MESTRADO

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DISSERTAÇÃO

DYNAMICS OF FINANCIAL MARKETS: STUDY OF AN AGENT-BASED MODEL

JOÃO FRANCISCO MAGRO MARQUES

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LISBOA SCHOOL OF ECONOMICS & MANAGEMENT

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Abstract

Over the past few decades, the global financial market has been facing multiple distresses and crashes which led to troubled years for the real economy and families. Dynamical systems emerged in the mathematical finance literature to help comprehending better the unique characteristics of these financial markets and the price dynamics over the time. This work consists mainly of a statistical approach of the one discontinuity point dynamical system market model introduced by Tramontana, Westerhoff and Gardini (2010). Using a model's version that produces chaotic orbits, we can observe stationary distributions under specific parameters. In other words, the dynamical system can be chaotic in a point-wise perspective, however, from a statistical approach, it can be asymptotically predictable, that is, most trajectories converge to an attractor which we can describe statistically. Still, under the proper parameters, the model may project an absolute erratic behavior, even in the statistical approach sense. For the latter, we conclude the price forecast is impossible because we can only restrict our prognoses to an invariant set sufficient large whose contain the whole price dynamic.

Keywords: Chaotic dynamical systems, Bull and bear market dynamics, Piecewise linear maps, Lorenz maps, Attractor, Ergodic theory

Resumo

Nas últimas décadas, o mercado financeiro mundial tem enfrentado vários problemas e colapsos que motivaram anos conturbados para a economia real e para as famílias. Os sistemas dinâmicos apareceram na literatura de matemática financeira para ajudar a compreender melhor as características únicas destes mercados financeiros e a dinâmica do preço ao longo do tempo. Este trabalho consiste principalmente numa aproximação estatística ao sistema dinâmico de modelo de mercado com um ponto de descontinuidade introduzido por Tramontana, Westerhoff e Gardini (2010). Usando uma versão do modelo que produz órbitas caóticas, podemos observar, para parâmetros específicos, distribuições estacionárias. Por outras palavras, o sistema dinâmico pode ser caótico do ponto de vista do estudo das órbitas, porém, em termos estatísticos, é assintoticamente previsível, isto é, a maioria das trajetórias converge para um atractor que nós conseguimos descrevê-lo estatisticamente. Ainda, para os parâmetros apropriados, o modelo pode projetar um comportamento absolutamente errático, mesmo numa aproximação estatística. Para este último, nós concluímos que a previsão do preço é impossível uma vez que só conseguimos restringir os nossos prognósticos a um intervalo invariante suficientemente grande que contém toda a dinâmica do preço.

Palavras-chave: Sistemas dinâmicos caóticos, Dinâmicas de mercado sobre e subvalorizado, Funções lineares por ramos, Funções de Lorenz, Atrator, Teoria ergódica

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Contents

1 Introduction

After the Great Depression (1929), some economists[1] didn't believe that was possible to experiment another financial crisis with such noxious and wicked consequences to the global economy and the mankind. Nevertheless, the exponential growth of technology and the fast development of emerging economies establish new demanding challenges to the global financial system. Over the last four decades, financial market flaws became more frequent and severe, jeopardizing the real economy. The first (1973) and the second Oil crisis (1979), the Black Wednesday (1992), the Asian financial crisis (1997), the Dot Com bubble (2000) and the Subprime crisis (2008) are a few remarkable examples of recent global financial disasters with catastrophic outcomes at a global scale.

Past experience gives some vague clues about the causes of financial crises. Just before a market crash, the price for a certain asset keeps growing, but, at some point, the market is no longer willing to keep paying more for that asset and then the price drives in a free falling. Such price dynamics is called bull-bear market dynamic. Bull markets are optimistic periods when prices are generally rising. On the other hand, bear markets are associated with pessimistic periods when prices are generally falling. It's also important to consider that in financial markets there is a very wide amount of participants, each one with his own perception of the market and reaction to the available information. The more the multiplicity and heterogeneity of the participants, the more unpredictable is the variation of the market prices. This complex behavior must be taken into account when designing mathematical models to forecast bull and bear dynamics, since participant's actions interfere direct or indirectly in the price of the assets.

The introduction of this kind of models had been made by Day and Huang (1990) [6], when they presented a simple one-dimensional nonlinear system with three market participants:

- Chartists - the noise traders; they believe in the persistence of bull or bear markets;
- Fundamentalists - they bear the price convergence to the fundamental price of the asset[2];
- Market maker - he adjusts the price according to the law of supply and demand.

This model absorbs the actions made by the participants changing the price dynamics in an unpredicted way (the price in the next period, say $n+1$, can increase or decrease). Bull and bear markets may appear and then we can measure how likely financial stress events emerge. Huang and Day (1993) [12] modified their initial work and created an one-dimensional continuous linear model to approach this issue. They have to assume the fundamentalists are only willing to play in the market if the difference between the asset price and his fundamental exceeds a certain critical value. Afterwards, several papers and publications regard this matter come to light. Even with simple linear systems, it's

[1] See Krugman (2009) [14].

[2] Fundamental price: sum of all discounted cash flows (net present value).

possible to check some stylized facts from financial markets and the randomness of the asset price evolution. Hence, these kind of models became important in the mathematical finance literature.

For the purposes of this thesis we are going to discuss the work which we believe that led to a good improvement of the models developed by Day and Huang. Tramontana et al (2010) [18] generalized the financial models introduced by Day and Huang (1990) [6] and Huang and Day (1993) [12] using piecewise systems rather than nonlinear or continuous linear systems: the paper uses a model with one discontinuity point to approach this issue. For each investment philosophy, the authors decided to split the market investors in two types:

- Type 1 - always active in the market, no matter the price of the asset;
- Type 2 - only active in the market if the price of the asset is above or below a certain critical value.

Across to the analysis of these financial models, we need to bring out some definitions and results from dynamical systems applied to the one-dimensional space $\mathbb{R}$. Dynamical systems play a crucial role on approaching problems related to the real economy and financial markets. As a consequence, they are essential and the core of this thesis.

The thesis is organized as follows. The Chapter 2 provides the basic mathematical tools towards studying dynamical systems from the point-wise perspective. The Chapter 3 approaches the dynamical systems by using statistical tools and the probability measure theory to describe them asymptotically. Both these chapters are the summary of the relevant literature, mainly from Chapters 4, 5 and 8 of Day (1994) [7]. In the Chapter 4 we tune the study of dynamical systems restricted to an important class of systems: Lorenz maps. Since the models studied by Tramontana, Westerhoff and Gardini are Lorenz maps, this chapter is particularly relevant to understand better these models. In the Chapter 5 we applied the essential theoretical work presented in previous chapters into the model of Tramontana et al (2010) [18] to study the attractors of price dynamics and how bull and bear appears in the dynamics. Finally, the last chapter sums up the main results of the thesis and points out some possible directions for future research.

2 One-Dimensional Dynamical Systems

2.1 Linear and Nonlinear Systems

Definition 2.1. *Consider a **map** (function) $\theta : D \to D$, where D is a close interval in $\mathbb{R}$ like $[a, b]$ or D is the whole set $\mathbb{R}$. Then $x_{n+1} = \theta(x_n)$ is a first-order difference equation. The pair (θ, D) is called a **system**.*

Example 2.1. *We introduce some functions that can be used to define θ or that are involved in the derivation of map θ for many cases. The domain sets are just demonstrative.*

Affine system:

$$\theta(x) = ax + b, \quad D = \mathbb{R}, \ a, b \in \mathbb{R} \tag{2.1a}$$

Quadratic system:

$$\theta(x) = \alpha + \beta x + \gamma x^2, \quad D = \mathbb{R}_0^+ \text{ and } \alpha, \beta, \gamma \in \mathbb{R} \tag{2.1b}$$

Piecewise linear system: *let $a_0, ..., a_n, b_0, ..., b_n$, be sequences of real numbers with $a_{i-1} < a_i, i = 1, ..., n$. Then $D := [a_0, a_n]$ and:*

$$\theta(x) = b_i + \beta_i(x - a_i), \quad a_i \leq x \leq a_{i+1}, \tag{2.1c}$$

where $\beta_i = (b_i - b_{i-1})/(a_i - a_{i-1}) = 1, ..., n$. θ combines n linear (affine) segments that are joint to form a continuous map on the interval $[a_0, a_n]$. Nevertheless, θ may be also built with n linear segments to define a discontinuous map on $[a_0, a_n]$ with continuous branches on each $[a_i, a_{i+1}]$.

Definition 2.2. *Let (θ_1, D_1) and (θ_2, D_2) be two systems. If there exists a bijective function $h : D_1 \to D_2$ such that $\theta_2 \circ h = h \circ \theta_1$, then θ_1 and θ_2 are **conjugated** and h is called the **conjugation** which is represented by the following diagram:*

$$\begin{array}{ccc} D_1 & \xrightarrow{\theta^1} & D_1 \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ D_2 & \xrightarrow{\theta^2} & D_2 \end{array}$$

In a non-theoretical environment, specific formulas as we introduce in the previous example are quite rare. They basically appear for illustration purposes. Fortunately, in most cases it is possible to derive the behavior of a system based exclusively on its qualitative properties. In terms of empirical work and forecasting, qualitative estimates are the best tools available and are also sufficiently powerful.

A class of systems with broadly application to economics and finance is the following:

Definition 2.3. *A map θ defined on D is **monotonic** if for all $x, y \in D$ either:*

$\theta(x) \leq \theta(y)$, for all $x < y$ *or* $\theta(x) \geq \theta(y)$, for all $x < y$

In the first (second) case, θ is ***monotonic increasing (decreasing)****. The map is* ***strictly monotonic*** *(respectively increasing or decreasing) if either:*

$\theta(x) < \theta(y)$, for all $x < y$ *or* $\theta(x) > \theta(y)$, for all $x < y$

Strictly monotonic systems play an especially important role in the economics growth theory.

Example 2.2. *See the complete example in the Appendix - Section A.1 (page 34).*

2.2 Semidynamical and Dynamical Systems

2.2.1 Iterated Maps and Semiflow

From the recursive application of the equation in Definition 2.1 we get this sequence:

$$\begin{aligned} x_0 &= x = \theta^0(x) \\ x_1 &= \theta(x) = \theta^1(x) \\ x_2 &= (\theta \circ \theta)(x) = \theta^2(x) \\ x_3 &= (\theta \circ \theta \circ \theta)(x) = \theta^3(x) \\ &\vdots \\ x_n &= (\theta \circ \cdots \circ \theta)(x) = \theta^n(x) \end{aligned} \tag{2.2}$$

Using the latter method, any state of the system (θ, D) can be obtained from an initial condition x. In that case, the state of the system at any time n is a well-defined function of the initial condition x and the period n. Then, in general, we define $\theta^{n+1} := \theta \circ \theta^n$ and the new function $\theta^n(x)$ is called the n**th iterated map**. The function $h : (n, x) \to h(n, x) := \theta^n(x)$, where n is an integer, is called the **semiflow** (two parameter-function: x and n). The latter specifies that for any initial condition x and a $n > 0$ it returns the subsequent state n periods later.

2.2.2 Trajectories and Orbits

A (finite) **trajectory** or **sequence** is the history from x until a period n:

$$\tau_n(x) := (x, \theta(x), \theta^2(x), ..., \theta^n(x)) \tag{2.3}$$

The infinite history of x is obtain recursively by the following formula:

$$\tau(x) := (x, \theta(x), \theta^2(x), ..., \theta^n(x), ...) \tag{2.4}$$

The **orbit** of a trajectory or $\gamma(x)$ is the set of points through which the trajectory takes place, i.e. $\gamma(x) = \left\{x, \theta(x), \theta^2(x), ..., \theta^n(x), ...\right\}$. From a trajectory with infinite history of x, $\gamma(x)$ may be a finite set if $\tau(x)$ repeats any point after a finite number of time intervals.

2.2.3 Semidynamical Systems

Let θ^s and θ^t be two iterated maps (semiflows) beginning from two points x and y with $z = \theta^s(y)$ and $y = \theta^t(x)$. By substitution, we have $z = \theta^s \circ \theta^t$. Due to y is the state that occurs t periods after x and z occurs s periods after y, then z will occur $s+t$ periods after x, i.e.:

$$z = \left(\theta^s \circ \theta^t\right)(x) := \theta^s(\theta^t(x)) = \theta^{s+t}(x), \tag{2.5}$$

where $\theta^{s+t}(x)$ is the $(s+t)$th iterated map. By this method iterated maps are composed to obtain other iterated maps.

The set of maps $\left\{\theta^0, \theta^1, \theta^2, ..., \theta^n, ...\right\}$ is a **semigroup**, with the group operation "$\circ$" defined by (2.5) and the identity element θ^0. This set of maps determines the unique trajectory from any initial condition.

Consequently, we define a **semidynamical system** as a system (θ, D) and its associated semigroup of iterated maps. The dynamical structure θ will represent the intrinsic semidynamical system that generates it.

2.2.4 Dynamical Systems

Suppose the map θ is invertible in D. Then θ^n is defined for any n restricted to the domain D. For the case when n is a positive integer we call θ^n a **forward iterate** (it gives the states of the system n periods after the initial condition). On the contrary, a **backward iterate** θ^n is defined when n is a negative integer (it gives the states of the system n periods before the initial condition). Using inverse elements of θ and the group operator we can define the identity: $(\theta^{-n} \circ \theta^n)(x) = \theta^{n-n}(x) = \theta^0(x) = x$. Now, the set of maps $\{\theta^n, n = 0, \pm 1, \pm 2, ...\}$ is a **group** which joins all the possible forward and backward iterates of θ. Therefore, a **dynamical system** can be seen as generalization of a semidynamical system by taking a θ invertible everywhere in D and considering a group.

In general, the backward iterates from nonlinear maps, for instance $\theta(x) = ax^2 + bx + c$, are not invertible since, for an initial condition x, it could have been reached by different points/paths. Let $h : (x, n) \to h(x, n)$ be a single-valued map such that $h(x, n) \in \theta^n(x)$, where n is a positive integer and we denote a **flow** of the dynamical system (θ, D) as $\theta^n(x) := \{\theta^n(x)\}$. Therefore, since semiflows doesn't require that θ is invertible, they always exists. The same is not valid for flows.

Example 2.3. ***(Semidynamical system)***

$$\theta_s(x) = \begin{cases} s(x - 1/2) + 1 & \text{if} \quad 0 \le x \le 1/2 \\ s(x - 1/2) & \text{if} \quad 1/2 < x \le 1 \end{cases}$$

If $s = \sqrt{2}$, then $\theta_{\sqrt{2}}$ is a semidynamical system because $\theta^{-1}_{\sqrt{2}}(x)$ is not single-valued at

$D = [0, 1]$. *For instance, the preimage of* $\frac{\sqrt{2}}{2}$ *is equal to* $\left\{1 - \frac{\sqrt{2}}{2}, 1\right\}$.

