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DISSERTATION

THE IMPACT OF INSURANCE IN INVESTMENT STRATEGIES:
A REAL OPTION APPROACH

SÍLVIA MONTEIRO OLIVEIRA

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Abstract

This study aims to understand what a firm's investment strategy should be if the firm considers purchasing insurance.

We consider an investment model with two sources of uncertainty. The firm's future revenue is assumed to depend on a random economic indicator, following a Geometric Brownian Motion. On the other hand, unexpected adverse events, that reduce a firm's future revenue, are introduced out, described by a compound Poisson Process.

The objective is to decide on the optimal moment for the firm to invest in the market and the insurance contract that it wants to buy. The decision to buy an insurance contract depends on the insurance premium and how the firm measures its risk. We formulate the model as a control problem that is solved using a dynamic programming approach.

JEL codes: C61; E22; G12; G22; G41.

Keywords: optimal stopping, insurance, investment, premium principle, risk.

Glossary

GBM - Geometric Brownian Motion

SDE - Stochastic Differential Equation

PP - Poisson Process

R.V - Random Variable

I.I.D - Independent and Identically Distributed

DP - Dynamic Programming

HJB - Hamilton-Jacobi-Bellman

AI - After Investment

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1 Introduction

The main goal of a firm is to maximize the expected profit and minimize the risks. Firms nowadays are threatened by several risks in the market, characterized by the uncertainty of various external and not controllable factors.

In this way, firms are increasingly exposed to unexpected and unfortunate events. This is where insurance comes in, as an important way to mitigate losses caused by these external factors. Costs related with damages to the firm's property, costs of lawsuits or even the income loss reported by a firm are examples of losses that can be partially or even totally covered by insurances. Thus, insurers can help the business survive or even thrive.

When the firm's management has the entitlement to take a decision that may lead change, expand or even abandon a project such as investments, they are faced with a real option. This projects usually entail tangible assets (Trigeorgis [1996]).

The field of research of real options has contributed to several other fields, such as financial economics and strategic management. It also provides approaches that help explain the greater flexibility in decision making by a firm under conditions of uncertainty. Using real options to organize the investment helps the firm in containing the investment's downside risk, while preserving the upside potential (Ragozzino et al. [2016]).

Over the last two decades, the investment approaches have changed a lot due to the application of the theory of financial options to study real assets investment strategies. Several works in the literature on real options study the valuation under uncertainty.

Among them, Trigeorgis [1993], combines real options and their interactions with financial flexibility; McDonald and Siegel [1985] examine the investment decision of firms undertaking risky investment projects; Majd and Pindyck [1987] develop an explicit model of investment projects; Pindyck [1990] presents some simple models of irreversible investment and shows how optimal investment rules and the valuation of projects can be obtained; Pindyck [1993] has derived a simple decision rule that maximizes the firm's value and shows how two types of uncertainty have very different effects on investment. Some problems were modeled within the study of the optimization of the firm's profit with two sources of uncertainty, including natural resource investments, environmental and new technology adoption, and strategic and competitive options (see, Dixit and Pindyck [1994] Farzin et al. [1998], Perlitz et al. [1999]).

Firms take advantage of their investment opportunities and try to realize the positive profits from the real options. This flexibility protects management against downside risk but gives the firm unlimited growth potential. Usually, real options extend financial options into investment opportunity analysis of real assets and often assign higher value to the investment opportunity because of time value (Hung and So [2010]).

As referred by Dixit and Pindyck [1994], most investment decisions have three characteristics in common. The investment is partially or totally irreversible, that is, the initial cost of the investment is a sunk cost, at least partially sunk. The costs cannot be recovered. Therefore, if there is a change in the decision, it is not possible to recover everything. There is also uncertainty about the future rewards of the investment. The best alternative for the firm's venture is to assess the probabilities of the outcomes and see if they might mean greater or lesser profit. Finally, there is room for manoeuvre regarding the timing of the investment. It is possible to postpone the decision to invest and, thus have access to more information about the future, without complete certainty about it. The firm has the opportunity to delay the investment to wait, for instance, for new information on prices, costs and market conditions. These three characteristics are crucial for investors to be able to make the optimal decision.

Risk is a broad, extremely important topic that greatly affects all firms. In this context, it is possible to define risk as the expected variance in profits, losses or cash flows arising from an uncertain event (Banks [2004]).

As Banks [2004] refers, firms are thus exposed to a wide range of risks that at any time can have negative and sometimes irreversible effects on the firm, such as business interruption or even property damage. These risks must be managed, for instance part of these risks can be transferred to an insurance company. There is a transfer of risk when the firm pays the insurer an extra cost, such as a risk premium, in exchange for covering uncertain losses. Thus, it is clear that the main purpose of insurance is to spread risk.

Hoyt and Khang [2000] note that several factors may lead to the need for insurance, one of the factors is the underinvestment problem, that consists in a conflict of interests between shareholders and debtholders which makes the firm become unable to make investments in opportunities. There are also conflicts of interests between business owners and managers, as their goals might differ, which could lead to decisions that are not optimal. Insurers have a comparative advantage in providing services that are risk-related, such as claims handling or loss prevention, which benefits the firms. The effects of the firm's expected tax liability are also a reason why companies should get insurance, as they can get protection for the uncertain tax costs. Insurance is also beneficial to businesses because of the costs that may come from an eventual bankruptcy. Furthermore, the regulatory status of a firm could also be a motivation for firms to obtain insurance.

As Centeno [2003] referred, in the insurance activity, the production cycle is completely changed. Insurance is sold upon payment of a premium and only after that, given the occurrence, or not, of claims, the costs will be known. In order to protect itself from the risk of very serious claims, the firm is willing to pay a price, the premium, that is higher than the average cost of the claims it will generate. The premium associated with a particular policy is a fixed amount received by the insurance company as a compensation for the risk assumed. The insurer will be responsible for part of the expenses arising from the claims (the part defined in the insurance contract) as well as the management expenses inherent to the policy. Thus, the premium associated with a policy contains, in addition to possible taxes, a loading to offset management costs and to cope with the risk the insurance company is undertaking.

The pure premium is the expected value of the claim costs transferred to the insurance company. The premium is the sum of the pure premium with the safety loading. As mentioned, the purpose of the loading is to protect the insurer against possible random deviations of the indemnities in relation to its average and takes the name of safety loading or loading for accident deviations. There are several ways to compute the safety loading, denoted premium calculation principles.

As Sibindi [2015] referred, a firm has two options, it can opt for partial insurance, where there is fractional risk coverage for a lower risk premium, or it can opt for full insurance where there is full coverage of a risk in exchange for a higher risk premium.

Centeno [2003] referred that there are various insurance contracts that can be acquired by firms. For instance, the proportional contract, in which the insurer is responsible for a certain percentage of the underlying risk. Other insurance contracts include an ordinary deductible or a limit loss, or both.

In this dissertation, we consider an investment model with two sources of uncertainty. The firm's future revenue is assumed to depend on a random economic indicator, following a Geometric Brownian motion (GBM). On the other hand, unexpected adverse events, that reduce a firm's future revenue, are introduced out, described by a compound Poisson Process (PP).

The objective is to decide on the optimal moment for the firm to invest in the market and the insurance contract that it wants to buy. The decision to buy an insurance contract depends on the insurance premium and how the firm measures its risk.

To solve this optimal stopping problem the Dynamic Programming (DP) approach is applied. Therefore, we solve a Hamilton–Jacobi–Bellman (HJB) equation in order to determine the solution of the problem. We include insurance in this model and aim at finding the optimal insurance contract. Thus, it was necessary to find the optimal parameters describing the insurance contract. We choose different risk functions, premium principles and the type of insurance contracts. Without considering the risk function, the losses are only evaluated by the firm in expected value. As the premium of an insurance contract is always given by the expected value of the risk plus a safety loading, which is positive. Thus, it would never be optimal to buy insurance. This study aims to understand what a firm's investment strategy should be if the firm considers purchasing insurance.

The dissertation is organized as follows. In Section 2 a baseline model for the optimal instant to invest, without considering any insurance strategy, is described. Some considerations about the economical meaning of the baseline model are discussed. The Hamilton-Jacobi-Bellman (HJB) equation for the optimization problem, and its solution based on Björk [2006] and Touzi [2004], are presented. In Section 3 the comparative statics are developed and analysed. In Section 4 the model with insurance and the optimal solution to the optimal stopping problem with investment, using different insurance contracts, premium principles and measures of risk, are deduced. The proofs of the propositions of Section 4 can be found in Appendix A. Finally, in Section 5, numerical results illustrating some of the several cases considered are presented. In Section 6 the main conclusions are summarized.

2 Baseline Model

We consider an investment model with two sources of uncertainty. Let X_t represent the firm's revenue at time t and let it be modeled by a GBM, satisfying the following Stochastic Differential Equation (SDE):

$$dX_t = \mu X_t dt + \sigma_t dW_t,$$

where $X_0 = x$ represents the initial wealth, $\mu \in \mathbb{R}$ is the drift, σ is the volatility, $\{W_t : t \geq 0\}$ is a Brownian Motion.

There are adverse events that can reduce a firm's future revenue. The times of occurrence of such events and their severity in the revenue reduction are both random. Such events are model by means of a compound PP. N is the number of negative occurrences, N_t is a $PP(\lambda)$ which denotes the events that can negatively affect the firm's revenue.

First, the problem where the firm has to decide the time to invest is considered. Therefore, the goal is to maximize the firm's expected return choosing the optimal time to invest. The firm's expected profit is then given by:

$$\begin{aligned} J(x,\tau) =& E_x\left[\int_\tau^\infty e^{-rt}\ X_t\left(1-\sum_{i=0}^{N_{t-\tau}} U_i - \sum_{i=0}^{N_{t-\tau}} f(U_i)\right)dt - e^{-r\tau}K\right] \\ =& E_x\left[\int_\tau^\infty e^{-rt}\ X_t\left(1-\sum_{i=0}^{N_{t-\tau}} (U_i + f(U_i))\right)dt - e^{-r\tau}K\right] \\ =& E_x\left[\int_\tau^\infty e^{-rt}\ E_x(X_t)\ E\left(1-\sum_{i=0}^{N_{t-\tau}} (U_i + f(U_i))\right)dt - e^{-r\tau}K\right] \\ =& E_x\left[e^{-r\tau}\ \left\{\frac{X_\tau}{r-\mu}\left(1-\frac{\lambda}{r-\mu}(u+E[f(U_i)])\right)-K\right\}\right], \end{aligned} \tag{2.1}$$

where U is the absolute percentage lost value in each occurrence, $U_i \overset{iid}{\sim} U$ a $r.v$, represents the percentage value that the firm loses, with $E(U) = u$, r is the interest rate, K the investment cost and $f(U)$ is a function that represents the firm's risk preferences and E_x is the conditional expectation on x. The value function, $J(x,\tau)$, denotes the value of the firm when the current level is x and we decide to take a decision at time τ. Assuming $(r-\mu) > \lambda(u + E[f(U)])$.

The optimised value of the investment, $V(x)$, is given by:

$$V(x) = \sup_{\tau \in S} J(x,\tau) = J(x,\tau^*).$$

The model is formulated as an optimal stopping problem, so the optimization problem is solved using a DP approach (see Touzi [2004]). To find a solution for this problem it is necessary to find the solution of the HJB equation, which is given by:

$$\min \{rV(x) - \mathcal{L}V(x), V(x) - g(x)\} = 0,$$

$$\text{where } g(x) = \frac{x}{r-\mu}\left(1 - \frac{\lambda}{r-\mu}(u + E(f(U)))\right) - K. \tag{2.2}$$

Here, $\mathcal{L}$ represents the infinitesimal generator of the process X_t. For a function $f \in C^2$, the infinitesimal generator is defined by:

$$\mathcal{L}f(x) = \lim_{t\to 0} \frac{E_x(f(X_t)) - f(x)}{t} = \mu x f'(x) + \frac{\sigma^2}{2}x^2 f''(x).$$

This means that the infinitesimal generator contains information about how the process evolves. The infinitesimal generator is in fact a differential operator which is one member of the HJB equation. In order to solve the HJB equation, it is necessary to solve a differential equation which is given by $\mathcal{L}V(x) = rV(x)$. This is an Euler-Cauchy equation and its general solution is given by:

$$Ax^{\beta_1} + Bx^{\beta_2}, \beta_1 \neq \beta_2,$$

in which β_1 and β_2 are the roots of the characteristic polynomial. The roots β_1 and β_2 are known and given by:

$$\beta_1 = \frac{-(\mu - \frac{1}{2}\sigma^2) + \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r}}{\sigma^2} > 1,$$

$$\beta_2 = \frac{-(\mu - \frac{1}{2}\sigma^2) - \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r}}{\sigma^2} < 0.$$

Proposition 1. *Let V be the the value function defined by:*

$$V(x) = \begin{cases} Ax^{\beta_1}, & x \in (0, x^*) \\ g(x), & x \geq x^* \end{cases} \tag{2.3}$$

The parameter A and the threshold x^* are given by:

$$x^* = \frac{K\beta_1(r-\mu)^2}{(\beta_1 - 1)\ (r-\mu) - (\beta_1 - 1)\ (u\lambda + E(f(U))\lambda)} \tag{2.4}$$

$$A = \frac{(x^*)^{1-\beta_1}}{r-\mu}\left(1 - \frac{\lambda}{r-\mu}(u + E(f(U)))\right) - K(x^*)^{-\beta_1} \tag{2.5}$$

The value function is composed by two regions, the stopping region and the continuation region. These two complement each other, where x^* is the threshold that determines when one region ends and the other begins. The firm invests as soon as X reaches the stopping region. $V(x)$ is a solution to $rv(x) - \mathcal{L}v(x) = 0$, for $x \in (0, x^*)$, i.e. $V(x) = Ax^{\beta_1} + Bx^{\beta_2}$, for $x \in (0, x^*)$, and $V(x) = g(x)$, for $x \in (x^*, \infty)$.

The general solution of the Euler-Cauchy equation is $Ax^{\beta_1} + Bx^{\beta_2}$, in which $B = 0$, if it is not like that, then $\lim_{x \to o^+} V(x) = \pm\infty$. To calculate the threshold x^* and the parameter A, it is necessary to use smooth-pasting conditions that admit that the value function of this problem is continuous and differentiable. The smooth-pasting conditions are given by:

$$\lim_{x \to x^*} V(x) = V(x^*)$$
$$\lim_{x \to x^*} V'(x) = V'(x^*).$$

2.1 Comparative Statics

The comparative statics allows us to understand the impact of a parameter on the firm's decision. For this, we fixed all the parameters and changed only the one we intend to study to see how the firm's strategy changes. Here, we analyse how a change in parameters λ, σ and μ, impacts on the optimal strategy. We assume that the parameters are exogenous, in the sense that their values are not controlled by the firm.

The following proposition shows the impact of each one of the parameters in the investment decision.

Proposition 2. *Let x^*, the threshold of the revenue, be defined by equation (2.4). Then $\frac{\partial x^*}{\partial \sigma} > 0$, $\frac{\partial x^*}{\partial \lambda} > 0$ and x^* is not monotonic with μ.*

The proof of Proposition 2 can be found in Appendix 2.

The threshold x^* increases with the market's volatility, therefore the investment decision is postponed if volatility increases. It also increases with the intensity of the PP, λ. Hence, if there is an increase in the average number of occurrences, the investment decision is also postponed.

