LISBON SCHOOL OF ECONOMICS & MANAGEMENT
UNIVERSIDADE DE LISBOA

MASTER
MATHEMATICAL FINANCE

MASTER'S FINAL WORK
DISSERTATION

THE LOTKA-VOLTERRA EQUATIONS IN FINANCE AND ECONOMICS

MAFALDA OLIVEIRA MARTINS BASTOS DE ALMEIDA

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SUPERVISOR:
JOÃO LOPES DIAS

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Acknowledgments

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Resumo

As equações de Lotka-Volterra, também conhecidas por equações de predador-presa, são um conjunto de equações diferencias não-lineares construídas para descrever a relação dinâmica entre espécies na natureza. No entanto, desde a sua publicação vários autores têm vindo a provar que estes sistemas dinâmicos têm diversas aplicações fora da área da biologia. Este trabalho tem como objetivo aprofundar as possíveis aplicações destas equações ao sistema bancário e à economia. Considerando o sistema bancário, estudamos três possíveis sistemas dinâmicos que podem descrever a relação entre o volume de depósitos e empréstimos num banco. Também apontamos as semelhanças entre um sistema bancário de três níveis e uma cadeia alimentar e estudamos a sua estabilidade. Olhando para as aplicações à economia, começamos por estudar o famoso modelo de Goodwin para ciclos de desemprego e crescimento dos ordenados. Para terminar, apresentamos um par de equações predador-presa que descrevem a relação entre bens capitais e bens de consumo, e concluímos que os ciclos económicos são endógenos, auto-sustentáveis e não-lineares.

Palavras-chave: Lotka-Volterra, Predador-Presa, Ciclos Económicos, Sistema Bancário, Sistema Ecológico.

Abstract

The Lotka-Volterra equations, frequently referred to as predator-prey equations, are a set of non-linear differential equations constructed to describe the interaction dynamics between different species in nature. Yet, since their publication many authors have proved that the applications of these equations go way beyond mathematical biology. The present work focuses on their application to the banking system and to economics. Regarding the banking system, we study three dynamical systems that may describe the relationship between deposit and loan growth in a bank's balance sheet. In addition, we look at the resemblance between a three level ecological food chain and a three level banking system, and study its stability. As for the applications to economics, we study the famous Goodwin's model for the cyclic behavior of wages and employment. To finish our work we present a pair of predator-prey equations that model the dynamical relationship between consumption and capital goods, finding that economic cycles are endogenous, self-sustained and non-linear.

Keywords: Lotka-Volterra Equations, Predator-Prey Dynamics, Economic Cycles, Banking System.

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Chapter 1

Introduction

The Lotka-Volterra equations, often referred to as the predator-prey equations, are first order non-linear differential equations that describe the dynamics of populations in systems where multiple species, with distinct characteristics, interact.

In the United States, 1925, Alfred Lotka proposed a model to describe a chemical reaction with oscillating concentrations. A year later, in Italy, the mathematician Vito Volterra, when trying to explain the observed increase in predator fish (that caused a decrease in prey fish) in the Adriatic Sea during World War I, independently arrived to the same set of equations proposed by Lotka. This model proposed by Lotka and Volterra is considered to be the simplest model for predator-prey interactions.

If we aim to study the populations' dynamics in a single species environment, we should concentrate on factors such as the natural growth rate and the environment's carrying capacity. Yet, if we aim for a more realistic approach we need to study interacting populations and remember that they affect each other's growth rates. Interacting populations affect one another's evolution, and predicting the result of their relationship is of high interest to understand how communities are organized and sustained.

Ever since the publication of the predator-prey population model, authors have used these equations not only in the study of ecological systems but also in other scientific areas. In the present work, we focus on the applications of the famous Lotka-Volterra equations to economics and to the banking system.

To start, in Chapter 2 we mention some of the most important studies presented throughout time regarding the Lotka-Volterra equations and their possible applications.

In Chapter 3 we present the Lotka-Volterra equations in some of their most common forms: First, the predator-prey model for two interacting species followed by a generalization for all Lotka-Volterra equations. Then we present the population dynamics of two species competing for the same resources, later generalizing it to a competitive system of n species. In the third part of this chapter we present the case where species interact in a way that benefits one another, once again starting with the two-species case and extending the results to an environment of n species.

In Chapter 4 we focus our attention in the applications of the Lotka-Volterra equations to the banking system. First, we present a dynamical system identical to the two-species predator-prey model that describes the relationship between deposit and loan growth inside the capital structure of a bank, as well as some other models constructed for the same porpoise but with different features. In the second part of this chapter we present a model that compares a three level banking system to a three species ecological food chain.

In Chapter 5, we look closely at the application of Lotka-Volterra equations to economics. In the first part of this chapter, we present the original Goodwin model and some of its properties followed by an improved model proposed by Vadasz. We also study a model presented by Palomba that describes the dynamical relationship between consumption and capital goods.

Chapter 2

Literature Review

The Lotka-Volterra equations for predator-prey models are very famous in mathematics as well as in biology. The biologist Umberto D'Ancona suggested that the interactions between various animal species could be mathematically modeled, and this resulted in the publication of [1] and later [2] by the mathematician Vito Volterra. Among the several models proposed by Volterra there is the predator-prey case, the most famous one. Since Volterra did not know the work of Alfred J. Lotka, who had independently anticipated some of Volterra's results in [3], these equations were named Lotka-Volterra equations.

There are many applications of the predator-prey model, including in Game Theory. For example, in 2010 Chen [4] proposed a model for predation behavior applied to Game Theory and concluded that the smaller predators tend to use a more passive strategy than large predators, and that preys always prefer an active strategy.

The derivation of dynamical models based on bank profit were first proposed in 2008 by Petersen and Shoeman [5], who analyzed the Return-on-Assets and the Return-on-Equity of a bank. Later that year, the economic aspects of the stochastic dynamics model of a bank's assets and liabilities were presented on [6]. In 2012, Comes proposed a three dimensional dynamical model to describe the interaction between levels of the banking system [7] and used the results on the work of Apreutesei [8][9] to study its stability. In 2013, the first dynamical system of deposit and loan volumes based on the Lotka-Volterra predator-prey dynamics [10] was presented, and in 2014 Sumarti, Nurfitriyana and Nurwenda studied the equilibria of this type of dynamical systems [11].

The first application of Lotka-Volterra equations to economics was made in 1939 when Giuseppe Palomba used the predator-prey equations to model the dynamics between consumption and capital goods [12]. Several years later, in 1967, Goodwin presented a simple model that describes the dynamical relationship between real wages and real employment using the predator-prey dynamics [13], and since this publication economists have concentrated in two types of research: Authors such as Desai (1973) [14], Wolfstetter (1982) [15] and Sportelli (1995) [16] focused on the development of more complex models by relaxing some of the original assumptions. On the other hand, Velupillai (1979) [17] and Flaschel (1984) [18] investigated the stability and other mathematical properties of the model; In addition, authors like Atkinson (1969) [19] and Harvie (2000) [20] have focused on verifying the model's validity by testing it.

Apedaille (1994) used the predator-prey paradigm in the modeling of long run trajectories for the shares of agriculture, industrial and ecospheric wealth in open interacting economy [21]. Chakraborty (2011) analyzed the predator-prey fishery taking into account the variation of the economic interest of harvesting [22]. Finally, in 2012 Michalakelis estimated the evolution of market concentration in high technology saturated markets with dominant players, using Lotka-Volterra equations [23].

Chapter 3

Theoretical Framework

Lotka-Volterra systems of equations model the dynamics of n interacting species (and they can also be applied in the study of financial institutions and economic behavior) According to this model, the populations change through time according to a system of differential equations of the form

$$\dot{x}_i = x_i\Big(r_i + \sum_{j=1}^{n} a_{ij}x_j\Big), \qquad i = 1, ..., n \tag{3.1}$$

where $r_i \in \mathbb{R}$ and $A = (a_{ij})$ is a matrix of real entries.

In this chapter we will start by studying a general predator-prey system of equations for two interacting species and some of its important features. Later, we will return to the general form of a Lotka-Volterra system and focus our attention on two specific types of interaction between species: competitive and cooperative. For each of these cases we analyze the two species model followed by a generalization to a system of n species.

3.1 The Predator-Prey Model

In the early 90's Volterra presented a model to describe the evolution of predator and prey fish populations in the Adriatic Sea. He based his model in two assumptions: First, in the absence of predators the per capita growth rate of the prey population is constant and positive. Otherwise, it decreases linearly as function of the predator population; Second, in the absence of prey the per capita growth rate of the predator population is constant and negative. Otherwise, it increases linearly as a function of the prey population.

Let $N(t)$ denote the prey population and $P(t)$ the predator population at time $t \geq 0$. The previous assumptions translate into

$$\begin{aligned} \frac{1}{N}\frac{dN}{dt} &= a - bP \\ \frac{1}{P}\frac{dP}{dt} &= cN - d \end{aligned} \tag{3.2}$$

where $a, b, c, d > 0$ are constants and $\big(N(0), P(0)\big) = (N_0, P_0)$. Also, if we manipulate the previous equations we get

$$\frac{d}{dt}\big\{d\log N - cN + a\log P - bP\big\} = 0. \tag{3.3}$$

Let us define $H : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ such that

$$H(N, P) = d\log N - cN + a\log P - bP.$$

It is easy to see that $\frac{d}{dt}H(N, P) = 0$ i.e H is constant along the trajectories $\big(N(t), P(t)\big)$. Since we aim to study $H(N, P)$ depending on the initial condition (N_0, P_0), we split the problem in two cases.

First, we assume that $(N_0, P_0) \in \mathbb{R}^+ \times \mathbb{R}^+$, so we have the guarantee that $H(N_0, P_0)$ is finite and all trajectories $(N(t), P(t))$ evolve so that $H\big(N(t), P(t)\big) = H(N_0, P_0)$. This happens because $H(N, P)$ is a strictly concave function which implies that the orbits are periodic and that the function has a unique maximum where $\nabla H = 0$, i.e

$$(N, P) = \Big(\frac{d}{c}, \frac{a}{b}\Big).$$

For the particular case where $a = 2,\ b = 2.5,\ c = 2.3$ and $d = 2.7$ the stability of the predator-prey system can be found in figure 3.1.