Example 2.4. ***(Dynamical system)***

$$\theta_\gamma(x) = \gamma x^2, \quad D = [0, 1]$$

Since $\theta_\gamma^{-1}(x)$ *is well-defined (all preimages has a single solution in* D*),* θ_γ *is a dynamical system.*

2.3 Explicit Solutions

An **explicit solution** can be derived using flows (if they exist) or semiflows. It's essentially a map which allows us to evaluate the trajectory of the states.

However, the deduction of explicit solutions is hard and not conventional in the dynamical system literature. Instead, a preferable way to study the trajectories of a system is the recursive method shown in (2.2). Describing a trajectory based on an initial condition x is sufficient and a better approach than obtaining explicit solutions.

2.4 Stationary Behavior

2.4.1 Fixed Points and Stationary States

A trajectory $\tau(x)$ is called **stationary** if for all n we have $\theta^n(x) = x$, where x is called a **stationary state**. Furthermore, let $\theta(x) = x$, then x is a **fixed point** of θ (the existence of stationary states for a dynamical system is equivalent to the existence of fixed points in D for the map θ). The trajectory for any stationary state is itself stationary. The existence of stationary states leads to a persistent situation which doesn't let θ to escape from the fixed point. The stationary states of a dynamical system can be graphically represented by the intersection between the graph of θ and the line $y = x$.

2.4.2 Existence of Stationary States

Recall this classical result from calculus:

Theorem 2.1. ***Bolzano's Theorem - Intermediate value theorem*** *Let* $\theta : D \to D$ *be a continuous function and* $x, z \in D$ *where* $\theta(x)\theta(z) < 0$*. Then, there is at least one point* $y \in (x, z]$ *such that* $\theta(y) = 0$*.*

Corollary 2.1. *Let* θ *be continuous on* D*. If there exist* $y, z \in D$ *such that* $\theta(y) \leq y$ *and* $\theta(z) \geq z$ *then there exists a stationary state* x *of the difference equation in Definition 2.1.*

Proof. See the Appendix - Section A.2 (page 35). □

2.5 Cycles

A point x is called p**-cyclic** or **periodic point of period** p, that is, the trajectory of an initial condition x will be repeated every p periods. Formally we define x as a p-cyclic if for an integer $p > 1$ we have:

$$\theta^p(x) = x \text{ and } \theta^n(x) \neq x, n = 1, ..., p-1 \tag{2.6}$$

Moreover, a p-cyclic state is obviously a fixed point of θ^p. If x is p-cyclic, then $\theta(x)$ is also p-cyclic (the idea is to apply θ to the both sides of (2.6) where we get: $\theta(x) = (\theta \circ \theta^p)(x) = (\theta^p \circ \theta)(x))$. The orbit $\gamma(x) := \{x, \theta(x), ..., \theta^{p-1}(x)\}$ is called a **cycle of period** p.

2.6 Stability Theory

2.6.1 Stable, Asymptotically Stable and Unstable

For a nonperiodic initial condition, the study of **stability** clears how the (infinite) trajectory behaves. The results presented in this section are also valid towards cycles of p-order: just replace θ for θ^p.

Definition 2.4. *A trajectory $\tau(x)$ is called **stable** if for all $\varepsilon > 0$, there is a $\delta > 0$ such that, for all $|y - x| < \delta$ implies $|\theta^n(y) - \theta^n(x)| < \varepsilon$.*

Definition 2.5. *A trajectory $\tau(x)$ is called **asymptotically stable** if there is a $\delta > 0$ such that, for all $|y - x| < \delta$ implies $\lim_{n\to\infty} |\theta^n(y) - \theta^n(x)| = 0$.*

Therefore, a system with asymptotically stable trajectories represents simple dynamics. That is, after enough iterates of θ the trajectories will converge to a p-periodic behavior of some period p and the system becomes more predictable.

However, stability doesn't imply asymptotic stability, but the reciprocal is true. The next example give us an illustration of this statement.

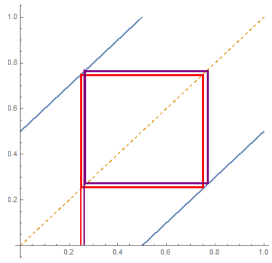


Figure 2.1: Plot of θ_1 (blue line); $\gamma(0.3)$ (red line) versus an orbit with initial condition very close to 0.3 (purple line)

Example 2.5. *Recall the $\theta_s(x)$ system introduced on Example 2.3. Considering $s = 1$:*

$$\theta_1(x) = \begin{cases} x + 1/2 & \text{if} \quad 0 \leq x \leq 1/2 \\ x - 1/2 & \text{if} \quad 1/2 < x \leq 1 \end{cases}$$

Since the slope of $\theta_1(x)$ is equal to 1, the graph of θ_1 is parallel with respect to the line $y = x$. It's also expected orbits with periodic behavior. From the previous graphic, it's easy to verify that orbit of $x = 0.3$ is a cycle of period 2. By choosing a x-point very close to $x = 0.3$ we come up with a similar orbit (purple line).

Consequently, $\theta_1(x)$ is stable because the distance between both orbits is bounded by a scalar $\varepsilon > 0$. In this particular example ε is equal to δ. Moreover, $\theta_1(x)$ is not asymptotically stable since the distance between both orbits doesn't converge to 0.

As opposed to stable, we now introduce the notion of unstable trajectories:

Definition 2.6. *A trajectory $\tau(x)$ is called* ***unstable*** *or* ***not stable*** *if there is a $\varepsilon > 0$ such that, for all $\delta > 0$, there is a $y \in D$ such that $|y - x| < \delta$ but $|\theta^n(y) - \theta^n(x)| \geq \varepsilon$ for some $n \geq 0$.*

In other words, no matter how close two trajectories start from each other, they will inevitable diverge.

The next theorem clarifies how we can categorize the different types of fixed points:

Theorem 2.2. *Let θ be a function of class C^1 and x be a fixed point of θ:*

(i) If $|\theta'(x)| < 1$, then x is ***asymptotically stable***

(ii) If $|\theta'(x)| > 1$, then x is ***unstable***

Proof. See the Appendix - Section A.3 (page 35). □

2.6.2 Expansivity

A map θ from an unstable system that satisfies the following theorem is named **expansive**. A map whose pth iterate θ^p is expansive, but $\theta^0, ..., \theta^{p-1}$ are not expansive, is known as p-expansive.

Theorem 2.3. *Let θ be differentiable almost everywhere on D and assume that there exists an integer $m \geq 1$ such that $|(\theta^m)'(x)| \geq \delta > 1$, for all $x \in D$ where the derivative is defined, then all trajectories in (θ, D) are unstable.*

Proof. Without loss of generality, suppose that $m = 1$ (otherwise repeat the argument for multiples of m). Therefore, the proof is equal to the Proof of Theorem 2.2. □

2.7 Chaos: an Informal Perspective

Chaos theory is the study of nonlinear dynamics with unstable trajectories. Such dynamics seems surprisingly random and unpredictable.

There isn't a single definition for *chaos*. Mathematicians diverge on the enough conditions to name a system as chaotic. Nevertheless, we'll introduce *chaos* according to Devaney (1989) [8].

Remark 2.1. *If a trajectory $\tau(x)$ is unstable in the sense of the Definition 2.6, then it has* ***sensitivity to initial conditions****.*

For instance, if a map has sensitivity to initial conditions, then small errors could emerge in the attempt to compute numerically the map's dynamic.

Definition 2.7. *Let $\theta : D \to D$. θ is* ***topologically transitive*** *if for any pair of non-empty open sets U and V in D, there is a non-negative integer k such that $\theta^k(U) \cap V \neq 0$.*

Vaguely, a topologically transitive map has points which eventually move under iteration from one arbitrarily small open set to any other. Consequently, such a dynamical system cannot be decomposed into two disjoint open sets (Cattaneo et al (1997) [3]).

Convention 2.1. *Denote by $cl(A)$ the* ***topological closure*** *of A.*

Definition 2.8. *A is* ***dense*** *(or a dense set) in B if $cl(A)$ contains B.*

Definition 2.9. *When $\theta(I) = I$, then I is called an* ***invariant set****.*

Finally we have all the proper mathematical tools to characterize a chaotic system.

Definition 2.10. ***(Devaney's chaos)*** *Let D be a set in $\mathbb{R}$ and $\theta : D \to D$. θ is* ***chaotic*** *on D if:*

(i) θ is sensitive to initial conditions;

(ii) θ is topologically transitive;

(iii) periodic points are dense in D.

Example 2.6. ***(Logistic map)*** *Let $\theta : [0, 1] \to [0, \alpha/4]$ be $\theta(x) = \alpha x(1-x)$. The map has two fixed points: $\theta(x) = 0$ and $\theta(x) = \frac{\alpha-1}{\alpha}$ (their stability rely on α). Denavey (1989) [8] and Holmgren (1994) [10] states that family of logistic maps are chaotic if $\alpha > 2 + \sqrt{5}$.*

Remark 2.2. *Another example of a chaotic map is the map introduced in the Example 2.3 with $s > \sqrt{2}$. This map is topologically transitive (as we shall see further along in the Lemma 4.3, the periodic points are dense in $[0, 1]$) and has sensitivity to initial conditions (because $s > 1$).*

3 Statistical Dynamics

In the previous chapter we were focus on the qualitative properties of deterministic dynamical systems and their point-wise orbits/trajectories when n drives to ∞. Nevertheless, in real life we are seldom capable to observe the precise states or the exact values of a system (θ, D). Statistics help us to approach this issue, recognizing that a state $x+\varepsilon$, where ε is the observational error, has an intrinsic probability of happening. Such measures are possible using **random variables** which are real-valued functions that gives a numerical quantity to any state of D, i.e. $X : D \to \mathbb{R}$ (later on we refer this function as an **observable**). The purpose of this chapter is to develop statistical indicators for evaluate the asymptotic behavior of the random variables by exploiting all the possible information inside (θ, D).

Before starting this section, we need to recall some results from Measure Theory:

Definition 3.1. *A σ-algebra is a collection of subsets $\mathcal{F}$ of a set D:*

(i) that contains D, i.e., $D \in \mathcal{F}$

(ii) that contains the complement of any set in $\mathcal{F}$, i.e., $A \in \mathcal{F}$ implies $A^c \in \mathcal{F}$

(iii) that contains the union of any countable collection of subsets in $\mathcal{F}$, i.e., let $\{A_n\}_{n=1}^{\infty}$ be a countable collection of sets with $A_n \in \mathcal{F}$ for all n; then $\bigcup_{n=0}^{\infty} A_n \in \mathcal{F}$

Definition 3.2. *Let $\mathcal{B}(A)$ denotes the Borel σ-algebra on A. $\mathcal{B}(A)$ is the smallest σ-algebra that contains all the open sets of A.*

Definition 3.3. *Let $\{A_n\}_{n=0}^{\infty}$ be a countable collection of disjoint sets in a σ-algebra $\mathcal{F}$. A measure is a map $\mu : \mathcal{F} \to \mathbb{R}_0^+$ such that:*

(i) $\mu(\emptyset) = 0$

(ii) $\mu\left(\bigcup_{n=0}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$

Definition 3.4. *A measure space is a triple $(D, \mathcal{F}, \mu)$ whose $\mathcal{F}$ is a σ-algebra of subsets of D and μ is a measure defined on $\mathcal{F}$. The sets in $\mathcal{F}$ are called measurable. The measure space is called finite if $\mu(A) < \infty$ for all $A \in \mathcal{F}$. The measure μ is called a probability measure if $\mu(D) = 1$ and therefore $(D, \mathcal{F}, \mu)$ is a probability space.*

Definition 3.5. *Let $(D, \mathcal{F})$ be a measurable set and $(\mathbb{R}, \mathcal{T})$ a topological set. A function $g : D \to \mathbb{R}$ is said $\mathcal{F}$-measurable if and only if the preimage of A under g belongs to $\mathcal{F}$ for all open set $A \in \mathbb{R}$.*

Definition 3.6. *Let $(D, \mathcal{F}, \mu)$ be a probability space and each random variable X_n is $\mathcal{F}$-mensurable for $n \geq 0$. A family of random variables $\{X_n\}_{n\geq 0}$ is called a* ***stochastic (or random) process****. Any stochastic process can be seen as a function of two variables: n and ω. For each fixed n, $\omega \to X_n(\omega)$ is a random variable. On the other hand, if we fix ω instead, we see that the stochastic process is a function mapping ω to the real-valued function $n \to X_n(\omega)$. These functions are called the* ***trajectories of*** *X.*

Definition 3.7. *Given a probability space $(D, \mathcal{F}, \mu)$, a $\mathcal{F}$-mensurable system (θ, D) and an integrable observable $X : D \to \mathbb{R}$, we define $X_n = X \circ \theta^n$ for all $n \geq 0$ as the stochastic process which evaluates the states over $\gamma(x)$, where $X = X_0$ is the initial condition and θ^n the deterministic transformation at the instant n.*

Definition 3.8. *A stochastic process $\{X_n\}_{n\geq 0}$ is **stationary** if the random vectors $(X_0, X_1, X_2, ..., X_m)$ and $(X_h, X_{h+1}, X_{h+2}, ..., X_{h+m})$ have the same joint distribution for all $h, m \geq 0$.*

Definition 3.9. *The support of a probability measure is the smallest closed set of full probability. We denote the support of μ by supp(μ).*

Example 3.1. *Let $(D, \mathcal{F})$ be a measurable space, $x \in D$ and δ the Dirac measure defined as:*

$$\delta_x(A) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Since $\delta_x(D) = 1$, the full probability set is $\{x\}$. Hence, the support of δ_x is also $\{x\}$.

Definition 3.10. *Suppose (θ, D) is $\mathcal{F}$-measurable. θ is called **measure preserving** and μ is called **invariant with respect to** θ if, for all $A \in \mathcal{F}_\theta = \left\{\theta^{-1}(B) : B \in \mathcal{B}\right\}$, the push-forward measure of μ to θ is exactly equal to the measure μ, i.e. $\theta_*\mu = \mu\left(\theta^{-1}(A)\right) = \mu(A)$.*

Remark 3.1. *It's easy to see that $\theta_*\mu$ is a measure in the sense of the Definition 3.3.*

Theorem 3.1. ***(Change of variable formula)** Let θ and ψ be $\mathcal{F}$-measurable functions in $\mathbb{R}$. ψ is integrable with respect to the push-forward measure $\theta_*\mu$ if and only if the composition $\psi \circ \theta$ is integrable with respect to the measure μ. Moreover the following formula holds:*

$$\int_A \psi \mathrm{d}(\theta_*\mu) = \int_{\theta^{-1}(A)} \psi \circ \theta \mathrm{d}\mu$$

Proposition 3.1. *If μ is an invariant measure to θ, then $X_n = X \circ \theta^n$ is stationary.*

Proof. See the Appendix - Section A.4 (page 36). □

Remark 3.2. *In particular, $E(X_n) = E(X)$.*

3.1 The Recurrence Theorem

Let $\sum_{n=0}^{N-1} \chi_A(\theta^n(x))$ be the number of times that the first N iterates of x will visit the set A. The next theorem states that after wait enough time the orbit of x will eventually enter in A, but not once: it will enter infinitely times.