The threshold x^* has an unusual behavior with variations in the drift μ, x^* first decreases and then increases as is depicted in figure 1. In this case the non monotony of x^* can be explained by the interaction between the PP and the GBM. When μ increases it leads to a raise in instantaneous profit. Therefore, the firm's decision is to anticipate the investment. However, when there is a negative occurrence, there are also greater drops in the revenue.

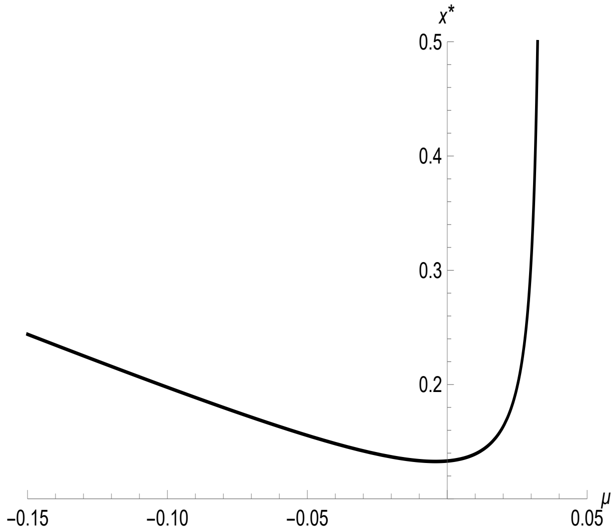

Figure 1: Monotony of x^* with μ, where $\sigma = 0.2$, $r = 0.05$, $\lambda = 0.05$, $u = 0.1$, $E(f(U)) = 0.2$, $K = 1$.

3 Model with insurance

Insurance contracts allow firms to mitigate the risk of occurrences that negatively impact the firm's revenue. By buying insurance, the firm shares its risk with an insurance company, which requires the payment of a premium.

The objective of this model is to maximize the firm's profits by finding the best time to invest, and the best insurance contract, Z, knowing that the firm can purchase insurance.

Once again, we consider an investment model with two sources of uncertainty, the future revenue which follows a GBM and adverse events that reduce a firm's future revenue. There are events modeled by means of a compound PP. In this case the parameters of the insurance contract are considered, hence the firm's expected profit is as follows:

$$
\begin{aligned}
J(x,\tau,Z) &= E_x\left[\int_\tau^\infty e^{-rt}\, X_t \left(1-\sum_{i=0}^{N_{t-\tau}}\Big(Z(U_i)+f(Z(U_i))+E(U-Z(U))+\alpha c\Big)\right)dt - e^{-r\tau}K\right] \\
&=E_x\left[E_\tau\left[\int_\tau^\infty e^{-rt}\, X_t \left(\sum_{i=0}^{N_{t-\tau}}\Big(1-f(Z(U_i))+E(U+\alpha c)\Big)\right)dt - e^{-r\tau}K\right]\right] \\
&=E_x\left[\int_\tau^\infty e^{-rt}\, E_x(X_t)\; E\left(\sum_{i=0}^{N_{t-\tau}}\Big(1-f(Z(U_i))+(U+\alpha c)\Big)\right)dt - e^{-r\tau}K\right] \\
&=E_x\left[e^{-r\tau}\left\{\frac{X_\tau}{r-\mu}\left(1-\frac{\lambda}{r-\mu}\Big(u+E(f(Z(U)))+\alpha c\Big)\right)-K\right\}\right],
\end{aligned}
\tag{3.1}
$$

where $Z(U)$ is the risk retained by the firm, $U-Z(U)$ is part of the risk transferred to the insurer, αc is the loading, it is what defines the premium principle, $E_{x,\tau}$ is the conditional expectation on x, τ and $f(Z(U))$ is a function that measures the firm's risk preferences given a certain insurance contract.

Without considering the risk function, the firm would never buy insurance. If we do not include this function in the model, then losses are only evaluated by the firm in expected value. As the premium of an insurance contract is always given by the expected value of the risk plus a safety loading, which is positive, it would never be optimal to buy insurance.

The value function denotes the value of the firm when the current revenue is x and it is decided to take a decision at time τ^*, knowing that the firm buys a certain insurance contract. The optimized value, $V(x)_{AI}$, is then given by:

$$V(x)_{AI} = \sup_{\tau\in S} J(x,\tau,Z) = J(x,\tau^*,Z)$$

This optimization model is again solved through a DP approach (see Touzi [2004]). Therefore, to find the solution to the problem it is necessary to solve the HJB equation which is given by:

$$\min\{rv(x)_{AI} - \mathcal{L}v(x)_{AI},\ v(x)_{AI} - \tilde{g}(x)\} = 0,$$

$$\text{where } \tilde{g}(x) = \frac{x}{r-\mu}\left(1 - \frac{\lambda}{r-\mu}(u + E(f(Z(U))) + \alpha c) - K\right). \tag{3.2}$$

In this equation we have $\mathcal{L}$, which represents an infinitesimal generator, a differential operator that has information about how the process evolves. In order to solve the HJB equation, it is necessary, once again, to solve the ODE.

Proposition 3. *Let $V(x)_{AI}$ be the value function defined by:*

$$V(x)_{AI} = \begin{cases} A_{AI}x^{\beta_1}, & x \in (0, x^*_{AI}) \\ \tilde{g}(x), & x \geq x^*_{AI} \end{cases} \tag{3.3}$$

The parameters A_{AI} and the threshold x^*_{AI} are given by:

$$x^*_{AI} = \frac{K\beta_1(r-\mu)^2}{(\beta_1 - 1)\left((r-\mu) - \lambda(u + E(f(Z(U))) + \alpha c)\right)} \tag{3.4}$$

$$A_{AI} = \frac{(x^*)^{1-\beta_1}}{r-\mu}\left(1 - \frac{\lambda}{r-\mu}(u + E(Z(f(U))) + \alpha c)\right) - K(x^*)^{-\beta_1} \tag{3.5}$$

There are two important regions. The stopping region and the continuation region. Additionally x^* is the threshold that determines when one region ends and the other begins.

The firm invests as soon as the process X reaches the stopping region. In this case, V is a solution to the ODE. In the investment region $V(x)_{AI} = \tilde{g}$ and in the stopping region $V(x)_{AI} = A_{AI}x^{\beta 1}$.

As already explained, the general solution of the Euler-Cauchy equation is $Ax^{\beta_1} + Bx^{\beta_2}$, in which $B = 0$, otherwise $\lim_{x\to o^+} V(x) = \pm\infty$. Using the smooth-pasting conditions it is possible to find the parameter A_{AI} and the threshold x^*_{AI}.

So far, we have no guarantee that buying insurance is optimal. In fact, we can guess that depending on the parameters of the model it may or may not be optimal to purchase insurance. To proceed with this analysis we have to introduce the premium principle, the risk function and the insurance policy.

We fix the premium principle first and vary the measure of risk, function f, and the insurance policy, since the goal is to understand the impact that each type of contract has on the firm's optimal strategy and also to understand which are the best contracts.

We use premium principles that are based on moments. These premium principles are computed as the expected value of the risk (the pure premium) plus a safety loading which is proportional to a moment (or a function of a moment).

First, we use the expected value premium principle, which means that a proportion of the expected value of the loss will be added to the pure premium. Where $U - Z(U)$ is the part of the risk ceded to the insurer. The premium is calculated as follows:

$$P = E(U - Z(U)) + \alpha E(U - Z(U)), \text{ where the loading is } \alpha E(U - Z(U)), \ \alpha > 0. \tag{3.6}$$

Then, we use the variance premium principle. In this case the premium depends not only on the expectation of the ceded risk but also on its variance. Therefore, the premium is calculated in the following way:

$$P = E(U - Z(U)) + \alpha Var(U - Z(U)), \text{ where the loading equals } \alpha Var(U - Z(U)), \ \alpha > 0. \tag{3.7}$$

Finally, we consider the standard deviation premium principle. Here the premium calculation depends not only on the expectation but also on the standard deviation of the loss. Hence the premium principle is given by:

$$P = E(U - Z(U)) + \alpha\sqrt{Var(U - Z(U))} \text{ where the loading equals } \alpha\sqrt{Var(U - Z(U))}, \ \alpha > 0. \tag{3.8}$$

The functions used to measure the risk are given by:

- $$f(Z(U)) = \nu\, Z(U), \tag{3.9}$$
- $$f(Z(U)) = \nu\, Z(U)^2, \tag{3.10}$$
- $$f(Z(U)) = \nu\, Z(U)1_{\{Z(U)>q\}}, \tag{3.11}$$
- $$f(Z(U)) = \nu\, \sqrt{Z(U)}, \tag{3.12}$$

where $\nu > 0$ is a positive parameter.

Regarding the insurance contract, first we used the proportional contract, a percentage of the risk is covered by the firm and the remaining percentage is covered by the insurance company.

Note that $Z(U)$ is the risk retained by the firm and $U - Z(U)$ is the risk transferred to the insurer. Therefore, the proportional contract is defined by:

$$Z(U) \ = \ a\,U, \tag{3.13}$$

$$U - Z(U) \ = \ (1-a)\,U. \tag{3.14}$$

Then, we consider a contract with a deductible, in which the firm cedes the risk above a certain limit M, i.e. the insurer only covers the losses in excess of M. The expense below this limit must be paid by the firm. Thus, the contract with a deductible is given by:

$$Z(U) \ = \ \begin{cases} U, & U \leq M \\ M, & U \ > \ M \end{cases}, \tag{3.15}$$

$$U - Z(U) \ = \ \begin{cases} 0, & U \leq M \\ U - M, & U \ > \ M \end{cases}. \tag{3.16}$$

Finally, we consider a contract with a limit, in which the firm cedes the risk below a certain limit L, i.e. the insurer only covers the losses for values smaller than L. The additional expense above this limit must be paid by the firm. Thus, the contract with a limit is given by:

$$Z(U) \ = \ \begin{cases} 0, & U \leq L \\ U - L, & U \ > \ L \end{cases}, \tag{3.17}$$

$$U - Z(U) \ = \ \begin{cases} U, & U \leq L \\ L, & U \ > \ L \end{cases}, \tag{3.18}$$

with $a, M, L \in (0, 1)$.

To find the optimal insurance policy for the firm, we have to proceed with the following maximization problem,

$$\max_{\gamma} \tilde{g}(x,\gamma) = \min_{\gamma} h(\gamma) = \min_{\gamma} \left\{ u + E(f(Z(U))) + \alpha c \right\} = h(\gamma), \tag{3.19}$$

where $Z(U;\gamma) \equiv Z(U)$ and γ represents a, M or L depending on the type of contract we have.

3.1 Expected Value Premium Principle

In this subsection, we assume that the premium principle is calculated according to the expected value principle (3.6). In the next propositions, we find the optimal contracts for each risk function.

In the next proposition we consider the risk function (3.9).

Proposition 4. *Let h be the function defined as in* (3.19), *f defined as in* (3.9), *and the insurance premium be defined as in* (3.6).

- *Proportional contract, $h(a)$*
  - *If $\nu > \alpha$, the function reaches the minimum at $\hat{a} = 0$, therefore the firm should transfer everything.*
  - *If $\nu < \alpha$, the function reaches the minimum at $\hat{a} = 1$, in this way the firm should not buy insurance.*
- *Contract with a deductible, $h(M)$*
  - *If $\nu > \alpha$ then the function reaches the minimum at $\hat{M} = 0$, the firm should transfer everything.*
  - *If $\nu < \alpha$ then the function reaches the minimum at $\hat{M} = 1$, hence the firm should retain everything.*
- *Contract with a limit, $h(L)$*
  - *If $\nu > \alpha$, then the function reaches the minimum at $\hat{L} = 1$, then the firm should transfer everything.*
  - *If $\nu < \alpha$, then the function reaches the minimum at $\hat{L} = 0$, thus the firm should retain everything.*

As we can see, when the expected value premium principle is used with the risk function (3.9), in each of the contracts, the best thing for the firm is either to insure everything or nothing, depending on the value of the parameters ν of the risk function and α of the safety loading.

Proposition 5. *Let h be the function defined as in* (3.19)*, f defined as in* (3.10)*, and the premium principle be defined as in* (3.6)*.*

- *Proportional contract, $h(a)$*
  - *If* [1] $2\nu\ u_2 - \alpha\ u \leq 0$*, then the function reaches the minimum at $\hat{a} = 1$ which means the firm should retain everything.*
  - *If $2\nu\ u_2 - \alpha\ u > 0$, then the function reaches the minimum at $\hat{a} = \frac{\alpha u}{2\nu u_2}$, and in that case it should retain that specific amount.*
- *Contract with a deductible, $h(M)$*
  - *If $2\nu - \alpha > 0$, then the function reaches the minimum at $\hat{M} = \frac{\alpha}{2\nu}$, the firm should retain up to that amount.*
  - *If $2\nu - \alpha \leq 0$, then the function reaches the minimum at $\hat{M} = 1$, in that case the firm should retain everything.*
- *Contract with a limit, $h(L)$*
  - *If $\alpha - 2\nu \geq 0$, the function reaches the minimum at $\hat{L} = 0$, in that case the firm should not acquire insurance.*
  - *If $\alpha - 2\nu < 0$, the function reaches the minimum at $\hat{L} = 0$ if $\nu u_2 < \alpha u$ and in that case the firm should not acquire insurance. However, if $\nu u_2 > \alpha u$, the function reaches the minimum at $\hat{L} = 1$, in that case the firm should transfer everything.*

Note that when the expected value premium principle is used with the function (3.10) for the proportional contract and the contract with a deductible it could be optimal to insure part of the risk but it is never optimal to insure everything.

Proposition 6. *Let h be the function defined as in* (3.19)*, f defined as in* (3.11)*, and the premium principle be defined as in* (3.6)*.*

- *Proportional contract, $h(a)$*
  - *When $a \leq q$ the function reaches the minimum at $\hat{a} = 1$, therefore it should retain everything.*
  - *When $a > q$ the firm should not insure values below q.*

[1] To simplify, from now on we assume $E(U^2) = u_2$

- *Contract with a deductible, $h(M)$*
  - *When $\alpha > \nu$ there are two hypotheses. The minimum is $\hat{M} = q$ when:*

$$^2\nu \; q \; S_U(q) > (\alpha - \nu) \int_q^1 S_U \; (u) \; du,$$

    in that case, the firm must retain up to the amount q. The minimum is $\hat{M} = 1$ when

$$\nu \; q \; S_U(q) < (\alpha - \nu) \int_q^1 S_U \; (u) \; du,$$

    in that case, the firm should retain everything.
  - *When $\alpha \leq \nu$, the minimum is $\hat{M} = q$, therefore the firm must retain up to the amount q.*
- *Contract with a limit, $h(L)$*
  - *In this case, the function reaches the minimum at $\hat{L} = 1 - q$, therefore the firm should acquire that amount of insurance.*

In the case of risk function (3.11) combined with the expected premium principle, we conclude that it can be optimal for the firm to retain everything or to transfer the specific amount q in the case of the contract with a deductible. In the case of the proportional contract it can be optimal to retain everything or just the amount q. In the contract with a limit it is always optimal to buy insurance.