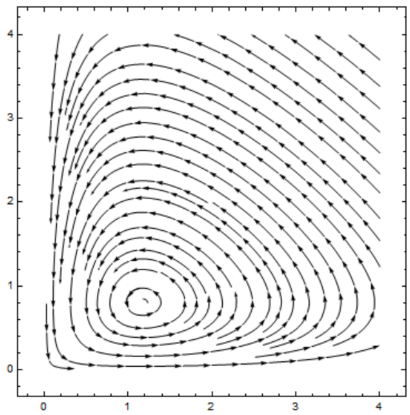

Figure 3.1: Stability of the Predator-Prey Model.

Second, we consider the possibility of either N_0 or P_0 being null, which allows us to obtain explicit solutions to the system in (3.1):

$$\begin{aligned} N_0 = 0 &\implies N(t) = 0\ ,\ P(t) = P_0e^{-dt} \\ P_0 = 0 &\implies N(t) = N_0e^{at}\ ,\ P(t) = 0 \end{aligned} \tag{3.4}$$

Therefore, as $t \to \infty$, the solution exponentially approaches the origin along the line $N = 0$ and exponentially goes to infinity along the line $P = 0$. Figure 3.2 shows the population of the predators across time in the absence of preys and figure 3.3 shows the evolution of the prey population in the absence of their predators.

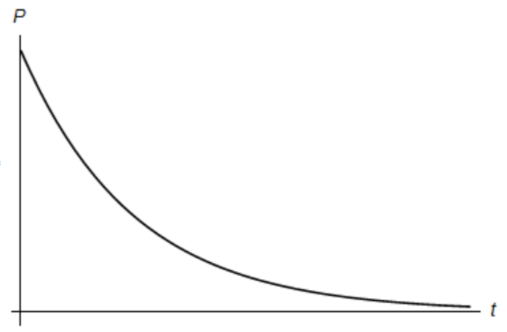

Figure 3.2: Predator population without Prey

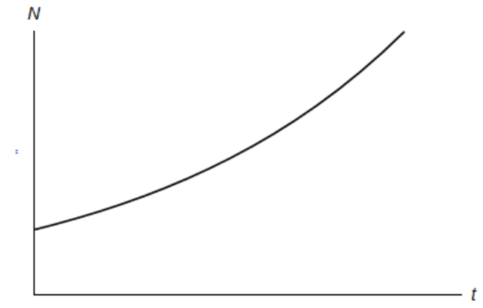

Figure 3.3: Prey population without Predators

After studying these two cases we can conclude that the model makes basic sense, since regardless of the choice of initial condition we obtain non-negative populations. At this point we find it useful to introduce Volterra's Principle: Suppose that $N_0 > 0$ and $P_0 > 0$. If T is the period of the closed orbit through (N_0, P_0) then the averages of $P(t)$ and $N(t)$ over T are given respectively by

$$\begin{aligned} \frac{1}{T}\int_0^T P(t)dt &= \frac{a}{b} \\ \frac{1}{T}\int_0^T N(t)dt &= \frac{d}{c}. \end{aligned} \tag{3.5}$$

Another important feature concerning the predator-prey equations is that system (3.2) is Hamiltonian, with H as its Hamiltonian function. Introducing the canonical coordinates $p = \log N$ and $q = \log P$ we get

$$H(N,P) = h(p,q) = dp - ce^p + aq - be^q$$

and the Lotka-Volterra equations are transformed into the canonical equations of Hamilton:

$$\begin{aligned} \frac{dp}{dt} &= \frac{1}{N}\frac{dN}{dt} = a - bP = a - be^q = \frac{dh}{dq} \\ \frac{dq}{dt} &= \frac{1}{P}\frac{dP}{dt} = cN - d = ce^p - d = -\frac{dh}{dp}. \end{aligned} \tag{3.6}$$

3.2 General Lotka-Volterra Systems

Before further discussing Lotka-Volterra systems, we need to present some important definitions and results.

Theorem 1 (Picard's existence theorem) Given an open set $U \subseteq \mathbb{R}^n$, a function $f : U \to \mathbb{R}^n$ that is locally Lipshitz in $x \in U$ and a point $x_0 \in U$, the differential equation $\dot{x} = f(x)$ with $x(t_0) = x_0$ has an unique solution $x : I \to U$ on some open interval I containing t_0.

Definition 1 We say that the vector field $f : U \to \mathbb{R}^n$ generates the flow $\varphi_t : U \to U$ where $\varphi_t(x) = \phi(x,t)$ for $x \in U$ and $t \in I = (a,b) \subseteq \mathbb{R}$ for some $a, b \in \mathbb{R}$ if

$$\left.\frac{d\phi(x,t)}{dt}\right|_{t=\tau} = f\big(\phi(x,\tau)\big),\ \forall x \in U,\ \tau \in I.$$

Definition 2 (Steady State) A Steady state of $\dot{x} = f(x)$ is a point $x \in U$ for which $f(x) = 0$.

Definition 3 (Forward invariant set) A set $S \subseteq U$ is a forward invariant set for φ_t if whenever $x \in S$ we have $\varphi_t(x) \in S$ for all $t \geq 0$.

Definition 4 (Omega limit point) A point $p \in U$ is an omega limit point of $x \in U$ if there are points $\varphi_{t_1}(x), \varphi_{t_2}(x), ...$ on the orbit of x such that $t_k \to +\infty$ and $\varphi_{t_k} \to p$ as $k \to \infty$.

Definition 5 (Alpha limit point) A point $p \in U$ is an alpha limit point of $x \in U$ if there are points $\varphi_{t_1}(x), \varphi_{t_2}(x), ...$ on the orbit of x such that $t_k \to -\infty$ and $\varphi_{t_k} \to p$ as $k \to \infty$.

Recall that the general Lotka-Volterra model is of the form (3.1). Applying Picard's existence theorem we conclude local existence and uniqueness of solutions for any initial condition. Suppose that $x(0) = (x_{01}, x_{02}, ..., x_{0n})$ has $x_{0k} = 0$ for $k \in J \subset \{1, ..., n\}$ so that some species are initially absent. Then, uniqueness of solution tells us that these species are absent for all t.

Theorem 2 For a model of the form (3.1) the coordinate axes and the subspaces spanned by them, and $(\mathbb{R}^+)^n$, are all forward invariant. In simpler words, populations that start non-negative remain non-negative throughout finite time.

Theorem 3 (Interior steady states) There exists an interior steady state $p \in (\mathbb{R}^+)^n$ if and only if (3.1) has omega or alpha limit points in $(\mathbb{R}^+)^n$.

Proof: The proof can be found in [24].

Theorem 4 (Time averages) Suppose that $x(t)$ is a periodic orbit of (3.1) of period T. If (3.1) has a unique interior steady state $x^* \in (\mathbb{R}^+)^n$, then

$$\frac{1}{T}\int_0^T x(t)dt = x^*.$$

Proof:

$$
\begin{aligned}
\dot{x}_i = x_i\Big(r_i + \sum_{j=1}^{n} a_{ij}x_j\Big) &\implies \frac{1}{T}\int_0^T \frac{\dot{x}_i(t)}{x_i(t)}dt = \frac{1}{T}\int_0^T r_i + \big(Ax(t)\big)_i dt \\
&\implies \frac{1}{T}\Big(\log x(T) - \log x(0)\Big) = r + \left[A\Big(\frac{1}{T}\int_0^T x(t)dt\Big)\right] \\
&\implies 0 = A^{-1}r + \frac{1}{T}\int_0^T x(t)dt \\
&\implies \frac{1}{T}\int_0^T x(t)dt = -A^{-1}r \\
&\implies \frac{1}{T}\int_0^T x(t)dt = x^* \quad \square
\end{aligned} \tag{3.7}
$$

3.3 Competitive Lotka-Volterra Systems

The competitive Lotka-Volterra equations describe a model for the population dynamics of species competing for some common resource. These equations cover both competition between members of the same species and competition between members of different species. While in the predator-prey equations the population model is exponential, in the competitive case it follows the logistic equations we will introduce throughout this section.

3.3.1 Two Species Dynamics

When considering an environment with a single population, the logistic equation is

$$
\frac{dN}{dt} = \rho N\Big(1 - \frac{N}{K}\Big), \tag{3.8}
$$

where $N(t)$ is the size of the population at time t, ρ is the per-capita growth rate and K is the maximum population density that the environment can carry, often referred to as the environmental carrying capacity. The quadratic term of the equation represents the competition between members of the same species for a common resource and we call this intraspecific competition. The previously presented logistic equation has an explicit solution of the form

$$
N(t) = \frac{N_0}{\frac{N_0}{K} + \big(1 - \frac{N_0}{K}\big)e^{-\rho t}}
$$

which implies that as $t \to \infty$ the population approaches its carrying, regardless the initial condition N_0. This tendency is illustrated in figure 3.4.

To write a model for competition in an environment with two species we need to take into account, as well as the intraspecific competition mentioned before, the competition between different species denoted as interspecific competition. Therefore we need two logistic equations similar to (3.8), one for the population

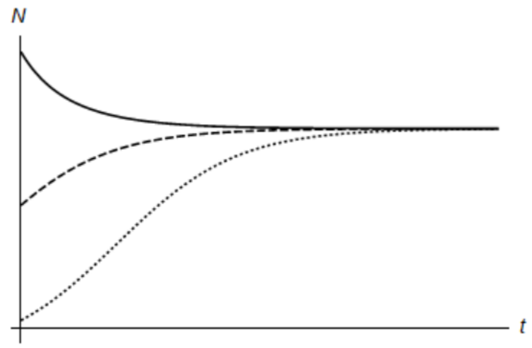

Figure 3.4: Unidimentional Logistic Equations.

of each species and both with an extra term representing the interspecific competition. This new model writes

$$\begin{aligned}\frac{dN_1}{dt} &= \rho_1 N_1\Big(1-\frac{N_1}{K_1}-c_1N_2\Big)\\ \frac{dN_2}{dt} &= \rho_2 N_2\Big(1-\frac{N_2}{K_2}-c_2N_1\Big)\end{aligned} \tag{3.9}$$

where $c_1, c_2 > 0$ are the relative sizes that measure the aggressiveness of the competition between the two species. Aiming to study the solution of (3.9) we define $N_1(0) = N_{10}$ and $N_2(0) = N_{20}$, and split the problem in two cases.

