility is generally higher. It comes to the conclusion that markets had difficulties in understanding the full economic impact of the pandemic. In this shock period, investors were not able to adapt their perspectives on the future path of monetary policy. The results show they barely anticipated the decisions of monetary policy agents or other officials, changing beliefs as authorities altered stance. Several are the applications proposed on this paper for the use of RND functions.

Contents

1 Introduction

Option markets are believed to contain important information about investor's future expectations through its implied RND. It is possible to assess significant changes in market implied expectations as a consequence of, in this case, the COVID-19 pandemic. As computational capability became more powerful and broadly accessible and option databases were made available, investors began to nurture interest on the estimation of the RND. Several techniques were developed to attain such goal. Which will be the focus of the following section.

In order to make a comparison between the period where the pandemic stressed the markets and the previous months, the studied span comprises 15 months, January 2019 to March 2020. However, the focus is between December 2019, when the first case of COVID-19 was confirmed, and March 2020, when the markets registered a significant crash.

In the beginning of 2020 the World Health Organization (WHO) and the European Center for Disease Prevention and Control published a flagship risk assessment on the Wuhan cluster of pneumonia. At the time, both were completely inconclusive as to the cause and consequences of the problem. Rapidly, advancements were being made. On the 11^{th} March 2020, two days after Italy declared a lockdown, the WHO made an assessment that COVID-19 could be characterized as a pandemic. This information came as several alarming reports on public health and safety and on the impacts on the economy were being made on news outlets all around the world. After this WHO announcement the VIX reached a level of around 75%, a surge not seen since late 2008, after the failure of Lehman Brothers.

The S&P 500 decreased by 9.5%, continuing on a downward trend up until March 23^{rd}. On the 25^{th} March the U.S. Senate passed its largest stimulus legislation to ease the impact of the COVID-19 pandemic, a $2 trillion stimulus package. These and other important events will be relevant when a analysing the results of the RND functions.

The following section provides the theoretical framework to be applied in section 3, data and methodology. The latter presents the data and how it will be used in the simulation of the method chosen. Finally, section 4 discusses the results and section 5 concludes.

2 Theoretical framework

2.1 Black Scholes model and RND

Black and Scholes (1973) built a model under assumptions about the market and the assets, frequently called the Black-Scholes economy. Among others, they assume that there is no arbitrage opportunity, the return of the riskless asset is given by a constant risk-free rate and the underlying price has a lognormal distribution evolving according to a stochastic process, the geometric Brownian motion (GBM). This GBM is

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{1}$$

where S_t represents the price of the underlying asset, which pays no dividends, with constant volatility σ, μ is the constant expected drift rate and dt are increments of a standard Wiener process, W_t, under the real world probability P.

Already, this dynamic implies that the underlying risk neutral density is lognormal. By applying Itô's Lemma to equation (1), suppose $f(x,t) = ln(S_t)$,

$$d(lnS_t) = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW_t \tag{2}$$

A variable has a lognormal distribution if the natural logarithm of the variable is normally distributed. So, the variable lnS_t is normally distributed, with mean $(\mu - \frac{\sigma^2}{2})\tau$ and standard deviation $\sigma\sqrt{\tau}$,

$$lnS_t \sim \xi[\ln S_t + (\mu - \frac{\sigma^2}{2})\tau, \sigma\sqrt{\tau}]$$

where $\xi(m,\nu)$ denotes a normal distribution with mean m, standard deviation ν

and time to maturity τ or $(T-t)$. The risk-neutral density of the underlying asset price, $q(S_t)$, is a lognormal distribution with parameters m and ν,

$$q(S_t) = \frac{1}{S_T \nu \sqrt{2\pi}} e^{-\frac{(lnS_T - m)^2}{2\nu^2}} \tag{3}$$

The following subsections lead to the closed-form solution of the Black-Scholes formula under the equivalent martingale measure and risk neutral valuation.

2.1.1 Change of measure

The Girsanov Theorem allows to change the Brownian motion from a P-measure, W_t, to another Brownian motion, $\overline{W}_t$, under the equivalent martingale measure Q.

Suppose W_t is a P-Brownian motion and θ_t a P-adapted process. Consider the Itô process

$$d\overline{W}_t = \theta_t dt + dW_t, \quad \overline{W}_t = 0$$

Then, an equivalent measure to P such that $\overline{W}_t$ is a Q-Brownian motion exists. This measure is characterized by a Radon-Nikodym derivative

$$\frac{dQ}{dP} = e^{-\int_0^T \theta_t dW_t - \frac{1}{2}\int_0^T \theta_t^2 dt}$$

Note that for any random variable X, $\mathbb{E}^Q(X) = \mathbb{E}^P(X \frac{dQ}{dP})$.

In this way it is possible, changing the probability measure, to change the drift of the Itô process

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sigma S_t dW_t \\
&= (\mu + r - r) S_t dt + \sigma S_t dW_t \\
&= r S_t dt + \sigma S_t (\frac{\mu - r}{\sigma} dt + dW_t) \\
&= r S_t dt + \sigma S_t d\overline{W}_t
\end{aligned} \tag{4}
$$

where W_t is the P-Brownian motion, S_t follows the GBM, $\overline{W}_t$ a Q-Brownian motion and $d\overline{W}_t = \theta_t dt + dW_t$, with the market price of risk constant $\theta_t = \frac{\mu - r}{\sigma}$.

2.1.2 Portfolio replication and Black-Scholes derivation

To find the value of $X = (S_T - K)^+$ at time $t < T$, let $\overline{S}_t = e^{-rt} S_t$ be the discounted stock price and $f(x, t) = e^{-rt} x$. By Itôs's Lemma,

$$
\begin{aligned}
d\overline{S}_t &= (-re^{-rt} S_t + \mu e^{-rt} S_t) dt + \sigma e^{-rt} S_t dW_t \\
&= (\mu - r) \overline{S}_t dt + \sigma \overline{S}_t dW_t
\end{aligned} \tag{5}
$$

$\overline{S}$ still follows a GBM. It has an expected growth rate of $(\mu - r)$. The growth rate in $\overline{S}$ is the excess return of S over the risk-free rate.

It follows three steps to get to X. Find the Q-measure such that $\overline{S}_t$ is a Q-martingale (I), consider a riskless asset $B_t = e^{rt}$ such that its dynamic follows $dB_t = r B_t dt$ (II) and construct a portfolio with value V_t at time t (III).

(I)

$$
\begin{aligned}
d\overline{S}_t &= (\mu - r)\overline{S}_t dt + \sigma \overline{S}_t dW_t \\
&= (\mu - r)\overline{S}_t dt + \sigma \overline{S}_t (d\overline{W}_t - \frac{\mu - r}{\sigma} dt) \\
&= \sigma \overline{S}_t d\overline{W}_t
\end{aligned}
$$

For $s < t$,

$$
\begin{aligned}
\mathbb{E}^Q(S_t|F_s) &= e^{rt}\mathbb{E}^Q(\overline{S}_t|F_s) \\
&= e^{rt}\overline{S}_s \\
&= e^{t-s}S_s
\end{aligned}
$$

Thus, under the Q-measure, the stock price has a continuously compounded growth rate given by the risk-free interest rate, r. Hence, the Q-measure is indeed a risk-neutral measure.

(II)

Consider a riskless asset $B_t = e^{rt}$, so that the dynamic of it follows the ODE $dB_t = rB_t dt$. Similar to $\overline{S}_t$, $M_t = \mathbb{E}^Q(\frac{X}{B_T}|F_t)$. It is a martingale with respect to F_t under Q.

(III)

Now, by the martingale representation theorem, $dM_t = \phi_t d\overline{S}_t$, for some F_t adapted ϕ_t.

Creating a portfolio, V_t, with ϕ_t stocks and ω_t bonds, where $\omega_t = M_t - \phi_t\overline{S}_t$. By definition of M_t, $M_T = \mathbb{E}^Q(\frac{X}{B_T}|F_t) = \frac{X}{B}$, so $V_T = X$. The value of the portfolio is given by

$$
\begin{aligned}
V_t &= \phi_t S_t + \omega_t B_t \\
&= \phi_t S_t + (M_t - \phi_t \overline{S}_t) B_t \\
&= B_t M_t
\end{aligned}
$$

The portfolio is self-financing,

$$
\begin{aligned}
dV_t &= B_t dM_t + M_t dB_t \\
&= B_t \phi_t d\overline{S}_t + (\omega_t + \phi_t S_t) dB_t \\
&= \phi_t d(B_t \overline{S}_t) + \omega_t dB_t \\
&= \phi_t dS_t + \omega_t dB_t
\end{aligned}
$$

Hence, the portfolio is self-financing and replicates the claim. The value of the portfolio is the same as the discounted expected payoff claim X at time t

$$
\begin{aligned}
V_t &= B_t M_t \\
&= e^{rt} \mathbb{E}^Q (e^{-rT} X | F_t) \\
&= e^{-r(T-t)} \mathbb{E}^Q (X | F_t)
\end{aligned}
\tag{6}
$$

With equation (6) is possible to derive the Black-Scholes formula. The discounted European call option pricing formula turns out to be

$$
V_t = e^{-r(t-t)} \mathbb{E}^Q ([S_T - K]^+ | F_t) \tag{7}
$$

Equation (4) has the explicit solution $S_t = S_0 e^{(r - \frac{1}{2}\sigma^2) + \sigma \overline{W}_t}$. With $\tau = T - t$

$$
\begin{aligned}
S_T &= S_t e^{(r - \frac{1}{2}\sigma^2)\tau + \sigma(\overline{W}_T - \overline{W}_t)} \\
&= S_t e^{(r - \frac{1}{2}\sigma^2)\tau + \sigma \sqrt{\tau} Y}
\end{aligned}
\tag{8}
$$

where,

$$\overline{W}_T - \overline{W}_t \sim \mathcal{N}(0,\tau)$$
$$Y = -\frac{\overline{W}_T - \overline{W}_t}{\sqrt{\tau}} \sim \mathcal{N}(0,1)$$