Proposition 7. *Let h be the function defined as in* (3.19)*, f defined as in* (3.12)*, and the premium principle be defined as in* (3.6)*.*

- *Proportional contract, $h(a)$*
  - *If [3] $\frac{\nu}{2\sqrt{a}} u_3 - \alpha u > 0$, the minimum of the function is $\hat{a} = 0$, the firm should transfer everything.*
  - *If $\frac{\nu}{2\sqrt{a}} u_3 - \alpha u < 0$, the minimum of the function is $\hat{a} = \frac{\nu^2 u_3^2}{\alpha^2 u^2}$, the firm should retain that amount.*
- *Contract with a deductible, $h(M)$*
  - *If $\frac{\nu}{2\sqrt{M}} > \alpha$, the minimum of the function is $\hat{M} = 0$, therefore the firm should transfer everything.*
  - *If $\frac{\nu}{2\sqrt{M}} < \alpha$, the minimum of the function is $\hat{M} = 1$, therefore the firm should retain everything.*

[2] $S_U(u)$ is the survival function; it is related to the cumulative distribution function $F_U(u)$ in the following way: $S_U\;(u) = 1 - F_U\;(u)$. Note that $F_U\;(u) = \frac{\partial f_U\;(u)}{\partial u}$, in which $f_U\;(u)$ is the probability density function.

[3] To simplify, from now on we assume $E(\sqrt{U}) = u_3$

- *Contract with a limit, $h(L)$*
  - *The function reaches the minimum when $\hat{L} = \frac{\nu^2}{4\alpha^2}$, therefore the firm should buy that amount of insurance.*

In this case, it is always optimal to acquire insurance in the proportional contract and in the contract with a limit.

3.2 Variance Premium Principle

In this subsection, we assume that the premium principle is calculated according with 3.7. In the next propositions, we find the optimal contracts for each risk function.

In the next proposition we consider the risk function (3.9).

Proposition 8. *Let h be the function defined as in* (3.19)*, f defined as in* (3.9)*, and the premium principle be defined as in* (3.7)*.*

- *Proportional contract, $h(a)$*
  - *If $\nu\ u - 2\alpha\ \sigma_U^2 < 0$ the function reaches the minimum at $\hat{a} = 1 - \frac{\nu u}{2\alpha\ \sigma_U^2}$, therefore the firm should retain that amount of insurance.*
  - *If $\nu\ u - 2\alpha\ \sigma_U^2 \geq 0$, the function reaches the minimum at $\hat{a} = 0$, the firm should transfer everything.*
- *Contract with a deductible, $h(M)$*
  - *If $Var_U < \nu\ u$, the function reaches the minimum at $\hat{M} = 0$, therefore the firm should transfer everything.*
  - *If $Var_U > \nu\ u$, the function reaches the minimum at $\hat{M} = 1$, the firm should retain everything.*
- *Contract with a limit, $h(L)$*
  - *If $(-\nu) + 2\alpha\ (1-u) < 0$, then the function reaches the minimum at $\hat{L} = 1$, therefore the firm should insure everything.*
  - *If $(-\nu) + 2\alpha\ (1-u) > 0$, there is a minimum when $\int_0^L S_U\ (u)\ du = \frac{\nu}{2\alpha}$ in that case the firm must insure that specific amount of insurance.*

Here, note that in the proportional contract and the contract with a limit it is always optimal to insure. However, in the contract with a deductible it can be optimal to retain or to transfer everything, depending on the value of the parameters.

Proposition 9. *Let h be the function defined as in* (3.19)*, f defined as in* (3.10)*, and the premium principle be defined as in* (3.7)*.*

- *Proportional contract, $h(a)$*
  - *The function reaches the minimum at $\hat{a} = \frac{\alpha\ \sigma_U^2}{\nu\ u_2\ +\ \alpha\ \sigma_U^2}$, therefore the firm should retain that amount.*
- *Contract with a deductible, $h(M)$*
  - *If $\nu \geq \alpha$, then the function reaches the minimum at $\hat{M} = 0$, then the firm should transfer everything.*
  - *If $\nu < \alpha$, there are two cases, if $\alpha\ Var_U < \nu\ u_2$, the function reaches the minimum at $\hat{M} = 0$, the firm should transfer everything as well and if $\alpha\ Var_U > \nu\ u_2$ the opposite happens, the function reaches the minimum at $\hat{M} = 1$ thus the firm should not buy insurance.*
- *Contract with a limit, $h(L)$*
  - *If $\alpha \leq \nu$, the function reaches the minimum at $\hat{L} = 1$, it means that the firm should transfer everything.*
  - *If $\alpha > \nu$, the function reaches the minimum when $L\ (\alpha - \nu) = \alpha \int_0^L S_U\ (u)\ du$, the firm should acquire that amount of insurance.*

In this case, it is always optimal to insure for both proportional contract and contract with a limit.

Proposition 10. *Let h be the function defined as in* (3.19)*, f defined as in* (3.11)*, and the premium principle be defined as in* (3.7)*.*

- *Proportional contract, $h(a)$*
  - *If $a \leq q$ the function reaches the minimum at $\hat{a} = 1$, thus the firm should retain everything.*
  - *The firm should not insure values below q.*
- *Contract with a deductible, $h(M)$*
  - *If $M > \int_M^1 S_U\ (u)\ du$ and $\nu < 2\alpha \left(\int_M^1 S_U\ (u)\ du - M \right)$, the minimum is $\hat{M} = q$ when $u + \nu \int_q^1 u\ S_U\ (u)\ du + \nu\ q\ S_U(q) > u + 2\alpha \int_q^1 uS_U\ (u)\ du - \alpha \left(\int_q^1 S_U\ (u)\ du \right)^2$, in this case the firm should retain the insurance up to amount q. The minimum is $\hat{M} = 1$ when $u + \nu \int_q^1 u\ S_U\ (u)\ du + \nu\ q\ S_U(q) < u + 2\alpha \int_q^1 u\ S_U\ (u)\ du - \alpha \left(\int_q^1 S_U\ (u)\ du \right)^2$, in this case the firm should retain everything.*

- *Contract with a limit,* $h(L)$
  - *If* $L > \int_0^L S_U\ (u)\ du$*, the function reaches the minimum at* $\hat{L} = 1 - q$*, thus the firm should buy that amount of insurance.*
  - *If* $L < \int_0^L S_U\ (u)\ du$*, the function reaches the minimum at* $\hat{L} = 1$*, thus the firm should transfer everything.*

Note that in the contract with a deductible it is never optimal to insure everything. In the contract with the limit it is always optimal to insure.

Proposition 11. *Let* h *be the function defined as in* (3.19)*,* f *defined as in* (3.12)*, and the premium principle be defined as in* (3.7)*.*

- *Proportional contract,* $h(a)$
  - *If alpha* $\sigma_U^2 > \nu\ u_3$*, the minimum is* $\hat{a} = 1$*, thus the firm should retain everything.*
  - *If alpha* $\sigma_U^2 < \nu\ u_3$*, the minimum is* $\hat{a} = 0$*, consequently the firm should transfer everything.*
- *Contract with a deductible,* $h(M)$
  - *If* $\int_M^1 S_U\ (u)\ du > M$*, the minimum of the function is* $\hat{M} = 0$*, consequently the firm should insure everything.*
  - *If* $\int_M^1 S_U\ (u)\ du < M$*, there are two hypotheses. When* $2\alpha\ Var_U < \frac{1}{2}\nu\ u_3$*, the function reaches the minimum at* $\hat{M} = 0$*, thus the firm should transfer everything. When* $2\alpha\ Var_U > \frac{1}{2}\nu\ u_3$*, the function reaches the minimum at* $\hat{M} = 1$*, therefore it should retain everything.*
- *Contract with a limit,* $h(L)$
  - *If* $L \leq \int_0^L S_U\ (u)\ du$*, the minimum of the function is* $\hat{L} = 1$*, consequently the firm should insure everything.*
  - *If* $L > \int_0^L S_U\ (u)\ du$*, the function reaches a minimum when* $2\alpha\left(L - \int_0^L\ S_U\ (u)\ du - L\right) = \frac{\nu}{2\sqrt{L}}$*, which means the firm should buy that specific amount of insurance.*

Here, in the contract with a limit it is always optimal to insure. In the other two contracts it can either be better to hold everything or to transfer everything, depending on the parameters.

3.3 Standard Deviation Premium Principle

In this subsection, we assume that the premium principle is calculated according with (3.8). In the next propositions, we find the optimal contracts for each risk function.

In the next proposition we consider the risk function (3.9).

Proposition 12. *Let h be the function defined as in* (3.19)*, f defined as in* (3.9)*, and the premium principle be defined as in* (3.8)*.*

- *Proportional contract, $h(a)$*
  - *If $u\ \nu > \alpha\ \sigma_U$, the function reaches the minimum in $\hat{a} = 0$, therefore the firm should transfer everything.*
  - *If $u\ \nu < \alpha\ \sigma_U$, the function reaches the minimum in $\hat{a} = 1$, therefore the firm should retain everything.*
- *Contract with a deductible, $h(M)$*
  - *The minimum is $\hat{M} = 0$ when $\alpha\ \sigma_U < \nu\ u$, therefore the firm should transfer everything.*
  - *The minimum is $\hat{M} = 1$ if $\alpha\ \sigma_U > \nu\ u$, consequently the firm should retain everything.*
- *Contract with a limit, $h(L)$*
  - *If $\frac{\alpha}{\sigma_U}(1-u) \leq \nu$ then the function reaches the minimum at $\hat{L} = 1$, therefore the firm should transfer everything.*
  - *If $\frac{\alpha}{\sigma_U}(1-u) > \nu$, $0 < \hat{L} < 1$. In this case there is no explicit expression for the minimum.*

In this case, we see that in the contract with a limit, it is always optimal to insure, everything or just part of it. In the other two contract it can either be better to hold everything or nothing, depending on the parameters.

Proposition 13. *Let h be the function defined as in* (3.19)*, f defined as in* (3.10)*, and the premium principle be defined as in* (3.8)*.*

- *Proportional contract, $h(a)$*
  - *If $2\nu\ u_2 - \alpha\ \sigma_U \leq 0$, then $\hat{a} = 1$ is the minimum of the function, hence the firm should retain everything.*
  - *If $2\nu\ u_2 - \alpha\ \sigma_U > 0$, then the function reaches the minimum at $\hat{a} = \frac{\alpha\ \sigma_U}{2\nu\ u_2}$, consequently the firm should retain that amount.*
- *Contract with a deductible, $h(M)$*
  - *If $\sigma_U \alpha < \nu\ u_2$, the function reaches the minimum at $\hat{M} = 0$, thus the firm should transfer everything.*
  - *If $\sigma_U \alpha > \nu\ u_2$, the function reaches the minimum at $\hat{M} = 1$, therefore the firm should not acquire insurance.*

- *Contract with a limit, $h(L)$*
  - *The function reaches a minimum $0 < \hat{L} < 1$, since $1 - S_U(L) > 0$. which means the firm should acquire a specific amount of insurance. In this case there is no explicit expression for the minimum.*

Here, we see that in the proportional contract it is optimal just to retain a part of the risks or to retain it all. In the contract with a limit it is always optimal to insure.

Proposition 14. *Let h be the function defined as in* (3.19)*, f defined as in* (3.11)*, and the premium principle be defined as in* (3.8)*.*

- *Proportional contract, $h(a)$*
  - *When $a \leq q$ the function reaches the minimum at $\hat{a} = 1$, thus the firm should retain everything.*
  - *When $a > q$ the firm should not insure values below q.*
- *Contract with a deductible, $h(M)$*
  - *If $M > \int_M^1 S_U\ (u)\ du$ and $\nu < 2\alpha\ (\int_M^1 S_U\ (u)\ du - M)$, the minimum is $\hat{M} = q$ when $u + \nu \int_q^1 u\ S_U\ (u)\ du + \nu\ qS_U(q) > u + 2\alpha \int_q^1 u\ S_U\ (u)\ du - \alpha\ (\int_q^1 S_U\ (u)\ du)^2$, in this case the firm should retain up to the amount q. The minimum is $\hat{M} = 1$ when $u + \nu \int_q^1 u\ S_U\ (u)\ du + \nu\ q\ S_U(q) < u + 2\alpha \int_q^1 u\ S_U\ (u)\ du - \alpha\ (\int_q^1 S_U\ (u)\ du)^2$, in this case the firm should retain everything.*
- *Contract with a limit, $h(L)$*
  - *If $L > \int_0^L S_U\ (u)\ du$, the function reaches the minimum at $\hat{L} = 1 - q$, thus the firm should transfer that amount of insurance.*
  - *If $L < \int_0^L S_U\ (u)\ du$, the function reaches the minimum at $\hat{L} = 1$, thus the firm should transfer everything.*

In this case, it is always optimal to insure in the contract with a limit.

Proposition 15. *Let h be the function defined as in* (3.19)*, f defined as in* (3.12)*, and the premium principle be defined as in* (3.8)*.*

- *Proportional contract, $h(a)$*
  - *If $\alpha\ \sigma_U < \nu\ u_3$, the minimum is $\hat{a} = 0$ which means the firm should transfer everything.*
  - *If $\alpha\ \sigma_U > \nu\ u_3$, the minimum is $\hat{a} = 1$ consequently the firm should not acquire insurance.*

- *Contract with a deductible,* $h(M)$
  - *If* $\alpha\ \sigma_U < \frac{1}{2}\nu\ u_3$, *the minimum is* $\hat{M} = 0$ *thus the firm should transfer everything.*
  - *If* $\alpha\ \sigma_U > \frac{1}{2}\nu\ u_3$, *the minimum is* $\hat{M} = 1$ *consequently the firm should retain everything.*
- *Contract with a limit,* $h(L)$
  - *The function reaches a minimum* $0 < \hat{L} < 1$, *since* $1 - S_U(L) > 0$. *which means the firm should acquire a specific amount of insurance. In this case there is no explicit expression for the minimum.*

In the proportional contract and in the contract with a deductible it is optimal either to insure everything, or nothing, depending on the parameters. In the contract with a limit it is always optimal to insure a certain amount.

4 Numerical results

In this section, we will see some numerical results. First, we consider the expected value premium principle, defined as in (3.6) and the risk function (3.10) to understand what the firm's strategy should be regarding the amount of insurance it should purchase. In this analysis, we will use two different distributions to see how the firm's strategy changes depending on the risk distribution. For this, we will establish the following parameters: $K = 1$, $r = 0.05$, $\mu = 0.01$, $\sigma = 0.2$, $\lambda = 0.05$, $u = 0.1$, $a = 0.5$, $M = 0.5$, $L = 0.5$, $\nu = 0.5$ and $\alpha = 0.3$.

Recall that K is the investment cost, r is the interest rate, μ is the drift, σ is the volatility, $lambda$ is intensity of the PP, $u = E(U)$, the mean value of the r.v U, ν is the coefficient of the risk function, α is the coefficient of the safety loading, a, M, L are the parameters of the proportional contract, contract with a deductible and contract with a limit, respectively.

The distributions used are defined by:

- $U \sim Uniform\ (0, 1)$.
- $U \sim F(x) = x^2,\ x \in (0, 1)$.

The function h, defined in (3.19), considering the proportional contract, is depicted in Figures 2 and 3.