First, we consider the case where either N_{10} or N_{20} is null. Using the solution for the single-species logistic equation, we conclude that in the absence of one species the other approaches exponentially its carrying capacity:

$$\begin{aligned} N_{10} = 0 &\implies N_1(t) = 0\ ,\ N_2(t) = \frac{N_{20}}{\frac{N_{20}}{K} + \big(1-\frac{N_{20}}{K}\big)e^{-\rho_2 t}}\\ N_{20} = 0 &\implies N_1(t) = \frac{N_{10}}{\frac{N_{10}}{K} + \big(1-\frac{N_{10}}{K}\big)e^{-\rho_1 t}}\ ,\ N_2(t) = 0.\end{aligned} \tag{3.10}$$

In the second case we assume both N_{10} and N_{20} to be positive numbers. For simplicity of calculations we set new variables and parameters: $u_1 = \frac{N_1}{K_1}$; $u_2 = \frac{N_2}{K_2}$; $a_{12} = c_1K_2$; $a_{21} = c_2K_1$; $\tau = \rho_1 t$; $\rho = \frac{\rho_1}{\rho_2}$. This gives us a new set of equations with less parameters but same behavior as before:

$$\begin{aligned}\frac{du_1}{d\tau} &= u_1(1-u_1-a_{12}u_2)\\ \frac{du_2}{d\tau} &= \rho u_2(1-u_2-a_{21}u_1).\end{aligned} \tag{3.11}$$

Note that the Jacobian of this system has sign structure

$$J = \begin{bmatrix} * & \leq 0\\ \leq 0 & * \end{bmatrix}$$

more specifically, it is of the form

$$J = \begin{bmatrix} 1 - 2u_1 - a_{12}u_2 & -a_{12}u_1 \\ -\rho a_{21} & \rho(1 - 2u_2 - a_{21}u_1) \end{bmatrix}$$

Our first step in studying (3.11) is finding the lines on which $\dot{u}_1 = 0$ and $\dot{u}_2 = 0$:

$$\begin{aligned} u_1 = 0 \quad &\vee \quad 1 - u_1 - a_{12}u_2 = 0 \\ u_2 = 0 \quad &\vee \quad 1 - u_2 - a_{21}u_1 = 0 \end{aligned} \tag{3.12}$$

and these conditions allow us to identify the equilibrium points of the system:

$$(0,0); \quad (1,0); \quad (0,1); \quad P = \left(\frac{1-a_{12}}{1-a_{12}a_{21}}, \frac{1-a_{21}}{1-a_{12}a_{21}}\right).$$

The second step in studying (3.11) is to assure that the coordinates of the steady points we found do not contradict the non-negativity of the populations. The first three points present no problem since they are in the non-negative part of the coordinate axes regardless the choice of parameters. As for P, we need to look at the values of a_{12} and a_{21}.

The third and final step consists in studying the stability of each of the equilibrium points of (3.11), depending on the choice of parameters:

- **Case 1:** $a_{12}, a_{21} < 1;$
  The steady state P is stable, $(0,0)$ is unstable and both $(1,0)$ and $(0,1)$ are saddles. These small values for a_{12} and a_{21} represent a competition that is not too aggressive and results in both populations coexisting in an equilibrium and never reaching their respective carrying capacities, as it shows in figure 3.5.

- **Case 2:** $a_{12}, a_{21} > 1;$
  The steady states P and $(0,0)$ are unstable nodes. Here, both $(1,0)$ and $(0,1)$ are stable and there is an invisible line that separates the plane in two regions. Above this line all trajectories go to $(1,0)$, otherwise they are attracted to $(0,1)$. Orbits that start in the separation line converge to the unstable node P. The higher values for a_{12} and a_{21} considered in this case represent strong interspecific competition which leads to one of the species eventually winning and the other being driven to extinction. The winner depends upon which has the starting advantage, as is shown on figure 3.6.

- **Case 3:** $a_{12} < 1, a_{21} > 1;$
  In this case there is no interior steady state P. The states $(0,0)$ and $(0,1)$ are unstable but $(1,0)$ is stable and attracts all trajectories. Here we have a_{12}, which represents how competitive species 2 is towards species 1, smaller that a_{21}. As it is illustrated in figure 3.7. the less competitive species is driven to extinction.

- **Case 4:** $a_{12} > 1, a_{21} < 1$;
  Once again there is no interior steady state P. Here $(0,0)$ and $(1,0)$ are unstable but $(0,1)$ is stable and all interior trajectories go to this state. In this case species 2 is more competitive and figure 3.8 shows that this leads to the extinction of species 1.

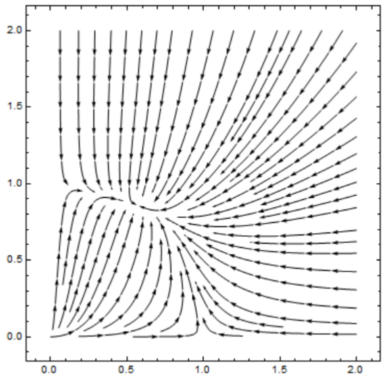

Figure 3.5: $a_{12} = 0.5$ and $a_{21} = 0.25$

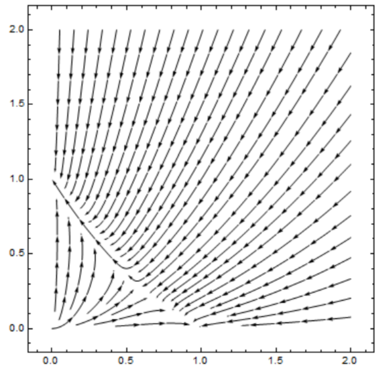

Figure 3.6: $a_{12} = 1.5$ and $a_{21} = 1.2$

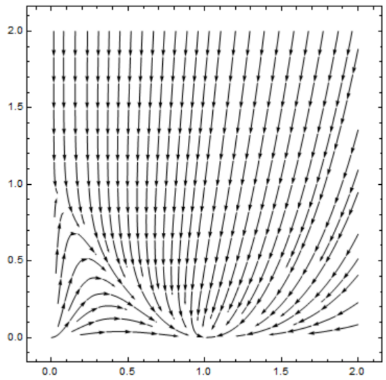

Figure 3.7: $a_{12} = 0.5$ and $a_{21} = 2.1$

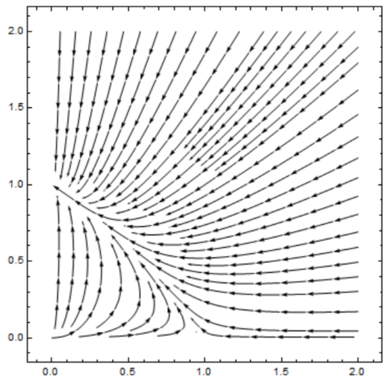

Figure 3.8: $a_{12} = 1.5$ and $a_{21} = 0.5$

3.3.2 N Species Dynamics

Now we consider the Lotka-Volterra system

$$\dot{x}_i = x_i\Big(r_i - \sum_{j=1}^{n} a_{ij}x_j\Big), \qquad i = 1, ..., n \tag{3.13}$$

under the special condition that $a_{ij} > 0$ for all $i, j \in \{1, ..., n\}$. This means that each species competes with all other species and that individuals of the same species compete with each other. Note that if $r_i \leq 0$, even in the absence of competitors, species i would lead itself to extinction. For that reason, from now on we consider $r_i > 0$ for each $i = 1, ..., n$. Also, note that the Jacobian of system (3.13) has sign structure:

$$J = \begin{bmatrix} * & \leq & \leq & \dots & \leq \\ \leq & * & \leq & \dots & \leq \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \leq & \leq & \leq & \dots & * \end{bmatrix}$$

Lemma 1 Since $a_{ij} > 0$ and $r_i > 0$, all orbits of (3.13) are bounded.

Proof: We know that $(\mathbb{R}_0^+)^n$ is invariant. Also, from

$$\begin{aligned} \dot{x}_i = x_i\Big(r_i + \sum_{j=1}^{n} a_{ij}x_j\Big) &\implies r_i x_i - x_i \sum_{j=1}^{n} a_{ij}x_j \\ &\implies \dot{x}_i \leq r_i x_i - a_{ii}x_i^2 \\ &\implies \dot{x}_i < 0 \text{ if } x_i > \frac{r_i}{a_i i} \end{aligned} \tag{3.14}$$

we can conclude that for each $i = 1, ..., n$ the population x_i is bounded. □

Lemma 2 Under the assumption that

$$\begin{cases} \frac{r_j}{ajj} < \frac{r_i}{a_{ij}}, & \text{if } 1 \leq i < j \leq n \\ \frac{r_j}{a_{jj}} > \frac{r_i}{a_{ij}}, & \text{if } n \geq i > j \geq 1 \end{cases}$$

the competitive system (3.13) has no interior steady state.

Proof: According to [24], if x^* is an interior steady state then:

$$\frac{a_{i1}}{r_i}x_1^* + \frac{a_{i2}}{r_i}x_2^* + ... + \frac{a_{in}}{r_i}x_n^* = 1, \quad i = 1, ..., n.$$

Therefore, for each i we can write

$$\left(\frac{a_{11}}{r1} - \frac{a_{i1}}{r_i}\right)x_1^* + \left(\frac{a_{12}}{r1} - \frac{a_{i2}}{r_i}\right)x_2^* + ... + \left(\frac{a_{1n}}{r1} - \frac{a_{in}}{r_i}\right)x_n^* = 0$$

and if we specify $i = n$ we get

$$\left(\frac{a_{11}}{r1} - \frac{a_{n1}}{r_n}\right)x_1^* + \left(\frac{a_{12}}{r1} - \frac{a_{n2}}{r_n}\right)x_2^* + ... + \left(\frac{a_{1n}}{r1} - \frac{a_{nn}}{r_n}\right)x_n^* = 0.$$

From the second equation on Lemma 2, we know that the only x^* that verifies this equality is the trivial equilibrium $x^* = 0$. Therefore, the system (3.12) has no interior steady states. □

Theorem 5 (Extinction in competitive Lotka-Volterra) Under the assumptions on Lemma 2, $(\frac{r_1}{a_{11}}, 0, ..., 0)$ is globally attracting on $(\mathbb{R}^+)^n$.