By the independence property of Brownian motion, under the Q-measure,

$$\begin{aligned} V_t &= e^{-r\tau}\mathbb{E}^Q([S_T - K]^+) \\ &= e^{-r\tau}\mathbb{E}^Q([S_t e^{(r-\frac{1}{2}\sigma^2)\tau-\sigma\sqrt{\tau}Y)} - K]^+) \\ &= \frac{1}{\sqrt{2\pi}}e^{-r\tau}\int_{-\infty}^{+\infty}([S_t e^{(r-\frac{1}{2}\sigma^2)\tau-\sigma\sqrt{\tau}y} - K]^+)e^{-\frac{1}{2}y^2}dy \end{aligned} \tag{9}$$

The integrand is greater than 0 thus, $y < d_2 := \frac{log\frac{S_t}{K}+(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$. So,

$$\begin{aligned} V_t &= \frac{1}{\sqrt{2\pi}}e^{-r\tau}\int_{-\infty}^{d_2}(S_t e^{(r-\frac{1}{2}\sigma^2)\tau-\sigma\sqrt{\tau}y} - K)e^{-\frac{1}{2}y^2}dy \\ &= I_1 - I_2 \end{aligned}$$

where,

$$\begin{aligned} I_1 &= \frac{1}{\sqrt{2\pi}}S_t\int_{-\infty}^{d_2} e^{-\frac{1}{2}\sigma^2\tau-\sigma\sqrt{\tau}y-\frac{1}{2}y^2}dy \\ &= \frac{1}{\sqrt{2\pi}}S_t\int_{-\infty}^{d_2+\sigma\sqrt{\tau}} e^{-\frac{1}{2}(y+\sigma\sqrt{\tau})^2}d(y+\sigma\sqrt{\tau}) \end{aligned}$$

$$I_2 = \frac{1}{\sqrt{2\pi}}e^{-r\tau}K\int_{-\infty}^{d_2} e^{-\frac{1}{2}y^2}dy$$

Rewriting equation (9) with the standard normal CDF,

$$\begin{aligned} V_t = C_t &= S_t\mathcal{N}(d_2+\sigma\sqrt{\tau}) - e^{-r\tau}K\mathcal{N}(d_2) \\ &= S_t\mathcal{N}(d_1) - e^{-r\tau}K\mathcal{N}(d_2) \end{aligned} \tag{10}$$

is the value of a call option at time t.

Similarly, for a put option,

$$P_t = e^{-r\tau} K \mathcal{N}(-d_2) - S_t \mathcal{N}(-d_1) \tag{11}$$

where,

$$d_1 = d_2 + \sigma\sqrt{\tau} = \frac{log(\frac{S_t}{K}) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$
$$d_2 = d_1 - \sigma\sqrt{\tau} = \frac{log(\frac{S_t}{K}) + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

To account for the dividend yield, q, on index options pricing the work by Garman and Kolhagen (1993) should be used. They extended the Black-Scholes model to account for a new interest rate in such way that not only the strike is discounted but also the underlying asset is. Although they did it for the pricing of foreign exchange rates it can be applied to index options where the payment of continuous dividend yields should be taken into account. Hence, the new closed form solutions for the pricing of European call and put options in this case are

$$C_t = S_t \mathcal{N}(d_1) e^{-q\tau} - e^{-r\tau} K \mathcal{N}(d_2) \tag{12}$$

$$P_t = e^{-r\tau} K \mathcal{N}(-d_2) - e^{-q\tau} S_t \mathcal{N}(-d_1) \tag{13}$$

Where,

$$d_1 = \frac{log(\frac{S_t}{K}) + (r - q + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$$
$$d_2 = \frac{log(\frac{S_t}{K}) + (r - q - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$$

All but one of the input parameters can be easily found and that is key to the extent of use of the Black-Scholes formula. However, the implied volatility can be estimated from historical stock returns by inverting the model.

Although this is a good solution it poses another snag. When the implied volatility is computed for options on the same underlying with different strikes, its level of moneyness makes the implied volatility change in a "smile" or "smirk" fashion. Out of the money options present a higher implied volatility than at the money options. This does not fall in with the assumption of the GBM where the implied volatility is constant across time and strike. Rubinstein (1994) argued that investors make more complex assumptions than those allowed by the GBM. However, this does not keep investors from using Black-Scholes formula as a pricing method. They use an *ad hoc* solution where, for short time intervals, different volatilities are applied to different options at different moneyness levels.

Not only the implied volatility can be extracted from market option prices but all of the risk-neutral probability distribution. The following section will show how to retrieve the probability distribution and the next one focuses on methods, both parametric and non-parametric, used to estimated the RND.

2.2 Option prices and RND

Arrow (1964) and Debreu (1959) developed an important framework that has been essential for the general theory of economic equilibrium with further applications to financial theory under uncertainty.

Arrow (1964) showed that it is possible to establish a relation between an elementary security and the risk-neutral probabilities of each state of a contingent claim. The Arrow-Debreu security or elementary claim (a state contingent claim), pays \$1 at maturity, T, if the underlying of the claim is of a particular state, S_T, or pays zero otherwise. The price of these state contingent claims show investors expectations on the probability of a certain state occurring in the future. However, since these securities are not traded it is not possible to use them in order to observe directly the RND function. The Arrow-Debreu security is then replicated by Breeden and Litzenberger (1978), under an assumption of perfect capital markets, as a portfolio, V_t, by a butterfly spread[1]

$$\frac{V_t}{\Delta K}\Big|_{K=S_T} = \frac{C_t(K+\Delta K) - 2C_t(K) + C_t(K-\Delta K)}{\Delta K}\Big|_{K=S_T} \tag{14}$$

Where C_t is the current price of a European call option with strike K and expiration date T.

Breeden and Litzenberger (1978) realized a general way to get to the state pricing function using the butterfly spread. It takes on the risk-neutral valuation approach used by Cox and Ross (1976) where the price of a European call option is $C(S_t, t) = e^{-r\tau}\int_0^\infty \max(S_T - K, 0)q(S_T|S_t, t)dS_T$, where $q(.)$ is the RND. By differentiating the function with respect to strike, K, the discounted cumulative distribution function is obtained

$$\begin{aligned}\frac{\partial C_t}{\partial K} &= (S_T - K)q_t(S_T)\Big|_{K=S_T} + e^{-r\tau}\int_K^\infty \frac{\partial (S_T - K)q_t(S_T)}{\partial K}dS_T \\ &= -e^{-r\tau}\int_K^\infty q_t(S_T)dS_T \end{aligned} \tag{15}$$

The second partial derivative of a European call option pricing function, again with respect to the exercise price, gives the discounted probability density function that corresponds to the discounted RND function,

$$\frac{\partial^2 C_t}{\partial K^2} = e^{-r\tau}q_t(S_T)\Big|_{K=S_T} = e^{-r\tau}q_t(K) \tag{16}$$

[1] A butterfly spread is a portfolio strategy consisting of two long European calls with strikes K-ΔK and K+ΔK and two short European calls with strike K, with ΔK>0.

The probability density function is given as in Breeden and Litzenberger (1978),

$$q_t(K) = e^{r\tau}\frac{\partial^2 C_t}{\partial K^2} \tag{17}$$

Thus, the future value of the second partial derivative of a call option with respect to strike is equivalent to the RND. The same holds to European put options.

On the case of the portfolio above mentioned, evaluated at S_T=K for a continuum of states or with very small ΔK, an estimate of q_t(K), obtained by approximating the partial derivative in equation (17), is[2]

$$q_t(K) = e^{r\tau}\frac{C_t(K+\Delta k) - 2C_t(K) + C_t(K-\Delta k)}{(\Delta K)^2} \tag{18}$$

To yield precise results it would be required to have liquid options across many strikes and that is not the case in real markets. If it were, it would be possible to derive, with a good degree of certainty, the probability density for all possible future states of the underlying. Also, taking differences twice, as illustrated by Jondeau et al. (2007), will exacerbate even tiny errors in the prices computed. Alternative methods that put more structure on the option prices were suggested.

[2]Note that no assumptions on the underlying price dynamics are made and it is possible to borrow at a risk-free rate, there are no restrictions on short sales and there are no transaction costs.

2.3 RND estimation methods

In recent decades, when option data became readily available, several approaches were developed to build the risk-neutral probability density functions from option prices. It became a fundamental concept in mathematical finance and is heavily used in the pricing of financial derivatives.

A simple way of obtaining the RND function is by discrete approximation of the implied risk-neutral probabilities of a given maturity across a continuum of adjacent strikes, based on the work of Breeden and Litzenberger (1978). This will yield a risk-neutral histogram as the functional form of the RND. However, options are not necessarily traded at equally spaced strike intervals. When strike prices are very high or very low liquidity is always low. Also, observed prices sometimes exhibit small but sudden changes in convexity across strike prices which result in large variations in the probabilities over adjoining strike intervals [Bahra (1997)].

From the mid 1990's onwards more worldly-wise methods were developed. Parametric methods, that use few variables to find the best fit for the RND and non-parametric methods that are more computationally intensive using often hundreds of variables to attain the same goal. The following two subsections mention both approaches and allude to some of the most mentioned authors on the subject.

2.3.1 Non-parametric methods

Non-parametric methods don't use any kind of parametric specification, including any assumption on the RND function. When strike prices are sparse, fitting a RND non-parametrically may not work well for the tails of the distribution imposing an additional step to deal with that problem. Although these methods use a large amount of variables making the simulation more cumbersome, they are more flexible. Jackwerth (2004) categorizes them in kernel methods, maximum entropy models and curve fitting methods.

The kernel method assumes that each observed value is in the centre of a distribution and that the likelihood of having the real function pass through a given point relates to the distance between that point and the observed data point. However, this strict non-parametric method, besides being data intensive, does not work well when observations are sparse, as in the case of discrete strike prices.