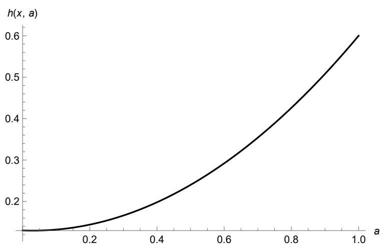

Figure 2: h function considering the proportional contract and $U \sim Uniform\ (0, 1)$.

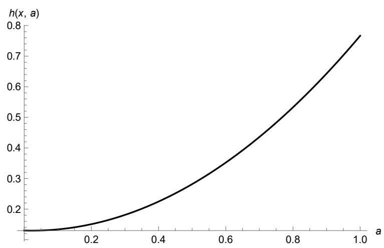

Figure 3: h function considering the proportional contract and $U \sim F(x) = x^2,\ x \in (0, 1)$.

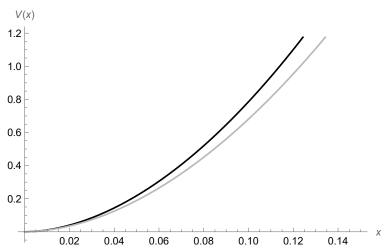

Figure 4: Value functions considering the proportional contract for both $U \sim Uniform\ (0, 1)$ (black line) and $U \sim F(x) = x^2,\ x \in (0, 1)$ (gray line).

When we minimize the function (3.19) with these parameters, it is possible to observe in the Figure 2 that the function reaches a minimum at $\hat{a} = 0.03$ and in Figure 3 reaches the minimum at $\hat{a} = 0.0225$. The amount the firm must insure is very similar on both distributions. As expected by Proposition 5, this happens since $2\nu\ u_2 - \alpha u > 0$. The firm's investment decision changes when the size of negative occurrences is modelled by different distributions. In Figure 4 the value functions are depicted. We know that the firm's investment decision is anticipated with the uniform distribution, since the threshold of the revenue, x^*, is smaller and the risk is evenly distributed as can be seen in Table 1.

The function h, defined in (3.19), considering an insurance policy with a deductible is depicted in Figures 5 and 6.

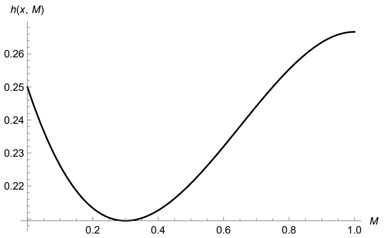

Figure 5: h function considering the contract with a deductible and $U \sim Uniform\ (0,1)$.

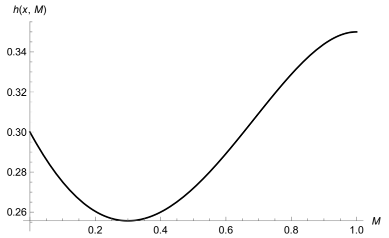

Figure 6: h function considering the contract with a deductible and $U \sim F(x) = x^2,\ x \in (0,1)$.

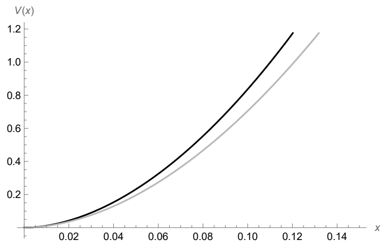

Figure 7: Value functions considering the contract with a deductible for both $U \sim Uniform\ (0,1)$ (black line) and $U \sim F(x) = x^2,\ x \in (0,1)$ (gray line).

We can verify that the functions present the same minimum. The minimizer is $\hat{M} = 0.3$, which means that the firm's strategy remains the same in both distributions. As expected by Proposition 5, this happens since $2\nu - \alpha > 0$.

The firm's investment decision will also change when the size of negative occurrences is modelled by different distributions. In Figure 7, we see the value functions in this case. Furthermore, we know that the firm's investment decision is anticipated with the Uniform distribution since the threshold of the revenue, x^*, is smaller, as you can see in Table 1.

The function h, defined in (3.19), considering an insurance policy with a limit is depicted in Figures 8 and 9.

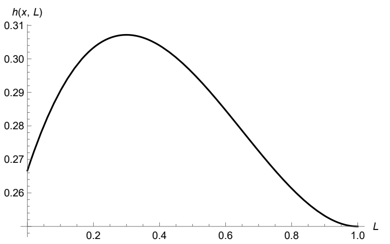

Figure 8: h function considering the contract with a limit and $U \sim Uniform\ (0,1)$.

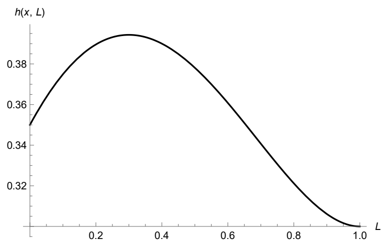

Figure 9: h function considering the contract with a limit and $U \sim F(x) = x^2,\ x \in (0,1)$.

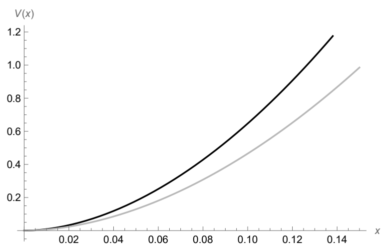

Figure 10: Value functions considering the contract with a limit for both $U \sim Uniform\ (0,1)$ (black line) and $U \sim F(x) = x^2,\ x \in (0,1)$ (gray line).

In the contract with a limit the functions have the same minimum. The minimizer is $\hat{L} = 1$, which means that the firm should transfer everything. As expected by Proposition 5, this happens because $\nu\ u_2 > \alpha\ u$.

The value functions are depicted in Figure 10. In this case, the firm takes the decision to anticipate the investment with $U \sim Uniform\ (0,1)$, since it is facing a lower threshold, x^*, than $U \sim F(x) = x^2,\ x \in (0,1)$ as you can see in Table 1.

Table 1: Optimal threshold: comparison between distributions for each contract.

Expected value premium principle			
x^*	Proportional contract	Contract with a Deductible	Contract with a Limit
$U \sim Uniform\ (0,1)$	0.124308	0.120194	0.138074
$U \sim F(x) = x^2,\ x \in (0,1)$	0.134301	0.131811	0.165007

So far, we have assumed that the premium principle used by the insurance company is the expected value premium principle. Next we will consider the standard deviation premium principle to proceed with a comparison between the two premium principles. We decided to fix the risk function (3.12) and the distribution $U \sim F(x) = x^2,\ x \in (0,1)$.

The function h, defined in (3.19), considering a proportional contract is depicted in Figures 11 and 12.

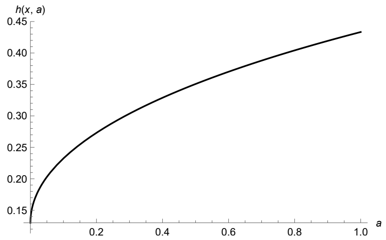

Figure 11: h function considering the proportional contract and the expected value premium principle.

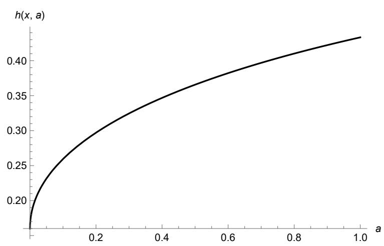

Figure 12: h function considering the proportional contract and the standard deviation value premium principle.

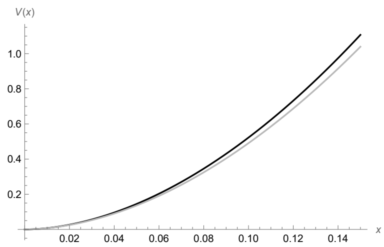

Figure 13: Value functions considering the proportional contract and the expected value (black line) and the standard deviation (gray line) premium principles.

In this case, the function reaches the minimum at $\hat{a} = 0$ on both premium principles. As expected in proposition 7, since $\frac{\nu}{2\sqrt{a}}\ u_3^2 > \alpha\ u$ and in proposition 15, since $\alpha\ \sigma_U < \nu\ u_3$. Thus the firm should insure everything.

The firm's investment decision will also change when the size of negative occurrences is modelled by different premium principles. In Figure 13, both value functions are depicted. Furthermore we know that the firm's investment decision is anticipated with the expected value premium principle although they do not differ much, as you can see in Table 2.

The function h, defined in (3.19), considering an insurance policy with a deductible is depicted in Figures 14 and 15.

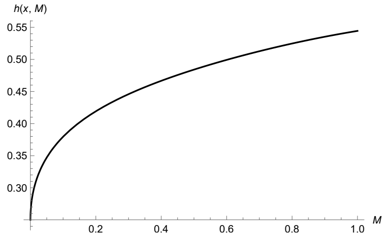

Figure 14: h function considering the contract with a deductible and the expected value premium principle.

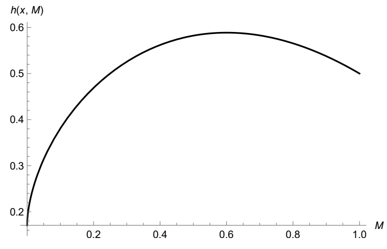

Figure 15: h function considering the contract with a deductible and the standard deviation value premium principle.

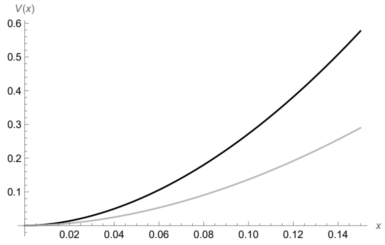

Figure 16: Value functions considering the contract with a deductible and both expected value and the standard deviation premium principles.

For both cases, the minimum is $\hat{M} = 0$, as expected by Proposition 7, since $\frac{\nu}{2\sqrt{M}} < \alpha$ and by Proposition 15, since $\alpha\ \sigma_U < \frac{1}{2}\nu\ u_3$. Therefore the firm should transfer everything.

The firm's investment decision will also change when the size of negative occurrences is modelled by different premium principles. In Figure 16, both value functions are depicted. In this case, the firm's investment decision is anticipated. The threshold of the revenue, x^*, is smaller for the expected value as depicted in Table 2.

The function h, defined in (3.19), considering an insurance policy with a limit is depicted in Figures 17 and 18.

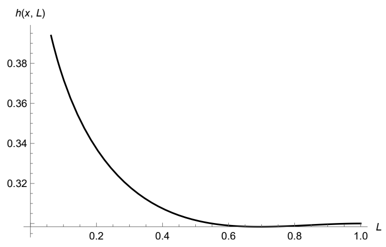

Figure 17: h function considering the contract with a limit and the expected value premium principle.

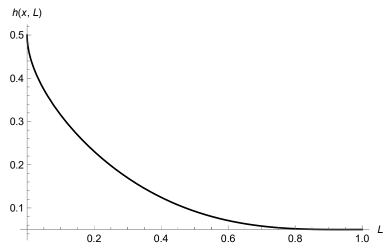

Figure 18: h function considering the contract with a limit and the standard deviation value premium principle.

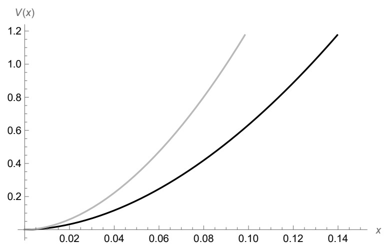

Figure 19: Value functions considering the contract with a limit and both expected value and the standard deviation premium principles.

The decision of how much insurance the firm should acquire diverges a lot between the premium principles. For the expected value the minimum is $\hat{L} = 0.69$ and for the standard deviation is $\hat{L} = 0.93$, therefore those are the amounts bellow which the firm should insure. As expected by Proposition 7 and in Proposition 15, both premium principles have a minimizer $0 < \hat{L} < 1$.

We can get a better insight into the firm's investment decision by looking at the value function which is depicted in Figure 19.

In this case, the conclusion is different to the previous one, we can see that the decision to invest is anticipated for the standard deviation. The threshold, x^*, when the standard deviation is used is smaller than the threshold, x^*, when the expected value is used, as presented in Table 2.

Table 2: Optimal threshold: comparison between premium principles for each contract.

$U \sim F(x) = x^2,\ x \in (0,1)$			
x^*	Proportional contract	Contract with a Deductible	Contract with a Limit
Expected value	0.154936	0.220369	0.139679
Standard Deviation	0.160287	0.319594	0.0983574

5 Conclusions

There are several parameters that ultimately intervene in the firm's decision on whether and how to buy insurance. Those are the premium used by the insurance company, the function to measure the risk of the firm and also the type of contract acquired by the firm.

When we fix the expected value premium principle, we conclude that with the risk function (3.12), it is always optimal to buy insurance in the case of the proportional contract and the contract with a limit. It is also always optimal to buy insurance in the contract with a limit with the risk function (3.11).

When we fix the variance premium principle and use the risk functions (3.9) and (3.10), it is always optimal to insure in the case of the proportional contract and the contract with a limit. Furthermore, when we use the risk functions (3.11) and (3.12), it is always optimal to acquire insurance in the contract with a limit.

When we fix the standard deviation premium principle it is always optimal to insure for all risk functions in the case of the contract with a limit.

The way risk is handled has a major impact on a firm's strategy. When risk is treated uniformly, with the $U \sim Uniform\ (0,1)$ distribution compared to the $U \sim F(x) = x^2,\ x \in (0,1)$ distribution, the firm's decision is always to anticipate the investment. The threshold, x^*, is smaller in the uniform distribution.

The firm's investment decision will also change when the size of negative occurrences is modelled by different premium principles. The standard deviation premium principle takes more into account the deviations, so the tail of the distribution has more weight. In the contract with a deductible we are ceding the tail to the insurance company. Then, the insurance is more expensive with the standard deviation premium principle. In this way, the firm anticipates the investment with the expected value premium principle. In the contract with a limit we are retaining the tail of the distribution, therefore the insurance contract becomes cheaper. Consequently, the firm anticipates the investment with the standard deviation premium principle.

A Proofs

Proof of Proposition 1. *To find the parameter A and the threshold x^* were used the following smooth pasting conditions to make sure the function is smooth and continuous.*

$$\begin{cases} A{x^*}^{\beta_1} &= g(x^*) \\ (A{x^*}^{\beta_1})' &= g(x^*)' \end{cases}.$$

The function $g(x)$ is given by 2.2, therefore the derivatives are given by:

$$\begin{aligned} (A{x^*}^{\beta 1})' &= \beta_1 A x^{*(\beta_1 - 1)}, \\ g(x^*)' &= \frac{1}{r-\mu} - \frac{u\ \lambda}{(r-\mu)^2} - \frac{E(f(U_i))\ \lambda}{(r-\mu)^2}. \end{aligned}$$

After solving the system of equations, it was possible to obtain the parameters A and x^*.