3.4 Cooperative Lotka-Volterra Systems

In contrast with the previous section, here we study the dynamics of an environment where species cooperate with each other but elements of the same species still compete for the same resources. Cooperative Lotka-Volterra logistic equations model the density of a pool of species taking into account that they work together for common benefit. This phenomenon, when considering only two species, is called mutualism.

3.4.1 Two Species Dynamics

Mutualism is the way two organisms of different species exist in a relationship where each individual benefits from the activity of the other. It contrasts with interspecific competition, where one species benefit at the "expense" of the other. The logistic equations for the case of mutualism are similar to the ones in the case of competition (3.9), the only difference being the sign of the interaction terms. For this type of problem we have

$$\begin{aligned} \frac{dN_1}{dt} &= \rho_1 N_1 \Big(1 - \frac{N_1}{K_1} + c_1 N_2\Big) \\ \frac{dN_2}{dt} &= \rho_2 N_2 \Big(1 - \frac{N_2}{K_2} + c_2 N_1\Big). \end{aligned} \tag{3.15}$$

Allowing one species to start with null population implies that it will remain null for all time, which leads us to the solution in (3.10) once again. On the other hand, when considering initial populations N_{10} and N_{20} positive we introduce the same parameters and variables as we did when simplifying the competitive two-species dynamics and obtain

$$\begin{aligned} \frac{du_1}{d\tau} &= u_1(1 - u_1 + a_{12}u_2) \\ \frac{du_2}{d\tau} &= \rho u_2(1 - u_2 + a_{21}u_1) \end{aligned} \tag{3.16}$$

with steady states

$$(0,0); \quad (1,0); \quad (0,1); \quad P = \Big(\frac{1 + a_{12}}{1 - a_{12}a_{21}}, \frac{1 + a_{21}}{1 - a_{12}a_{21}}\Big).$$

Note that P only exists when $a_{12}a_{21} < 1$, since we need to guarantee the non-negativity of the populations. The Jacobian of this system has sign structure

$$J = \begin{bmatrix} * & \geq 0 \\ \geq 0 & * \end{bmatrix}$$

more specifically

$$J = \begin{bmatrix} 1 - 2u_1 + a_{12}u_2 & a_{12}u_1 \\ \rho u_2 a_{21} & \rho(1 - 2u_2 + a_{21}u_1) \end{bmatrix}$$

To study the stability of the two species cooperative equilibrium we consider the following cases:

- **Case 1:** $a_{12}a_{21} < 1$;
  There are four steady states and all trajectories converge to the stable node P, the only one which does not lie on the coordinate axes. This behavior holds regardless of the choice of a_{12} and a_{21}, which is illustrated in figures 3.9 and 3.10.

- **Case 2:** $a_{12}a_{21} > 1$;
  There are only three steady states : $(0,0)$, $(0,1)$ and $(1,0)$. In the absence of an interior state P all orbits diverge to infinity as can be observed in figures 3.11 and 3.12. This behavior holds for all values of a_{12} and a_{21}.

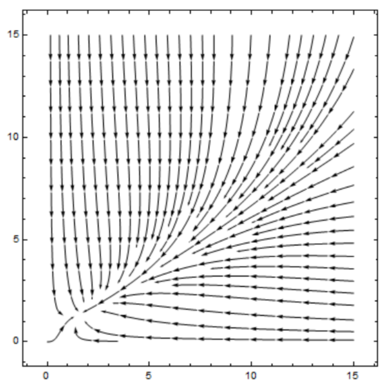

Figure 3.9: $a_{12} = 0.5$ and $a_{21} = 0.25$

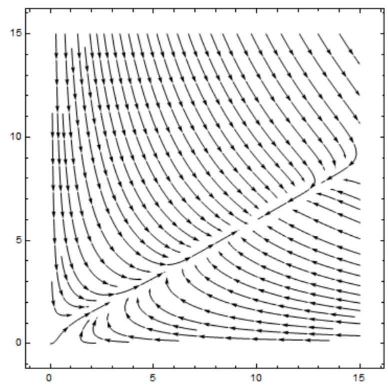

Figure 3.10: $a_{12} = 1.5$ and $a_{21} = 0.5$

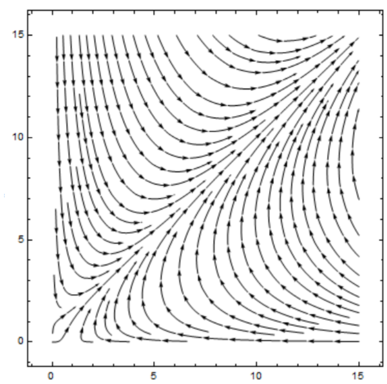

Figure 3.11: $a_{12} = 1.5$ and $a_{21} = 1.2$

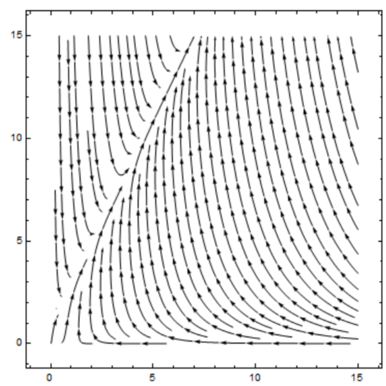

Figure 3.12: $a_{12} = 0.5$ and $a_{21} = 2.1$

3.4.2 N Species Dynamics

To study Lotka-Volterra cooperative systems for the interaction of n species, we simply write

$$\dot{x}_i = x_i\Big(r_i + \sum_{j=1}^{n} a_{ij}x_j\Big), \qquad i = 1, ..., n \tag{3.17}$$

where $r_i \in \mathbb{R}$, $A = (a_{ij})$ is a matrix of real entries and $a_{ij} \geq 0$ when $i \neq j$. This is very similar to the general case, but with the restriction that the term representing interspecific interaction is non-negative due to each species benefiting from the existence of the others. Therefore, the interaction matrix A has off-diagonal elements greater or equal than zero and the Jacobian of (3.17) has the sign structure :

$$J = \begin{bmatrix} * & \geq & \geq & \cdots & \geq \\ \geq & * & \geq & \cdots & \geq \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \geq & \geq & \geq & \cdots & * \end{bmatrix}$$

Definition 6 (Cooperative matrix) We say that any real $n \times n$ matrix with the previous sign structure is cooperative.

Definition 7 (Negatively diagonally dominant) A matrix A is negatively diagonally dominant if there exists $d \in \mathbb{R}^n$ such that $d_i > 0$ and $a_{ii}d_i + \sum_{i \neq j} |a_{ij}d_j| < 0$ for all $i = 1, ..., n$. When A is cooperative, this translates to $(Ad)_i < 0$ for all $i = 1, ..., n$.

Definition 8 (Stable Matrix) A square matrix A is said to be stable if every eigenvalue of A has strictly negative real part.

Lemma 3 Let A be a cooperative matrix. Then A is stable if and only if it is negatively diagonally dominant.

Proof: First, let us assume that A is a stable cooperative matrix. Then, for a sufficiently large positive c we get a non-negative matrix of the form $B = A + cI$. Therefore, by the Perron-Frobenius theorem we can guarantee the existence of a spectral radius $\lambda = \rho(B) \geq 0$ and $v > 0$ such that $Bv = \lambda v = \rho(B)v$, which means $Av = (\rho(B) - c)v$ where $\rho(B) < c$ (because we are assuming A to be a stable matrix). Then, the series

$$A^{-1} = -\frac{1}{c}\Big(I + \frac{1}{c}B + \frac{1}{c^2}B^2 + ...\Big)$$

converges and all tranches of the sum are ≤ 0. If we set $d = -A^{-1}(1, ..., 1)^T$, it is easy to see that $d > 0$ and that $Ad = -(1, ..., 1)^T < 0$ meaning that A is negatively diagonally dominant.

On the other hand, assuming that A is negatively diagonally dominant, we can guarantee the existence of a vector $d > 0$ such that $Ad < 0$. Note that $a_{ii} < 0$ for each $i = 1, ..., n$. Let λ be an eigenvalue of

A with an eigenvector x to the right. Also, let $y_i = \frac{x_i}{d_i}$ and we choose m such that $|y_m| = \max_i |y_i| > 0$. Then,

$$\lambda d_i y_i = \sum_{j=1}^{n} a_{ij} d_j y_j$$

and in particular, taking $i = m$ and dividing by y_m we can write

$$\lambda d_m = d_m a_{mm} + \sum_{j \neq m}^{n} d_j a_{mj} \frac{y_j}{y_m}.$$

Since by hypothesis we have that

$$a_{ii} d_i + \sum_{i \neq j} |a_{ij} d_j| < 0$$

we can conclude

$$|\lambda d_m - d_m a_{mm}| \leq \sum_{j \neq m}^{n} d_j a_{mj} \Big| \frac{y_i}{y_m} \Big| \leq \sum_{j \neq m}^{n} d_j a_{mj} < -d_m a_{mm}$$

which means that $|\lambda - a_{mm}| < -a_{mm}$, implying that for every choice of λ it has negative real part. Therefore, the cooperative matrix A is stable. $\square$

Corollary 1 If A is cooperative and $r_i > 0$ for all $i = 1, ..., n$ then $Ax + r = 0$ has an unique interior solution $x \in (\mathbb{R}^+)^n$ if and only if A is stable.

Definition 9 (Lyapunov stability) A steady state x^* is said to be Lyapunov stable if for any $\epsilon > 0$ there is $\delta > 0$ such that for all x_0 with $|x^* - x_0| < \delta$ we have $|\varphi(x_0, t) - x^*| < \epsilon$ for all $t \geq 0$. A steady state is said to be unstable if it is not Lyapunov stable.

Definition 10 (Asymptotic stability) A steady state x^* is said to be locally asymptotically stable if it is Lyapunov stable and there is $\rho > 0$ such that for all x_0 with $|x^* - x_0| < \rho$ we have $|\varphi(x_0, t) - x^*| < \epsilon$ for all $t \geq 0$.