Curve fitting methods try to fit the data with some flexible function or curve, often by means of sums of squared differences. Several are the authors that use this method as Jackwerth and Rubinstein (1996) that fit the curve to observed S&P 500 option prices for the period of the 1987 market crash. However, Shimko (1993) had previously proposed an alternative way. Instead of interpolating in the call price domain, he transformed the market option prices in the implied volatility domain and transformed again the interpolated implied volatility curve back to the former domain to compute the RND. He fitted a quadratic polynomial to the implied volatilities that present a smoother curve than option prices. With those, and inverting the smile curve from the Black-Scholes formula, he obtained the call prices. After that, he employed Breeden and Litzenberger's (1978) approach to get to the risk-neutral distribution. He assumed the tails to be lognormal and added them beyond the traded strikes due to the lack of options data for far-right and far-left strike values. He matched the frequency and cumulative frequency of the distribution with a lognormal distribution in each tail. At the minimum and maximum strike prices he calculated the density and distribution values and searched for lognormal distributions that had the same values, reaching to a final distribution by grafting these three pieces. Bahra (1997) used Shimko's method in LIFFE options in the short sterling future. Because he attaches a lognormal distribution to the tails, Bahra argues that there's not always a smooth transition from the observable part of the distribution to the tails.

2.3.2 Parametric methods

The basic idea behind a simple parametric case is to use a small set of parameters to describe the distribution, price all options based on that, confront the results with observable data and minimize the error between those. With parametric methods, assumptions need to be made regarding the functional form of the RND function or the price process of the underlying.

Melick and Thomas (1997) resort to the first way above mentioned.[3] [3] They assume the functional form of the terminal price distribution and use a three lognormal mixture approach to American options pricing. On their paper

[3]Because Melick and Thomas (1997) use American options, due to the possible early exercise, other rules are crafted before proceeding to the estimation of the RND. These are out of the scope of this paper.

they reason that by doing so, the method turns more flexible, general and direct. Assuming a functional form allows for different many stochastic processes while the opposite does not hold. However, Bahra (1997) points out that using three independent lognormal distributions could exceed the limits of the number of parameters to be used, loosing its tractability. Therefore, he uses a mixture of two lognormals with exactly five different parameters. This is discussed further.

On the other hand, Bates (1991) resorts to the second way mentioned above. He assumes a price process for the underlying, an asymmetric jump-diffusion. His work on expectations for the 1987 crash concluded that the RND for the S&P 500 options did not show a predictive power for the month the crash happened. Another example is the known case of the Black and Scholes. It assumes the underlying evolves according to a GBM with constant drift rate and volatility, implying a lognormal RND function. Other authors used different parametric methods to find the most suitable RND. Jackwerth (2004) categorizes these in expansion methods, generalized distribution methods and mixture methods.

The following section and the model used in this paper will focus on a mixture method, the two-lognormal mixture method. It encompasses the normal distribution, offers flexibility in the computation of the RND, is relatively fast to simulate and allows for a bimodal distribution which is useful when the market is in doubt between two possible states.

2.4 Two-lognormal mixture method

The mixture of distributions to describe the RND function of European options and other derivatives was worked by Ritchey (1990), Melick and Thomas (1997), Bahra (1997), Söderlind and Svensson (1997) and Gemmill and Saflekos (2000). The approach used follows Bahra (1997) that applied the mixture of two-lognormal densities to the pricing of, among others, LIFFE equity European options. As he points out, assuming daily prices are lognormal then, for any arbitrary holding period, price distributions must also be lognormal. The method consists of estimating the functional form of the RND by minimizing the squared difference between the prices generated by the assumed parametric form and the observed prices in the market. The probability is built as a weighted average of tow lognormal distributions. The mixture method does not impose an overly restrictive functional form for the density. It allows to capture skewness and excess kurtosis for many functional forms, including a bimodal shape [Bahra (1997)].

The prices of European call and put options are taken with respect to the risk-neutral probabilities and using the risk-free rate as a discounting factor. It can be written as the discounted sum of all expected future payoffs on the expiration date:

$$C(K,\tau) = e^{-r\tau} \int_{K}^{\infty} q(S_T)(S_T - K)dS_T \qquad (19)$$

$$P(K,\tau) = e^{-r\tau} \int_{0}^{K} q(S_T)(K - S_T)dS_T \qquad (20)$$

Bahra assumes that price distributions are close to a lognormal distribution and, as suggested by Ritchey (1990), assumes that $q(S_T)$ is the weighted sum of k lognormal distributions, given by Black and Scholes (1973). In this case two lognormals:

$$q_t(S_T;\mu_1,\mu_2,\sigma_1,\sigma_2,\theta) = \theta\frac{e^{-\frac{(log(S_T)-\mu_1)^2}{2\sigma_1^2}}}{S_T\sigma_1\sqrt{2\pi}} + (1-\theta)\frac{e^{-\frac{(log(S_T)-\mu_2)^2}{2\sigma_2^2}}}{S_T\sigma_2\sqrt{2\pi}}, \quad S_T > 0 \quad (21)$$

Where $\theta \in [0,1]$ and its sum is one.

So, the closed form solution for the European call and put options are written as:

$$C_t = e^{-r\tau}(\theta(e^{\mu_1+\frac{1}{2}\sigma_1^2}\Phi(d_1) - K\Phi(d_2)) + (1-\theta)(e^{\mu_2+\frac{1}{2}\sigma_2^2}\Phi(d_3) - K\Phi(d_4))) \quad (22)$$

$$P_t = e^{-r\tau}(\theta(K\Phi(-d_2) - e^{\mu_1+\frac{1}{2}\sigma_1^2}\Phi(-d_1)) + (1-\theta)(K\Phi(-d_4) - e^{\mu_2+\frac{1}{2}\sigma_2^2}\Phi(-d_3)))(23)$$

Where,

$$d_1 = \frac{-log(K)+\mu_1+\sigma_1^2}{\sigma_1}$$

$$d_2 = d_1 - \sigma_1$$

$$d_3 = \frac{-log(K)+\mu_2+\sigma_2^2}{\sigma_2}$$

$$d_4 = d_3 - \sigma_2$$

Now, given the closed pricing formulas above, it is possible to use them in the minimization problem. The closed form solutions of C_t and P_t will be used for the minimization of the sum of squared errors between the option prices computed by the model and the actual market prices, across all strikes, fitting the two log-normal RND function to the real prices.

$$\min_{\mu_1,\mu_2,\sigma_1,\sigma_2,\theta} \quad \sum_{i=1}^{n}(C_t(K_i) - \hat{C}_t(K_i))^2 + \sum_{i=1}^{n}(P_t(K_i) - \hat{P}_t(K_i))^2 + \\ (\theta e^{\mu_1+\frac{1}{2}\sigma_1^2} + (1-\theta)e^{\mu_2+\frac{1}{2}\sigma_2^2} - e^{r\tau}S_t)^2 \tag{24}$$

Also, to ensure absence of arbitrage the mean of the implied RND function should equal the forward price of the underlying asset, as assure by the last term of the equation.

3 Data and methodology

The Standard and Poor's 500 Index (S&P 500) is a market-capitalization-weighted index on approximately the 500 largest public traded companies, from a broad range of industries, in the United States. Because this index is a good gauge of the U.S. large-cap equities it is considered to be a good reflection of the U.S. financial and economic shape. Options on the S&P 500 are the ones used in this paper.

The Chicago Board of Exchange offers a comprehensive suite of listed options on the S&P 500. For the purpose of this paper the SPX Options Traditional were used. These options are of European style with up to twelve near-term expiration months and expiring on the third Friday of the respective expiration month. These are AM settled – the index value is usually computed with the opening price of the bundle of the index's component securities on the day of exercise.

Three-month maturity options were chosen. Short-term options have higher liquidity, but only up to a certain point, since when it gets closer to expiration uncertainty falls and so does liquidity. With 3-month maturity options there should be a good balance between uncertainty and liquidity. After doing several simulations it was found that the results around the exact 3-month maturity mark were not significantly different. So, the dates were chosen assuring the best liquidity possible around that same mark.

The data comprises the period from January 2019 to March 2020. The reason for this span is to have an idea of how the functional form of the RND behaves in months not affected by the consequences of the pandemic, or other major events, and to understand when, in fact, the functional form of the RND starts to change.[4] Therefore, for this purpose, one RND was estimated for each of those months. Notwithstanding, other RND's were computed in light of events mentioned further on.

The data set was exported from Thomson Reuters' Datastream, provided by the Option Price Reporting Authority. To make sure the information retrieved was well grounded, the options were filtered guaranteeing that the total cumulative volume and/or the total open positions for all individual option series were above zero. The options' price used was the last traded price, provided last trade is within bid/ask range at the end of the closing session. The puts were disregarded for the evaluation at hand since, according to Birru and Figlweski (2010), calls and puts at the same strike price trade at different implied volatilities, creating jumps in the implied volatility curves and badly behaved RND functions.

Since many announcements and measures made public from the beginning of the COVID-19 outbreak and some exceptional market activities were observed, several are the dates that should be taken into more attentive consideration. The introduction of this paper mentioned several of these key moments, but it will be giving special thought to the following weeks: 20^{th} January 2020,[5] when the first U.S. case was confirmed (20^{th} January) and the Chinese government enforced several large-scale interventions to control the spread of the virus,

[4]One should note that the second semester of 2019 and beginning of 2020 was also marked by the US-China phase one agreement on trade, weakening international trade and bringing uncertainty on the two major global partners relationship.