Proof of Proposition 2. *It is possible to see the behavior of the function after there is a change in one of its parameters, making an analysis of the sign of the first derivative in order to that parameter. The first derivative in order to λ is given by:*

$$\frac{\partial x^*}{\partial \lambda} = \frac{K\ \beta 1\ (r-\mu)^2\ (\beta 1 - 1)\ \big(u + E(f(U))\big)}{\Big[(\beta 1 - 1)\ (r-\mu) - (\beta 1 - 1)\ \Big(\lambda\ \big(u + E(f(U))\big)\Big)\Big]^2}.$$

Note that $\beta_1 > 1$, therefore it is possible to see that $\frac{\partial x^*}{\partial \lambda} > 0$, since:

$$\begin{aligned} &K\ \beta 1\ (r-\mu)^2\ (\beta 1 - 1)\ \big(u + E(f(U))\big) > 0 \\ &\Big[(\beta 1 - 1)\ (r-\mu) - (\beta 1 - 1)\ \big(\lambda\ (u + E(f(U)))\big)\Big]^2 > 0. \end{aligned}$$

It is possible to perceive that the functions $\beta i(\sigma)$, with i = 1, 2, are such that the function $\beta 1(\sigma)$ is decreasing and $\beta 2(\sigma)$ is increasing. This follows in view of the following derivative:

$$\frac{\partial \beta i}{\partial \sigma} = (-1)^i \frac{\sigma\ \beta i\ (\beta i - 1)}{\sqrt{\left(u - \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 r}}.$$

We can write the derivative in order to σ as follows:

$$\frac{\partial x^*}{\partial \sigma} = \frac{K\ (r-\mu)^2\ \frac{\partial \beta 1}{\partial \sigma}}{(\frac{\partial \beta 1}{\partial \sigma}-1)\ (r-\mu) - (\frac{\partial \beta 1}{\partial \sigma}-1)\ \big(\lambda\ (u + E(f(U)))\big)}.$$

Since $K\ (r-\mu)^2\ \frac{\partial \beta 1}{\partial \sigma} < 0$ and $(\frac{\partial \beta 1}{\partial \sigma}-1)\ (r-\mu) - (\frac{\partial \beta 1}{\partial \sigma}-1)\ \big(\lambda\ (u+E(f(U)))\big) < 0$, $\frac{\partial x^*}{\partial \sigma} > 0$.

Proof of Proposition 3. *To find the parameter A_{AI} and the threshold x_{AI} were used the following smooth pasting conditions once again to make sure the function is smooth and continuous.*

$$\begin{cases} A_{AI} x_{AI}^{*(\beta 1-1)} &= \tilde{g}(x^*) \\ (A_{AI} x_{AI}^{*(\beta 1-1)})' &= (\tilde{g}(x^*))' \end{cases}.$$

The function $\tilde{g}$ is given by 3.2, therefore the derivatives are as follows:

$$\begin{aligned} (A_{AI} x_{AI}^{*\beta 1})' &= \beta 1 A_{AI} x_{AI}^{*(\beta 1-1)}, \\ g(x^*)' &= \frac{1}{r-\mu} - \frac{u\ \lambda}{(r-\mu)^2} - \frac{E\big(f(U_i)\big)\ \lambda}{(r-\mu)^2}. \end{aligned}$$

After solving the system of equations, it was possible to obtain A_{AI} and x_{AI}^*.

Proof of Proposition 4. *We are using the maximization problem defined as in* (3.19). *Note that f is defined as in* (3.9), *and the premium principle is defined as in* (3.6).

- *Proportional contract*

  In this case, the contract is defined as in (3.13) *and* (3.14), *thus* $\nu\ E\big(f(Z(U)\big) = \nu\ E(a\ U) = \nu a u$ *and* $\alpha c = \alpha\ E\big(U - Z(U)\big) = \alpha\ E\big((1-a)\ U\big) = \alpha\ (1-a)\ u$. *Thus, the function h defined by* (3.19) *is as follows:*

  $$\sup_{0<a<1} h(x,a) = \min_{0<a<1} \big\{u + \nu\ a\ u + \alpha\ (1-a)\ u\big\}.$$

  The derivative of the function is as follows:

  $$\frac{\partial h}{\partial a} = u\ (\alpha - \nu).$$

  If $\nu > \alpha$, the function reaches the minimum at $\hat{a} = 0$ and if $\nu < \alpha$, the function reaches the minimum at $\hat{a} = 1$.

- *Contract with a deductible*

  First, note that the expected value of $Z(U)$ and $U - Z(U)$ is computed as follows:

$$
\begin{aligned}
E\big(Z(U)\big) &= \int_0^M u \ f_U \ (u) \ du + M \int_0^M f_U \ (u) \ du \\
&= \Big[-u \ S_U \ (u) \Big]_0^M + \int_0^M S_U \ (u) \ du + M \ P(U > M) \\
&= -M \ S_U(M) + \int_0^M S_U \ (u) \ du + M \ S_U(M) \\
&= \int_0^M S_U \ (u) \ du.
\end{aligned}
$$

$$
\begin{aligned}
E\big(U - Z(U)\big) &= \int_0^M 0 \ du + \int_M^1 (U - M) f_U \ (u) \ du \\
&= \int_M^1 u \ f_U \ (u) \ du - M \ P(U > M) \\
&= \Big[-u \ S_U \ (u) \Big]_M^1 + \int_M^1 S_U \ (u) \ du - M \ P(U > M)) \\
&= \int_M^1 S_U \ (u) \ du.
\end{aligned}
$$

In this case, the contract is defined as in (3.15) *and* (3.16), *thus* $\nu E\big(f(Z(U))\big) = \nu \int_0^M S_U(u) du$ *and* $\alpha c = \alpha \int_M^1 S_U \ (u) \ du$. *Therefore, the function* h *defined by* (3.19) *is as follows:*

$$
\sup_{0<a<1} h(x, M) = \min_{0<M<1} \left\{ u + \nu \int_0^M S_U \ (u) \ du + \alpha \int_M^1 S_U \ (u) \ du \right\}.
$$

The derivative of the function is as follows:

$$
\frac{\partial h}{\partial M} \ = \ S_U(M) \ (\nu - \alpha).
$$

If $\nu > \alpha$ *the function reaches the minimum at* $\hat{M} = 0$ *and if* $\nu < \alpha$ *the function reaches the minimum at* $\hat{M} = 1$.

- *Contract with a limit*

  In this case the insurance policy is defined as in (3.17) *and* (3.18), *thus* $\nu E\big(f(Z(U))\big) = \nu \int_L^1 S_U\ (u)\ du$ *and* $\alpha c = \alpha \int_0^L S_U\ (u)\ du$. *Therefore it follows that:*

$$\sup_{0<L<1} h(x,L) = \min_{0<L<1} \left\{ u + \nu \int_L^1 S_U(u)du + \alpha \int_0^L S_U(u)du \right\}.$$

  To study the behavior of the function we calculate the derivative, which is given by:

$$\frac{\partial h}{\partial L} \;=\; S_U(L)\ (\alpha - \nu).$$

If $\alpha > \nu$ the function reaches the minimum at $\hat{L} = 0$ and if $\alpha < \nu$ the function reaches the minimum at $\hat{L} = 1$.

Proof of Proposition 5. *Note that f is defined as in* (3.10), *and the premium principle is defined as in* (3.6).

- *Proportional contract*

  In this case the insurance policy is defined as in (3.13) *and* (3.14), *thus* $\nu\ E\big(f(Z(U)^2\big) = \nu\ E(a\ U)^2 = \nu a^2 u_2$, *and* $\alpha c = \alpha E\big(U - Z(U)\big) = \alpha E\big((1-a)\ U\big) = \alpha\ (1-a)\ u$. *Therefore, the function h defined by* (3.19) *is as follows:*

$$\sup_{0<a<1} h(x,a) = \min_{0<a<1} \big\{ u + \nu a^2\ u_2 + alpha\ (1-a)\ u \big\}.$$

  To find the minimum of the function we calculate the following derivative:

$$\frac{\partial h}{\partial a} = 0. \tag{A.1}$$

  Therefore, the minimum is given by:

$$\hat{a} = \frac{\alpha\ u}{2\nu\ u_2}. \tag{A.2}$$

  If $\frac{\partial h}{\partial a} < 0$, *then* $a < \hat{a}$ *and if* $\frac{\partial h}{\partial a} > 0$, *then* $a > \hat{a}$. *Therefore,* $\hat{a} = 1$ *if* $2\nu\ u_2 \leq 0$ *and* $\hat{a} = \frac{\alpha\ u}{2\nu\ u_2}$ *if* $2\nu\ u_2 > 0$.

- *Contract with a deductible*

  In this case, the contract is defined as in (3.17) *and* (3.18), *thus* $E\big(f(Z(U)\big) = 2\nu\ \big(\int_0^M u\ S_U\ (u)\ du\big)$ *and* $\alpha c = \alpha \int_M^1 S_U\ (u)\ du$. *Therefore, the function* h *defined by* (3.19) *is as follows:*

$$\sup_{0<M<1} h(x,M) = \min_{0<M<1}\left\{u + 2\nu\left(\int_0^M u\ S_U\ (u)\ du\right) + \alpha\int_M^1 S_U(u)du\right\}.$$

  We calculated the derivative to perceive the comportment of the function:

$$\frac{\partial h}{\partial M} = S_U(M)\ (2M\nu - \alpha).$$

  If $2\nu - \alpha > 0$, *then the function reaches the minimum at* $\hat{M} = \frac{\alpha}{2\nu}$ *and if* $2\nu - \alpha < 0$, *the function reaches the minimum at* $\hat{M} = 1$.

- *Contract with a limit*

  Here, the insurance policy is defined as in (3.17) *and* (3.18), *therefore* $E\big(f(Z(U)\big) = 2\nu\ \big(\int_L^1 u\ S_U\ (u)\ du\big)$ *and* $\alpha c = \alpha\ \int_0^L S_U\ (u)\ du$. *Thus, the function* h *defined by* (3.19) *is as follows:*

$$\sup_{0<L<1} h(x,L) = \min_{0<L<1}\left\{u + 2\nu\ \left(\int_0^L u\ S_U\ (u)\ du\right) + \alpha\int_L^1\ S_U(u)\ du\right\}.$$

  The derivative of the equation follows from:

$$\frac{\partial h}{\partial L} = S_U(L)\ (\alpha - 2\nu\ L).$$

  When $\alpha - 2\nu \geq 0$, *the function reaches the minimum at* $\hat{L} = 0$ *and when* $\alpha - 2\nu < 0$ *the function increases and decreases so the minimum can be* $\hat{L} = 0$ *or* $\hat{L} = 1$. *So, when* $\nu u_2 < \alpha u$ *the minimum is* $\hat{L} = 0$ *and when* $\nu u_2 > \alpha u$ *the minimum is* $\hat{L} = 1$.

Proof of Proposition 6. *Note that* f *is defined as in* (3.11), *and the premium principle is defined as in* (3.6).

- *Proportional contract*

  Here, the insurance policy is defined as in (3.13) *and* (3.14). *First, note that:*

$$\begin{aligned} E(f(Z(U))) =& E\Big(\nu a\ U\ 1_{\{a\ U>\ q\}}\Big) \\ =& \nu a \int_0^1 U\ 1_{\{a\ U>\ q\}} f_U\ (u)\ du. \end{aligned}$$

To calculate the expected value of the function, were made the following steps:

$$1_{\{a\ U\ >\ q\}} = \begin{cases} 1, & U > \frac{q}{a} \\ 0, & U < \frac{q}{a} \end{cases}.$$

Therefore we have that:

$$E(\nu\ a\ U\ 1_{\{a\ U\ >\ q\}} = \begin{cases} 0, & a \leq q \\ \nu a \int_{\frac{q}{a}}^{1} u\ f_U\ (u)\ du, & a > q \end{cases}.$$

Thus, the function h defined by (3.19) *is as follows:*

$$h(x,a) = \begin{cases} u + \alpha\ (1-a)\ u, & a \leq q \\ u + \nu a \int_{\frac{q}{a}}^{1} u\ f_U\ (u) du + \alpha\ (1-a)\ u, & a > q \end{cases}.$$

When $a \leq q$ the function decreases since $u + \frac{\partial h}{\partial a} = -\alpha\ u$, therefore the function reaches the minimum at $\hat{a} = 1$. When $a > q$, $\frac{\partial h}{\partial a} = \nu\ \left(\int_{\frac{q}{a}}^{1} u\ f_U\ (u)\ du + \frac{q^2}{a^2}\ f_U\left(\frac{q}{a}\right)\right) - \alpha\ u$.

- *Contract with a deductible*

  Here, the insurance policy is defined as in (3.15) *and* (3.16)*. First notice that,*

$$Z(U)1_{\{Z(U)\}} = 0, \quad \textit{when } M \leq q.$$

Therefore, it follows that:

$$Z(U)1_{\{Z(U)>q\}} = \begin{cases} 0, & U < q \\ U, & q < U < M, \\ M, & U \geq M \end{cases} \qquad \textit{when } M > q\ .$$

Note that in this case, $E(Z(U))$ is as follows:

$$\begin{aligned} E(Z(U)) = E(Z(u)1_{\{Z(U)>q\}}) &= \int_q^M u\ f_U\ (u)\ du + \int_M^1 M\ f_U\ (u)\ du \\ &= \Big[-u\ S_U\ (u)\ du \Big]_q^M + \int_q^M S_U\ (u)\ du + M\ P(U > M) \\ &= \int_q^M S_U\ (u)\ du + q\ S_U(q). \end{aligned}$$

Thus, the function h defined by (3.19) is as follows:

$$h(x,a) = \begin{cases} u + \alpha \int_M^1 S_U\ (u)\ du, & M \leq q \\ u + \int_q^M S_U\ (u)\ du + q\ S_U(q) + \alpha \int_M^1 S_U\ (u)\ du, & M > q \end{cases}.$$

When $M \leq q$, $\frac{\partial}{\partial M} = -\alpha\ S_U(M)$ and when $M > q$ $\frac{\partial h}{\partial M} = S_U(M)(\nu - \alpha)$. Therefore, there are two hypotheses, if $\alpha > \nu$ the function is decreasing with M and if $\alpha < \nu$ the function is increasing with M.

Therefore when $\alpha > \nu$:

- *The minimum is $\hat{M}$=q when :*

$$u + \nu \int_q^1 S_U\ (u)\ du + \nu q\ S_U(q) > u + \alpha \int_q^1 S_U\ (u)\ du$$
$$\Leftrightarrow \quad \nu\ q\ S_U(q) > (\alpha - \nu) \int_q^1 S_U\ (u)\ du.$$

- *The minimum is $\hat{M} = 1$ when :*

$$\nu\ q\ S_U(q) > (\alpha - \nu) \int_q^1 S_U\ (u)\ du.$$

When $\alpha \leq \nu$, the minimum is $\hat{M} = q$. For $\alpha = \nu$, the function is constant for values of $M > q$.