Theorem 6 (Global convergence for cooperative Lotka-Volterra) Suppose that the system (3.15), with each $r_i > 0$, has an unique interior steady state x^* and that A has non-negative off diagonal elements. Then, x^* is globally asymptotically stable on $(\mathbb{R}^+)^n$ and all (boundary) orbits are uniformly bounded as $t \to \infty$.

Proof: The proof can be found in [24].

Chapter 4

Lotka-Volterra Equations in the Banking System

Banking regulations can be governmental and non-governmental and they are managed with specific requirements, restraints and guidance. This guarantees transparency between banks and the institutions with whom they work. Financial institutions have high influence on the economy and their behavior can have great impact regarding global instability.

4.1 Deposit and Loan Growth

Banks use the deposits made by clients as funds to administer loans and they benefit from the difference between the interest rates agreed to on the deposits and loans. For this reason, it is important to understand the interaction between deposit and loan volume.

Sumarti, Nurfitriyana and Nurwenda (2014) [10] proposed a dynamical system for deposits and loan volumes in a bank using the predator-prey equations. They argued that, as the existence of predators depends on the existence of prey, the existence of loans depends on the existence of deposits since the bank's loan volume is defined as a portion of its deposit volume.

In a bank's balance we find its liabilities, assets and equity. The bank's liabilities are the obligations that must be paid in the future, and the bank's assets consist on the investments and resources expected to give income in the future. The bank's equity is calculated by making the difference between its assets and liabilities.

The main liabilities of a bank are the deposits made by costumers (D) while its main assets are the loans (L), the primary reserve (R_1) and the secondary reserve (R_2). Their interaction is described by

$$L + R_1 + R_2 = D \tag{4.1}$$

which means that there is a relationship between the volume of loans and reserves, and the deposits. Next, we will explore this dependence starting by comparing it with a predator-prey interaction where the loans and reserves represent the predators and the deposits represent the prey.

4.1.1 A Simple Model

The model proposed in [10] for the interaction between loan and deposit volumes based on the Lotka-Volterra predator-prey equations writes:

$$\begin{aligned} \frac{1}{D}\frac{dD}{dt} &= \alpha - pL \\ \frac{1}{L}\frac{dL}{dt} &= pD - \beta \end{aligned} \tag{4.2}$$

where α is the interest rate of the deposit and β is the interest rate of the loan, which makes both parameters positive constants. Furthermore, p is the maximum mixture rate between deposit and loan volumes. Looking at this system we can observe that an increase in α will cause and increase in deposit volume, as an increase in β will cause a decrease in loan volume. On the other hand an increase on pDL, which represents the mixture between the deposit and loan volumes, will cause a decrease in the growth of deposit volume and an increase in loan volume growth though time.

System (4.2) has three equilibrium points:

$$P_1 = (0,0); \quad P_2 = \left(\frac{\beta}{p}, \frac{\alpha}{p}\right);$$

In the context of our problem, P_1 is a saddle point and P_2 is a stable node if $\alpha < \beta$, i.e the interest rate on the deposit is lower than the interest rate of the loan, which makes sense according to reality.

Assuming $\alpha = 0.3$, $\beta = 0.5$ and $p = 20$ the stability of this model for the dynamics of deposit and loan growth is illustrated in figure 4.1

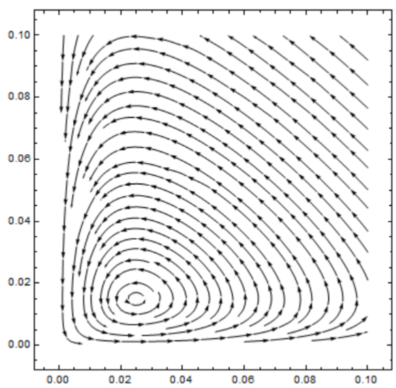

Figure 4.1: Stability of the Simple Model for Deposit and Loan Growth.

Note that, just like the predator-prey system in (3.2), this model can be written in Hamiltonian coordinates. Considering $m = \log D$ and $n = \log L$ and denoting by H the Hamiltonian function we can write

$$H(D, L) = h(m, n) = \beta m - pe^m + \alpha n - pe^n.$$

Then, the equations in (4.2) become

$$\begin{aligned} \frac{dm}{dt} &= \frac{1}{D}\frac{dD}{dt} = \alpha - pL = \alpha - pe^n = \frac{dh}{dn} \\ \frac{dn}{dt} &= \frac{1}{L}\frac{dL}{dt} = pD - \beta = pe^m - \beta = -\frac{dh}{dm}. \end{aligned} \tag{4.3}$$

which proves that the system is Hamiltonian.

4.1.2 A model with Michaelis-Menten Response

Kar (2005) [25] proposed a model on this subject based on a study presented by Michaelis and Menten (1913) [26] concerning the saturation curve for enzyme reactions. Kar's model was based on the assumption that the deposit volume is limited and the loan volume approaches a constant as the deposits increase. Note that, under these conditions, the model no longer follows a Lotka-Volterra system of equations:

$$\begin{aligned} \frac{dD}{dt} &= \alpha D\Big(1 - \frac{D}{k}\Big) - \frac{pDL}{1 + bD} \\ \frac{dL}{dt} &= \frac{pDL}{1 + bD} - \beta L \end{aligned} \tag{4.4}$$

where α, β, k, p and b are positive constants. Furthermore, k is the carrying capacity of the deposit volume and $\frac{p}{b}$ is the maximum portion of the deposits that can be invested in the loans.

This model has three equilibrium points:

$$P_1' = (0, 0); \quad P_2' = (k, 0); \quad P_3' = \left(\frac{\beta}{p - b\beta}, \frac{\alpha\big(pk - (1 + bk)\beta\big)}{k(b\beta - p)^2}\right);$$

Here P_1' is once again a saddle point and P_2' is stable if $\beta = \frac{pk}{1+bk}$, i.e the interest rate of the loan will rise with an increase in the deposit's carrying capacity k. Furthermore, P_3' only exists if $\beta \neq \frac{p}{b}$ and is a stable equilibrium if

$$\frac{\alpha}{2pk(b\beta - p)}\big((bk + 1)b\beta^2 + (1 - bk)p\beta\big) < 0$$

and

$$\beta < \frac{p(bk - 1)}{b(bk + 1)}.$$

4.1.3 A model with Reserve Requirement

Kar (2005) [25] also proposed a model including reserve requirements, i.e requirements regarding the amount of cash a bank must hold in reserve considering the deposits made by customers. This money must be in the bank's vaults or at a Federal Reserve bank. If we consider mD the protected reserve, the model is of the form

$$\begin{aligned} \frac{dD}{dt} &= \alpha D\Big(1-\frac{D}{k}\Big) - \frac{p(1-m)DL}{1+b(1-m)D} \\ \frac{dL}{dt} &= \frac{p(1-m)DL}{1+b(1-m)D} - \beta L \end{aligned} \tag{4.5}$$

with equilibrium points:

$$P_1'' = (0,0); \quad P_2'' = (k,0); \quad P_3'' = \left(\frac{\beta}{(1-m)(p-b\beta)}, \frac{\big((bk(1-m)-1)\beta + (1-m)pk\big)\alpha}{k(1-m)^2(b\beta-p)^2}\right);$$

As in the previous models P_1'' is a saddle point. On the other hand, P_2'' is a stable equilibrium as long as we guarantee that, for $\alpha < 1$

$$\beta > \frac{k(1-m)p}{1+k(1-m)b}.$$

Furthermore, P_3'' will be stable if

$$\frac{\alpha\Big(\big(-(1-m)b^2k - b\big)\beta^2 + \big((1-m)pbk - p\big)\beta\Big)}{2(1-m)pk(b\beta+p)} < 0$$

and

$$\beta < \frac{p\big((1-m)bk-1\big)}{b\big((1-m)bk+1\big)}.$$

4.2 Three Level Banking System

Thompson (2011) [27] argued that the financial system behaves like an ecological network in the following way: "Considering a three level food chain in ecological systems, we have biomass transfer from Herbivorous to Carnivorous and from plants to Herbivorous. On the other hand, in a financial system we have capital transfer from Mother Bank to Subsidiary Bank, and from Subsidiary Bank to Individuals or Companies." A subsidiary bank is a type of foreign bank that is located in a country but is owned by a foreign mother bank. Subsidiary banks are regulated by their mother banks and at the same time have to perform under the rules of the host country.

If we consider $x_1(t)$ and $x_2(t)$ the number of subsidiary and mother banks respectively, and $x_3(t)$ the number of clients of the subsidiary banks, the banking system in figure 4.3 translates into the following

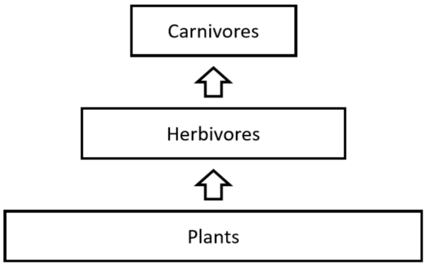

Figure 4.2: Three Level Ecological System

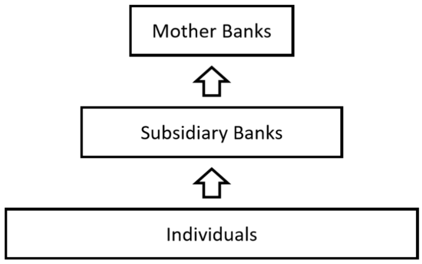

Figure 4.3: Three Level Banking System

dynamics:

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)(a_1 - b_1x_2(t) + c_1x_3(t)) \\ \dot{x}_2(t) &= x_2(t)(-a_2 + b_2x_1(t)) \\ \dot{x}_3(t) &= x_3(t)(a_3 - b_3x_1(t)) \end{aligned} \tag{4.6}$$

where $a_i, b_i, c_i > 0,\ i \in \{1, 2, 3\}$. This system represents a three-level banking system where a top predator Mother Bank ”feeds” on a intermediate consumer Subsidiary Bank , which ”feeds” on the Individuals that use its services. Furthermore, from Haimovici (1980) [28] and Apreutesei (2006) [9] we know that for every initial solution $(x_1(0), x_2(0), x_3(0))$ system (4.6) has an unique solution which is continuous in $\mathbb{R}_0^+$.