[5]The 20^{th} January is a U.S. national holiday, for that reason no data is available on this day.

namely a major lockdown in Wuhan affecting 9 million people (23^{rd} January); 9^{th} March 2020,[6] when the Italian government expanded its quarantine restrictions to an area affecting over 16 million residents (9^{th} March), the World Health Organization briefed the media that the outbreak could be categorized as a pandemic (11^{th} March) and the VIX surged to 75 points (12^{th} March) and 16^{th} March 2020, when the VIX surged to its second all time peak, 82.69 points at close, while the major U.S. indexes kept their receding trends.

These were not the only factors taken into consideration when deciding where to place a more particular interest. Analysing the data around this period, it was noticed that the week of 24^{th} February 2020 manifested some visible change. Hence, this week is also examined closer. Daily RND's were retrieved for these four weeks.

The mixture of lognormal distributions allows to capture very flexible distributions. This could be considered as a furtherance of the BS model, since it uses two log-normal densities. The implied RND functions were retrieved using it. For that purpose MATLAB was chosen. The model was translated to code allowing to minimize the sum of squared residuals.

[6] Although the energy industry on S&P 500, at the time considered, only accounts for less than 3%, one should be mindful of the price war Saudi Arabia started with Russia on the 8^{th} of March 2020 that encourage a great decrease in oil prices.

4 Results

In this section the results of the methods used are examined. It was thought appropriate to outline some crucial announcements from policy makers and news outlets, since these usually bear a relevant weight on investors' decisions. Looking at the shape of the various RND functional forms and some of its relevant statistics conclusions can be withdrawn. Appendix A presents the method's minimized parameters. The functions can be observed in Appendix B. While the table on Appendix C summarizes the monthly RND functions' key descriptive statistics, Chart 1, on this section, displays it on a chart and Table 1 exhibits the daily RND functions' descriptive statistics of four of the weeks closer examined.

The period chosen for the purpose of comparing the functional form of the RND, starting at January 2019, shows that the RND for three month maturity options do not show a significant variation. Throughout the year, the monthly RND computed always presents a clear left tail with no noteworthy protuberance. At least until August 2019. Although more relevant changes will be seen further on, looking at the August RND is obvious that a small hump occurs on the left tail of the distribution, changing its skewness from -0.19983 to -0.29874 and lowering kurtosis values to below 5.[7] This change could possibly be explained by the escalation of the U.S.-China trade war and the talks of Chinese currency manipulation that coincided with a VIX surge to 24.59 on the 5^{th} of August 2019. Also, the U.S. 2-10 year yield curve inverted, signalling for many investors a looming

[7]Because kurtosis is sensitive to values far on the tails and since such observations are sparser, these considerations should be taken into account when analysing its values.

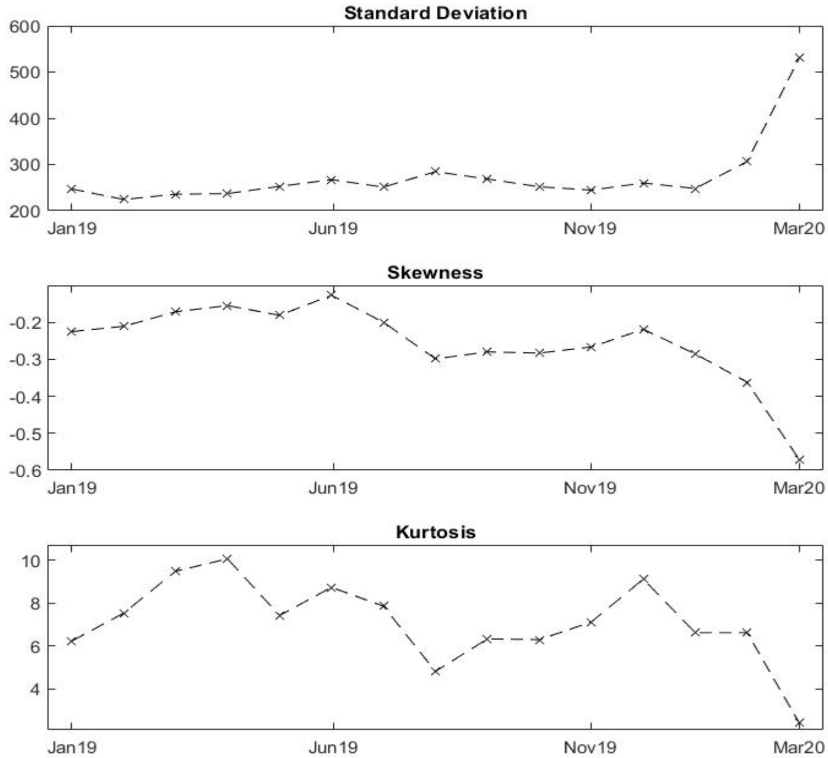

Chart 1: Standard deviation, skewness and kurtosis of the monthly RND functions from January 2019 to March 2020. The corresponding values can be seen in Appendix C

recession. As a consequence investors apparently tend to loose some confidence in the markets and start dumping stocks and taking positions on long-term U.S. debt and other safer investments. In due course, by the end of the third quarter and beginning of the last quarter, U.S. and China displayed intentions of softening their rhetoric on trade discussions and the yield of the 10-year bond exceeded that of 2-year bond. Indeed, by the end of the year the RND function sprang back to is preceding form.

Meanwhile, as China imposes its first lockdown and the U.S. confirms its

Descriptive Statistics									
Date	$\mathbb{E}$	$\mathbb{D}$	$\mathbb{S}$	$\mathbb{K}$	Date	$\mathbb{E}$	$\mathbb{D}$	$\mathbb{S}$	$\mathbb{K}$
					9MAR20	2731.5	545.73	-0.59822	2.4689
21JAN20	3311	241.42	-0.23619	7.9007	10MAR20	2681.6	568.61	-0.64608	2.5432
22JAN20	3312.4	240.66	-0.2311	7.9142	11MAR20	2726.4	551.96	-0.61341	2.4515
23JAN20	3318.3	238.6	-0.24172	7.8828	12MAR20	2560.6	597.21	-0.62815	2.2326
24JAN20	3286.2	248.09	-0.28538	6.6169	13MAR20	2846.4	526.61	-0.56195	2.7572
24FEB20	3220	306.67	-0.36205	4.7829	16MAR20	2404	665.46	-0.71684	1.9718
25FEB20	3126.9	319.05	-0.39213	3.9829	17MAR20	2490.7	643.65	-0.69968	2.0616
26FEB20	3104.2	314.62	-0.39994	3.8206	18MAR20	2483.1	644.46	-0.69406	2.1874
27FEB20	2951.9	375.91	-0.50577	3.0665	19MAR20	2394.9	601.31	-0.63552	2.1926
28FEB20	2960.1	392.99	-0.49095	3.4129	20MAR20	2296.4	530.33	-0.59064	2.4265

Table 1: Descriptive statistics [mean ($\mathbb{E}$), standard deviation ($\mathbb{D}$), skewness ($\mathbb{S}$) and kurtosis ($\mathbb{K}$)] of the daily RND functions for the weeks of 20^{th} January 2020, 24^{th} February 2020, 9^{th} March 2020 and 16^{th} March 2020. Note that 20^{th} January 2020 corresponds to a national holiday.

first cases, news on the coronavirus were starting to flood media outlets. However, looking at the RND functions for the week of 20^{th} January it seems the markets did not realize the great threat the virus would pose to the economy just yet. The RND functions kept its form with the left skewness and kurtosis alike the ones aforementioned, between -0.28538 and -0.2311 and 6.6169 and 7.9142 respectively.

Things started to change when February came to an end. When analysing the RND functions, this period revealed significant differences in its RND functional forms. A small context may be drawn.

In Italy, 21^{th} February 2020, Giulio Gallera, Lombardy's Welfare Councilor, invited more than 50 thousand people to stay home for a week.[8] Recommendation that was followed by law-decree on the 23^{rd}.[9] On the 25^{th} February, 22 days after the White House administration declared the coronavirus outbreak a public health emergency, the director of CDC's National Center for Immunization and Respiratory Diseases said that the Covid-19 met, at the time, two of the three required factors to be categorized as a pandemic.[10] Worldwide spread was the criteria not yet met.

Nonetheless, it was not just the vocal concerns of health authorities but also regulatory and economic actors of international recognition that started weighing on market's sentiment. ECB's Vice-president, Luis de Guindos, aknowledged on the 20^{th} February that "the outbreak of the coronavirus and its potential effect on global growth add a new layer of uncertainty"[11]. On the 25^{th} February, Richard H. Clarida, Fed's Vice Chair, said that the coronavirus "is likely to have a noticible impact on Chinese growth [...] that could spill over to the rest of the

[8] *Coronavirus in Italia: tutte le notizie di febbraio*, la Repubblica, 1 March 2020, https://www.repubblica.it/cronaca/2020/02/22/news/coronavirus_in_italia_aggiornamento_ora_per_ora-249241616/, (accessed 28 December 2020)

[9] *Decreto-legge 23 febbraio 2020, n.6*, followed by further implementation procedures in *Decreto del Presidente del Consiglio dei Ministri 25 febbraio 2020*

[10] Nancy Messonnier, MD, *CDC (Centers for Disease Control and Prevention) media telebriefing update on COVID-19* , [media briefing], National Center for Immunization and Respiratory Diseases, 25 February 2020

[11] Luis de Guindos, Vice President of ECB, *The Euro Area Economic Outlook and the Current Monetary Policy Stance*, [speech], Frankfurt am Main, 20 February 2020

global economy"[12]. Also, on the 28^{th} of the same month, Jerome H. Powell, Fed's Chair, released a statment declaring that "the coronavirus poses evolving risks to [US'] economic activity"[13]. These statements, although slightly hazy, could be a possible reason for a change in the perspective of future outcomes.

Up until this moment the RND functional forms exhibited a modest left hump. Whereafter, as seen on Appendix B, Figures 21 to 25, the RND functions for the period in question display a much more pronounced left hump, increasing skewness to values below -0.36, and standard deviation to values above 300. Also, kurtosis decreases significantly this week, presenting a lower peak around the mean. A significant lower kurtosis from 8.5984 and 6.7948 on the two preceding Fridays to values between 3.0665 and 4.7829 on the week at hand. It would seem safe to say that when February came to an end the first signs of the pandemic were perceived by investors.