Theorem 3.2. ***(Poincaré Recurrence Theorem)** Let $(X, \mathcal{F}, \mu)$ be a probability space and let the measure μ be invariant under θ. Let A be any set of positive measure. Then* *almost all points of A return to A infinitely often. That is, for almost $x \in A$:*

$$\lim_{N\to\infty} \sum_{n=1}^{N} \chi_A(\theta^n(x)) = \infty$$

Proof. See the Appendix - Section A.5 (page 37). □

This theorem appeared in the statistical physics and it's related to some of the most famous paradoxes in the mechanic physics field.

3.2 Limit Sets and Attractors

In the Chapter 2, stable periodic trajectories were introduced and their limit points are simply the elements of the periodic orbit to which trajectories converge (recall Definition 2.5). However, a general definition for limit set can emerge to consider chaotic systems.

Definition 3.11. *The **limit set** $\omega(x)$ of the trajectory $\tau(x)$ is defined to be the set of all limit points of $\tau(x)$, i.e., $\omega(x) := \bigcap_{n=1}^{\infty} cl\,[\gamma(\theta^n(x))]$, where $\gamma(y)$ is the orbit from y.*

Definition 3.12. *An **attractor** for θ is a closed set $L \subset D$ such that $\omega(x) = L$ for x in a set B of positive Lebesgue measure. The set B is called the **basin of attraction** of L.*

Remark 3.3. *Note that $\omega(x)$ is closed and $\theta(\omega(x)) = \omega(x)$. Obviously, $L \subset B$ and $\theta(L) = L$.*

3.3 Ergodic Dynamical Systems

Ergodicity relates the notion of recurrence introduced by Theorem 3.2 and the existence of invariant sets. A system is said **ergodic** if its trajectories enter in an unique invariant set without leaving it. In particular, that set cannot be decomposed in different parts with similar properties, which leads us to the next definition:

Definition 3.13. *Let $(D, \mathcal{F}, \mu)$ be a probability space. The dynamical system (θ, D) is **μ-ergodic** if the measure of every undecomposable invariant set is either 0 or 1, i.e., if $A \in \mathcal{F}$, then $\theta^{-1}(A) = A$ implies either that $\mu(A) = 0$ or that $\mu(A) = 1$.*

As a consequence, ergodic systems imply that **time average** is equal to **space average**. In general, this is not true, because the measure μ can take any value from 0 to 1. In this case, μ can only take the values 0 or 1, furthermore it's also invariant to the system.

Theorem 3.3. ***(The Birkhoff-von Neuman Mean Ergodic Theorem)*** *Let $(D, \mathcal{F}, \mu)$ be a probability space and let θ be a transformation which is measure preserving and ergodic. Let X be an integrable observable, then:*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} X \circ \theta^i = E(X), \quad \textit{for almost all } x \in D$$

Remark 3.4. *In the conditions of the previous theorem, $X_n = X \circ \theta^n$ is an* ***ergodic process****.*

The left side of the equation from last theorem is the time average value of X_n and the right side is the space average of X_n defined explicitly as: $\frac{1}{\mu(D)} \int_D (X \circ \theta^n)(x) \mathrm{d}\mu$.

Corollary 3.1. *Let $(D, \mathcal{F}, \mu)$ be a probability space and let θ be measure preserving and ergodic on D. Then for μ-almost all x in D, $\tau(x)$ will visit every measurable set proportionally to its measure.*

Proof. Let $A \in \mathcal{F}$ be a set with positive measure. If a trajectory $\tau(x)$ starts in D, how much time does τ spend in A?

Let $X = \chi_A(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \notin A \end{cases}$ defines if a point $x \in \tau$ "enters" (or not) in A.

In general, $(X \circ \theta^i)(x) = \chi_A(\theta^i(x)) = \begin{cases} 1 & \text{if} \quad \theta^i(x) \in A \\ 0 & \text{if} \quad \theta^i(x) \notin A \end{cases}$

According to Theorem 3.2, the sum of all points in τ that enters in the set A will be infinite because A has positive measure. But by the Theorem 3.3 the average time spent in the set A can be determined by:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_A\left(\theta^i(x)\right) = \int_D \chi_A \mathrm{d}\mu = \int_A \mathrm{d}\mu = \mu(A)$$

□

3.4 Distributions for Dynamical Systems

3.4.1 Absolute Continuity: Density and Distribution

Now we'll explore what kind of measures can be intimately related to distributions or density functions.

Convention 3.1. *We denote the Lebesgue measure in $\mathbb{R}$ by λ.*

Definition 3.14. *A measure is said to be* ***absolutely continuous with respect to*** *λ if for all $A \in \mathcal{F}$ there exists an integrable function f, called the density of μ, such that:*

$$\mu(A) := \int_A f(x)\mathrm{d}x = \int_{\theta^{-1}(A)} f(x)\mathrm{d}x = \theta_*\mu(A)$$

Remark 3.5. *If μ is absolutely continuous the Theorem 3.1 is equivalent to the Definition 3.14.*

Such measures are differentiable in the sense of the **Radon-Nikodym derivative** (λ-almost everywhere), that is $f = \frac{d\mu}{dx} = \frac{d\mu}{d\lambda}$ and for any interval $[a, b]$ in $\mathbb{R}$ $\mu([a, b]) =$

$\int_a^b f(x)\mathrm{d}x$. Note that under these conditions f is unique as well as continuous λ-almost everywhere.

Definition 3.15. *Let μ_1 and μ_2 be two measures on the same measure space $(D, \mathcal{F})$. We say μ_1 is absolutely continuous with respect to μ_2 or $\mu_1 << \mu_2$ if for any set $A \in D$, such that $\mu_2(A) = 0$ always implies $\mu_1(A) = 0$.*

Definition 3.16. *Let $f : D \to \mathbb{R}$ be a density function and D a subset of $\mathbb{R}$. The set-theoretic **support of** f is the closure set of points in D where f is non-zero, i.e.:*

$$supp(f) = cl\left(\{x \in D | f(x) \neq 0\}\right)$$

Convention 3.2. ***Acip** is the abbreviation for absolutely continuous invariant probability measure and its density is called **invariant density**.*

Remark 3.6. *If μ is an acip, then $supp(\mu)$ is equal to $supp(f)$.*

Example 3.2. *Recall the map from Example 2.3 where $s = 2$:*

$$\theta_2(x) = \begin{cases} 2(x - 1/2) + 1 & if \quad 0 \leq x \leq 1/2 \\ 2(x - 1/2) & if \quad 1/2 < x \leq 1 \end{cases} = \begin{cases} 2x & if \quad 0 \leq x \leq 1/2 \\ 2x - 1 & if \quad 1/2 < x \leq 1 \end{cases}$$

Let $f = \chi_{[0,1]}$, where f is the density function of Lebesgue measure restricted to $[0, 1]$. We'll use the results from Theorem 3.1 and Definition 3.14 to show that μ is an acip:

$$\theta_*\mu\left([a, x]\right) = \int_{\theta^{-1}([a,x])} \chi_{[0,1]}\mathrm{d}\lambda = \int_{\frac{[a,x]}{2}} 1\mathrm{d}\lambda + \int_{\frac{[a,x]}{2}+\frac{1}{2}} 1\mathrm{d}\lambda = x - a = \int_{[a,x]} \chi_{[0,1]}\mathrm{d}\lambda = \mu\left([a, x]\right)$$

Definition 3.17. ***(Strongly ergodic dynamical systems)** A dynamical system (θ, D) that is ergodic with respect to an absolutely continuous measure μ defined on $\mathbb{R}$ will be called **strongly ergodic**. For a strongly ergodic system with density f, the measure is equivalent to the **cumulative distribution function**:*

$$\mathrm{F}(x) \coloneqq \mu([a, b]) = \int_a^x f(u)\mathrm{d}u = \int_{\inf D}^x f(u)\mathrm{d}u$$

If we extend θ to $\mathbb{R}$ in the usual way, then we obtain $\mathrm{F}(x) = \int_{-\infty}^x f(u)\mathrm{d}u$.

Theorem 3.4. ***(Lasota-Yorke, 1973)** Let $\theta : D \to D$ be a piecewise function of class C^2 where D is an interval. If $|\theta'(x)| \geq \delta > 1$ λ-almost everywhere in D, then there exists an acip for θ.*

Example 3.3. *This result applies to piecewise linear systems like the system presented on* (2.1c) *with $\beta_i > 1$.*

Corollary 3.2. *If a map θ does not satisfy the assumption of Theorem 3.4 but there exists an integer, say p, such that $\left|\frac{d}{dx}\theta^p\right| > 1$, then the theorem holds.*

3.4.2 The Number of Absolutely Continuous Invariant Ergodic Measures

The next theorem sets up sufficient conditions to establish a maximum number of ergodic acips in D. In addition, it tells us what we should expect for the shape of their supports. The extension of the following work is available on Section 8.2 from Boyarsky & Góra (1997) [1].

Theorem 3.5. *Let (θ, D) be a dynamical system where D is an interval and the map θ is piecewise with d discontinuity points and strictly monotonic on each piece of the partition D of interval pieces D_i, $i = 1, ..., d+1$. Assume that for each $i = 1, ..., d+1$, θ is restricted to the interior of D_i and it is continuously differentiable and expansive. Then there exists at most d (could be less) ergodic acips μ_i whose supports are union of finitely many intervals.*

As an immediate consequence of Theorem 3.5 and Definition 3.12 we get:

Corollary 3.3. *There exists a partition $\{B_i, i = 1, ..., m\}$ of D such that B_i is a basin of attraction of supp(μ_i). Moreover supp(μ_i) is an attractor for θ and this implies at most d attractors.*

Remark 3.7. *If the map has only one discontinuity point ($d = 1$), then there exists an unique ergodic acip with an unique attractor that is the union of closed intervals.*

3.4.3 The Mean and Variance of a Trajectory

Let $\{B_i, i = 1, ..., m\}$ be the partition imposed by Corollary 3.3. From Theorems 3.3 and 3.5 we have:

Corollary 3.4. *Let $f_i = \frac{d\mu_i}{d\lambda}$ for each ergodic acip μ_i. Then for all $x \in B_i$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left(X \circ \theta^k\right)(x) = \int_D X(u) f_i(u) \mathrm{d}u$$

In particular, for all $x \in B_i$, the expected or mean value of states in the trajectory is:

$$\bar{\mu}_i = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \theta^k(x) = \int_D u f_i(u) \mathrm{d}u$$

and the variance of states in the trajectory is:

$$\bar{\sigma_i}^2 = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left[\theta^k(x) - \bar{\mu}\right]^2 = \int_D (u - \bar{\mu})^2 f_i(u) \mathrm{d}u$$

3.5 Constructing Densities

Motivated by the last section, we will describe two methods for compute the density function.

3.5.1 The Frobenius-Perron Operator

Let (θ, D) be a dynamical system, where D is a finite interval $[a, b]$ and suppose that μ is an ergodic acip with an invariant density f.

Definition 3.18. *The Frobenius-Perron Operator is defined by:*

$$Pf(x) = \frac{d}{dx} \int_{\theta^{-1}([a,x])} f \mathrm{d}\lambda$$

Proposition 3.2. ***(Properties of** P**)***

(i) $P : \mathcal{L}^1 \to \mathcal{L}^1$ *is linear, where* $\mathcal{L}^1$ *is the space of integrable functions.*

(ii) $Pf \geq 0$ *if* $f \geq 0$

(iii) $\int Pf \mathrm{d}\lambda = \int f \mathrm{d}\lambda$

(iv) $Pf = f$ *if and only if* $\mu(A) = \int_A f \mathrm{d}\lambda$ *for all* A *is invariant under* θ

Proof. See the Appendix - Section A.6 (page 38). □

3.5.2 The Frobenius-Perron Operator for Piecewise Monotonic Systems

Now, we will show how the Pf operator can be used for piecewise monotonic maps.

Proposition 3.3. *Let θ be a piecewise, strictly monotonic map satisfying Theorem 3.4. Let L_i be an attractor which is the support of an ergodic acip μ_i. Then μ_i has an unique invariant density function $f(x)$ that satisfies:*

$$Pf(x) = \sum_{i=1}^{n} f(\theta_i^{-1}(x)) \cdot \left| \frac{d\theta_i^{-1}(x)}{dx} \right| \cdot \chi_{[\theta(a_{i-1}), \theta(a_i)]}(x)$$

Proof. It comes from simple calculations and basic Lebesgue integration rules. For the complete proof see Section 4.3, page 85 and 86 from Boyarsky & Góra (1997) [1]. □

3.5.3 Empirical Approach to Density Functions

Despite the fact that Frobenius-Perron operator offers a more precise formula for the density function f, in most cases we are not able to compute it analytically due to the complexity of the map. Instead, we may approach f numerically by collecting experimental data. Recall Definition 3.17 for the cumulative distribution function (denoted by F). Let X be a random variable, then: $\mathrm{F}(z) = \mathrm{Prob}[X \leq z] = \int_{-\infty}^{z} f(x) \mathrm{d}x$, where f is the density function of X. See more in Evans et al (2011) [9].

The previous expression is somehow similar to the space average (recall Theorem 3.3). Let $X = \chi_{(-\infty, z]}$, then from the ergodic theory we know the time average will converge to $\mu((-\infty, z])$. Since μ is an absolutely continuous measure with respect to λ, we have: $\mu((-\infty, z]) = \mathrm{F}(z)$. This result allows us to use the time average to build our empirical distribution.

Lemma 3.1. *The* ***empirical cumulative distribution function*** *for a random variable X is denoted by* ecdf *and it has the following expression:*

$$\mathrm{ecdf}(z) = \mu((-\infty, z]) = \frac{1}{n} \sum_{i=0}^{l-1} \chi_{(-\infty, z]} \circ \theta^i(x),$$

where n determines how many elements are inside $\tau(x)$ and l is the number of points used to build the empirical cumulative distribution function.

In principle, ecdf should be a function of z and x. However, by the ergodic theorem $\mathrm{ecdf}(z, x)$ is equal to $\mathrm{ecdf}(z)$ for μ-almost every x. This means that from a probabilistic point of view we can remove the dependence on x.

The use of empirical data should lead to a cumulative distribution function quite similar to the real underlying distribution. However, we don't have the same ease for the density function. In order to get a fair replicate of the true density function f, we need a large amount of data. Otherwise, for close values of z the cumulative distribution function has similar values and therefore the density function near those points goes (wrongly) to 0. Plus, the function χ is not smooth because it isn't differentiable everywhere. This is why in the expression of ecdf we need to declare a variable l distinct of n. Note that with a large l, on the one hand, we have more points to draw our ecdf, but the density function will require lots of data to distinguish consecutive points. Hence, choosing a small but not too small l and a large n (inducing some sort of smoothness) we can simulate f numerically and see some resemblances to the true density function.