- *Contract with a limit*

Here, the insurance policy is defined as in (3.17) and (3.18). First note that,
$1_{\{Z(U)>q\}} = 0$, $q \geq 1 - L \Leftrightarrow L \geq 1 - q$.
Therefore, we have the following conditions:

$$1_{\{Z(U)>q\}} = \begin{cases} 0, & U \leq L + q \\ 1, & U > L + q \end{cases}, \qquad L < 1 - q.$$

Once $U - L = q$, this is equivalent to $U = L + q$. So the risk function is then given by:

$$\nu Z(U) 1_{\{Z(U)>q\}} = \begin{cases} 0, & U \leq L + q \\ \nu\ (U - L), & U > L + q \end{cases}, \qquad L < 1 - q.$$

To calculate the expected value of the risk function and the premium principle we use the method of integration by parts as follows:

$$
\begin{aligned}
E(Z(U)) &= \int_0^{L+q} 0\ du + \int_{L+q}^{1} (u-L)\ f_U\ (u)\ du \\
&= \int_{L+q}^{1} u\ f_U\ (u)\ du - L\ P(U > L+q) \\
&= \Big[-u\ S_U(du) \Big]_{L+q}^{1} + \int_{L+q}^{1} S_U\ (u)\ du - L\ P(U > L+q) \\
&= q\ S_U(L+q) + \int_{L+q}^{1} S_U\ (u)\ du.
\end{aligned}
$$

Therefore, the function h defined by (3.19) is as follows:

$$
h(x,L) = \begin{cases} u + \alpha \int_0^L S_U\ (u)\ du, \ L \geq 1-q \\ u + \alpha \int_0^L S_U\ (u)\ du + \nu q\ S_U(L+q) + \nu \int_{L+q}^{1} S_U\ (u)\ du, \quad L < 1-q \end{cases}.
$$

When $L < 1-q$ the derivative of the function is as follows:

$$
\alpha\ S_U(L) - \nu\ S_U(L+q) - \nu\ q\ f_U(L+q) < 0.
$$

When $L \geq 1-q$, the derivative of the function is as follows:

$$
\frac{\partial h}{\partial L} = \alpha\ S_U(L) > 0.
$$

Furthermore, to study the function in the critical point we see the following limits:

$$
\begin{aligned}
&\lim_{L \to (1-q)^+} u + \alpha \int_0^L S_U\ (u)\ du \\
&\lim_{L \to (1-q)^-} u + \alpha \int_0^L S_U\ (u)\ du + \nu q\ S_U(L+q) + \nu \int_{L+q}^{1} S_U(u)\ du \\
&= u + \alpha \int_0^{1-q} S_U\ (u)\ du + \nu\ q\ S_U(1-q+q).
\end{aligned}
$$

The $\lim_{L\to(1-q)^+} = \lim_{L\to(1-q)^-}$ *therefore, the function is continuous.*

The function decreases until $\hat{L} = 1-q$ and then it increases. Thus, the function reaches the minimum at $\hat{L} = 1-q$.

Proof of Proposition 7. *Note that f is defined as in* (3.12)*, and the premium principle is defined as in* (3.6)*.*

- *Proportional contract*

  The insurance policy is defined as in (3.13) *and* (3.14)*. Thus, the function h defined by* (3.19) *is as follows:*

$$\sup_{0<a<1} h(x,a) = \min_{0<a<1} \Big\{ u + \sqrt{a}\ \nu\ u_3 + \alpha\ (1-a)\ u \Big\}.$$

  The derivative of the function to understand how it behaves is given by:

$$\frac{\partial h}{\partial a} = \frac{\nu\ u_3}{\sqrt{a}} - \alpha\ u > 0.$$

  The function is convex, therefore $0 < \hat{a} < 1$*, as we can see in the following derivative:*

$$\frac{\partial^2 h}{\partial a^2} = -\frac{\nu\ u_3}{2a^{\frac{3}{2}}} < 0.$$

  Thus, the minimum of the function is given by:

$$\hat{a} \ = \ \frac{\nu^2\ u_3^2}{\alpha^2\ u^2}.$$

- *Contract with a deductible*

  Here, the insurance policy is defined as in (3.15) *and* (3.16)*. Thus, the function h, defined by* (3.19) *is as follows:*

$$\sup_{0<M<1} h(x,M) = \min_{0<M<1} \Big\{ u + \frac{1}{2}\nu \int_0^M u^{\frac{-1}{2}}\ S_U\ (u)\ du + \alpha\ \int_M^1\ S_U\ (u)\ du \Big\}.$$

  To understand the behavior of the function, we study the derivative which is given by:

$$\frac{\partial h}{\partial M} = S_U(M)\left(\frac{\nu}{2\sqrt{M}} + \alpha\right).$$

  Thus, $\frac{\partial h}{\partial M} > 0$ *if* $\frac{\nu}{2\sqrt{M}} > \alpha$*. Therefore the function reaches the minimum at* $\hat{M} = 0$ *and* $\frac{\partial h}{\partial M} < 0$ *if* $\frac{\nu}{2\sqrt{M}} < \alpha$ *so the function reaches the minimum at* $\hat{M} = 1$*.*

- *Contract with a limit*

  Here the insurance policy is defined as in (3.17) *and* (3.18)*. Thus, the function h defined by* (3.19) *is as follows:*

$$\sup_{0<L<1} h(x,L) = \min_{0<L<1} \Big\{ u + \frac{1}{2}\nu \int_L^1 u^{\frac{-1}{2}}\ S_U\ (u)\ du + \alpha \int_0^L\ S_U\ (u)\ du \Big\}$$

To understand the behavior of the function, we study the derivative which is given by:

$$\frac{\partial h}{\partial L} = S_U(L)\left(\alpha - \frac{\nu}{2\sqrt{L}}\right).$$

Note that, $\left(\alpha - \frac{\nu}{2\sqrt{L}}\right)' = \frac{\nu}{4\alpha^{\frac{3}{2}}} > 0$, *so we know that the function is convex, therefore* $0 < \hat{L} < 1$. *The function reaches the minimum at* $\hat{L} = \frac{\nu^2}{4\alpha^2}$.

Proof of Proposition 8. *Note that* f *is defined as in* (3.9)*, and the premium principle is defined as in* (3.7)*.*

- *Proportional contract*

  Here, the insurance contract is defined as in (3.13) *and* (3.14)*. Thus,* $\nu E(f(Z(U)) = \nu a u$ *and* $\alpha c = \alpha\ Var\big((1-a)\ U\big) = \alpha\ (1-a)^2\ \sigma_U^2$. *Therefore, the function* h *defined by* (3.19) *is as follows:*

  $$\sup_{0<a<1} h(x,a) = \min_{0<a<1}\Big\{u + \nu\ a\ u + \alpha\ (1-a)^2\ \sigma_U^2\Big\}.$$

  The derivative of the function which is increasing with a*:*

  $$\frac{\partial h}{\partial a} = \nu\ u - 2\alpha\ \sigma_U^2 + 2\alpha\ a.$$

  If $\nu\ u - 2\alpha\ \sigma_U^2 < 0$, *then the function reaches a minimum at* $\hat{a} = 1 - \frac{\nu\ u}{2\alpha\ \sigma_U^2}$ *and if* $\nu\ u - 2\alpha\ \sigma_U^2 > 0$, *then the function reaches the minimum at* $\hat{a} = 0$.

- *Contract with a deductible*

  The insurance contract is defined as in (3.15) *and* (3.16)*. Therefore,* $\alpha c = \alpha\ Var\big(U - Z(U)\big)$. *Since* $Var\big(U - Z(U)\big) = E\big((U-Z(U))^2\big) - E\big(U - Z(U)\big)^2$, $E\big((U-Z(U))^2\big) = 2\int_M^1 S_U\ (u)\ du$ *and* $E\big(U - Z(U)\big)^2 = \int_M^1 S_U\ (u)\ du)^2$, *then* $\alpha c = \alpha\ \big(2\int_M^1 S_U\ (u)\ du - \big(\int_M^1 S_U\ (u)\ du\big)^2\big)$. *In this way, the function* h *defined by* (3.19) *is as follows:*

  $$\sup_{0<M<1} h(x,M) = \min_{0<M<1}\left\{u + \nu\int_0^M S_U\ (u)\ du + \alpha\left(2\int_M^1 S_U\ (u)\ du - \left(\int_M^1 S_U\ (u)\ du\right)^2\right)\right\}.$$

  The first derivative is as follows:

  $$\frac{\partial h}{\partial M} = S_U(M)\left(\nu + \alpha\left(2\int_M^1 u\ S_U\ (u)\ du - \left(\int_M^1 S_U\ (u)\ du\right)^2\right)^{\frac{-1}{2}}\left(\int_M^1 S_U\ (u)\ du - M\right)\right).$$

The second derivative of the function is given by:

$$\frac{\partial^2 h}{\partial M^2} = -\frac{\alpha}{2}\left(\left(2\int_M^1 u\ S_U\ (u)\ du - \left(\int_M^1 S_U\ (u)\ du\right)^2\right)^{\frac{-3}{2}}\right.$$
$$\left(-2M\ S_U(M) + S_U(M)\ 2\int_M^1 S_U\ (u)\ du\right)\left(\int_M^1 S_U\ (u)\ du - M\right)$$
$$\left. -\alpha\left(2\int_M^1 u\ S_U\ (u)\ du - \left(\int_M^1 S_U\ (u)\ du\right)^2\right)^{\frac{-1}{2}}\left(1 + S_U(M)\right)\right) < 0.$$

It is known that $\big(\int_M^1 S_U\ (u)\ du - M\big)' = -S_U(M) - 1 < 0$, *so we need to check for both cases, when* $M = 0$ *and* $M = 1$.

When $M = 0$, *the minimizer is* $\nu + 2\alpha\ \big(\int_M^1 S_U\ (u)\ du - M\big) = \nu + 2\alpha\ E(M) > 0$, *therefore this is a maximum value.*

When $M = 1$, *the minimizer is* $\nu + 2\alpha\ \big(\int_M^1 S_U\ (u)\ du - M\big) = \nu - 2\alpha$ *which is a minimum value. Since the goal is to find a minimum value it is necessary to study this case.*

If $\nu - 2\alpha \geq 0$ *then* $M = 0$ *and if* $\nu - 2\alpha < 0$ *then the derivative is positive in* $M = 0$, *and negative in* $M = 1$, *therefore the monotonic derivative has a maximum.*

If we minimize the function in $M = 0$, *the function is* $\alpha\ Var_U$ *and if we minimize the function in* $M = 1$ *it is* $\nu\ U$. *Therefore, if* $\alpha\ Var_U < \nu\ u$ *then the function reaches the minimum at* $\hat{M} = 0$ *and if* $\alpha\ Var_U > \nu\ u$ *then the function reaches the minimum at* $\hat{M} = 1$.

- *Contract with a limit*

  Here, the policy insurance is defined as in (3.17) *and* (3.18). *In this case,* $\alpha c = Var\big(U - Z(U)\big) = E\big((U - Z(U))^2\big) - E\big(U - Z(U)\big)^2$, $E\big((U - Z(U))^2\big) = 2\int_0^L S_U\ (u)\ du$ *and* $E\big(U - Z(U)\big)^2 = \big(\int_0^L S_U\ (u)\ du\big)^2$, *then* $\alpha c = \alpha\ \big(2\int_0^L S_U\ (u)\ du - \big(\int_0^L S_U\ (u)\ du\big)^2\big)$. *In this way, the function* h *defined by* (3.19) *is as follows:*

$$\sup_{0<L<1} h(x, L) = \min_{0<L<1}\left\{u + \nu\int_L^1 S_U\ (u)\ du + \alpha\left(2\int_0^L S_U\ (u)\ du - \left(\int_0^L S_U\ (u)\ du\right)^2\right)\right\}.$$

The derivative of the function is given by:

$$\frac{\partial h}{\partial L} = S_U(L)\left(-\nu + 2\alpha\left(L - \int_0^L S_U\ (u)\ du\right)\right).$$

Note that $\big(L - \int_0^L S_U\ (u)\ du\big)' = 1 - S_U(L) > 0$, *so the function is increasing with* L.

When $L=0$ the minimizing function is given by $-\nu+2\alpha\ \big(L-\int_0^L S_U\ (u)\ du\big)=-\nu<0$ and when $L=1$ it is given by $-\nu+2\alpha\ \big(L-\int_0^L S_U\ (u)\ du\big)=-\nu+2\alpha\ \big(1-u\big)$.

If it is the case that $-\nu+2\alpha\ \big(1-u\big)<0$ then $L=1$. If $-\nu+2\alpha\ \big(1-u\big)<0$ then the derivative is negative in $L=0$ and positive in $L=1$, so the function is monotonic and exists a minimum when $L-\int_0^L S_U\ (u)\ du=\frac{\nu}{2\alpha}$.

Proof of Proposition 9. *Note that f is defined as in* (3.10)*, and the premium principle is defined as in* (3.7)*.*

- *Proportional contract*

  In this case, the insurance contract is defined as in (3.13) *and* (3.14)*. Thus, $\nu E\big(f(Z(U)^2\big)=\nu\ E(a\ U)^2=\nu\ a^2\ u_2$ and $\alpha c=\alpha\ E\big((U-Z(U))^2\big)=\alpha\ E\big(((1-a)\ U)^2\big)=\alpha\ (1-a)^2\ \sigma_U^2$. Thus, the function h defined by* (3.19) *is as follows:*

$$\sup_{0<a<1} h(x,a)=\min_{0<a<1}\Big\{u+\nu\ a^2\ u_2+\alpha\ (1-a)^2\ \sigma_U^2\Big\}.$$

  To see the behavior of the function that we want to minimize, the result of the derivative follows as:

$$\begin{aligned}&\frac{\partial h}{\partial a}=0\\ \Leftrightarrow\ & 2a\ \big(\nu\ u_2+\alpha\ \sigma_U^2\big)-2\alpha\ \sigma_U^2=0\\ \Leftrightarrow\ & \hat{a}=\frac{\alpha\ \sigma_U^2}{\nu\ u_2+\alpha\ \sigma_U^2}.\end{aligned}$$

  Assuming that $\nu\ u_2+\alpha\ \sigma_U^2\neq 0$, we note that $\hat{a}$ is a minimum.

- *Contract with a deductible*

  In this case, the insurance policy is defined as in (3.15) *and* (3.16)*. Therefore, $E\big(f(Z(U))\big)=2\nu\ \big(\int_0^M u\ S_U\ (u)\ du\big)$ and $\alpha c=\alpha\ \big(2\int_M^1 S_U\ (u)\ du-\big(\int_M^1 S_U\ (u)\ du\big)^2\big)$. Thus, the function h defined by* (3.19) *is as follows:*

$$\sup_{0<M<1} h(x,M)=\min_{0<M<1}\left\{u+2\nu\left(\int_0^M u\ S_U\ (u)\ du\right)+\alpha\ \left(2\int_M^1 S_U\ (u)\ du-\left(\int_M^1 S_U\ (u)\ du\right)^2\right)\right\}.$$

  In order to see how is the behavior of the function, we have the following derivative:

$$\frac{\partial h}{\partial M}=2S_U(M)\ \left(\alpha\int_M^1 S_U\ (u)\ du+M\ (\nu-\alpha)\right).$$

If $\nu \geq \alpha$ then the derivative of the minimizer is increasing so it reaches the minimum at $\hat{M} = 0$.

If $\nu < \alpha$ then $\left(\alpha \int_M^1 S_U\ (u)\ du + M(\nu - \alpha)\right)' = -\alpha\ S_U(M) + (\nu - \alpha)$ which is decreasing.

Therefore, the minimum can be in $M = 0$ or $M = 1$, checking both hypothesis: when $M = 0$, the minimizer is $\alpha\ u > 0$ and when $M = 1$, the minimizer is $M\ (\nu - \alpha) < 0$.

So the derivative increases and then decreases, so it has a maximum. The minimizer in $M = 0$ is $u + \alpha\ Var_U$ and the minimizer in $M = 1$ is $u + \nu\ u_2$.

If $\alpha\ Var_U < \nu\ u_2$ then the minimum is $\hat{M} = 0$. If $\alpha\ Var_U > \nu\ u_2$ then the minimum is $\hat{M} = 1$.