To study system (4.6) we start by solving each planar system in its respective coordinate plane. First, we notice that in the absence of mother banks $(x_2 = 0)$ the equations are reduced to a two-species predator prey model similar to the one studied in Section 3.1. In this case, we have closed orbits around the equilibrium $\left(\frac{a_1}{b_1}, \frac{a_3}{b_3}, 0\right)$ for all possible values of the parameters.

When considering the solution of (4.6) on the plane $x_1 = 0$ (when the subsidiary banks are absent from the system) we arrive to the system

$$\begin{aligned} \dot{x_1}(t) &= 0 \\ \dot{x_2}(t) &= -a_2x_2 \\ \dot{x_3}(t) &= a_3x_3 \end{aligned} \tag{4.7}$$

The equation $\dot{x_2}(t) = -a_2x_2$ implies that $x_2(t) \to 0$ exponentially as $t \to \infty$, while $\dot{x_3}(t) = a_3x_3$ means that $x_3(t) \to \infty$ exponentially as $t \to \infty$. The behavior of the three level system in the absence of subsidiary banks is illustrated in figure 4.4, for the particular case where $a_2, a_3 = 1$.

As for the scenario where the individuals/companies that use the services of the subsidiary banks are missing from the equations $(x_3 = 0)$, the system we need to solve becomes

$$\begin{aligned} \dot{x_1}(t) &= x_1(-a_1 - c_1 x_2) \\ \dot{x_2}(t) &= x_2(-a_2 + b_2 x_1) \\ \dot{x_3}(t) &= 0 \end{aligned} \tag{4.8}$$

Looking at system (4.8), we can easily see that $\dot{x_1} \leq -a_1 x_1$. This means that, as t grows to infinity, $x_1(t)$ will become null causing $x_2(t)$ to exponentially decay to zero. This tendency makes sense when we think of the three level banking system: in the absence of clients for the subsidiary banks, they will disappear since there is no demand for their services and that will cause the mother banks to also go extinct.

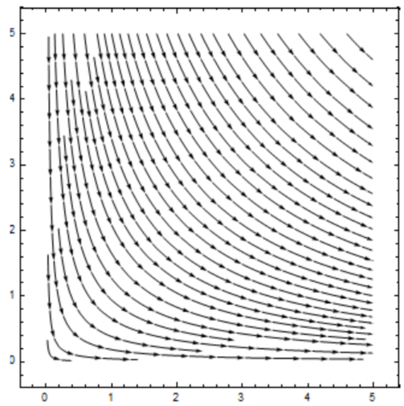

Figure 4.4: Solution of System (4.6) on the plane $x_1 = 0$.

When studying the equilibrium of system (4.6), we first need to find the steady states by solving

$$\begin{aligned} \dot{x_1}(t) &= 0 \\ \dot{x_2}(t) &= 0 \\ \dot{x_3}(t) &= 0. \end{aligned} \tag{4.9}$$

In the context of our problem, we find two equilibrium points:

$$P_1 = (0,0,0); \quad P_2 = \Big(\frac{a_2}{b_2}, \frac{a_1}{b_1}, 0\Big);$$

Next, to find the stability of this steady states, we compute the Jacobian of (4.6):

$$J = \begin{bmatrix} a_1 - b_1 x_2 + c_1 x_3 & -b_1 x_1 & c_1 x_1 \\ b_2 x_2 & -a_2 + b_2 x_1 & 0 \\ -b_3 x_3 & 0 & a_3 - b_3 x_1 \end{bmatrix}$$

Looking at J, we conclude that P_1 is an unstable equilibrium and that P_2 is stable if $a_3 b_2 - a_2 b_3 < 0$, otherwise it is also unstable.

Chapter 5

Lotka-Volterra Equations in Economics

5.1 Goodwin's Model

The behavior of economic systems has been described in three different ways in the literature. The first models considered the markets to be in a stable equilibrium, i.e even with random shocks the equilibrium is always be restored. Later in time, models started to be constructed based on the assumption that growth is cyclical and its equilibrium is affected by past changes. More recently, with the arrival of modern statistical methods, economists found that random shocks create what features chaotic behavior: economic relations resemble white noise and economic motion is random.

Goodwin (1967) [13] presented a model describing the dynamic relationship between wages and employment. Later on, this model was improved so it would incorporate the three behaviors of economic systems mentioned before: The economy would have stable wages and employment but small perturbations could lead cycles. Furthermore, a drastic change would cause the economy exhibit chaotic behaviour.

Throughout this section we will study Goodwin's original model as well as some of its shortfalls, and to finish we will present an improved model that incorporates some important features missing in the original one.

5.1.1 The Original Model

Goodwin's economic model resembles the Lotka-Volterra predator-prey model, where wages correspond to the predators and employment to the prey. When the employment levels are high, the employed workers have more bargaining power which rises the wages and diminishes the profits. This decrease of profits will cause less workers to be hired and employment will fall, which then leads to an increase of profits. With more profit more workers will be hired causing a rise in employment levels and so a cycle arises.

These cycles are not the same as business cycles but they are related, since a recession can affect the employment cycle and,contrarily, changes in wages can cause a recession. Goodwin's main goal with his study is to understand the cyclical behavior of employment, using the predator-prey equations to dynamically model income distribution and employment levels.

Goodwin's model has its origin in an idea in Marx (1887) [29]. Marx believed that "capitalism's alternate ups and downs are a result of the dynamic interaction between profits, wages and employment". To mathematically express this variations, Goodwin used the predator-prey equations and wrote his model based on the following assumptions:

Assumption 1: Both technological progress and growth in labor force are constant.

Assumption 2: There are only two factors of production (labor and capital).

Assumption 3: All quantities mentioned are real and net.

Assumption 4: All wages are consumed and all profits are saved and invested.

Assumption 5: The ratio between capital and output is constant.

Assumption 6: The real wage rate rises in the neighborhood of full employment.

Constant technological progress means growth in labor productivity of the form

$$a = a_0 e^{\alpha t}, \quad \alpha > 0$$

where a is the labor productivity that grows at a constant rate α. On the other hand, assuming constant growth in labor force we can write

$$n = n_0 e^{\beta t}, \quad \beta > 0$$

where n represents the labor force growing at a constant rate β. Also, denoting as k the capital and as q the output, we obtain a constant capital-output ratio of

$$\sigma = \frac{k}{q}.$$

If we consider w the wage rate, then the worker's share of output will be given by

$$u = \frac{w}{a}$$

which makes the capitalists' share of output $1 - u$. As was assumed, all wages are consumed and all profits are saved and invested, meaning that the growth rate of capital is the same as investment which is equal to the profit and therefore

$$\dot{k} = (1-u)q.$$

Through some easy computations we find that the growth rate of capital over time is of the form

$$\frac{\dot{k}}{k} = \frac{(1-u)}{\sigma}$$

and since we have a fixed capital-output ratio the growth rate of output over time will be the same:

$$\frac{\dot{q}}{q} = \frac{(1-u)}{\sigma}.$$

At this point it makes sense to consider the employment as

$$l = \frac{q}{a}$$

and after some manipulation of the formulas we find that the change in employment over time is of the form

$$\frac{\dot{l}}{l} = \frac{(1-u)}{\sigma} - \alpha.$$

The real employment rate is calculated by dividing employment by labor force

$$v = \frac{l}{n}$$

implying that the growth rate of real employment over time changes according to

$$\frac{\dot{v}}{v} = \frac{(1-u)}{\sigma} - (\alpha + \beta). \tag{5.1}$$

As was mentioned before, Goodwin assumes that the real wage rate rises in the neighborhood of full employment. Therefore, we can describe the wage growth as

$$\frac{\dot{w}}{w} = \rho v - \gamma$$

and since the real wage rate is

$$u = \frac{w}{a}$$

we finally obtain

$$\frac{\dot{u}}{u} = -(\alpha + \gamma) + \rho v. \tag{5.2}$$

Equations (5.1) and (5.2) together describe Goodwin's employment-wage cycle model:

$$\begin{aligned} \dot{v} &= \Big[\big(\frac{1}{\sigma} - (\alpha + \beta)\big) - \frac{u}{\sigma}\Big]v \\ \dot{u} &= \big[-(\alpha + \gamma) + \rho v\big]u. \end{aligned} \tag{5.3}$$

Goodwin arrived to a Lotka-Volterra predator-prey model of the form

$$\begin{aligned}\dot{v} &= (\eta_1 - \theta_1 u)v \\ \dot{u} &= (-\eta_2 + \theta_2 v)u\end{aligned} \tag{5.4}$$

where

$$\begin{aligned}\eta_1 &= \frac{1}{\sigma} - (\alpha + \beta) \\ \eta_2 &= \alpha + \gamma\end{aligned} \tag{5.5}$$

and

$$\begin{aligned}\theta_1 &= \frac{1}{\sigma} \\ \theta_2 &= \rho.\end{aligned} \tag{5.6}$$

In the original predator-prey system we identify the predator and the prey by the fact that the predator population grows faster with an increase of prey population, while the prey population grows faster with a decrease of predator population. Looking at (5.4) it is clear that the employment v represents the prey and the wages u the predators of this system.

We already know from the previous chapter that, in the presence of both species, the predator-prey model's solution is a family of closed cycles with a common equilibrium point. For the system in (5.4) that point is

$$(v^*, u^*) = \left(\frac{\eta_2}{\theta_2}, \frac{\eta_1}{\theta_1}\right).$$

Using the original parameters, we can write the center of the economic model as

$$(v^*, u^*) = \left(\frac{(\alpha + \gamma)}{\rho}, 1 - (\alpha + \beta)\sigma\right). \tag{5.7}$$

5.1.2 A Revised Model

Goodwin's model has some shortfalls, some of them even pointed out by Goodwin himself in his later work.