Lemma 3.2. *The invariant density function f may be numerically obtained by:*

$$f(z_1) \approx \frac{\mathrm{ecdf}(z_2) - \mathrm{ecdf}(z_1)}{\delta}, \quad z_2 > z_1, \delta = \frac{1}{l-1}$$

Example 3.4. *See the complete example in the Appendix - Section A.7 (page 39).*

Remark 3.8. *The code used to compute the empirical distribution in the previous example is available in the Appendix - Section A.8 (page 41).*

3.6 Other Statistical Properties

Despite the "sensation" of randomness intrinsic to chaotic dynamical systems, they are not actually stochastic processes because for an initial condition x_n, the value for the next iteration x_{n+1} is exactly known by a deterministic map. However, this kind of trajectories shares some resemblances to stochastic processes in the sense that a trajectory appears to be a realization of a stochastic process (generating a series of independent, identically distributed random variables). Now we'll see how the ergodic theory can be related (in a certain way) to a few important results in Probability Theory.

Theorem 3.6. ***(Strong Law of large number)*** *Recall the definition of the stochastic process $\{X_n\}_{n\geq 0}$, where all $X_n = X\circ\theta^n$ represent a sequence of independent and identically distributed random variables. Then, by the strong law of large numbers, the sample average (time average) converges with probability 1 to the expected value $E(X)$ (space average):*

$$\mu\left(\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}X_k = E(X)\right) = 1, \quad \mu\text{-almost surely}$$

Proof. It's an immediate consequence of the ergodic theorem (see Theorem 3.3). □

Note that the values of trajectories are not random and not necessarily independent among them. Therefore the Theorem 3.3 must be seen as a generalization of the law of large numbers.

From now on, let (θ, D) be in the conditions of Theorem 3.5 and suppose $d = 1$. By Remark 3.7 there exists only one acip μ. In addition, suppose that (θ, D) is topological mixing, i.e.:

Definition 3.19. *Let $\theta : D \to D$ be a real-valued function. θ is topological mixing if for all interval I subset of D there is a non-negative N such that $\theta^N(I) = D$.*

In order to evaluate how fast the observable $X_n = X\circ\theta^n$ becomes independent of the initial $X_0 = X$, Boyarsky & Góra (1997) [1] established the following result:

Theorem 3.7. ***(Decay of correlations)*** *For any bounded observable $X : D \to \mathbb{R}$ and $m \geq 0$, we have:*

$$\rho(X_m, X_n) = \lim_{n\to\infty}|E(X_m, X_n) - E(X_m)E(X_n)| = 0$$

Proof. See Definition 8.3.1 and Theorem 8.3.2, page 148 from Boyarsky & Góra (1997) [1]. □

When time average and space average are similar[3], the sum of the observations under the process X_n converges in limit to a standard normal distribution $\mathcal{N}(0,1)$.

Theorem 3.8. ***(Central Limit Theorem)*** *For any bounded observable $X : D \to \mathbb{R}$, there is a $\sigma > 0$ such that:*

$$\lim_{n\to\infty}\mu\left[\frac{1}{\sigma/\sqrt{n}}\left(\sum_{i=0}^{n-1}X_i - nE(X_0)\right) < z\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z}\exp\left(-\frac{x^2}{2}\right)\mathrm{d}x$$

Proof. See Theorem 8.5.1, page 157 from Boyarsky & Góra (1997) [1]. □

[3]If n goes to ∞, the quotient $\frac{1}{\sigma/\sqrt{n}}$ goes equally to ∞; then, the limit is only bounded by z when the difference between the time average and space average is sufficient close to zero.

4 Dynamics of Lorenz Maps

Before diving in the discussion of the market model of Tramontana et al (2010) [18], we need to declare some results regarding the class of maps present in their work. The next definition comes from Hubbard & Sparrow (1990) [13]:

Definition 4.1. *Let $\theta : [0,1] \to [0,1]$ be a function of class C^1 satisfying:*

(i) There is a $d \in (0,1)$ such that θ is continuous and strictly increasing on $[0,d)$ and on $(d,1]$;

(ii) $\lim_{x\to d^-} \theta(x) = 1$ and $\lim_{x\to d^+} \theta(x) = 0$;

(iii) θ is (topologically) expansive;

then θ is a ***Lorenz map****.*

Let θ_s take the form of the map from Example 2.3:

$$\theta_s(x) = \begin{cases} s(x - 1/2) + 1 & \text{if} \quad 0 \leq x \leq 1/2 \\ s(x - 1/2) & \text{if} \quad 1/2 < x \leq 1 \end{cases} \tag{4.1}$$

θ_s is a particular form of a Lorenz map called **symmetric piecewise linear Lorenz map** because it is a symmetric function on the discontinuity $d = 1/2$. Moreover, θ_s is an odd function[4].

Lemma 4.1. *Let $\varphi(x) = 1 - x$, then θ_s is $\varphi-$symmetric, i.e.: $\theta_s \circ \varphi = \varphi \circ \theta_s$.*

Theorem 4.1. *Let θ_s be a symmetric piecewise linear Lorenz map. θ_s has an unique ergodic acip μ. Moreover, this measure is φ-symmetric, i.e. $\varphi_*\mu = \mu$.*

Proof. From the Theorem 3.5, θ_s is a class C^1 expansive and a continuous function on the partitions D_1 and D_2. By the first condition of the theorem we have $d = 1$. Then it follows μ_i is equal to μ_1. Therefore, μ_1 is the unique admissible acip. With the result from Remark 3.7, the proof is straightforward. □

Remark 4.1. *We say that an interval I is φ-symmetric if $\varphi(I) = I$, which means the interval I is symmetric with respect to $x = 1/2$.*

Lemma 4.2. *Suppose $s \in \left(1, \sqrt{2}\right]$. Let $J = \left[1 - \frac{1}{2}s, \frac{1}{2}s\right]$ be φ-symmetric and $h_s : J \to [0,1]$ the affine transformation $h_s(x) = \frac{1}{s-1}\left(x + \frac{1}{2}s - 1\right)$, then $h_s \circ \theta_s^2 \circ h_s^{-1} = \theta_{s^2}$ holds.*

Proof. First we need to calculate an explicit expression for θ_s^2. Due to the fact θ_s is a piecewise function the composition $\theta_s^2 = \theta_s \circ \theta_s$ won't be a straight forward expression. θ_s has a discontinuity at $x = 1/2$, so we will expect that θ_s^2 also has a discontinuity at

[4] A function θ is odd if $\theta(x) + \theta(-x) = 0$.

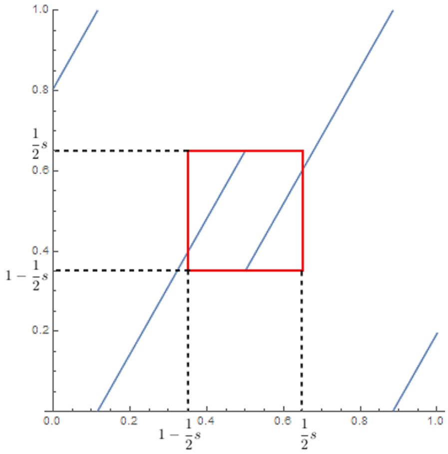


Figure 4.1: Graph of θ_s^2 with $s = 1.3$ (blue line) and the set J (red box)

$\theta_s(x) = 1/2$ and two other discontinuity points. After some calculus, we come up with this expression (where $u = \frac{1}{2} - \frac{1}{2s}$):

$$\theta_s^2(x) = s^2(x - 1/2) + \begin{cases} \frac{1}{2}s + 1 & \text{if} \quad 0 \leq x \leq u \\ \frac{1}{2}s & \text{if} \quad u < x \leq 1/2 \\ -\frac{1}{2}s + 1 & \text{if} \quad 1/2 < x \leq 1 - u \\ -\frac{1}{2}s & \text{if} \quad 1 - u < x \leq 1 \end{cases}$$

The Figure 4.1 shows when θ_s^2 is restricted to J, there is a striking resemblance to the plot of θ_s. In some sense, the function inside of the red box may be understood as a smaller version of θ with a slope equals to s^2.

The inverse function of h_s has the following expression:

$$h_s^{-1}(x) = (s-1)x - \frac{1}{2}s + 1$$

Now, we'll get the expression $h_s \circ \theta_s^2 \circ h_s^{-1}$ step-by-step. First compute $\theta_s^2 \circ h_s^{-1}$:

$$s^2\left((s-1)x - \frac{1}{2}s + 1 - 1/2\right) + \begin{cases} \frac{1}{2}s + 1 & \text{if} \quad 0 \leq h_s^{-1}(x) \leq u \\ \frac{1}{2}s & \text{if} \quad u < h_s^{-1}(x) \leq 1/2 \\ -\frac{1}{2}s + 1 & \text{if} \quad 1/2 < h_s^{-1}(x) \leq 1 - u \\ -\frac{1}{2}s & \text{if} \quad 1 - u < h_s^{-1}(x) \leq 1 \end{cases}$$

Whereas $h_s^{-1} : [0,1] \to J$, the range of h_s^{-1} is only defined on $J \subset [0,1]$. Plus, by visual proof we already know from the plot of θ_s^2 that $u < 1 - \frac{1}{2}s$ (the complete proof is obtained by verifying $1 - \frac{1}{2}s - u > 0$ is true for all $s \in \left(1, \sqrt{2}\right]$). As a result we exclude the outer branches because they don't lie on J. Then, we replace u by a stricter condition $1-\frac{1}{2}s$.

$$\left(\theta_s^2 \circ h_s^{-1}\right)(x) = s^2\left((s-1)x - \frac{1}{2}s + 1/2\right) + \begin{cases} \frac{1}{2}s & \text{if } \ 1-\frac{1}{2}s \leq h_s^{-1}(x) \leq 1/2 \\ -\frac{1}{2}s+1 & \text{if } \ 1/2 < h_s^{-1}(x) \leq \frac{1}{2}s \end{cases}$$

Finally,

$$\left(h_s \circ \theta_s^2 \circ h_s^{-1}\right)(x) = \begin{cases} s^2\left(x-1/2\right)+1 & \text{if } \ 0 \leq x \leq 1/2 \\ s^2\left(x-1/2\right) & \text{if } \ 1/2 < x \leq 1 \end{cases} = \theta_{s^2}(x)$$

□

Intuitively the previous lemma states the existence of a suitable affine function h_s which allows us to transform the information provided by θ_s^2 in a smaller scale, θ_{s^2}. Thus, θ_s^2 and θ_s are somehow related. The goal of the next results is to deepen that relation under the set J.

Remark 4.2. *Note that if* $J = \left[1-\frac{1}{2}s, \frac{1}{2}s\right]$ *is* φ*-symmetric and invariant, then the following inequality must hold:*

$$1-\frac{1}{2}s < -\frac{1}{2}\left(s^3-s^2-s\right)$$

Lemma 4.3. *Suppose* $s \in (\sqrt{2}, 2]$*, then* θ_s *is topological mixing, i.e. for all* I *subset of* $[0,1]$*, there is a* $n \in \mathbb{N}$ *such that* $\theta_s^n(I) = [0,1]$.

Proof. See the Appendix - Section A.9 (page 42). □

Corollary 4.1. *If* $s \in \left(\sqrt{2}, 2\right]$ *and* f *is the ergodic invariant density of* θ_s*, then supp*$(f) = [0,1]$ *and the attractor is also* $[0,1]$.

Proof. This follows from the fact that *supp*(f) is invariant and contains an interval I. By iterating this interval and due to the previous lemma we get *supp*$(f) = [0,1]$. □

Definition 4.2. *If* θ *is topological mixing, then* θ *is called prime (in the sense that is not divisible).*

Lemma 4.4. *Let* $\theta_1 : D_1 \to D_1$ *and* $\theta_2 : D_2 \to D_2$ *be two arbitrary functions and* $D_2 \subset D_1$*. Suppose there exists a bijective function* h *such that* $\theta_2 \circ h = h \circ \theta_1$*. Then* θ_1 *is prime if and only if* θ_2 *is also prime (the property of being prime is invariant under conjugacy).*

Proof. By the lemma's hypothesis, for any subset of D_1, say I, we have $\theta_1^n(I) = D_1$. If we choose an interval J as a subset of D_2, then there exists a $n > 0$ such that $(\theta_1^n \circ h^{-1})(J) = D_1$. Now, we need to verify if θ_2 is prime. It follows from the lemma that $\theta_2 = h \circ \theta_1 \circ h^{-1}$.

If we take the nth iterated of $\theta_2(J)$ we come up with: $\theta_2^n(J) = (h \circ \theta_1 \circ h^{-1})^n(J) = (h \circ \theta_1^n \circ h^{-1})(J) = h(D_1) = D_2$. $\square$

Definition 4.3. *We call the order of renormalization of θ_s the number:*

$$n = \max\left\{k \geq 0 : s^{2^k} \leq 2\right\},$$

and we say θ_s is n-times renormalizable.

Directly from the previous definition, θ_s is n-times renormalizable if the value s lies in the interval $\left(2^{\frac{1}{2^{n+1}}}, 2^{\frac{1}{2^n}}\right]$. Nonetheless θ_s is prime and also 0-times renormalizable when s belongs to $\left(\sqrt{2}, 2\right]$. In the latter case we may also say θ_s is not renormalizable ($n = 0$). Furthermore the set J, introduced on Lemma 4.2, depends on the order of renormalization n and we will henceforth denote it as J_n.

Example 4.1. *Suppose θ_s is 2-times renormalizable, that is $s \in \left(2^{\frac{1}{8}}, 2^{\frac{1}{4}}\right]$. Then:*

$$\begin{cases} h_s \circ \theta_s^2 \circ h_s^{-1} = \theta_{s^2} \\ h_{s^2} \circ \theta_{s^2}^2 \circ h_{s^2}^{-1} = \theta_{s^4} \end{cases}, \tag{4.2}$$

where h_s is the function introduced by the Lemma 4.2.

The goal is to find a bijective function which relates θ_s^4 with θ_{s^4} restricted to a certain set, say J_2. Rewriting the expression of $\theta_{s^2}^2$ as the composition $\theta_{s^2} \circ \theta_{s^2}$:

$$\theta_{s^2}^2 = \theta_{s^2} \circ \theta_{s^2} = \left(h_s \circ \theta_s^2 \circ h_s^{-1}\right) \circ \left(h_s \circ \theta_s^2 \circ h_s^{-1}\right) = h_s \circ \theta_s^4 \circ h_s^{-1}$$

Now we'll use this result to replace in the second equation of (4.2):

$$\theta_{s^4} = h_{s^2} \circ \theta_{s^2}^2 \circ h_{s^2}^{-1} = h_{s^2} \circ h_s \circ \theta_s^4 \circ h_s^{-1} \circ h_{s^2}^{-1} = (h_{s^2} \circ h_s) \circ \theta_s^4 \circ (h_{s^2} \circ h_s)^{-1}$$

The bijective function we are looking for is defined as $h_{s^2} \circ h_s$. Therefore this function is our conjugation:

$$\begin{array}{ccc} J_2 & \xrightarrow{\theta_s^4} & J_2 \\ {\scriptstyle h_{s^2} \circ h_s}\downarrow & & \downarrow{\scriptstyle h_{s^2} \circ h_s} \\ [0,1] & \xrightarrow{\theta_{s^4}} & [0,1] \end{array}$$

where $J_2 = (h_{s^2} \circ h_s)^{-1}([0,1])$. Note that $s^4 \in (\sqrt{2}, 2]$. Hence $\theta_{s^4} : [0,1] \to [0,1]$ is prime, then θ_s^4 restricted to J_2 is prime.