- *Contract with a limit*

  Here, the insurance contract is defined as in (3.17) *and* (3.18)*. Thus, $E(f(Z(U)) = 2\nu\ \left(\int_L^1 u\ S_U\ (u)\ du\right)$ and $\alpha c = \alpha\ \left(2 \int_0^L S_U\ (u)\ du - (\int_0^L S_U\ (u)\ du)^2\right)$. Therefore, the function h defined by* (3.19) *is as follows:*

$$\sup_{0<a<1} g(x, L) = \min_{0<L<1} \left\{ u + 2\nu \left(\int_0^L u\ S_U\ (u)\ du \right) + \alpha \left(2 \int_L^1 S_U\ (u)\ du - \left(\int_L^1 S_U\ (u)\ du)^2 \right) \right) \right\}.$$

The derivative of the function follows as:

$$\frac{\partial h}{\partial L} = 2 S_U(L) \left(L\ (\alpha - \nu) - \alpha \int_0^L S_U\ (u)\ du \right).$$

If $\alpha \leq \nu$ then the function reaches the minimum at $\hat{L} = 1$. If $\alpha > \nu$, then see that $\big(L\ (\alpha - \nu) - \alpha \int_0^L S_U\ (u)\ du\big)' = (\alpha - \nu) - \alpha\ S_U(L)$, therefore the function is increasing with L. The function starts in $-\nu$ and it ends in $(\alpha - \nu) > 0$, it means that the function decreases and then increases, therefore there is a minimum $0 < \hat{L} < 1$ when $L\ (\alpha - \nu) = \alpha \int_0^L S_U\ (u)\ du$.

Proof of Proposition 10. *Note that f is defined as in* (3.11)*, and the premium principle is defined as in* (3.7)*.*

- *Proportional contract*

  Here, the insurance contract is defined as in (3.13) *and* (3.14)*. Thus, the function h defined by* (3.19) *is as follows:*

$$h(x, a) = \begin{cases} u + \alpha\ (1-a)^2\ \sigma_U^2, & a \leq q \\ u + \nu\ a \int_{\frac{q}{a}}^1 u\ f_U\ (u)\ du + \alpha\ (1-a)^2\ \sigma_U^2, & a > q \end{cases}.$$

When $a \leq q$ we have the following derivatives:

$$\frac{\partial h}{\partial a} = -2\alpha\ \sigma_U^2\ (1-a) < 0,$$

$$\frac{\partial^2 h}{\partial a^2} = \alpha\ \sigma_U^2 > 0.$$

The function reaches the minimum at $\hat{a} = 1$.

When $a > q$ the function is increasing since:

$$\frac{\partial h}{\partial a} = \nu \left(\int_{\frac{q}{a}}^{1} u\ f_U\ (u)\ du + \frac{q^2}{a^2}\ f_U \left(\frac{q}{a}\right) \right) - 2\alpha\ \sigma_U^2\ (1-a).$$

- *Contract with a deductible*

  Here, the insurance contract is defined as in (3.15) *and* (3.16). *Thus,* $E(Z(U)) = \int_q^M S_U\ (u)\ du + q\ S_U(q)$, $\alpha c = \alpha\ \left(2 \int_M^1 S_U\ (u)\ du - \left(\int_M^1 S_U\ (u)\ du\right)^2\right)$. *In this way, the function h defined by* (3.19) *is as follows:*

$$h(x,M) = \begin{cases} u + \alpha\left(2 \int_M^1 S_U\ (u)\ du - \left(\int_M^1 S_U\ (u)\ du\right)^2\right), & M \leq q \\ u + \nu \int_q^M S_U\ (u)\ du + q\ S_U(q) + \alpha\ \left(2 \int_M^1 S_U\ (u)\ du - \left(\int_M^1 S_U\ (u)\ du\right)^2\right), & M > q \end{cases}.$$

To understand the behavior of the function we should study the derivatives and the limits of the function since it is not continuous.

When $M \leq q$, we have the following derivative:

$$\frac{\partial}{\partial M} = 2\alpha\ M\ S_U(M)\ \left(\int_M^1 S_U\ (u)\ du - M\right).$$

Therefore if $M > \int_M^1 S_U\ (u)\ du$, the function is decreasing with M and if $M < \int_M^1 S_U\ (u)\ du$, the function is increasing with M.

$$\text{When } M = q \begin{cases} u + \alpha\ \left(2 \int_q^1 S_U\ (u)\ du - \left(\int_q^1 SU\ (u)\ du\right)^2\right) \\ u + q\ S_U(q) + \alpha\ \left(2 \int_q^1 Su\ (u)\ du - \left(\int_q^1 S_U\ (u)\ du\right)^2\right) \end{cases}.$$

Since $\nu\ q\ SU(q) > 0$, the following holds:

$$u{+}q\ S_U(q){+}\alpha\left(2\int_q^1 S_U(u)du{-}\left(\int_q^1 S_U\ (u)\ du\right)^2\right) > u{+}\alpha\left(2\int_q^1 S_U\ (u)\ du{-}\left(\int_q^1 S_U\ (u)\ du\right)^2\right).$$

If $M > \int_M^1 S_U\ (u)\ du$, $\nu < 2\alpha\ \left(\int_M^1 S_U\ (u)\ du - M\right)$ and $u + \nu \int_q^1 S_U\ (u)\ du + q\ S_U(q) > u + \alpha\ \left(2\int_q^1 S_U\ (u)\ du - \left(\int_q^1 S_U\ (u)\ du\right)^2\right)\right)$, the function reaches the minimum at $\hat{M} = q$.

If $M > \int_M^1 S_U\ (u)\ du$, $\nu < 2\alpha\ \left(\int_M^1 S_U\ (u)\ du - M\right)$ and $u + \nu \int_q^1 S_U\ (u)\ du + q\ S_U(q) < u + \alpha\ \left(2\int_q^1 S_U(u)du - \left(\int_q^1 S_U\ (u)\ du\right)^2\right)$, the function reaches the minimum at $\hat{M} = 1$.

- *Contract with a limit*

  In this case, the insurance contract is defined as in (3.17) *and* (3.18)*. Thus, the function h defined by* (3.19) *is as follows:*

$$h(x,L) = \begin{cases} u + \nu\ q\ S_U(L+q) + \nu \displaystyle\int_{L+q}^1 S_U\ (u)\ du + \\ \alpha\left(2\displaystyle\int_0^L u\ S_U\ (u)\ du - \left(\displaystyle\int_0^L S_U\ (u)\ du\right)^2\right),\ L < 1-q \\ u + \alpha\left(2\displaystyle\int_0^L u\ S_U\ (u)\ du - \left(\displaystyle\int_0^L S_U\ (u)\ du\right)^2\right),\ L \geq 1-q \end{cases}.$$

  The derivative of the function when $L < 1-q$ is as follows:

$$\frac{\partial h}{\partial L} = 2\alpha\ S_U(L)\left(L - \int_0^L S_U\ (u)\ du\right) - \nu\ S_U(L+q) - \nu\ q\ f_U(L+q).$$

  The derivative of the function when $L \geq 1-q$ is as follows:

$$\frac{\partial h}{\partial L} = 2\alpha\ S_U(L)\left(L - \int_0^L S_U\ (u)\ du\right).$$

  The function decreases until $\hat{L} = 1-q$.

  Therefore, if $L > \int_0^L S_U\ (u)\ du$ the function is increasing with L, therefore the function reaches the minimum at $\hat{L} = 1-q$. If $L < \int_0^L S_U\ (u)\ du$ the function is decreasing with L, therefore the function reaches the minimum at $\hat{L} = 1$.

  Here, it is also possible to see that the function is continuous, since $\lim_{L\to(1-q)^+} = \lim_{L\to(1-q)^-}$.

Proof of Proposition 11. *Note that f is defined as in* (3.12)*, and the premium principle is defined as in* (3.7)*.*

- *Proportional contract*

  Here, the insurance contract is defined as in (3.13) *and* (3.14)*. Thus, the function h defined by* (3.19) *is as follows:*

$$\sup_{0<a<1} h(x,a) = \min_{0<a<1} \left\{ u + \sqrt{a}\ \nu\ u_3 + \alpha\ (1-a)^2\ \sigma_U^2 \right\}.$$

  The derivative of the function to understand how it behaves is as follows:

$$\frac{\partial h}{\partial a} = \frac{\nu\ u_3}{2\sqrt{a}} - 2\alpha\ \sigma_U^2\ (1-a) > 0.$$

  The second derivative of the function is as follows:

$$\frac{\partial^2 h}{\partial a^2} = \frac{-\nu\ u_3}{4a^{\frac{3}{2}}} + 2\alpha\ \sigma_U^2 < 0.$$

  The function has a maximum, therefore the minimum is in $M = 0$ *or* $M = 1$*. The minimizer in* $M = 0$ *is* $u + \sigma_U^2$ *and in* $M = 1$ *is* $\nu\ u_3$*. If* $\sigma_U^2 > \nu\ u_3$*, the minimum will be* $\hat{M} = 1$ *and if* $\sigma_U^2 < \nu\ u_3$*, the minimum will be* $\hat{M} = 0$*.*

- *Contract with a deductible*

  In this case, the insurance contract is defined as in (3.15) *and* (3.16)*. Thus, the function h defined by* (3.19) *is as follows:*

$$\sup_{0<M<1} h(x,M) = \min_{0<M<1} \left\{ u + \frac{1}{2}\nu \int_0^M u^{\frac{-1}{2}} S_U\ (u)\ du + \alpha \left(2\int_M^1 u\ S_U\ (u)\ du - \left(\int_M^1 S_U\ (u)\ du \right)^2 \right) \right\}$$

  The derivative of the function to understand how it behaves is as follows:

$$\frac{\partial h}{\partial M} = S_U(M) \left(\frac{\nu}{2\sqrt{M}} + 2\alpha \left(\int_M^1 S_U(u)du - M \right) \right).$$

  The function is convex, therefore the minimum is in $\hat{M} = 0$ *or* $\hat{M} = 1$ *since the following derivative is negative:*

$$\frac{\partial^2 h}{\partial M^2} = \frac{-\nu}{4M^{\frac{3}{2}}} - 2\alpha\ (S_U(M) + 1)).$$

When $\int_M^1 S_U(u)\,du > M$, *the minimum is* $\hat{M} = 0$. *When* $\int_M^1 S_U(u)\,du < M$, *note that the function to be minimized in* $M = 0$ *is* $u + 2\alpha\, Var_U$ *and the function to be minimized in* $M = 1$ *is* $\frac{1}{2}\nu\, u_3$.

If $u + 2\alpha\, Var_U < \frac{1}{2}\nu\, u_3$, *the minimum of the function is* $\hat{M} = 0$ *and if* $u + 2\alpha\, Var_U > \frac{1}{2}\nu\, u_3$, *the minimum of the function is* $\hat{M} = 1$.

- *Contract with a limit*

  In this case, the insurance contract is defined as in (3.17) *and* (3.18). *Thus, the function h defined by* (3.19) *is as follows:*

$$\sup_{0<L<1} g(x, L) = \min_{0<L<1} \left\{ u + \frac{1}{2}\nu \int_L^1 u^{\frac{-1}{2}} S_U(u)\,du + \alpha \left(2\int_0^L u\, S_U(u)\,du - \left(\int_0^L S_U(u)\,du \right)^2 \right) \right\}.$$

  The derivative of the function to understand how it behaves is as follows:

$$\frac{\partial h}{\partial L} = S_U(L) \left(2\alpha \left(L - \int_0^L S_U(u)\,du \right) \right) - \frac{\nu}{2\sqrt{L}} \Bigg).$$

  The function is convex since $\left(2\alpha \left(L - \int_0^L S_U(u)\,du \right) - \frac{\nu}{2\sqrt{L}} \right)' = 2\alpha\,(1 - S_U(L)) + \frac{\nu}{8L^{\frac{3}{2}}} < 0$ *therefore* $0 < \hat{L} < 1$. *There is a minimum when* $2\alpha \left(L - \int_0^L S_U(u)\,du \right) = \frac{\nu}{2\sqrt{L}}$. *If* $L \leq \int_0^L S_U(u)\,du$, *the minimum is* $\hat{L} = 1$.

Proof of Proposition 12. *Note that f is defined as in* (3.9), *and the premium principle is defined as in* (3.8).

- *Proportional contract*

  In this case, the insurance contract is defined as in (3.13) *and* (3.14). *Thus, the function h defined by* (3.19) *is as follows:*

$$\sup_{0<a<1} h(x, a) = \min_{0<a<1} \{ u + \nu a\, u + \alpha\, (1 - a)\, \sigma_U \}.$$

  The following derivative shows that if $u\,\nu > \alpha\,\sigma_U$ *the slope of the function is positive, therefore the minimum is* $\hat{a} = 0$ *and if* $u\nu < \alpha\sigma_U$ *the slope of the function is negative, therefore the minimum is* $\hat{a} = 1$.

$$\frac{\partial h}{\partial a} = \nu\, u - \alpha\, \sigma_U.$$

- *Contract with a deductible*

  In this case, the insurance contract is defined as in (3.15) *and* (3.16). *Thus, the function h defined by* (3.19) *is as follows:*

$$\sup_{0<M<1} h(x,M) = \min_{0<M<1} \left\{ u + \nu \int_0^M S_U\ (u)\ du + \alpha \left(2\int_M^1 S_U\ (u)\ du - \left(\int_M^1 S_U\ (u)\ du \right)^2 \right)^{\frac{1}{2}} \right\}$$

  In order to see the behavior of the function, we have the following derivatives:

$$\frac{\partial h}{\partial M} = S_U(M) \left(\nu + \alpha \left(2\int_M^1 u\ S_U\ (u)\ du - \left(\int_M^1 S_U\ (u)\ du \right)^2 \right)^{\frac{-1}{2}} \left(\int_M^1 S_U\ (u)\ du - M \right) \right).$$

$$\frac{\partial^2 h}{\partial M^2} = -\frac{\alpha}{2} \left(\left(2\int_M^1 u\ S_U\ (u)\ du - \left(\int_M^1 S_U\ (u)\ du)^2 \right)^{\frac{-3}{2}} \left(-2MS_U(M) + 2\int_M^1 S_U\ (u)\ du\ S_U(M) \right) \right)$$
$$\left(\int_M^1 S_U\ (u)\ du - M \right) - \alpha \left(\left(2\int_M^1 u\ S_U\ (u)\ du - \left(\int_M^1 S_U\ (u)\ du \right)^2 \right)^{\frac{-1}{2}} \left(1 + S_U(M)\right) \right) < 0.$$

  The function for $M = 0$ *is* $\nu + \frac{\alpha}{\sigma_U} u$ *and for* $M = 1$ *we have that* $\lim_{M \to 1} g(M) = \infty$. *Therefore, the derivative is positive in* $M = 0$ *and it is negative in* $M = 1$, *so it is monotonic and* $0 < \hat{M} < 1$. *When* $\alpha\sigma_U < \nu E(\nu)$ *the function reaches the minimum at* $\hat{M} = 0$ *and when* $\alpha\sigma_U > \nu E(\nu)$ *the function reaches the minimum at* $\hat{M} = 1$.