Unlimited Growth

The first problem with this model arises when we look at (5.4) and see that, in the absence of wages u, the employment rate is given by

$$v = v_0 e^{\eta_1 t}$$

which means it grows exponentially, passes full employment and continues to grow without boundaries. In reality, employment cannot increase without limits and drawbacks on productivity, since additional workers are unlikely to be as productive as employed workers. Furthermore, labor market downsizing often excludes the less trained workers first, as an upsizing may lead to the acceptance of who is available regardless their training and skills. To make this model more realistic, one should consider a logistic saturation so that at $u = 0$ we have

$$\dot{v} = \eta_1 \Big(1 - \frac{v}{K}\Big) v.$$

In this model, we should consider $K = 1$, since the employment rate can't be bigger than 100%. Including this feature, the model would write:

$$\begin{aligned} \dot{v} &= \eta_1(1 - v)v - \theta_1 vu \\ \dot{u} &= (-\eta_2 + \theta_2 v)u \end{aligned} \tag{5.8}$$

Wages and Employment

The second problem with Goodwin's original model is the reaction of wages to employment, since any changes in wages as a result of changes in employment cannot be instantaneous as they are assumed to be. Wage contracts planed ahead and usually not don't into account future demand for labor, causing a delay in the reaction of wages to employment. This delay can be introduced in (5.8) by replacing in the second equation

$$v = \int_0^t v(\tau) G(t - \tau) d\tau$$

where G is a non negative integrable weight function that verifies

$$\int_{-\infty}^{t} G(t - \tau) d\tau = \int_{t}^{\infty} G(s) ds = 1.$$

Therefore, the way wages depend on the past employment levels can be set according to the choice of function G. With this modification, the new model writes

$$\begin{aligned} \dot{v} &= \eta_1(1 - v)v - \theta_1 vu \\ \dot{u} &= -\eta_2 u + \theta_2 u \int_0^t v(\tau) G(t - \tau) d\tau. \end{aligned} \tag{5.9}$$

Structural Instability

The structural instability of Goodwin's model is considered to be its worst fault. This instabililty comes from the fact that small changes in the initial condition (v_0, u_0) may lead to a much different behavior of solutions. Cushing (1977) [30] and MacDonald (1977) [31] managed to stabilize the solutions by choosing as weight function

$$G(s) = a e^{-as}, \quad a > 0.$$

This means that these authors considered that employers take into account changes in the company's profits before making the wage contracts. Due to the constant capital-output ratio and the fact that output is strongly connected to employment, employers discount past employment levels using a as a discount rate. Using this weight function, the authors guarantee that the furthest in the past an employment level is the less it affects the wages. Furthermore, if we consider a time period s large enough the influence becomes practically null, as can be observed in figure 5.1.

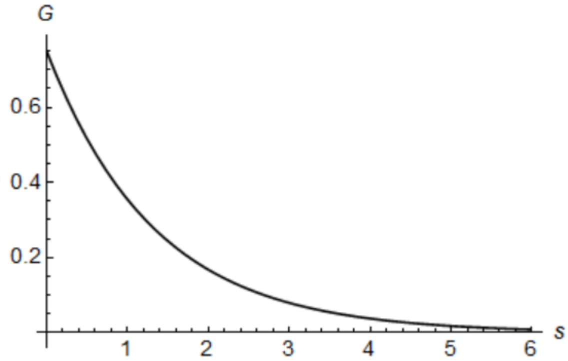

Figure 5.1: Weight Function $G(s) = ae^{-as},\ a = 0.75.$

These authors wrote a variation of system (5.9) without structural instability:

$$\begin{aligned} \dot{v} &= \eta_1(1-v)v - \theta_1 vu \\ \dot{u} &= -\eta_2 u + \theta_2 u \int_0^t v(\tau) a e^{-a(t-\tau)} d\tau. \end{aligned} \tag{5.10}$$

At this point we have an improved model for the dynamic relationship of wage and employment, making our next step the study of its solution and stability properties.

5.1.3 An Improved Model

Solution and Stability

Looking at the topics mentioned before, Vadasz (2008) [32] proposed an improved model of the form

$$\begin{aligned} \dot{x} &= \eta_1(1-x)x - \theta_1 xy \\ \dot{y} &= -\eta_2 y + \theta_2 yz \\ \dot{z} &= a(x-z) \end{aligned} \tag{5.11}$$

where x represents the employment, y the wages and z stands for the expectations of future employment levels based on past employment levels. The third equation of the system shows that this expectations change continuously and correct themselves.

The system in (5.11) has three equilibrium points:

$$\begin{aligned} S_1 &= (0,0,0) \\ S_2 &= (1,0,1) \\ S_3 &= \left(\frac{\eta_2}{\theta_2}, \left(1-\frac{\eta_2}{\theta_2}\right)\frac{\eta_1}{\theta_1}, \frac{\eta_2}{\theta_2}\right) \end{aligned} \tag{5.12}$$

Vadasz studied the stability of this steady states and he reached the following conclusions:

- The equilibrium S_1 is a saddle point regardless of the choice of parameters.
- The equilibrium S_2 is asymptotically stable only if $\eta_2 > \theta_2$. This condition implies that there is a wage decrease, but since S_2 represents the case where zero wages correspond to full employment, it is invalid for this case. As a result, S_2 becomes an unstable node in the context of the model studied;
- To study the stability of S_3 we consider three cases. Let $\mu = \frac{1}{a}$.
  - If $\theta_2(\theta_2-\theta_1)-\eta_1\eta_2 < 0$ then S_3 is asymptotically stable, regardless the delay;
  - If $\theta_2(\theta_2-\theta_1)-\eta_1\eta_2 > 0$ and $\mu\Big(\theta_2-\eta_2-\frac{\eta_1\eta_2}{\theta_2}\Big) < 1$ then S_3 is asymptotically stable;
  - If $\theta_2(\theta_2-\theta_1)-\eta_1\eta_2 > 0$ and $\mu\Big(\theta_2-\eta_2-\frac{\eta_1\eta_2}{\theta_2}\Big) > 1$ then S_3 is unstable;

Further Improvements

Although the changes mentioned before improve Goodwin's model by making it more realistic, the weight function chosen doesn't incorporate the phenomenon of rising wages during a recession. Instead, it slows the growth of wages when employment levels start decreasing. To get a model closer to reality, one should choose a weight function of the form

$$G(s) = a^2 s e^{-as}, \quad a > 0.$$

This choice of function introduces a delay on how wages react to employment, and assumes a maximum at $s = \frac{1}{a}$ such that $G(\frac{1}{a}) = \frac{a}{e}$. For the particular case where $a = 0.75$ the weight function is illustrated in figure 5.2.

For this particular choice of parameter, the bigger weight corresponds to employment at $s = 1,33$ with $G = 0.276$. Choosing $a = 0.75$ means that the employment level of nine months ago has the largest influence on the present wages, and as can be seen in figure 5.2 the employment levels far back in the past have almost no effect on current salaries.

According to Vadasz, choosing this type of weight function would transform the system in (5.11) in

$$\begin{aligned} \dot{x} &= \eta_1(1-x)x - \theta_1 xy \\ \dot{y} &= -\eta_2 y + \theta_2 y z_1 \\ \dot{z_1} &= a(z_2 - z_1) \\ \dot{z_2} &= a(x - z_2). \end{aligned} \tag{5.13}$$

Looking at this model, one can see that the real wages in the second equation will be set according to past employment with dynamic relationship described by last two equations of the system.

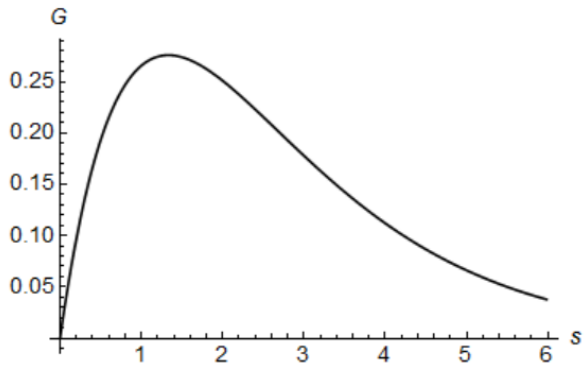

Figure 5.2: Weight Function $G(s) = a^2 s e^{-as},\ a = 0.75.$

5.2 Palomba's Model

The author famous for introducing the Lotka-Volterra dynamics in economics is Goodwin (1967). Yet, as pointed out by Massimo di Mateo (1988) [26], the economist Giuseppe Palomba had used these equations in a book published in 1939. Throughout this section we will study Palomba's work and results concerning the application of Lotka-Volterra dynamics in economics.

Palomba (1939) [12] considers an economy where there are only two types of goods: consumption goods, such as clothing and food, and capital goods, such as buildings and machinery. The model is built under the following assumptions:

Assumption 1: There are two types of goods: goods of type a, which consist of goods ready for immediate consumption and goods that directly enter into their production; goods of type b, namely capital goods that directly enter into the production of other capital goods and only indirectly into the production of consumption goods.

Assumption 2: The economy is in a dynamic situation tending to increase its capital equipment. This means that it is necessary to reduce consumption in order to invest and increase capital stock. Therefore, some of the commodities of type a are allocated to category b.

Assumption 3: In any given time goods of type a have a coefficient of increase equal to ϵ_1, and this growth may be caused by long-term forces such as productivity and labor force growth. On the other hand, goods of type b have a coefficient of increase equal to $-\epsilon_2$. Since ϵ_1 and ϵ_2 are positive constants, these coefficients imply that if there is no change in destination as mentioned in assumption 2, then goods of type a would increase continuously while goods of type b would decrease towards zero.

Assumption 4: The coefficient of decrease of goods of type a, due to changes in destination, is equal to $-\gamma_1$, while the coefficient of increase of the goods of type b, due to the same reason, is equal to γ_2. Here, γ_1 and γ_2 are positive constants.