Generalizing the previous example we can introduce the following lemma:

Lemma 4.5. *Let θ_s be n-times renormalizable. Moreover, let $g_n = h_{s^{2(n-1)}} \circ \cdots \circ h_{s^2} \circ h_s$, then the map $\theta_s^{2^n} : J_n \to J_n$, where $J_n = \left(h_{s^{2(n-1)}} \circ \cdots \circ h_{s^2} \circ h_s\right)^{-1}([0,1])$, is conjugated to $\theta_{s^{2^n}} : [0,1] \to [0,1]$ through g_n, i.e.:*

$$\begin{tikzcd} J_n \arrow[r, "\theta_s^{2^n}"] \arrow[d, "g_n"'] & J_n \arrow[d, "g_n"] \\ {[0,1]} \arrow[r, "\theta_{s^{2^n}}"] & {[0,1]} \end{tikzcd}$$

From the previous lemma and by the fact that $\theta_s^{2^n}$ restricted to J_n is prime, the following corollary arises:

Corollary 4.2. *If $s \in \left(2^{\frac{1}{2^{n+1}}}, 2^{\frac{1}{2^n}}\right]$, then the support of the ergodic acip is the union of $J_n, \theta_s(J_n^{\pm}), ..., \theta_s^{2^n-1}(J_n^{\pm})$, where $J_n^- = J_n \cap \left[0, \frac{1}{2}\right]$ and $J_n^+ = J_n \cap \left[\frac{1}{2}, 1\right]$.*

Remark 4.3. *By the previous corollary, the supp(f) is an union of $2^{n+1} - 1$ intervals.*

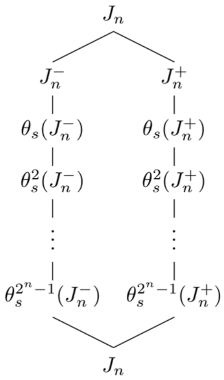


Figure 4.2: The graph explains for an initial condition on J_n how the path of the trajectory is

Proposition 4.1. *The set J_n has the following neat representation:*

$$J_n = [a_n, b_n]\,, \ \textit{where} \ a_n = \sum_{i=0}^{n-1}\left(1 - \frac{s^{2^i}}{2}\right)\left(\prod_{j=0}^{i-1} s^{2^j} - 1\right), \quad b_n = \left(\prod_{i=0}^{n-1} s^{2^i} - 1\right) + a_n$$

Proof. See the Appendix - Section A.10 (page 42). □

Corollary 4.3. *The length of interval J_n tends to zero when s drives to 1.*

Proof. If we assume s goes to 1, by the Definition 4.3 the order of renormalization n will never be upper bounded. Hence n goes to ∞ and then we have:

$$\lim_{s\to 1} |b_n - a_n| = \prod_{i=0}^{\infty}\left(s^{2^i} - 1\right) = 0,$$

where each $\left(s^{2^i} - 1\right)$ is less than 1. □

5 Applications: a Simple Financial Market Model

The model proposed by Tramontana, Westerhoff and Gardini is a combination of several papers and contributes from the literature of agent-based financial market models. According to their formulation, the interactions between heterogeneous agents, bounded by simple technical and fundamentalist trading rules, may generate very interesting and complex price dynamics, containing the appearance of financial distress events such as bubbles and crashes. This model is closely related to the models found on Day and Huang (1990) [6] and Huang and Day (1993) [12].

A survey study presented by Menkhoff & Taylor (2006) [16] proves market speculators believe in technical and fundamental analysis to forecast prices on financial markets. Fundamentalists support their trading strategies on the idea the price of an asset will eventually revert to its (estimate) fundamental value (constant value known to all market contestants) and stays there for awhile. In the bear (undervalued) market, when the market prices are smaller than their fundamental value, fundamentalists seek for investment opportunities as buyers and the lower the price, more aggressive fundamentalists are. Due to their conventional approach, they usually contribute to the stability of the market. On the other hand, chartists, technical analysts or just noise traders disregard the hypothesis of the prices revert to their fundamental value. Instead, they evaluate the future prices of the market based on the chart analysis, which consists studying historical price patterns and exploit them to make (destabilizing) investment decisions. Chartists are more comfortable to explore investment opportunities as buyers in the bull (overvalued) market because they believe the prices will continue rising. Fundamentalists and chartists may responds with asymmetrically aggressiveness, different trading horizons/volume and market entry levels whether they are facing a bull or bear market. Thus, the model distinguishes fundamentalists or chartists in two types. Type 1 speculators are always active in the market regardless the price. Type 2 speculators are more conservative and they are only able to interact in the market if the mispricing (absolute difference between the asset price and its fundamental) reaches to a certain critical value. Type 2 fundamentalists believe the investment opportunities close to the fundamental value are worthless due to the slim chances to be profitable. Type 2 chartists don't trust the persistence of bull or bear markets when the mispricing is close enough to the fundamental value.

In order to reduce the positive or negative excess of demand, the market maker adjusts the prices to reach a classic market equilibrium (in the sense of the basic hypotheses of the law of supply and demand). Therefore, the market maker quotes the market prices under the following rule: $P_{n+1} = P_n + a\left(D_n^{C,1} + D_n^{C,2} + D_n^{F,1} + D_n^{F,2}\right)$, where P is the log price, a is a positive price adjustment factor, and $D_n^{C,1}, D_n^{C,2}, D_n^{F,1}$ and $D_n^{F,2}$ are the investment orders of the four types of speculators. Positive excess demand (loosely speaking, more buyers than sellers) makes prices rising and negative demand forces prices to fall. Without loss of generality, we'll set the positive factor a equal to 1.

Surprisingly, this whole financial plot can be represented as an one-dimensional discontinuous piecewise-linear system. Depending on the form of $D_n^{C,1}, D_n^{C,2}, D_n^{F,1}$ and $D_n^{F,2}$, the system may (or not) have chaotic dynamic (led by unstable orbits) and multiple discontinuity points. Further in this chapter, we will demonstrate even with such a simple mathematical setup the possibility to generate very interesting dynamics (which allows us to study the bull-bear market phenomena). In this way, this deterministic model is already capable to incorporate some stylized facts from financial markets like bubbles, crashes and excess of volatility. In the next section we are disclosure one possible shape for P_{n+1}.

5.1 Setup with One Discontinuity Point

The one discontinuity model is presented according to Tramontana et al (2010) [18]. **Type 1 chartists** believe in the persistence of bull (+) and bear markets (−), then their orders are specified as:

$$D_n^{C,1} = \begin{cases} c_1^-(P_n - F) & \text{if} \quad P_n - F < 0 \\ c_1^+(P_n - F) & \text{if} \quad P_n - F \geq 0 \end{cases}, \tag{5.1}$$

where c_1^+ and c_1^- are positive reaction factors and F is the log of the fundamental value. This type of speculator will take buying (selling) positions if the prices are above (bellow) the fundamental value. If $c_1^+ > c_1^-$, the chartists are trading more aggressive in the bull market than the bear market. On the other hand, the chartists are submitting orders with larger size in the bear market if $c_1^+ < c_1^-$.

Type 2 chartists submit their orders based on the following rule:

$$D_n^{C,2} = \begin{cases} -c_2^- & \text{if} \quad P_n - F < 0 \\ 0 & \text{if} \quad P_n - F = 0 \\ c_2^+ & \text{if} \quad P_n - F > 0 \end{cases}, \tag{5.2}$$

where c_2^+ and c_2^- are positive reaction factors. This type of chartists still believe in the persistence of bull-bear markets but they assume an idle position when the price is equal to its fundamental value. The size of the orders are now only subordinate to the reaction factors c_2^+ (order size in the bull market) and c_2^- (order size in the bear market), no matter how far or close the price is to the fundamental value.

On the contrary, **type 1 fundamentalists** believe the prices will converge to their fundamental value in the long run, then their orders are placed according to:

$$D_n^{F,1} = \begin{cases} -u_1^-(P_n - F) & \text{if} \quad P_n - F < 0 \\ -u_1^+(P_n - F) & \text{if} \quad P_n - F \geq 0 \end{cases}, \tag{5.3}$$

where u_1^+ and u_1^- are positive reaction factors. In contrast to the chartist trade strategy, fundamentalists take buying (selling) positions in the bear (bull) market hopping the price will eventually rise up (fall) to the fundamental value. Moreover, type 1 fundamentalists may respond asymmetrically to the bull and bear markets: when $u_1^- > u_1^+$ fundamentalists are more aggressive in the bear market and $u_1^- < u_1^+$ otherwise.

Lastly, **type 2 fundamentalists** submit their orders under the following rule:

$$D_n^{F,2} = \begin{cases} u_2^- & \text{if} \quad P_n - F < 0 \\ 0 & \text{if} \quad P_n - F = 0 \\ -u_2^+ & \text{if} \quad P_n - F > 0 \end{cases}, \tag{5.4}$$

where u_2^+ and u_2^- are positive reaction factors. Type 2 fundamentalists are in an idle state when the price is equal to the fundamental value. Otherwise, they are buying (selling) orders by the size of u_2^- (u_2^+).

After a few variable changes, we come up with the following dynamical system which expresses the model in terms of the deviations from the fundamental value:

$$x_{n+1} = \psi(x_n) = \begin{cases} \psi_L(x_n) = s_L x_n + m_L & \text{if} \quad x_n < 0 \\ \psi_C(x_n) = 0 & \text{if} \quad x_n = 0 \\ \psi_R(x_n) = s_R x_n + m_R & \text{if} \quad x_n > 0 \end{cases}, \tag{5.5}$$

where $x_n = P_n - F$, $s_L = 1 + c_1^- - u_1^-$, $s_R = 1 + c_1^+ - u_1^+$, $m_L = u_2^- - c_2^-$ and $m_R = c_2^+ - u_2^+$. This map formulation is generally an one-dimensional discontinuous map, except for the very particular case when we omit type 2 speculators ($m_R = 0$ and $m_L = 0$) which has limited interest for our applications. Note that s_L, s_R, m_L and m_R are linear combinations of positive factors, hence they can take any value in $\mathbb{R}$. To simplify the work in the next sections and without loss of generality, let the mathematical model in (5.5) be reduced to[5]:

$$x_{n+1} = \psi(x_n) = \begin{cases} \psi_L(x_n) = s_L x_n + m_L & \text{if} \quad x_n < 0 \\ \psi_R(x_n) = s_R x_n + m_R & \text{if} \quad x_n \geq 0 \end{cases} \tag{5.6}$$

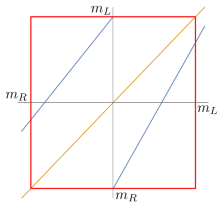
As we explained in the Chapter 3, there are many cases where it's very hard to observe the states under their natural form. For instance, in a financial market, how many speculators can predict the exact asset's closing price? Probably none. Without access to the real prices, how they decide to issue their orders? Some of them are able to get fair forecasts: instead of predict the exact prices, they perform prognoses based on price intervals by assuming an observational error[6], say ε. That's exactly the approach we are seeking. We assume an initial condition X which is an observable such that the states in D are assigned to a real number. Later we use the deterministic transformation of ψ,

[5]From the statistical standpoint, the branch ψ_C is irrelevant because the probability of reaching $x = 0$ is zero.

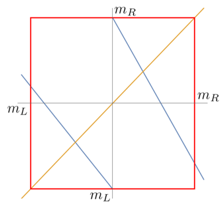
[6]The price prediction is often associated with an observational error, the difference between the real price and the initial prediction of the real price.

which is our *law of the market*, to iterate X. This process is a particular application of the stochastic process X_n already introduced in the Definition 3.7.

5.2 Bounded Instability Regime



(a) Orientation preserving case



(b) Orientation reverse case

Figure 5.1: Plot of two versions of ψ presented in this section (blue line) and the invariant interval I (red box); $y = x$ line (yellow line)

Different agent behavior and market price fluctuations are derived by manipulating the reaction factors. For the purposes of the thesis, we'll restrict the model variables such that we get two distinct cases:

- For the **orientation preserving case** (see Figure 5.1a), type 1 chartists trade more aggressively in the bull/bear market than type 1 fundamentalists. This means the slopes s_L and s_R must be positive and greater than 1. On the other hand, for type 2 speculators, the fundamentalists trade more aggressively in the bull/bear market than chartists. This statement implies that m_L is positive (intercept of the left branch) and m_R is negative (intercept of the right branch). Inside the bull or bear market, when the current price increases (decreases), the future price increases (decreases).

- The **orientation reverse case** (see Figure 5.1b) may be seen as a negation of the previous case. For the type 1 speculators, fundamentalists are now trading more aggressively than chartists, therefore we assume that both slopes s_L and s_R are negative and less than -1. Simultaneously, type 2 chartists are trading more heavily than fundamentalists which suggests m_L is negative and m_R is positive. In this case, whenever the current price inside the bull or bear market increases (decreases), the future price decreases (increases).

Since the slopes for both cases are greater than 1 in absolute value, ψ is an expansive map and so we expect its orbits be unstable. In these scenarios, which will be henceforth referred as the instability regime, only chaotic dynamics can occur.

Remark 5.1. *The methods studied for the orientation preserving case can be applied to the orientation reverse case. Note that the second iteration for both cases (ψ^2) is the same when the model parameters are symmetric, i.e. each slope and intercept for both cases are equal in absolute value.*

Therefore, we will only keep studying the orientation preserving case. Considering all the restrictions, two unstable fixed points can be determined: $x_*^- = \frac{m_L}{1-s_L} < 0$ and $x_*^+ = \frac{m_R}{1-s_R} > 0$. Any initial condition outside the interval (x_*^-, x_*^+) drives the orbit to ∞. From an economic point of view, the explosion of the dynamic gives practically no information regarding the evolution of the price because in the real markets prices won't indefinitely rise or fall. Thus, we need to find more restrictive conditions to determine when bounded behavior is indeed a reality.

Lemma 5.1. *ψ has bounded orbits if any initial condition lies on (x_*^-, x_*^+) and m_R belongs to the interval $\left(\frac{m_L}{1-s_L}, m_L(1-s_R)\right)$ or, alternatively, m_L lies on $\left(m_R(1-s_L), \frac{m_R}{1-s_R}\right)$. There is also an invariant interval $I = [m_R, m_L]$ which absorbs the dynamic and don't never let it exit from I.*

Proof. If any initial condition belongs to the interval $(-\infty, x_*^-) \cup (x_*^+, \infty)$, the orbit of x_n is divergent towards ∞. In addition, we know there exists an interval $I = [m_R, m_L]$ such that it's an invariant absorbing interval. In that case the following conditions must hold: $x_*^- < m_R$ and $x_*^+ > m_L$. Then, we obtain the desire condition for $m_R = \left(\frac{m_L}{1-s_L}, m_L(1-s_R)\right)$ (or for m_L) by replacing x_*^- and x_*^+ for their respective expressions.