- *Contract with a limit*

  In this case, the insurance contract is defined as in (3.17) *and* (3.18). *Thus, the function h defined by* (3.19) *is as follows:*

$$\sup_{0<L<1} h(x,L) = \min_{0<L<1} \left\{ u + \nu \int_L^1 S_U\ (u)\ du + \alpha \left(2\int_0^L S_U\ (u)\ du - \left(\int_0^L S_U\ (u)\ du \right)^2 \right)^{\frac{1}{2}} \right\}.$$

  The derivative of the function follows as:

$$\frac{\partial}{\partial L} = S_U(L) \left(-\nu + \alpha \left(2\int_0^L u\ S_U\ (u)\ du - \left(\int_0^L S_U\ (U)\ du \right)^2 \right)^{\frac{-1}{2}} \right) \left(L - \int_0^L S_U\ (u)\ du \right).$$

  Both $\alpha \left(2\int_0^L u\ S_U\ (u)\ du - (\int_0^L S_U\ (u)\ du)^2 \right)^{\frac{-1}{2}} > 0$ *and* $L - \int_0^L S_U\ (u)\ du > 0$ *are increasing.*

Let $\hat{h}$ *be* $(-\nu)+\alpha\left(2\int_0^L uS_U(u)\,du-\left(\int_0^L S_U(u)\,du\right)^2\right)^{\frac{-1}{2}}\left(L-\int_0^L S_U(u)\,du\right)$*, then in* 1*,* $\hat{h}=(-\nu)+\frac{\alpha}{\sigma_U}(1-u)$ *and in* 0*,* $\hat{h}=-\nu$ *.*
If $\frac{\alpha}{\sigma_U}(1-u)\leq\nu$*, the derivative is always negative, then the function reaches the minimum at* 1*. If* $\frac{\alpha}{\sigma_U}(1-u)>\nu$*, the derivative is negative and then positive, therefore* $0<\hat{L}<1$*.*

Proof of Proposition 13. *Note that f is defined as in* (3.10)*, and the premium principle is defined as in* (3.8)*.*

- *Proportional contract*

  In this case, the insurance contract is defined as in (3.13) *and* (3.14)*. Thus, the function h defined by* (3.19) *is as follows:*

$$\sup_{0<a<1} g(x,a)=\min_{0<a<1}\left\{u+\nu a^2\ u_2+\alpha(1-a)\ \sigma_U\right\}.$$

  The derivative of the function follows as:

$$\frac{\partial h}{\partial a}=0\Leftrightarrow 2\nu a u_2-\alpha\ \sigma_U=0\Leftrightarrow a=\frac{\alpha\ \sigma_U}{2\nu\ u_2}.$$

  If $2\nu\ u_2-\alpha\leq 0$ *then the minimum is* $\hat{a}=1$ *and if* $2\nu\ u_2-\alpha>0$ *then the minimum is* $\hat{a}=\frac{\alpha\ \sigma}{2\nu\ u_2}$*, assuming that* $2\nu\ u_2\neq 0$*.*

- *Contract with a deductible*

  In this case, the insurance contract is defined as in (3.15) *and* (3.16)*. Thus, the function h defined by* (3.19) *is as follows:*

$$\sup_{0<a<1} g(x,M)=\min_{0<M<1}\left\{u+2\nu\left(\int_0^M u\ S_U\ (u)\ du+\alpha\left(2\int_M^1 S_U\ (u)\ du-\left(\int_M^1 S_U\ (u)\ du\right)^2\right)^{\frac{1}{2}}\right)\right\}.$$

  The derivative of the function is given by:

$$\frac{\partial h}{\partial M}=S_U(M)\left(2M\ \nu+\alpha\left(2\int_M^1 u\ S_U\ (u)\ du-\left(\int_M^1 S_U\ (u)\ du\right)^2\right)^{\frac{-1}{2}}\left(\int_M^1 S_U\ (u)\ du-M\right)\right).$$

  The function increases and then it decreases. In $M=0$ *the function is* $u+\sigma_U\ \alpha$ *and in* $M=1$ *is* $u+\nu\ u_2$*. Therefore, if* $\sigma_U\alpha<\nu u_2$ *then the function reaches the minimum at* $\hat{M}=0$ *and if* $\sigma_U\alpha>\nu u_2$ *then the function reaches the minimum at* $\hat{M}=1$*.*

- *Contract with a limit*

  Here, the insurance policy is defined as in (3.17) *and* (3.18). *Thus, the function* h *defined by* (3.19) *is as follows:*

$$\sup_{0<L<1} h(x,L) = \min_{0<L<1}\left\{u+2\nu\left(\int_L^1 u\ S_U\ (u)\ du\right)+\alpha\left(\left(2\int_0^L S_U\ (u)\ du-\left(\int_0^L S_U\ (u)\ du\right)^2\right)^{\frac{1}{2}}\right)\right\}.$$

  The derivative of the function is as follows:

$$\frac{\partial h}{\partial L} = S_U(L)\left(\nu+\alpha\left(\int_0^L u\ S_U\ (u)\ du-\left(\int_0^L S_U\ (u)\ du\right)^2\right)^{\frac{-1}{2}}\left(L-\int_0^L S_U\ (u)\ du\right)\right).$$

  Note that $\left(L-\int_0^L S_U\ (u)\ du\right)' = 1-S_U(L)>0$, *so* $0<\hat{L}<1$.

Proof of Proposition 14. *Note that* f *is defined as in* (3.11), *and the premium principle is defined as in* (3.8).

- *Proportional contract*

  Here, the insurance policy is defined as in (3.13) *and* (3.14). *Thus, the function* h *defined by* (3.19) *is as follows:*

$$h(x,a) = \begin{cases} u+\alpha\ (1-a)\ \sigma_U, & a\leq q \\ u+\nu\ a\int_{\frac{q}{a}}^1 u+\nu\ a\int_{\frac{q}{a}}^1 u\ f_U\ (u)\ du+\alpha\ (1-a)\ \sigma_U, & a>q \end{cases}.$$

  When $a\leq q$, *the function reaches the minimum at* $\hat{a}=1$, *since the derivative of the function is decreasing,*

$$\frac{\partial h}{\partial a} = \alpha\ \sigma_U - a\ \alpha\ \sigma_U.$$

  When $a>q$ *the derivative is as follows:*

$$\frac{\partial h}{\partial a} = \nu\ \int_{\frac{q}{a}}^1 u\ f_U\ (u)\ du+\nu\ \frac{q^2}{a^2}f_U\left(\frac{q}{a}\right)-\alpha\ \sigma_U.$$

- *Contract with a deductible*

  Here, the insurance policy is defined as in (3.15) *and* (3.16)*. Thus, the function h defined by* (3.19) *is as follows:*

$$h(x,M)=\begin{cases} u+\alpha\Bigg(2\displaystyle\int_M^1 S_U\ (u)\ du-\Bigg(\int_M^1 S_U\ (u)\ du)^2\Bigg)^{\frac{1}{2}}\Bigg),\ M\le q \\ u+\nu\displaystyle\int_q^M S_U\ (u)\ du+q\ S_U(q)+\alpha\Bigg(\Bigg(2\int_M^1 S_U\ (u)\ du-\Bigg(\int_M^1 S_U\ (u)\ du\Bigg)^2\Bigg)^{\frac{1}{2}}\Bigg),\ M>q \end{cases}.$$

We know that $\alpha\ \Big(Var\big(Z(U)\big)\Big)' > 0$*, therefore* $\alpha\ \Big(Var\big(Z(U)\big)^{\frac{1}{2}}\Big)' > 0$

To understand the behavior of the function we should study the derivatives and the limits of the function since it is not continuous.

When $M\le q$ *we have the following derivative:*

$$\frac{\partial h}{\partial M}=\alpha\ Var\big(Z(U)\big)^{\frac{-1}{2}}\ S_U(M)\ \left(\int_M^1 S_U\ (u)\ du-M\right)$$

When $M>q$*, the derivative of the function is given by:*

$$\frac{\partial h}{\partial M}=\nu\ S_U(M)+\alpha\ Var\big(Z(U)\big)^{\frac{-1}{2}} S_U(M)\ \left(\int_M^1 S_U\ (u)\ du-M\right)$$

Therefore, if $M>\int_M^1 S_U\ (u)\ du$*, the function is decreasing with* M *and if* $M<\int_M^1 S_U\ (u)\ du$*, the function is increasing with* M*.*

If $M>\int_M^1 S_U\ (u)\ du$*,* $\nu<2\alpha\ \left(\int_M^1 S_U\ (u)\ du-M\right)$ *and* $u+\nu\int_q^1 S_U\ (u)\ du+q\ S_U(q)>u+\alpha\ \left(2\int_q^1 S_U\ (u)\ du-\left(\int_q^1 S_U\ (u)\ du\right)^2\right)$ *, the function reaches the minimum at* $\hat{M}=q$*.*

If $M>\int_M^1 S_U\ (u)\ du$*,* $\nu<2\alpha\ \left(\int_M^1 S_U\ (u)\ du-M\right)$ *and* $u+\nu\int_q^1 S_U\ (u)\ du+q\ S_U(q)<u+\alpha\ \left(2\int_q^1 S_U\ (u)\ du-(\int_q^1 S_U\ (u)\ du)^2\right)$ *, the function reaches the minimum at* $\hat{M}=1$*.*

- *Contract with a limit*

  Here, the insurance policy is defined as in (3.17) *and* (3.18). *Thus, the function* h *defined by* (3.19) *is as follows:*

$$h(x,L)\begin{cases} u+2\left(\alpha\int_0^L u\ S_U\ (u)\ du-\alpha\left(\int_0^L S_U\ (u)\ du\right)^2\right)^{\frac{1}{2}}, L\geq 1-q \\ u+\nu\ q\ S_U(L+q)+\nu\int_{L+q}^1 S_U\ (u)\ du\ + \\ \alpha\left(2\int_0^L u\ S_U\ (u)\ du-\left(\int_0^L S_U\ (u)\ du\right)^2\right)^{\frac{1}{2}},\ L>1-q \end{cases}.$$

When $L<1-q$*, the derivative is as follows:*

$$\frac{\partial h}{\partial L}=-\nu\ q\ f_U(L+q)-\nu\ S_U(L+q)\ +$$
$$\alpha\left(2\int_0^L u\ S_U\ (u)\ du-\left(\int_0^L S_U\ (u)\ du\right)^2\right)^{\frac{1}{2}}\left(L-\int_0^L S_U\ (u)\ du\right).$$

When $L\geq 1-q$*, the derivative is as follows:*

$$\frac{\partial h}{\partial L}=S_U(L)\left(\alpha\left(2\int_0^L u\ S_U\ (u)\ du-\left(\int_0^L S_U\ (u)\ du\right)^2\right)^{\frac{1}{2}}\right)\left(L-\int_0^L S_U\ (u)\ du\right).$$

To study the behavior of the function in the critical point, we look into the limits of the function:

$$\lim_{L\to(1-q)^+} u+\alpha\left(2\int_0^L S_U\ (u)\ du-\left(\int_0^L u\ S_U\ (u)\ du\right)^2\right)^{\frac{1}{2}}$$
$$\lim_{L\to(1-q)^-} u+\nu\ q\ P\big(U>1-q+q\big)+alpha\left(2\int_0^L S_U\ (u)\ du-\left(\int_0^L u\ S_U\ (u)\ du\right)^2\right)^{\frac{1}{2}}.$$

Therefore the following holds:

$$\lim_{L\to(1-q)^+}-\lim_{L\to(1-q)^-}=0.$$

The minimizer is then continuous and if $L>\int_0^L S_U\ (u)\ du$*, the function decreases and then increases, therefore, it reaches the minimum at* $\hat{L}=1-q$ *and if* $L>\int_0^L S_U\ (u)\ du$*, the function decreases and reaches the minimum at* $\hat{L}=1$.

Proof of Proposition 15. *Note that f is defined as in* (3.12)*, and the premium principle is defined as in* (3.8)*.*

- *Proportional contract*

  In this case, the insurance policy is defined as in (3.13) *and* (3.14)*. Thus, the function h defined by* (3.19) *is as follows:*

$$\sup_{0<a<1} h(x,a) = \min_{0<a<1}\left\{u + \sqrt{a}\ \nu\ u_3 + \alpha\ (1-a)\ \sigma_U\right\}.$$

  The derivative of the function to understand how it behaves is as follows:

$$\frac{\partial h}{\partial a} = \frac{\nu\ u_3}{2\sqrt{a}} - \alpha\ \sigma_U\ >\ 0.$$

  The function has the minimum at $a = 0$ or at $a = 1$, since :

$$\frac{\partial^2 h}{\partial a^2} = -\frac{\nu\ u_3}{4a^{\frac{3}{2}}}.$$

  Therefore, the minimizer in $a = 0$ is $u + \alpha\ \sigma_U$ and in $a = 1$ is $u + \nu\ u_3$. If $\alpha\ \sigma_U > \nu\ u_3$ the function reaches the minimum at $\hat{a} = 1$ and if $\alpha\ \sigma_U < \nu\ u_3$ the function reaches the minimum at $\hat{a} = 0$.

- *Contract with a deductible*

  Here, the insurance policy is defined as in (3.15) *and* (3.16)*. Thus, the function h defined by* (3.19) *is as follows:*

$$\sup_{0<M<1} h(x,M) = \min_{0<M<1}\left\{u+\frac{1}{2}\nu \int_0^M u^{\frac{-1}{2}} S_U\ (u)\ du+\alpha \left(2\int_M^1 u\ S_U\ (u)\ du-\left(\int_M^1 S_U\ (u)\ du\right)^2\right)^{\frac{1}{2}}\right\}.$$

  The derivative of the function to understand how it behaves is as follows:

$$\frac{\partial h}{\partial M} = S_U(M)\left(\frac{\nu}{2\sqrt{M}}+\alpha\left(2\int_M^1 u\ S_U(u)du\left(\int_M^1 S_U\ (u)\ du\right)^2\right)^{\frac{-1}{2}}\right)\left(\int_M^1 S_U\ (u)\ du-M\right).$$

  Through the derivative it is possible to understand that the function increases and then decreases, therefore, we know that the minimizer at $M = 0$ is $u + \alpha\ \sigma_U$ and in $M = 1$ is $u + \frac{1}{2}\nu\ u_3$.

  If $\alpha\ \sigma_U < \frac{1}{2}\nu\ u_3$, the minimum of the function is $\hat{M} = 0$ and if $\alpha\ \sigma_U > \frac{1}{2}\nu\ u_3$, the minimum of the function is $\hat{M} = 1$.

- *Contract with a limit*

  In this case, the insurance policy is defined as in (3.17) *and* (3.18). *Therefore, the function h defined by* (3.19) *is as follows:*

$$\sup_{0<L<1} h(x,L) = \min_{0<L<1}\left\{u+\frac{1}{2}\nu\int_L^1 u^{\frac{-1}{2}} S_U(u)\,du+\alpha\left(2\int_0^L u\,S_U(u)\,du-\left(\int_0^L S_U(u)\,du\right)^2\right)^{\frac{1}{2}}\right\}.$$

  The derivative of the function to understand how it behaves is as follows:

$$\frac{\partial h}{\partial L} = S_U(L)\left(-\frac{\nu}{2\sqrt{L}}+\alpha\left(2\int_0^L u\,S_U(u)du\left(\int_0^L S_U(u)\,du\right)^2\right)^{\frac{-1}{2}}\right)\left(L-\int_0^L S_U(u)\,du\right).$$

  Since $\left(L-\int_0^L S_U(u)\,du\right)' = 1-S_U(L)>0$, *it means* $0 < \hat{L} < 1$.

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