Palomba denoted as C_1 the volume of the goods of type a, and as C_2 the volume of the goods of type b. The assumptions previously made translate into the following equations:

$$\begin{aligned} \frac{dC_1}{dt} &= C_1(\epsilon_1 - \gamma_1 C_2) \\ \frac{dC_2}{dt} &= -C_2(\epsilon_2 - \gamma_2 C_1) \end{aligned} \tag{5.14}$$

For the simplicity of of future computations we define $\alpha_1 = \epsilon_1$, $\alpha_2 = -\epsilon_2$, $\beta_1 = -\gamma_1$ and $\beta_2 = \gamma_2$. With this notation we obtain a system equivalent to the previous one

$$\begin{aligned} \frac{dC_1}{dt} &= C_1(\alpha_1 + \beta_1 C_2) \\ \frac{dC_2}{dt} &= C_2(\alpha_2 + \beta_2 C_1) \end{aligned} \tag{5.15}$$

Manipulating system (4.14) we find that

$$\begin{aligned} &\beta_2 \frac{dC_1}{dt} - \beta_1 \frac{dC_2}{dt} = \beta_2 C_1 \alpha_1 - \alpha_2 \gamma_1 C_1 \\ &\alpha_2 \frac{1}{C_1}\frac{dC_1}{dt} - \alpha_1 \frac{1}{C_2}\frac{dC_2}{dt} = \beta_1 C_2 \alpha_2 - \alpha_1 \beta_2 C_1. \end{aligned} \tag{5.16}$$

and if we sum these conditions we get

$$\beta_2 \frac{dC_1}{dt} + \alpha_2 \frac{1}{C_1}\frac{dC_1}{dt} - \beta_1 \frac{dC_2}{dt} - \alpha_1 \frac{1}{C_2}\frac{dC_2}{dt} = 0.$$

Integrating both sides of this equality we obtain

$$\beta_2 C_1 + \alpha_2 \log C_1 - \beta_1 C_2 - \alpha_1 \log C_2 = k',$$

where k' is a constant. Therefore, applying the exponential function to both sides of the equation we get

$$e^{\beta_2 C_1} C_1^{\alpha_2} = k e^{\beta_1 C_2} C_2^{\alpha_1}$$

where $k = e^{k'}$. If we go back to the original notation, the previous equality writes

$$e^{\gamma_2 C_1} C_1^{-\epsilon_2} = k e^{-\gamma_1 C_2} C_2^{\epsilon_1}. \tag{5.17}$$

Furthermore, if we let

$$\begin{aligned} Y &= e^{\gamma_2 C_1} C_1^{-\epsilon_2} \\ X &= e^{-\gamma_1 C_2} C_2^{\epsilon_1} \end{aligned} \tag{5.18}$$

we finally obtain

$$Y = kX. \tag{5.19}$$

Palomba studied the behavior of this system following closely the work of Lotka and Volterra on their predator-prey model, where one species feeds on the other. He then made two important observations.

First, he states that the economy's ondulatory behavior depends solely on the variables involved and their interaction. In other words, he says that cycles are endogenous, self-sustained and consequently non-linear. Considering the year when this study was published this was a surprising statement for the scientific community, since in that period the models proposed by mathematical economists for business cycles were linear and required exogenous factors, such as random shocks, to keep their cyclic behavior. Only in the 1950's did Goodwin start to develop non-linear models for economic cycles, which lead him to the construction of his famous model for wages and employment presented in the previous section.

Second, he points out that the parameters ϵ_1, ϵ_2, γ_1 and γ_2 should be considered general functions of time, instead of static values. With this new feature, Palomba's model would write

$$\begin{aligned} \frac{dC_1}{dt} &= C_1[\epsilon_1(t) + \gamma_1(t) C_1] \\ \frac{dC_2}{dt} &= C_2[\epsilon_2(t) + \gamma_2(t) C_1] \end{aligned} \tag{5.20}$$

Unlike the predator-prey model, where in the presence of both species the orbits are closed curves, the orbits of this improved model do not have a known shape since they depend on the nature of the functions $\epsilon_1(t)$, $\epsilon_2(t)$, $\gamma_1(t)$ and $\gamma_2(t)$.

Chapter 6

Conclusions

In nature, species compete, expand and seek resources to continue their existence. The Lotka-Volterra equations model these loss-win interactions and can be used in many fields of study besides the equilibrium of ecosystems. Throughout our work, we have studied some of their possible applications to the banking system and economics.

Studying the application of Lotka-Volterra equations to the banking system, we first looked at three possible models that can describe the relationship between deposit and loan volume on a bank's balance sheet. The stability of these systems has been analyzed and in the three cases the trivial equilibrium behaves like a saddle point, while the non-trivial equilibria are unstable in the simple model, and conditionally stable in the model with Michaelis-Menten response and in the model with reserve requirement. Also, the relationship between mother banks, subsidiary banks and the individuals or companies that use their services has been compared to a three level ecologic food chain. Studying the equilibrium of this dynamical system we found that, under a specific set of conditions involving the parameters of the equations, it is possible to find a stable equilibrium for the system.

Regarding the applications of Lotka-Volterra equations to economics, Vadasz found that the original Goodwin model, after some modifications, is able to accurately predict changes in wages and employment as well as employment cycles. In addition, Palomba modeled the dynamic relationship between the volumes of consumption and capital goods. He concluded that economic cycles are non-linear and endogenous, and suggested that the behavior of these systems should be studied using, as coefficients for the Lotka-Volterra equations, functions of time rather than the constants on the original model.

Bibliography

[1] V. Volterra. Variazioni e fluttuazioni del numero di individui in specie animali conviventi. *Lincei. Mem.*, pages 31–113, 1926.

[2] V. Volterra. Leçons sur la théorie mathématique de la lutte pour la vie. *Gauthiers Villar*, 1931.

[3] A. J. Lotka. Elements of mathematical biology. *Dover Publications*, 1925.

[4] S. Chen and S. Bao. A game theory based on predation behavior model. *International Conference on Game Theory*, 2010.

[5] M. A. Petersen and I. Schoeman. Modeling of banking profit via return-on-assets and return-on-equity. *WCE*, 2:2–4, 2008.

[6] J. Mukuddem-Petersen, M. A. Petersen, I. M. Schoeman, and B. A. Tau. Dynamic modelling of bank profits. *Applied Financial Economics Letters*, pages 157–161, 2008.

[7] C. A. Comes. Banking system: Three level Lotka-Volterra model. *Procedia Economics and Finance*, 3:251–255, 2012.

[8] N. C. Apreutesei. An optimal control problem for Lotka-Volterra system with diffusion. *Bul. Inst. Polytechnic*, pages 31–41, 1998.

[9] N. C. Apreutesei. Necessary optimality conditions for a Lotka-Volterra three species system. *Mathematical Modelling of Natural Phenomena*, pages 123–125, 2006.

[10] N. Sumarti and I. Gunadi. Reserve requirement analysis using a dymacial system of a bank based on Monti-Klein model of bank's profit function. *Computers and Mathematics with Application*, 2013.

[11] N. Sumarti, R. Nurfitriyana, and W. Nurwenda. A dynamical system of deposit and loan volumes based on the Lotka-Volterra model. *AIP Conference Proceedings*, pages 92–94, 2014.

[12] G. Palomba. Introduzione allo studio della dinamica economica. *Napoli Jovene*, 1939.

[13] R. M. Goodwin. A growth cycle. *Cambridge University Press*, pages 54–58, 1965.

[14] M. Desai. Growth cycles and inflation in a model of the class struggle. *Journal of Economic Theory*, 6:527–545, 1973.

[15] E. Wolfstetter. Fiscal policy and the classical growth cycle. *Journal of Economics*, pages 93–375, 1982.

[16] M. C. Sportelli. A Kolmogoroff generalized predator-prey model of Goodwin's growth cycle. *Journal of Economics*, pages 35–64, 1995.

[17] K. V. Velupillai. Linear and nonlinear dynamics in economics: The contributions of R. Goodwin. *Economic Notes*, pages 73–91, 1982.

[18] P. Flaschel. Some stability properties of Goodwin's growth cycle. *Journal of Economics*, pages 63–69, 1984.

[19] D. E. Atkinson. Regulation of enzyme function. *Annu. Rev. Microbiol.*, pages 47–68, 1969.

[20] D. Harvie. Testing Goodwin: growth cycles in ten OECD countries. *Cambridge Journal of Economics*, pages 349–376, 2000.

[21] L. Apedaille, H. Freedman, S. Schilizzi, and M. Solomonovich. Equilibria and dynamics in an economic predator-prey model in agriculture. *Mathl Comput Modelling*, 19:1–15, 1994.

[22] K. Chakraborty, M. Chakraborty, and T. Kar. Bifurcation and control of a bioeconomic model of a prey-predator system with a time delay. *Nonlinear Analysis: Hybrid Systems*, pages 613–625, 2011.

[23] C. Michalakelis, C. Christodoulos, D. Varoutas, and T. Sphicopoulos. Dynamic estimation of markets exhibiting a prey-predator behaviour. *Expert Systems with Applications*, 2012.

[24] S. Baigent. Lotka-Volterra dynamics: an introduction. *Preprint*, 2010.

[25] T. K. Kar. Stablilty analysis of a prey-predator model incorporating a prey refuge. *Communications in Nonlinear Science and Numerical Simulation*, pages 681–691, 2005.

[26] M. di Matteo. Goodwin and the evolution of a capitalistic economy: An afterthought. *Berlin Heidelberg*, 1988.

[27] G. Thompson. Sources of financial sociability: Networks, ecological systems or diligent risk preparednes? *CRESC Working Paper Series*, pages 1–17, 2011.

[28] A. Haimovici. A control problem for a Volterra three species system. *Mathematica-Revue d'Analyse Numérique et de Théorie de l'Approx.*, pages 35–41, 1980.

[29] K. Marx. A contribution to the critique of political economy. *Progress Publishers*, 1859.

[30] J. M. Cushing. Periodic time-dependent predator-prey systems. *SIAM Journal of Applied Mathematics*, pages 82–95, 1977.

[31] D. D. Macdonald. The mathematics of diffusion. *Transient techniques in Electrochemistry*, 1977.

[32] V. Vadasz. Economic motion: An economic application of the Lotka-Volterra predator-prey model. 2007.

[33] E. Chauvet, J. Paullet, J. Previte, and Z. Walls. A Lotka-Volterra three-species food chain. *Mathematics Magazine*, 2002.

[34] L. Michaelis and M. L. Menten. Die kinetik der invertinwirkung. *Biochem*, pages 333–369, 1913.

[35] G. Gandolfo. The Lotka-Volterra equations in economics: an italian precursor. *Accademia dei Lincei*, pages 347–357, 2017.