Getting to the week of 9^{th} March, the media briefing of the World Health Organization,[14] categorizing the outbreak as a pandemic, was a pivotal moment to how the world would deal with this crisis as a whole.

Since late February the VIX had registered a steady increase and this week, on the 12^{th} of March, reached a level of 76.83. It is evident when looking to the results on Appendix C that the uncertainty sparked volatility to levels not

[12] Richard H. Clarida, Vice Chair of Fed, *U.S. Economic Outlook and Monetary Policy*, [speech], 6th Annual NABE Economic Policy Conference, Washington, D.C., 25 February 2020

[13] Jerome H. Powell, Chair of Fed, *Statement from Federal Reserve Chair Jerome H. Powell*, [press release], 28 February 2020, https://www.federalreserve.gov/newsevents/pressreleases/other20200228a.htm, (accessed 17 December 2020)

[14] Tedros Adhanom,MD, Director-General of WHO, [media briefing], 11 March 2020

yielded before. From 9 th March onwards the values for standard deviation never fell below 500. Yet another evident change is the functional form of the RND that presents a much more evident bimodal shape. The first table on Appendix A reveals that, before March, the final distribution is much more influenced by only one of the mixture distributions. However, on this month the weights of the distributions that comprise the final RND are almost identical. That affects the moments, as seen in Figure 26, Appendix B, and Table 1, kurtosis is now 2.4689. Value that did not change much during this week but that is certainly lower than that of previous periods. As for skewness, it is now more left pronounced, running up to -0.72815 on Thursday.

Concurrently with the changes on results, several were the significant monetary policies made public this week to address market uncertainties. The U.S. Federal Reserve had already announced on the beginning of the month a cut of 50 basis point to interest rates.[15] On Thursday, 12th of March, the N.Y. Federal Reserve Bank announced an update on the monthly schedule of repurchase agreement operations to address temporary market disruptions. The repo operations meant an offer of $\sim$ \$1.5 Trillion in short-term loans to banks.[16] While, internationally, the Bank of England announced a cut on interest rates and other monetary measures[17] and the European Central Bank revealed a €120 billion

[15] Federal reserve issues FOMC statement, [press release, 3 March 2020, https://www.federalreserve.gov/newsevents/pressreleases/monetary20200303.htm, (accessed 29 December 2020)]

[16] *Statement Regarding Treasury Reserve Management Purchases and Repurchase Operations*, 12 March, 2020, https://www.newyorkfed.org/markets/opolicy/operating_policy_200312a, (accessed 28 December 2020)

[17] *Bank of England measures to respond to the economic shock from Covid-19*, 11 March, https://www.bankofengland.co.uk/news/2020/march/boe-measures-to-respond-to-the-

expansion to its assets purchase program and surprisingly kept rates unchanged.[18]

Comparing Thursday's with Friday's results it may well be argued that the definite action of regulatory authorities were influential. The VIX dropped, closing at 57.83 on the end of the week. As for the results of the simulation,

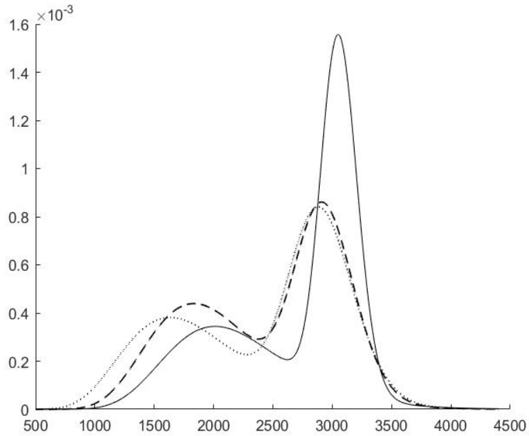

Figure 1: RND functions for three month maturity options for 12^{th} March 2020 (- -), 13^{th} March 2020 (—) and 16^{th} March 2020 (· ·).

standard deviation and skewness decreased, from 597.21 to 526.61 and from -0.62815 to -0.56195, respectively. While kurtosis increased from 2.2326 to 2.7572. Were the actions taken a prelude for a less doubtful market the following week?

The answer is a resonant no. The Fed, confirming market's distress, announced just on Sunday several measures that remembered those taken in a several month period on the previous financial crisis. Interest rates were cut to essentially zero, a quantitative easing program was launched with a target of $700 billion,

economic-shock-from-covid-19, (acessed 28 December 2020)

[18]ECB Directorate General Communications,[Press Release], Frankfurt am Main, 12 March 2020

thousands of banks were contemplated with a cut on reserve requirements to zero and agreements were made with other central banks to increase dollar liquidity.[19] Despite these aggressive moves, the market's response was negative. On Monday, 16^{th} March the markets' forecast was that the S&P 500 would fluctuate 82.69% over the next 30 days. This was the VIX's close second all time peak. The S&P 500 plunged about 8% after the opening bell and ended up dropping 12%. Very similar trends were seen on the Dow Jones Industrial Average and Nasdaq Composite. The results on the RND functions' descriptive statistics go in hand with this scenario. For a clear view of the differences between 12^{th} and 16^{th} March 2020, Figure 1 comprises the RND functions of those three days. There was definitely a sudden change on the RND for Friday, 13^{th} March 2020, but after the weekend it regained the previous shape. The values for kurtosis on this month are probably the most striking when looking at the first table of Appendix C. During this period the kurtosis kept its values always bellow 2.5, reaching 1.9718 on Monday, 16^{th} March. Also, skewness is far from the results of preceding months. Overall, this week presented a more left skewed distribution than the previous ones with a figure of -0.71684 on Monday and decreasing to -0.59064 on Friday.

[19] Federal Reserve issues FOMC statment, [press release], 15 March 2020, https://www.federalreserve.gov/newsevents/pressreleases/monetary20200315a.htm, (accessed 29 December 2020)

Federal Reserve Actions to Support the FLow of Credit to Households and Businesses, [press release], 15 March 2020, https://www.federalreserve.gov/newsevents/pressreleases/monetary20200315b.htm, (accessed 29 December 2020)

Coordinated Central Bank Action to Enhance the Provision of U.S. Dollar Liquidity, [press release], 15 March 2020, https://www.federalreserve.gov/newsevents/pressreleases/monetary20200315c.htm, (accessed 29 December 2020)

5 Conclusion

The objective of this work is to take conclusions on the information value of RND from investors expected future outcomes using the prices of S&P 500 options, particularly to understand whether it is possible to anticipate future outcomes on the index. By using a mixture of two log normals to estimate the markets RND it was possible to infer on how the option markets behaved getting to and during the initial phase of the COVID-19 pandemic.

It should be taken into consideration that by estimating risk-neutral probabilities there are assumptions that are not consistent with real world probabilities. The economy is not risk-neutral and investors show different non-linear utility functions, requiring a premium to overcome future price uncertainty. So, one should be cautious when interpreting the forecast ability of these densities. For a more reliable estimate Liu, Shackleton, Taylor and Xu (2003) accounted for premia earned for bearing risk by making a risk adjustment motivated by a representative utility function and statistical calibration. They came to the conclusion that the transformed real world densities are slightly more robust and informative. Nonetheless, risk-neutral probabilities still are a good reflection of the real world economy for the purpose of this paper.

Since confirmation of the first cases of COVID-19 on December 2019, in China, and January 2020, in the US, several weeks gone through until the RND started to show an inconsistent functional form. Because of the nature of this crisis and how it unfolded it would not be sensible to pinpoint an exact day as a pivot for outcomes. Investors were steadily made aware of the effects COVID-19 would have on the economy. Hence, the several references made previously.

Alike macroeconomic news were adjunct to this paper, Brown and Jackwerth (2000) studied specifically the impact of eight different macroeconomic announcements on the RND. They found that it was not affected by those announcements. However, they studied month-on-month indicators. Since this paper deals with an exceptional period it was chosen the most relevant extraordinary actions taken by central banks and respective policy makers' statements during the period studied. Concurrent with the latter section it seems the changes on the RND were always in unison with the reactions of news outlets and economic policy agents.

Despite the inability to anticipate those changes relative to the forthcoming shock, descriptive statistics of the densities seem to be consistent. It shows that in the distressed period the risk is higher, investors attach greater weight to the possibility of future negative returns and they show weaker confidence in the current price level. The RND is always left skewed. This result is consistent with most literature on the subject [see, for example, Bates (1991), Gemmill and Saflekos (2000) and *Äijö* (2006)]. The large change in skewness from February to March indicates a surprise or uncertainty among investors. Also, as it was evident on the month of March, it is possible to conclude that negatively perceived events are associated with an increase in left skewness. The left tail of the distribution responds considerably more than the right one. Which is even more evident when analysing the plots of the RND, where an almost bimodal shape arises. In the previous months the higher weight on one of the mixture distributions could indicate that investors had similar expectations. In this cases the other, lower weight, mixture distribution is merely useful to ensure the added flexibility this method allows.

Now, the March's similar distribution weights on the minimization problem, or the bimodal shape, can be interpreted as investors being fearful of extreme moves of the underlying but not sure of its direction. Figlewski (2008) and Gemmill and Saflekos (2000) reached the same conclusion. Kurtosis also diminishes for the most critical period. However, this measure is sensitive to observations far away from the mean, where there are fewer transactions. It should be interpreted with care. Apart from August 2019 that presents a 4.819 kurtosis, possibly for the reasons afore mentioned, it gets lower values in March 2020, 2.1926. Gleaning the conclusions of literature in general and of *Äijö* (2006) particularly, comparing the results of volatility with those of kurtosis it may be deemed that a rise in the former and a fall in the latter means greater risk on price changes and weaker confidence in the current price levels. In summary, for a distressed period volatility rises, the RND takes on an almost bimodal shape while skewness increases, becoming more left pronounced, and kurtosis diminishes.