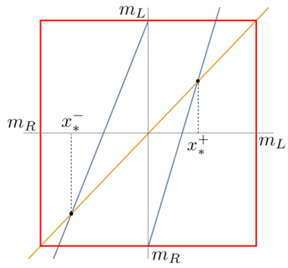


Figure 5.2: Plot of ψ (blue line) where I (red box) is not an invariant interval

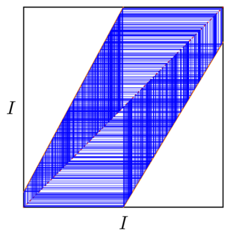
Otherwise, once the orbit of x_n is inside I, it could escape from I. Figure 5.2 allows us to see what happen when the fixed points are inside I. There are two intervals which don't verify the condition of invariance and they are $[m_R, x_*^-]$ and $[x_*^+, m_L]$. □

The previous lemma is a direct consequence of the *Theorem 3* from Tramontana et al (2010) [18]. The authors were more concerned to extensively exhibit the qualitative properties of the model's orbits as well as show strong evidences of chaos (for instance, see section *3.3. Case III* from Tramontana et al (2010) [18]). The existence of chaos with such simple trading market rules is indeed intriguing, but does this mean the prices in this regime are unpredictable? The answer depends whether we are investigate the dynamics in the point-wise or statistical standpoint. Note that when chaos emerges, by the ergodic theory it's possible to asymptotically obtain a stationary distribution. In a certain way, dynamics with erratic point-wise behavior may lead to well-behaved and predictable distributions. But first, recall the expression of the measure absolutely continuous μ with respect to λ, that is, $\mu(A) = \int_A f(x)\mathrm{d}x$.

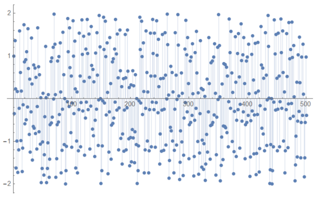
Theorem 5.1. *Under the hypotheses of Lemma 5.1, the stochastic process $X_n = X \circ \psi^n$ is an ergodic and stationary process.*

Proof. From Theorem 3.5, it follows that ψ has only one ergodic acip μ. Since in stochastic processes notation we replace the concept of invariant measure by stationary distribution, the proof is complete. □

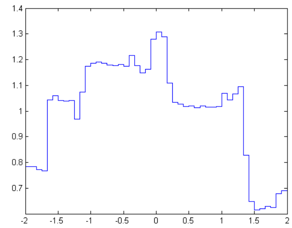
Lemma 5.2. *Under the hypotheses of Lemma 5.1 and let s_L and s_R be greater than $\sqrt{2}$ and less than 2 (topological mixing regime). Given a point $x_0 \in I$ and a small positive error, say ε, then there is a $n > 0$ such that $\psi^n([x_0 - \varepsilon, x_0 + \varepsilon]) = I$*



(a) Cobweb plot



(b) Time series for 500 iterations



(c) Empirical density function

Figure 5.3: Simulation of $\gamma(1)$ with the parameters $s_L = 1.85$, $s_R = 1.65$, $m_L = 2 = -m_R$

When the differences of behavior between chartists and fundamentalists of the same type are very wide, the prediction of future prices becomes an useless exercise, even with a very small observational error. This result comes from the Lemma 4.3 (this lemma doesn't require symmetric functions and it can be generalized for functions with different slope branches). Furthermore, from Figures 5.3a-5.3b, we conclude that the trajectories of ψ are spread all over the entire interval I without any pattern. It's unquestionably a topological mixing scenario where prices change between bull and bear market without any logical sequence, with high unpredictability. Therefore, the attractor of ψ becomes the whole interval I. Moreover, considering an initial condition X (an observable) and this deterministic transformation ψ, then the following results must hold:

Corollary 5.1. *Assume the hypotheses from Lemma 5.2. By the Theorem 3.7, the initial observable X becomes more independent of its next iterations when the deviance between X and the next iterations increases. If we try to predict prices for an enough distant future, those predictions won't be related to our start point X.*

Corollary 5.2. *Under the hypotheses of Lemma 5.2 and by the Theorem 3.8, the sum of the observations $X_n = X \circ \psi^n$ (properly normalized) follows a standard normal distribution $\mathcal{N}(0, 1)$.*

5.3 Symmetric Speculator Behavior

We now assume more restrictive conditions than the instability regime: the speculators behave identically in the bull and bear market. In this case the following conditions must hold:

$$s_L = s_R = s \in (1, \sqrt{2}] \text{ and } m_L = -m_R = m > 0 \tag{5.7}$$

Hence, ψ can be redefined under the new conditions:

$$\psi_s(x_n) = \begin{cases} sx_n + m & \text{if} \quad x_n < 0 \\ sx_n - m & \text{if} \quad x_n \geq 0 \end{cases} \tag{5.8}$$

The symmetric speculator behavior expression derives from the fact that ψ_s, under these new restrictions, is in fact symmetric with respect to the origin (more specifically, ψ_s is an odd function). Unlike the instability regime, we are now capable to use all the results from Chapter 4 about symmetric piecewise linear Lorenz maps.

Lemma 5.3. *Let h be the unique orientation preserving affine map that maps the interval I to $[0,1]$. Then $h \circ \psi_s \circ h^{-1}$ is a Lorenz map with slope s.*

Proof. See the Appendix - Section A.11 (page 43). ☐

Lemma 5.4. *The map ψ_s is n-times renormalizable where n only depends on the slope s and lies on $\left(\log_2\left(\frac{1}{\log_2 s}\right) - 1, \log_2\left(\frac{1}{\log_2 s}\right)\right]$.*

Proof. See the Appendix - Section A.12 (page 44). ☐

Definition 5.1. *We call the interval F_{n_s} the **fundamental region** where it contains the fundamental price value F and has the order of renormalization n. Explicitly, F_{n_s} is equal to $h^{-1}(J_n)$.*

Proof. Since θ_s is equal to (4.1), we are now able to cite the work from Chapter 4. Recall the neat formula for J_n (see Proposition 4.1). But the expression of J_n itself is not enough because this interval only exists in $[0,1] \to [0,1]$. However, we can send J_n to I using the inverse of h. Therefore $F_{n_s} = h^{-1}(J_n)$. ☐

Remark 5.2. *Bull and bear market regions can be seen as a complementary set of F_{n_s}.*

Lemma 5.5. *Given an initial condition, say x_0, the orbit of x_0 will visit the interval F_{n_s} with probability $\mu(F_{n_s})$.*

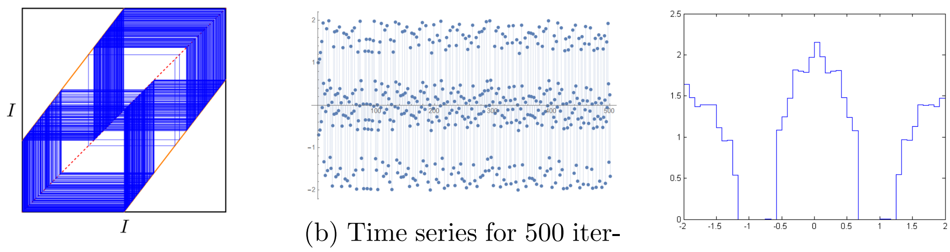
Proof. Using the Theorem 3.3 and assuming $X = \chi_{F_{n_s}}(x)$ (the function takes the value 1 when the orbit of x_0 enters in F_{n_s} and is valued 0 otherwise), the proof is quite similar to the proof of Corollary 3.1: the time average converges to the space average which is equal to $\mu(F_{n_s})$. □

Remark 5.3. *Due to $\mu << \lambda$ and from the Corollary 4.3, if the slope s goes to 1, the length of F_{n_s}, i.e. $\lambda(F_{n_s})$, goes to zero. The latter implies that $\mu(F_{n_s})$ converges to zero.*

Theorem 5.2. *Under the hypotheses of Lemma 5.1 and the symmetric speculator behavior, the density of the stationary ergodic process X_i is supported in $2^{n+1}-1$ intervals for all $i \in \mathbb{N}_0$.*

Proof. By Remark 4.3, the proof is straightforward. □

Remark 5.4. *The length of the market cycle is 2^n. After 2^n periods, the market returns to its fundamental region F_{n_s}.*



(a) Cobweb plot

(b) Time series for 500 iterations

(c) Empirical density function

Figure 5.4: Simulation of $\gamma(1)$ with the parameters $s = 1.3$, $m = 2$

Unlike the instability regime, in this case the orbit seems to stay focus roughly in three regions (the fundamental region and two extreme regions whose represent the bull and the bear market) and exchanging among them. The dispersion inside these regions becomes much smaller when s goes to 1 and much bigger when s goes in the opposite direction to $\sqrt{2}$. Analytically this means the support (or the attractor) is divided in a few but large intervals or in many tiny intervals (by Theorem 5.2, it depends on how many times the map is renormalizable). Consequently, when chartists and fundamentalists of type 1 have less differences in matters of aggressiveness to the market (note that s varies between 1 and $\sqrt{2}$), the price may change abruptly between regions or remain in the same market region where it was before.

In this scenario, price forecast could be achievable. Future prices will attain to those three regions almost surely (which is the attractor of ψ_s). If we consider, for instance, a 1-time renormalizable ψ_s we can divide the fundamental region in the usual way, the three regions are given by three distinct intervals: $\psi_s(F_{n_s} \cap [-m, 0])$ (bull market), F_{n_s} (fundamental region) and $\psi_s(F_{n_s} \cap [0, m])$ (bear market). If we instead take a 2-times renormalizable ψ_s, the three regions are composed by 7 different intervals and so on.

5.4 Extreme Cases of Symmetric Speculator Behavior

(a) $\gamma(-0.25)$ for $s = 1$, $m = 2$ (b) $\gamma\left(\frac{1+\sqrt{5}}{2}\right)$ for $s = 2$, $m = 2$

Figure 5.5: Time series of 50 iterations of 2 distinct orbits

Here we will briefly show two other regimes by assuming the symmetry of the speculators ($s_L = s_R = s$ and $m_L = -m_R = m > 0$). When type 1 speculators are in an idle position ($s = 1$), it's the case where the map ψ_1 is parallel with respect to the line $y = x$. Therefore its orbits are 2-cyclic. Using an initial condition x_0, it's hard to get fair price prediction with a such simple dynamic (only binary decisions: x_0 or $\psi_1(x_0)$).

However, assuming the scenario that type 1 chartists are much more aggressive than type 1 fundamentalists ($s = 2$), we have completely the opposite. We are facing the most chaotic and unpredictable scenario from all we presented so far (even more unpredictable than the topological mixing regime). With no surprise, in these conditions the forecast of future prices is useless.

The easiest way to prove if ψ_2 is a highly chaotic map[7] is to determine what's the empirical distribution for its orbits (in particular if the ECDF converges to an uniform distribution). Here we found an inconsistency: for multiple initial conditions, the orbits of ψ_2 after awhile will converge to a fixed point. But how is this happen for an expansive map? The problem is we need to recognize the limitations of the empirical method by using the computer. The set of (eventually) periodic points of ψ_2 is given by $\mathbb{Q} \cap [-m, m]$ which has zero Lebesgue measure. Nevertheless, the computer cannot work with irrational numbers because it has finite memory (more specifically, a number of fixed decimal places). Therefore, it converts irrational numbers into rationals which could force orbits to converge wrongly to an impossible path or point (in the sense that ψ_2 would never take that trajectory if it was computed analytically).

To draw the ECDF of ψ_2 instead of considering $s = 2$ we take $s = 1.999$ (solving our issue with the periodic points) and we may conclude that there is convergence to an uniform distribution (similar arguments like Example 3.4 show that the uniform distribution is the unique acip of ψ_2). This implies that the probability of the price jump to (or stay in) the bull or bear market is equal to $1/2$.

[7]Note that ψ_2 shows strong resemblances with Dyadic transformations but with the discontinuity at $x = 0$.

6 Discussion and Conclusions

This thesis approaches the financial model proposed by Tramontana el al (2010) [18] from a statistical standpoint which gives us more insights about the behavior of the speculators and how the price may vary in the long term inside the invariant interval I. The study is mainly centered in two different regimes[8]: the topological mixing regime and the symmetric speculator behavior. For the first one, we assume the type 1 chartists are much more aggressive than type 1 fundamentalists ($s_L, s_R \in (\sqrt{2}, 2)$). In such conditions, it's impossible to predict the prices (even with a very small observational error) to a sufficient distant period. Due to the price dynamic is so chaotic inside I, we don't get any information about the possible outcome. In the topological mixing regime, I becomes the attractor of the map's regime. On the other hand, in the symmetric speculator behavior we assume symmetry between bull and bear markets ($s_L = s_R = s$ and $m_L = -m_R = m$), maintaining the dominance of type 1 chartists over type 1 fundamentalists but with less intensity ($1 < s \leq \sqrt{2}$). With this formulation, the map's renormalization and the support of the acip under this map rely on s. In this regime, the price dynamic is still chaotic but the chaotic region is small, i.e. the chaotic attractor is a finite union of tiny intervals generated by a fundamental region for which there is a precise description. Moreover, the price dynamic is recurrent to the fundamental region and there is a concrete characterization of its period depending on the parameters of the model.

Other two cases analyze the price dynamics in the extreme conditions of symmetric speculator behavior. When type 1 agents are not contestants in the financial market ($s = 1$), the price dynamic takes a cycle of period 2. No useful knowledge can be taken by this particular model, once the market hardly rely on such simple price predictions. At last, we consider the most chaotic case ($s = 2$) which implies type 1 chartists to be much more aggressive than type 1 fundamentalists. In this case, the price dynamic is highly unstable and unpredictable. Like the topological mixing regime, the pricing forecast has no interest because we cannot narrow it down to a reasonable small subset of I.

For further investigation, we point out some possible directions. Firstly, we may consider to introduce a white noise in the model. The point is to transform the deterministic model in a stochastic model such that $X_n^\varepsilon = X \circ \psi^n + \varepsilon_n$, where $\varepsilon_n \sim \mathcal{N}(0, \sigma^2)$ and ε is an iid process. It would be also interesting to compare the results obtained in this thesis and in the stochastic version when σ^2 drives to 0 (for instance, the study of the stationary processes X_n versus X_n^ε). Secondly, we believe the study of the model with two discontinuity points will better capture what's happening in the real financial markets (for example, see Tramontana & Westerhoff (2013) [19]). For certain conditions, we expect the map has 2 attractors (2 acips), one when $x \in \mathbb{R}^-$ and other when $x \in \mathbb{R}^+$. However, a particular case of this model is the overlap of the attractors which will lead to an unique attractor and the price dynamic is allowed to jump freely between the bull and the bear market.

[8]For both regimes we assume that type 2 fundamentalists are more aggressive than type 2 chartists ($m_L > 0$ and $m_R < 0$).

Appendix

A.1. Example 2.2 - Strictly Monotonic System

In economics, the Cobb-Douglas Production Function is a well-known particular form of the production function and widely used to represent the relationship between an output and their inputs (usually between labor and capital as inputs and the amount of output that can be produced by those inputs):

$$\Pi(K, L) = \kappa K^{\alpha} L^{\beta}$$

- Π = total production (the monetary value of all goods produced in a year)
- K = capital input (the monetary worth of all machinery, equipment and buildings)
- L = labor input (the total number of person-hours worked in a year)
- κ = total factor productivity
- α and β are the output elasticities of capital and labor, respectively and they are also constants between 0 and 1

For this example, we will consider the constant returns to scale form of Cobb-Douglas Production Function, which means output will increase by the same proportional change as all inputs change ($\alpha + \beta = 1$).

A discrete version of the Solow-Swan can be introduced using the work found in Brida & Pereyra (2008) [2]. Our production function $\Pi(K, L)$ satisfies the required properties:

- $\Pi(\lambda K, \lambda L) = \lambda \Pi(K, L), \forall \lambda, K, L \in \mathbb{R}^+$ (constant return to scale)
- $\Pi(K, 0) = \Pi(0, L) = 0, \forall K, L \in \mathbb{R}^+$
- $\frac{\partial \Pi}{\partial K} > 0, \frac{\partial \Pi}{\partial L} > 0; \frac{\partial^2 \Pi}{\partial K^2} < 0, \frac{\partial^2 \Pi}{\partial L^2} < 0$
- $\lim_{K \to 0^+} \frac{\partial \Pi}{\partial K} = \lim_{L \to 0^+} \frac{\partial \Pi}{\partial L} = \infty; \lim_{K \to \infty} \frac{\partial \Pi}{\partial K} = \lim_{L \to \infty} \frac{\partial \Pi}{\partial L} = 0$

The next system explains the changes on capital stock where, in this particular model, part of savings are used for investment purposes to replace depreciated capital.

$$\begin{cases} K_{t+1} - K_t = s\Pi(K_t, L_t) - \delta K_t \\ L_{t+1} = (1+n)L_t \end{cases} \tag{6.1}$$

- s = fraction of output is saved
- $s\Pi(K_t, L_t)$ = gross investment at t or total saving at t
- δ = rate of depreciation
- δK_t = capital depreciation at t
- n = positive constant growth rate of labor

Let $u = \frac{K}{L}$. Therefore u is the capital per worker and $\pi(u) = \Pi(u,1)$ is the production function in the intensive form with these properties:

$$\pi(0) = 0, \quad \pi'(u) > 0 \; \forall u \in \mathbb{R}^+, \quad \pi''(u) < 0 \; \forall u \in \mathbb{R}^+, \quad \lim_{u \to 0^+} \pi'(u) = \infty, \quad \lim_{u \to \infty} \pi'(u) = 0 \tag{6.2}$$

From the system (6.1), we are now able to describe how capital per worker changes over time:

$$u_{t+1} = \theta(u_t) = \frac{s\pi(u_t) + (1-\delta)u_t}{1+n}$$

The function θ is the sum of two components: $\frac{s}{1+n}\pi(u_t)$ and $\frac{1-\delta}{1+n}u_t$. In general, θ will be characterized by the same properties in (6.2) which define π. If the first derivative of π is always positive, then $\theta'(u_t) > 0$ and θ is strictly monotonic increasing.

Now we'll check if this general result can be applied for the Cobb-Douglas Production Function. Replacing $\pi(u)$ for the Cobb-Douglas Production Function in the intense form, thus θ takes the following form:

$$\theta(u_t) = \frac{s\pi(u_t) + (1-\delta)u_t}{1+n} = \frac{s\kappa(u_t)^{\alpha}(1)^{\beta} + (1-\delta)u_t}{1+n} = \frac{u_t}{1+n}\left(s\kappa u_t^{\alpha-1} + 1 - \delta\right)$$

By the model hypotheses, α lies on the interval $(0,1)$ and the numbers $\frac{1-\delta}{1+n}$, $\frac{s\kappa}{1+n}$ are always positive. Hence, from the Definition 2.3, θ is strictly monotonic increasing, i.e. $\theta(u_t) < \theta(u_{t+1})$ for all $u_t < u_{t+1}$.

A.2. Proof of Corollary 2.1

Proof. A stationary state x is obtained if $\theta(x) = x$.

Suppose $g(w) = \theta(w) - w$. By the limit laws, g is still a continuous function on D.

Using the points y and z, we obtain: $g(y) = \theta(y) - y \Rightarrow g(y) < 0$ and $g(z) = \theta(z) - z \Rightarrow g(z) > 0$

By the Bolzano's Theorem result, we know that there is a number x such that $g(x) = 0$.

Therefore: $g(x) = 0 \Leftrightarrow \theta(x) - x = 0 \Leftrightarrow \theta(x) = x$ □

A.3. Proof of Theorem 2.2

Proof. We will prove the *case (ii)*. By analogy, the proof for *case (i)* is identical.

Recall Definition 2.6 and since $|\theta'(x)| \geq \delta > 1$ for all $x \in D$, by the chain rule we have:

$$\left|(\theta^n)'(x)\right| = \underbrace{\left|\theta'(\theta^{n-1}(x))\right| \cdot \left|\theta'(\theta^{n-2}(x))\right| \cdots \left|\theta'(x)\right|}_{n \text{ terms}} > \delta \cdot \delta \cdot \delta \cdots \delta = \delta^n$$

For every small ϵ:

$$(\theta^n)'(x) = \lim_{\epsilon \to 0} \frac{\theta^n(x+\epsilon) - \theta^n(x)}{\epsilon}$$

Now we pick a $n \geq 0$ such that the next condition holds:

$$|\theta^n(x+\epsilon) - \theta^n(x)| \geq \delta^n \epsilon > 1$$

From the Definition 2.6, for all $\epsilon > 0$ there exists a $n \geq 0$ such the orbit of x is unstable. □

A.4. Proof of Proposition 3.1

Proof. Let B be a Borel set such that $B \in \mathcal{B}$. We start by computing the joint distribution function of the random vector $(X_h, X_{h+1}, X_{h+2}, ..., X_{h+m})$ for B:

$$p_{X_h, X_{h+1}, X_{h+2}, ..., X_{h+m}}(B) \coloneqq \mu\left(\{\omega \in D : (X_h(\omega), X_{h+1}(\omega), X_{h+2}(\omega), ..., X_{h+m}(\omega)) \in B\}\right)$$

For each $i = 0, ..., m$, replace X_{h+i} by $X \circ \theta^{h+i}$:

$$\mu\left(\left\{\omega \in D : (X \circ \theta^h)(\omega), (X \circ \theta^{h+1})(\omega), (X \circ \theta^{h+2})(\omega), ..., (X \circ \theta^{h+m})(\omega)) \in B\right\}\right)$$

Now let $A = \{\bar{\omega} \in D : (X_h(\bar{\omega}), X_{h+1}(\bar{\omega}), X_{h+2}(\bar{\omega}), ..., X_{h+m}(\bar{\omega})) \in B\}$

Then:

$$\mu\left(\left\{\omega \in D : \theta^h(\omega) \in A\right\}\right) \coloneqq \theta^h_* \mu(A)$$

Since μ is invariant to θ, it follows:

$$\theta^h_* \mu(A) = \mu(A) = \mu\left(\left\{\omega \in D : (X \circ \theta^0)(\omega), (X \circ \theta^1)(\omega), (X \circ \theta^2)(\omega), ..., (X \circ \theta^m)(\omega)) \in B\right\}\right)$$

Therefore we reached the desire identity:

$$p_{X_h, X_{h+1}, X_{h+2}, ..., X_{h+m}}(B) = p_{X_0, X_1, X_2, ..., X_m}(B)$$

□

A.5. Proof of Theorem 3.2

Proof. This proof is based on Day (1994), pages 138-140 [7].

Consider the set of points A_k that is eventually mapped into A after at least k periods for any $k \geq 0$, that is: $A_k \coloneqq \{x \mid \text{there exists a } n \geq k \text{ such that } \theta^n(x) \in A\}$

Iterating the inverse image of θ until n (that is the set of points which enters in A after the nth-period):

$\theta^{-1}(A) \Rightarrow x \in \theta^{-1}(A) \Leftrightarrow \theta(x) \in \theta^1(A)$

$\theta^{-2}(A) \Rightarrow x \in \theta^{-2}(A) \Leftrightarrow \theta(x) \in \theta^2(A)$

$\theta^{-3}(A) \Rightarrow x \in \theta^{-3}(A) \Leftrightarrow \theta(x) \in \theta^3(A)$

$\vdots$

$\theta^{-n}(A) \Rightarrow x \in \theta^{-n}(A) \Leftrightarrow \theta(x) \in \theta^n(A)$

So, equivalently we have: $A_k \coloneqq \{x \mid \text{there exists a } n \geq k \text{ such that } \theta^n(x) \in A\} = \bigcup\limits_{n=k}^{\infty} \theta^{-n}(A)$

Of course, A_0 is the set of all points eventually mapped into A, which obviously includes A. Notice that $\theta^{-1}(A_k)$ is the set of points that maps into the set A_k, which maps itself into A after k periods. Therefore,

$A_0 \supset A_1 \supset A_2 \supset \cdots \supset A_n \supset \ldots$ and $A_{k+1} = \theta^{-1}(A_k)$

Because, $A_k = \bigcup_{n=k}^{\infty} \theta^{-n}(A) = \theta^{-k}(A) \cup \theta^{-(k+1)}(A) \cup \theta^{-(k+2)}(A) \cup \ldots$

Then, $\theta^{-1}(A_k) = \theta^{-1}(\bigcup_{n=k}^{\infty} \theta^{-n}(A)) = \theta^{-1}(\theta^{-k}(A) \cup \theta^{-(k+1)}(A) \cup \theta^{-(k+2)}(A) \cup \ldots) = \theta^{-(k+1)}(A) \cup \theta^{-(k+2)}(A) \cup \theta^{-(k+3)}(A) \cup \cdots = A_{k+1}$

Let $A_* = \bigcap\limits_{k=0}^{\infty} A_k = \{x \mid \exists (n_k) \text{ increasing sequence with } \theta^{n_k}(x) \in A\}$

Since $A_* = A_0 \cap A_1 \cap A_2 \cap ... \Rightarrow A_* \subset A \subset A_0$

By assumption μ is invariant with respect to θ. Therefore, $\mu(A_k) = \mu\left(\theta^{-1}(A_k)\right) = \mu(A_{k+1})$, for all k.

The goal is to prove that all point which belongs to A is recurrent in this sense: $\mu(A_*) = \mu(A)$

The non-recurrent points form the set $A \backslash A_*$ and it must be also a null measure set:

$A \backslash A_* = A \backslash \left(\bigcap\limits_{k=0}^{\infty} A_k\right) = \bigcup\limits_{k=0}^{\infty} (A \backslash A_k)$

However, $A \backslash A_k \subset A_0 \backslash A_k$

Using the measure properties and the fact μ is invariant:

$\mu(A_0) = \mu(A_k + A_0 \backslash A_k) \Leftrightarrow \mu(A_0) = \mu(A_k) + \mu(A_0 \backslash A_k) \Leftrightarrow \mu(A_0 \backslash A_k) = \mu(A_0) -$

$\mu(A_k) \Leftrightarrow \mu(A_0 \backslash A_k) = 0$

This implies that (measure monotonicity): $0 = \mu(A_0 \backslash A_k) \geq \mu(A \backslash A_k) = 0$

This means the set of points that both belong to A and eventually "return" to A has positive measure. □

A.6. Proof of Proposition 3.2

Proof. The argument of item *(iv)* will be proven bellow. The other properties are trivial to check. This proof is based on Day (1994), page 151 [7].

By invariance we know that:

$$\mu(\theta^{-1}(A)) = \mu(A), \quad \forall A \in \mathcal{F}$$

In particular:

$$\mu\left(\theta^{-1}([a,x])\right) = \mu([a,x]), \quad \forall x \in [a,b]$$

By absolute continuity we meant:

$$\int_{\theta^{-1}([a,x])} f(u)\mathrm{d}u = \int_a^x f(u)\mathrm{d}u$$

Taking the derivative on the both sides, we get:

$$\frac{d}{dx}\int_{\theta^{-1}([a,x])} f(u)\mathrm{d}u = \frac{d}{dx}\int_a^x f(u)\mathrm{d}u = f(x)$$

From the expression on the left arises the *Frobenius-Perron Operator*, denoted as follow:

$$Pf(x) = \frac{d}{dx}\int_{\theta^{-1}([a,x])} f\mathrm{d}\lambda$$

Combining the last two equations we therefore have:

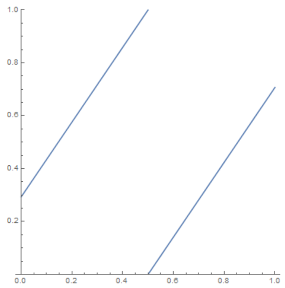
$$Pf(x) = f(x), \quad \forall x \in D$$

□

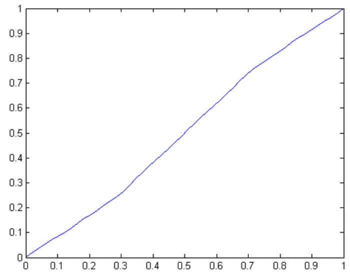
A.7. Example 3.4 - Invariant Density Function

Recall the map from Example 2.3 where $s = \sqrt{2}$. To avoid any confusion, let $\theta = \theta_{\sqrt{2}}$:

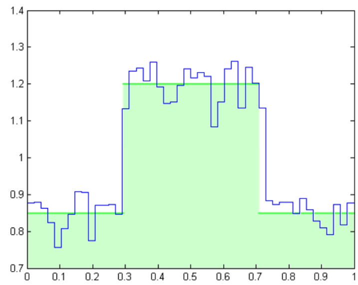
$$\theta(x) = \begin{cases} \sqrt{2}(x - 1/2) + 1 & \text{if } \ 0 \le x \le 1/2 \\ \sqrt{2}(x - 1/2) & \text{if } \ 1/2 < x \le 1 \end{cases} = \begin{cases} \theta_1(x) & \text{if } \ 0 \le x \le 1/2 \\ \theta_2(x) & \text{if } \ 1/2 < x \le 1 \end{cases}$$



(a) Plot of θ



(b) ECDF of θ



(c) Empirical (blue) versus Real (green) density function of θ

Figure 6.1: Simulation with $s = \sqrt{2}$, $x_0 = \frac{1}{\sqrt{3}}$, $n = 1000000$, $l = 50$

We want to find an explicit expression for the density function of θ. We have to apply the Frobenius-Perron Operator:

$$Pf(x) = \sum_{i=1}^{n} f(\theta_i^{-1}(x)) \cdot \left| \frac{d\theta_i^{-1}(x)}{dx} \right| \cdot \chi_{[\theta(a_{i-1}),\theta(a_i)]}(x) = f(\theta_1^{-1}(x)) \cdot \left| \frac{d\theta_1^{-1}(x)}{dx} \right| \cdot \chi_{\theta_1([0,1/2])}(x) +$$
$$+ f(\theta_2^{-1}(x)) \cdot \left| \frac{d\theta_2^{-1}(x)}{dx} \right| \cdot \chi_{\theta_2([1/2,1])}(x) = f\left(\frac{\sqrt{2}(x-1)+1}{2}\right) \cdot \left| \frac{\sqrt{2}}{2} \right| \cdot \chi_{\theta_1([0,1/2])}(x) +$$
$$+ f\left(\frac{\sqrt{2}x+1}{2}\right) \cdot \left| \frac{\sqrt{2}}{2} \right| \cdot \chi_{\theta_2([1/2,1])}(x) = \frac{\sqrt{2}}{2} \left[f\left(\frac{\sqrt{2}(x-1)+1}{2}\right) \chi_{[1-\frac{\sqrt{2}}{2},1]}(x) + \right.$$
$$\left. + f\left(\frac{\sqrt{2}x+1}{2}\right) \chi_{[0,\frac{\sqrt{2}}{2}]}(x) \right]$$

Before we keep further on the calculation of the Pf operator, we must find the suitable form of f. It can be obtained by computing the trajectory of the discontinuity (i.e. $x = 1/2$):

$$\tau(1/2) = \left(\theta^0(1/2), \theta^1(1/2), \theta^2(1/2), \theta^3(1/2), \dots\right) = \left(\frac{1}{2}, 1, \frac{\sqrt{2}}{2}, 1 - \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1 - \frac{\sqrt{2}}{2}, \dots\right)$$

Note that the trajectory of $x = 1/2$ enters in a cycle of period 2. To build f, we'll consider all unique values from $\tau(1/2)$, i.e. $\gamma(1/2)$ (5 unique values plus the origin $x = 0$).