RND functions would not be wisely used to gauge market's sentiment in order to make policy measures at a monetary level. The information on the shape and moments of the RND functions is not indicative of market's future turbulence in a prompt way. If regulators reacted to changes in RND functions they would be signalling the prevention of a further shock. However, it still produces information on future expectations of market participants. If an investor were to follow a strategy that resulted from his/her forecast, by looking at the densities he/she could assess the differences and decide to take another course. Also, central banks can use RND functions to figure whether new measures were expected by market participants and by doing so, assessing its effectiveness and credibility.

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Appendix

A - Method's Minimized Parameters

The tables below present the parameters retrieved from the minimization with the two lognormal method. The first table for every month of the period between January 2019 and March 2020 an the second for every trading day of the weeks 20^{th} January, 24^{th} February, 9^{th} March and 16^{th} March.

Method's Minimized Parameters					
Date	μ_1	μ_2	σ_1	σ_2	w
JAN19	7.9116	7.7215	0.052671	0.15155	0.85049
FEB19	7.7736	7.9463	0.14533	0.045118	0.13021
MAR19	7.7627	7.9601	0.17606	0.046324	0.1034
APR19	7.9836	7.7818	0.047555	0.18481	0.90909
MAY19	7.7881	7.9809	0.16306	0.051927	0.11951
JUN19	7.7528	8.0037	0.18564	0.057492	0.078788
JUL19	7.8079	8.0147	0.14868	0.047346	0.1129
AUG19	7.9984	7.8172	0.053238	0.13562	0.79303
SEP19	7.8507	8.0264	0.14272	0.042958	0.1821
OCT19	8.0255	7.864	0.040721	0.13179	0.81581
NOV19	8.0659	7.9106	0.037815	0.13121	0.8345
DEC19	8.0938	7.902	0.040352	0.15586	0.88098
JAN20	8.1207	7.9706	0.035813	0.11641	0.82504
FEB20	7.9612	8.1395	0.13846	0.040072	0.12909
MAR20	7.9402	7.4862	0.086761	0.27653	0.56985

Method's Minimized Parameters					
Date	μ_1	μ_2	σ_1	σ_2	w
21JAN20	7.9658	8.1238	0.12578	0.037247	0.13679
22JAN20	7.9657	8.1239	0.12554	0.03754	0.13465
23JAN20	8.1262	7.9719	0.035963	0.12305	0.85924
24JAN20	8.1207	7.9706	0.035813	0.11641	0.82504
24FEB20	8.1138	7.9439	0.04249	0.12719	0.75499
25FEB20	7.923	8.0909	0.1259	0.04764	0.29051
26FEB20	7.925	8.0845	0.12336	0.04763	0.31097
27FEB20	7.8637	8.0584	0.14204	0.05265	0.39559
28FEB20	7.8445	8.0601	0.15827	0.04899	0.35679
09MAR20	7.7213	8.0352	0.21716	0.072894	0.46064
10MAR20	7.7522	8.059	0.21065	0.06357	0.40503
11MAR20	8.0361	7.7089	0.069364	0.21945	0.54695
12MAR20	7.988	7.5772	0.082951	0.24501	0.4771
13MAR20	8.0261	7.6671	0.049657	0.24525	0.56104
16MAR20	7.9784	7.4853	0.091364	0.29389	0.52065
17MAR20	7.9957	7.5441	0.079065	0.27454	0.52713
18MAR20	7.9829	7.6002	0.07847	0.27212	0.54911
19MAR20	7.9402	7.4862	0.086761	0.27653	0.56985
20MAR20	7.8751	7.4796	0.089138	0.26701	0.58011

B - Risk Neutral Density Functions

Here are represented the three month maturity RND functions for the periods studied. The doted line assumes the first set of parameters, the dashed line the second set of parameters and the bold line corresponds to the RND function as a weighted sum of both sets.

Three month RND functions for every month of the period between January 2019 and March 2020:

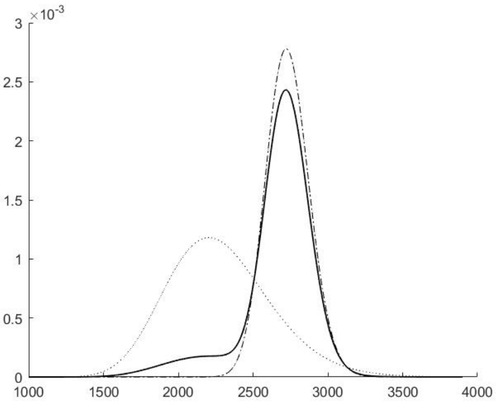

Figure 2: January 2019

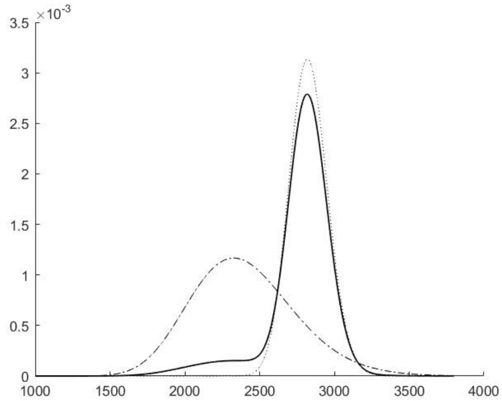

Figure 3: February 2019

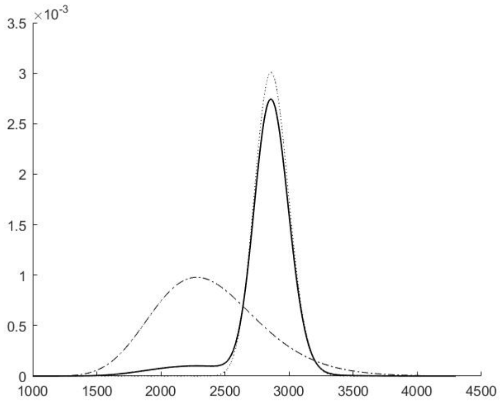

Figure 4: March 2019

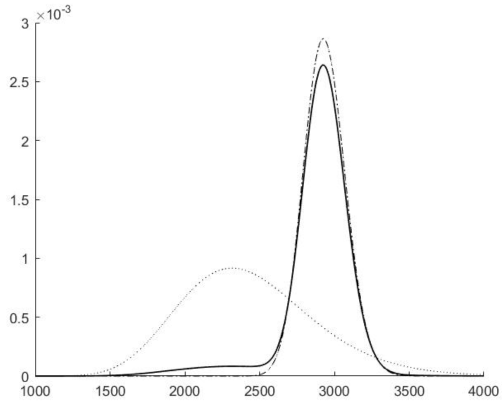

Figure 5: April 2019

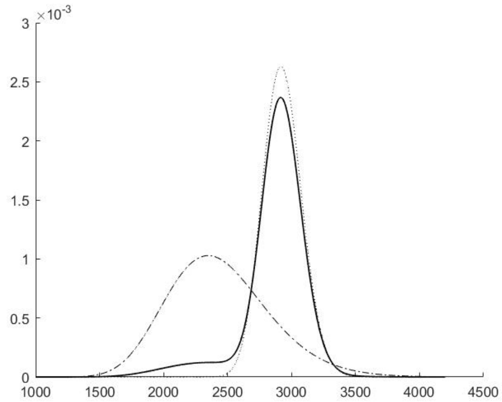

Figure 6: May 2019

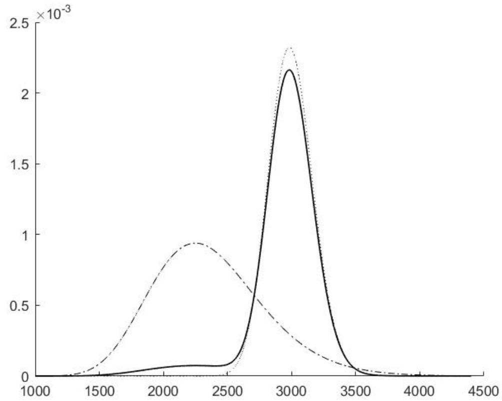

Figure 7: June 2019

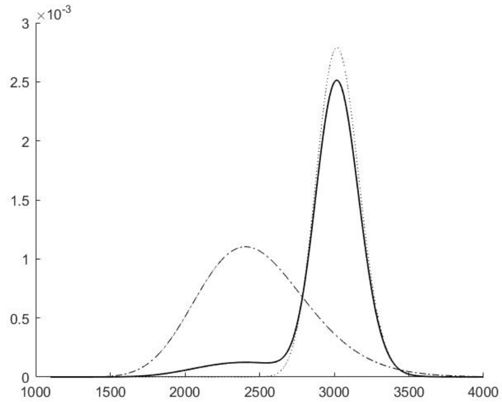

Figure 8: July 2019

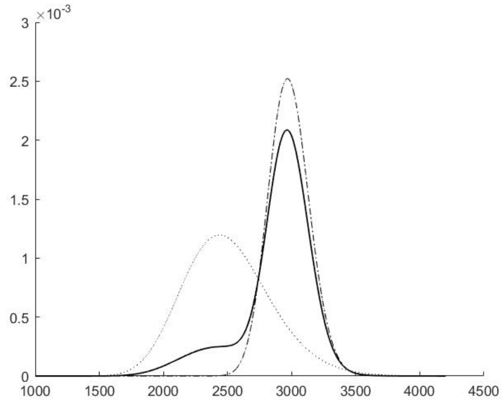

Figure 9: August 2019

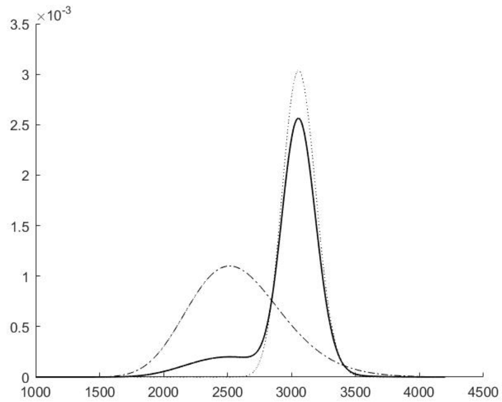

Figure 10: September 2019

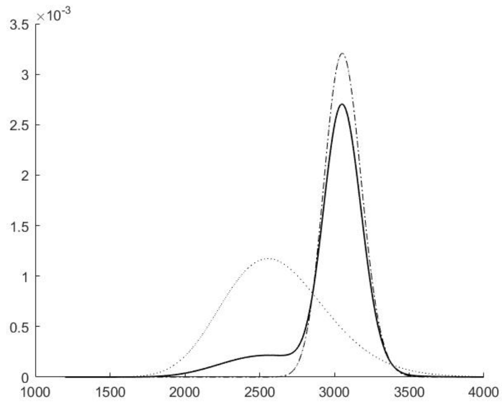

Figure 11: October 2019

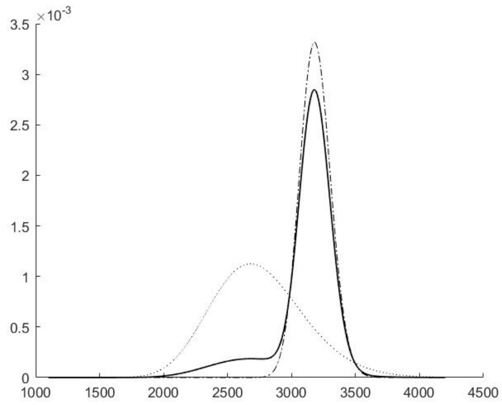

Figure 12: November 2019

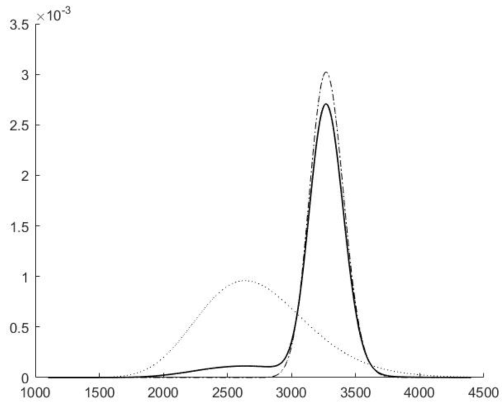

Figure 13: December 2019

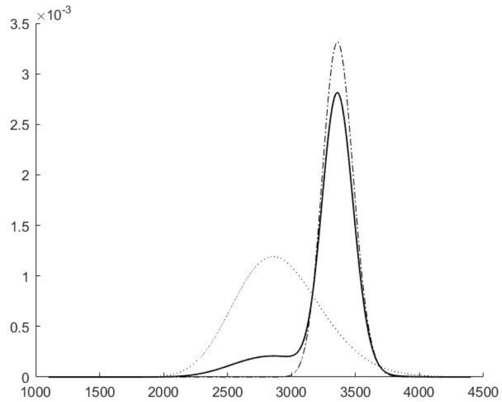

Figure 14: January 2020

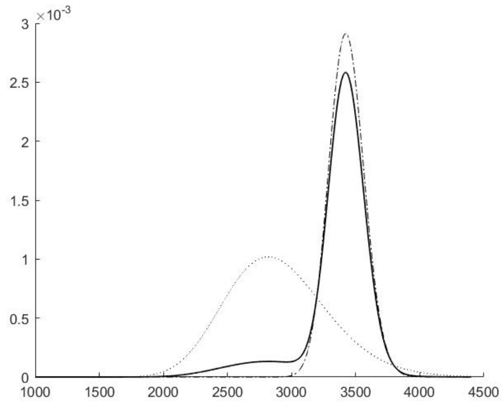

Figure 15: February 2020

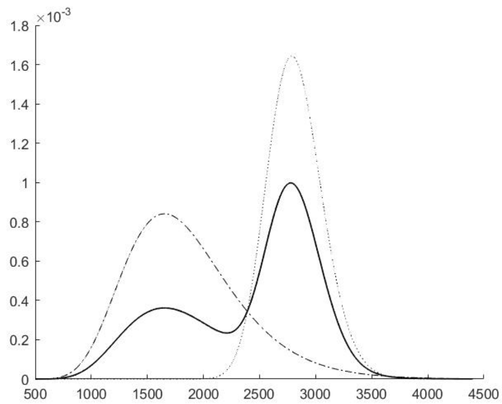

Figure 16: March 2020

Three month RND functions for every trading day of the weeks 20^{th} January, 24^{th} February, 9^{th} March and 16^{th} March:

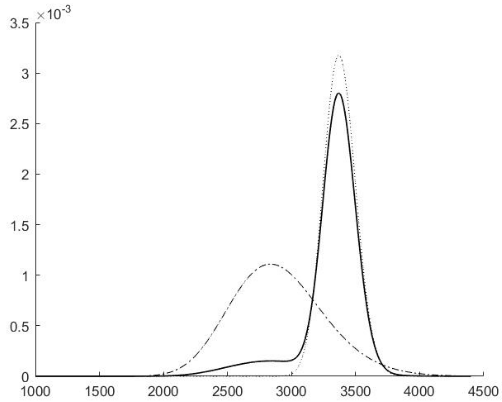

Figure 17: 21 January 2020

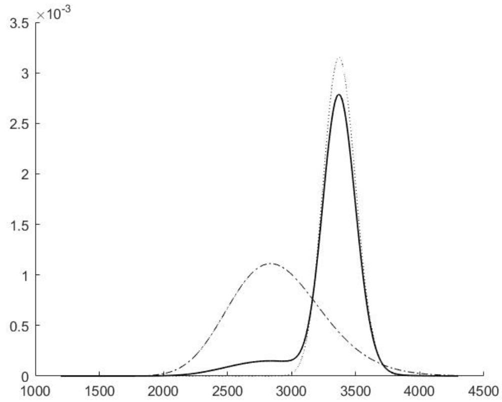

Figure 18: 22 January 2020

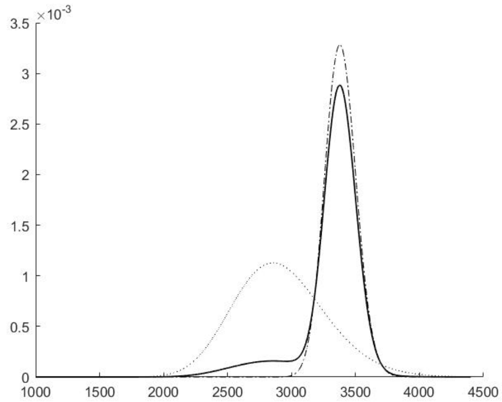

Figure 19: 23 January 2020

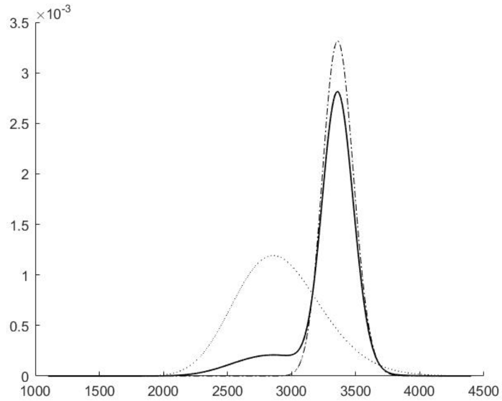

Figure 20: 24 January 2020

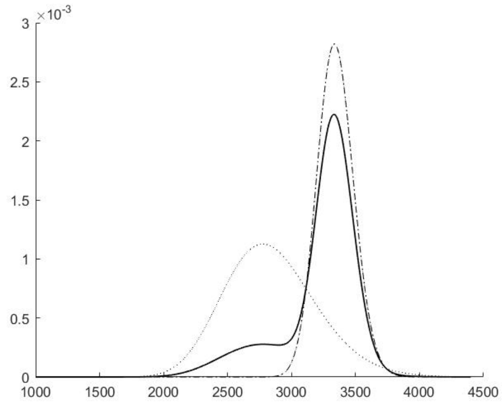

Figure 21: 24 February 2020

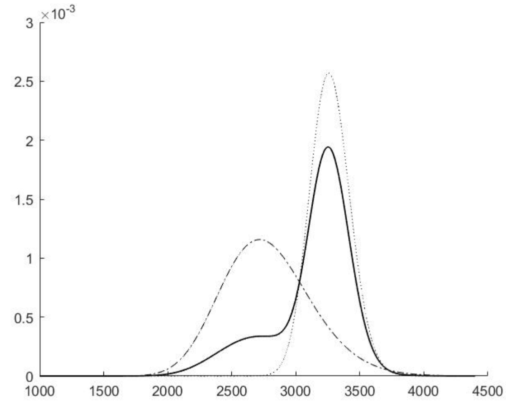

Figure 22: 25 February 2020

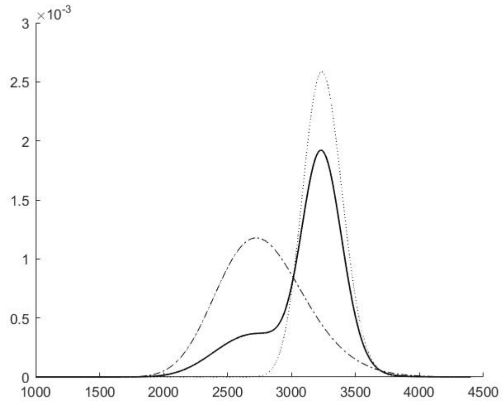

Figure 23: 26 February 2020

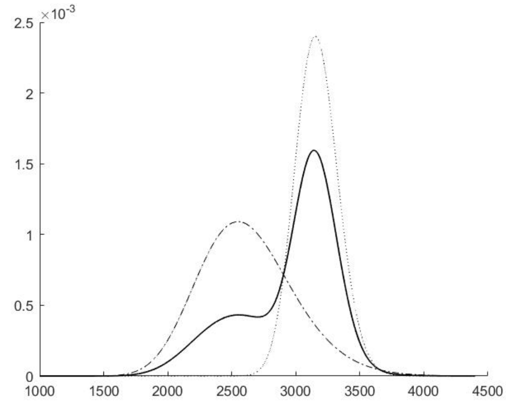

Figure 24: 27 February 2020

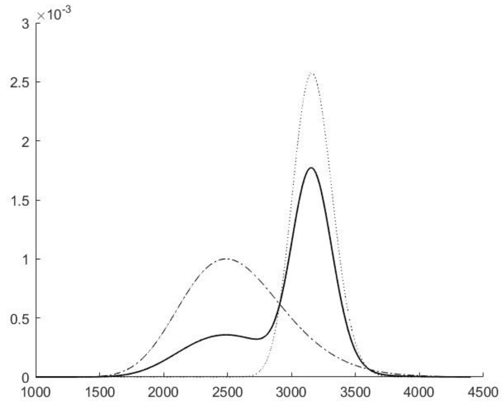

Figure 25: 28 February 2020

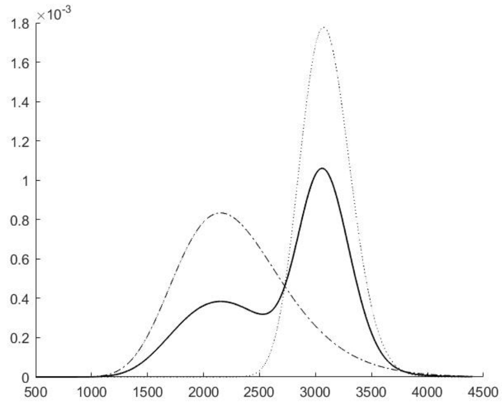

Figure 26: 9 March 2020

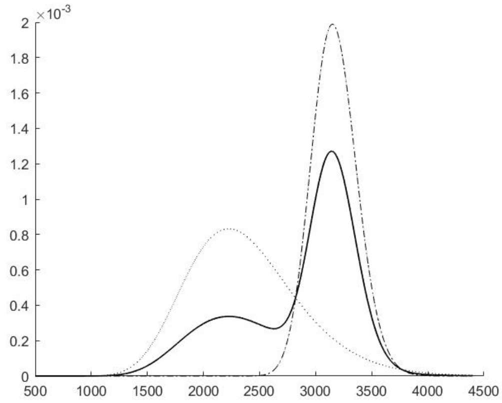

Figure 27: 10 March 2020

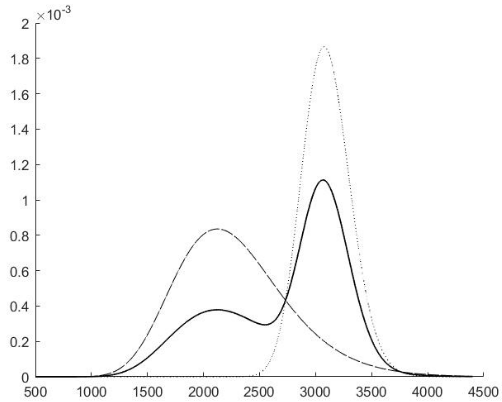

Figure 28: 11 March 2020

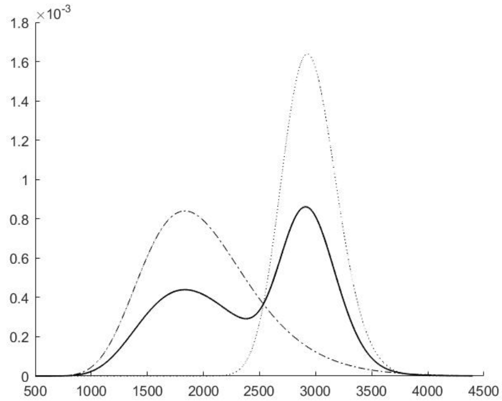

Figure 29: 12 March 2020

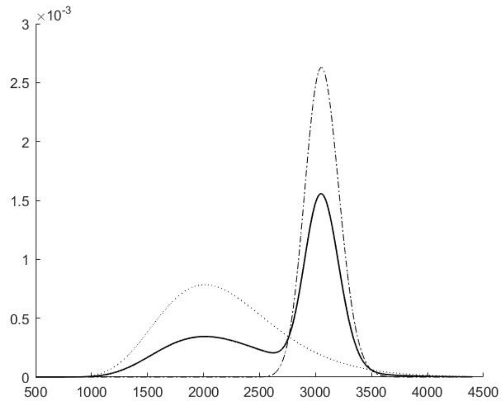

Figure 30: 13 March 2020

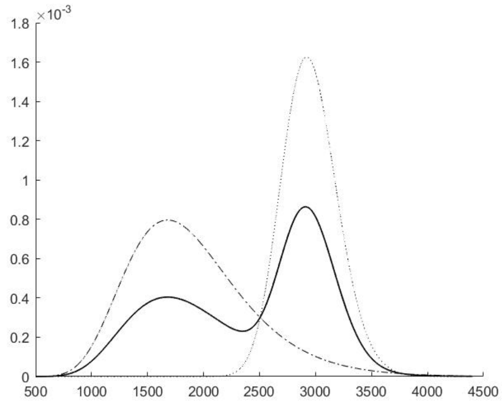

Figure 31: 16 March 2020

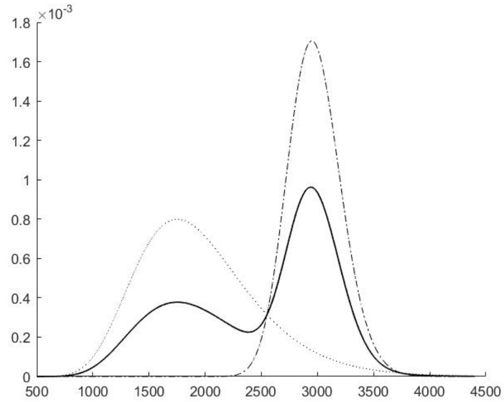

Figure 32: 17 March 2020

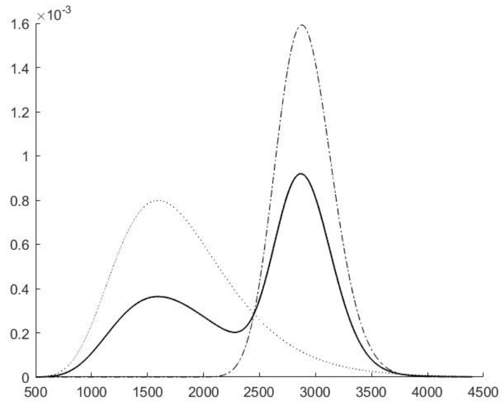

Figure 33: 18 March 2020

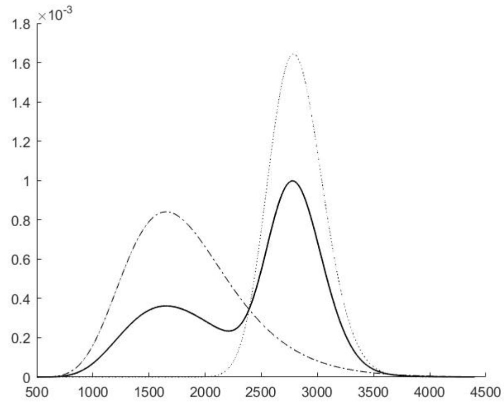

Figure 34: 19 March 2020

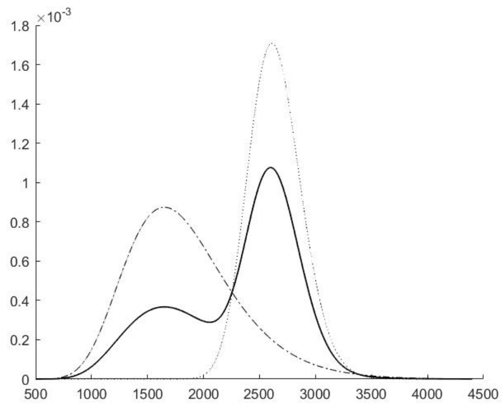

Figure 35: 20 March 2020

C - Descriptive Statistics

The descriptive statistics [mean ($\mathbb{E}$), standard deviation ($\mathbb{D}$), skewness ($\mathbb{S}$) and kurtosis ($\mathbb{K}$)] of the three month maturity RND functions are presented in the table below for each month between January 2019 and March 2020.

Descriptive Statistics				
Date	$\mathbb{E}$	$\mathbb{D}$	$\mathbb{S}$	$\mathbb{K}$
JAN19	2663.4	247.12	-0.22502	6.2261
FEB19	2770.7	224.34	-0.21068	7.5266
MAR19	2817.6	235.66	-0.17142	9.477
APR19	2888.3	237.12	-0.15466	10.054
MAY19	2869.2	252.77	-0.181	7.417
JUN19	2947.1	267.23	-0.12689	8.7385
JUL19	2966.8	251.11	-0.19983	7.8517
AUG19	2881.9	284.5	-0.29874	4.819
SEP19	2977.8	268.65	-0.2798	6.3293
OCT19	2980	251.81	-0.28198	6.2956
NOV19	3112.8	244.64	-0.26659	7.1146
DEC19	3211.1	259.99	-0.21889	9.112
JAN20	3286.2	248.09	-0.28538	6.6169
FEB20	3360.4	263.42	-0.23003	8.1259
MAR20	2394.9	601.31	-0.63552	2.1926