Rearranged by ascendant order, we have the following expression:

$$f(x) = a_1\chi_{[0,1-\frac{\sqrt{2}}{2}]}(x) + a_2\chi_{[1-\frac{\sqrt{2}}{2},1/2]}(x) + a_3\chi_{[1/2,\frac{\sqrt{2}}{2}]}(x) + a_4\chi_{[\frac{\sqrt{2}}{2},1]}(x)$$

Now, replacing f on the $Pf(x)$ equation:

$$\begin{aligned}Pf(x) = \frac{\sqrt{2}}{2}\Bigg[&\Bigg(a_1\chi_{[0,1-\frac{\sqrt{2}}{2}]}\left(\frac{\sqrt{2}(x-1)+1}{2}\right) + a_2\chi_{[1-\frac{\sqrt{2}}{2},1/2]}\left(\frac{\sqrt{2}(x-1)+1}{2}\right) + \\ &+a_3\chi_{[1/2,\frac{\sqrt{2}}{2}]}\left(\frac{\sqrt{2}(x-1)+1}{2}\right) + a_4\chi_{[\frac{\sqrt{2}}{2},1]}\left(\frac{\sqrt{2}(x-1)+1}{2}\right)\Bigg)\chi_{[1-\frac{\sqrt{2}}{2},1]}(x) + \\ &+\Bigg(a_1\chi_{[0,1-\frac{\sqrt{2}}{2}]}\left(\frac{\sqrt{2}x+1}{2}\right) + a_2\chi_{[1-\frac{\sqrt{2}}{2},1/2]}\left(\frac{\sqrt{2}x+1}{2}\right) + a_3\chi_{[1/2,\frac{\sqrt{2}}{2}]}\left(\frac{\sqrt{2}x+1}{2}\right) + \\ &+a_4\chi_{[\frac{\sqrt{2}}{2},1]}\left(\frac{\sqrt{2}x+1}{2}\right)\Bigg)\chi_{[0,\frac{\sqrt{2}}{2}]}(x)\Bigg]\end{aligned}$$

After some calculus we come up with this expression:

$$Pf(x) = \frac{\sqrt{2}}{2}\left[a_3\chi_{[0,1-\frac{\sqrt{2}}{2}]}(x) + (a_1+a_4)\chi_{[1-\frac{\sqrt{2}}{2},1/2]}(x) + (a_1+a_4)\chi_{[1/2,\frac{\sqrt{2}}{2}]}(x) + a_2\chi_{[\frac{\sqrt{2}}{2},1]}(x)\right]$$

To find the a_i values we'll reduce the problem to a matrix:

$$\frac{\sqrt{2}}{2}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \Leftrightarrow \begin{cases} a_1 = \frac{\sqrt{2}}{2}a_3 \\ a_2 = a_3 \\ a_3 = a_3 \\ a_4 = \frac{\sqrt{2}}{2}a_3 \end{cases}$$

Using the general properties of density functions:

$$\int f\,\mathrm{d}\lambda = 1 \Leftrightarrow a_1\left(1-\frac{\sqrt{2}}{2}-0\right)+a_2\left(1/2-1+\frac{\sqrt{2}}{2}\right)+a_3\left(\frac{\sqrt{2}}{2}-1/2\right)+a_4\left(\frac{1-\sqrt{2}}{2}\right) = 1 \Leftrightarrow$$

$$\Leftrightarrow a_3 = \frac{1}{2(\sqrt{2}-1)}$$

Then, we finally obtain an explicit expression for the invariant density $f(x)$:

$$f(x) = \frac{\sqrt{2}}{4(\sqrt{2}-1)}\chi_{[0,1-\frac{\sqrt{2}}{2}]}(x) + \frac{1}{2(\sqrt{2}-1)}\chi_{[1-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}]}(x) + \frac{\sqrt{2}}{4(\sqrt{2}-1)}\chi_{[\frac{\sqrt{2}}{2},1]}(x)$$

From Theorem 3.3 and Corollary 3.4, the mean and variance are known:

$$\bar{\mu} = \lim_n \frac{1}{n} \sum_{i=0}^{n-1} \theta^i(x) = \int_{\mathbb{R}} x f(x) \mathrm{d}x = \frac{1}{2}$$

$$\bar{\sigma}^2 = \lim_n \frac{1}{n} \sum_{i=0}^{n-1} \left(\theta^i(x) - \bar{\mu}\right)^2 = \int_{\mathbb{R}} (x - \bar{\mu})^2 f(x) \mathrm{d}x = \frac{3\sqrt{2} - 4}{8(\sqrt{2} - 1)}$$

A.8. Example 3.4 - MATLAB Code for the Empirical Approach

Expression for $\theta_s(x)$:

```
function y = theta(x,s)
    if x<=0.5
        y=s*(x-0.5)+1;
    else
        y=s*(x-0.5);
    end
 end
```

Orbit of x_0 using the function $\theta_s(x)$:

```
function X = orbit(x0,dimOrbit,s)
    X = zeros(1,dimOrbit);
    X(1) = x0;
    for i = 2:dimOrbit
        X(i) = theta(X(i-1),s);
    end
end
```

Empirical cumulative distribution function:

```
function Y = cdf(s,x0,dimOrbit,ecdfScale)
    Y = zeros(1,ecdfScale);
    X = orbit(x0,dimOrbit,s);
    for i = 1:ecdfScale-1
        Y(i+1) = sum(X<=i/(ecdfScale-1))/dimOrbit;
    end
    Z = linspace(0,1,ecdfScale);
    plot(Z,Y);
end
```

Empirical invariant density function:

```
function dX = pdf(s,x0,dimOrbit,ecdfScale)
    X = cdf(s,x0,dimOrbit,ecdfScale);
    delta = 1/(ecdfScale-1);
    dX = zeros(1,ecdfScale-1);
    for i = 2:ecdfScale
        dX(i-1) = (X(i)-X(i-1))/delta;
```

```
    end
    Z = linspace(0,1,ecdfScale-1);
    stairs(Z,dX);
end
```

A.9. Proof of Lemma 4.3

Proof. Start by choosing an arbitrary interval I such that $I = [a, b] \subset [0, 1]$, hence the length of I is $b - a$.

After each iteration of $\theta_s^2(I)$, the length of the interval will grow at a rate of s^2 which is larger than 2. At some point, say an even number k, $\theta_s^k(I)$ will intercept a discontinuity point d. In this case, the next iteration will return two disjoint intervals, say J_1 and J_2.

For the purpose of the proof, it makes sense to disregard the smaller intervals in the next iteration. The trajectory of the largest interval will attain faster to $[0, 1]$ rather than the smaller intervals, hence we'll only keep the largest interval.

After p iterations of I by θ_s^2, we'll inevitably face the dilemma of finding two discontinuity points in an interval J_p, say d_1 and d_2. One of those discontinuity points is equal to $1/2$ and the other one takes the value of u or $1 - u$.

Suppose $d_1 = u$ and $d_2 = 1/2$ (the other case is analogous), therefore $J_p = [a_1, u] \cup [u, 1/2] \cup [1/2, a_2]$, where $0 \leq a_1 < u$ and $1/2 < a_2 < 1 - u$. It's easy to check the interval $[u, 1/2]$ is the largest, then we take it as J_{p+1}.

Now we want to prove that $\theta_s^4(J_{p+1}) = [0, 1]$. We know that $\theta_s^2(J_{p+1}) = \left[0, \frac{1}{2}s\right]$ and since s belongs to the interval $\left(\sqrt{2}, 2\right]$, the number $\frac{1}{2}s$ is always greater than $\frac{1}{2}$, i.e. $\theta_s^2(J_{p+1})$ contains $\left[0, \frac{1}{2}\right]$. With no surprise, we conclude that $\theta_s^4(J_{p+1})$ contains $\theta_s^2\left(\left[0, \frac{1}{2}\right]\right) = [0, 1]$. □

A.10. Proof of Proposition 4.1

Proof. First, we need to deduct the expression of $h_k \circ h_{k-1} \circ \cdots \circ h_0$ by recursion where for all $i = 0, ..., k$ we have a function $h_i(x) = \alpha_i x + \beta_i$ and the real numbers α_i and β_i:

$$h_0 = \alpha_0 x + \beta_0$$

$$h_1 \circ h_0 = \alpha_1 (\alpha_0 x + \beta_0) + \beta_1 = \alpha_1\alpha_0 x + \alpha_1\beta_0 + \beta_1$$

$$h_2 \circ h_1 \circ h_0 = \alpha_2 (\alpha_1\alpha_0 x + \alpha_1\beta_0 + \beta_1) + \beta_2 = \alpha_2\alpha_1\alpha_0 x + \alpha_2\alpha_1\beta_0 + \alpha_2\beta_1 + \beta_2$$

$$\vdots$$

$$h_k \circ h_{k-1} \circ \cdots \circ h_0 = \alpha_k \cdots \alpha_0 x + \alpha_k \cdots \alpha_1\beta_0 + \alpha_k \cdots \alpha_2\beta_1 + \alpha_k \cdots \alpha_3\beta_2 + ... + \alpha_k\beta_{k-1} + \beta_k =$$

$$= \prod_{i=0}^{k} \alpha_i x + \sum_{i=0}^{k} \beta_i \prod_{j=i+1}^{k} \alpha_j = F_k x + G_k = \left(\prod_{i=0}^{k} \alpha_i\right) x + \sum_{i=0}^{k} \beta_i \left(\frac{\prod_{i=0}^{k} \alpha_i}{\prod_{j=0}^{i} \alpha_j}\right) =$$

$$= \prod_{i=0}^{k} \alpha_i \left(x + \sum_{i=0}^{k} \frac{\beta_i}{\alpha_0 \dots \alpha_i} \right)$$

Now, recall this result:

$$h_s(x) = \frac{1}{s-1}\left(x + \frac{1}{2}s - 1\right) = \frac{1}{s-1}x + \frac{1}{s-1}\left(\frac{1}{2}s - 1\right) = \alpha_s x + \beta_s$$

Therefore, the expression for h with slope equals to s^{2^i} is given as follow:

$$h_{s^{2^i}}(x) = \alpha_{s^{2^i}} x + \beta_{s^{2^i}} = \frac{1}{s^{2^i}-1}x + \frac{1}{s^{2^i}-1}\left(\frac{1}{2}s^{2^i} - 1\right)$$

From Lemma 4.5 and combining the results we obtained so far, we have:

$$\begin{aligned}
g_n &= \prod_{i=0}^{n-1} \frac{1}{s^{2^i}-1} \left(x + \sum_{i=0}^{n-1} \frac{\frac{1}{s^{2^i}-1}\left(\frac{1}{2}s^{2^i}-1\right)}{\prod_{j=0}^{i} \frac{1}{s^{2^j}-1}} \right) = \\
&= \prod_{i=0}^{n-1} \frac{1}{s^{2^i}-1} \left(x + \sum_{i=0}^{n-1} \left(\frac{s^{2^i}}{2} - 1\right) \frac{\prod_{j=i}^{n-1} \frac{1}{s^{2^j}-1}}{\prod_{i=0}^{n-1} \frac{1}{s^{2^i}-1}} \right) = \\
&= \prod_{i=0}^{n-1} \frac{1}{s^{2^i}-1} \left[x + \sum_{i=0}^{n-1} \left(\frac{s^{2^i}}{2} - 1\right) \left(\prod_{j=0}^{i-1} s^{2^j} - 1 \right) \right]
\end{aligned}$$

We already know that $J_n = g_n^{-1}([0,1]) = \left(h_{s^{2(n-1)}} \circ \dots \circ h_{s^2} \circ h_s\right)^{-1}([0,1])$, then:

$$g_n^{-1}([0,1]) = \left[\left(\prod_{i=0}^{n-1} s^{2^i} - 1 \right) x + \sum_{i=0}^{n-1} \left(1 - \frac{s^{2^i}}{2}\right) \left(\prod_{j=0}^{i-1} s^{2^j} - 1 \right) \right] ([0,1]) =$$

$$= \left[\sum_{i=0}^{n-1} \left(1 - \frac{s^{2^i}}{2}\right) \left(\prod_{j=0}^{i-1} s^{2^j} - 1 \right), \left(\prod_{i=0}^{n-1} s^{2^i} - 1 \right) + \sum_{i=0}^{n-1} \left(1 - \frac{s^{2^i}}{2}\right) \left(\prod_{j=0}^{i-1} s^{2^j} - 1 \right) \right] = [a_n, b_n]$$

□

A.11. Proof of Lemma 5.3

Proof. The aim is to construct a function h such that maps the invariant set $I = [-m, m]$ into the interval $[0,1]$. Then, we have the following diagram:

$$\begin{array}{ccc} I & \xrightarrow{\psi_s} & I \\ h \downarrow & & \downarrow h \\ [0,1] & \xrightarrow{\theta_s} & [0,1] \end{array}$$

where $h(-m) = 0$ and $h(m) = 1$. Now it's clear what we want to accomplish: the idea is to prove that θ_s is the symmetric piecewise linear Lorenz map defined in (4.1) for us to apply the results found in Chapter 4.

The complete expressions for h and h^{-1} are:

$$h(x) = \frac{x+m}{2m} \quad \text{and} \quad h^{-1}(x) = m(2x-1)$$

From the conjugation diagram, we know that $\theta_s = h \circ \psi_s \circ h^{-1}$. It's easier if we break the conjugation in two parts. First consider $\psi_s \circ h^{-1}$:

$$\left(\psi_s \circ h^{-1}\right)(x) = \begin{cases} m[s(2x-1)-1] & \text{if} \quad h^{-1}(x) \geq 0 \\ m[s(2x-1)+1] & \text{if} \quad h^{-1}(x) < 0 \end{cases}$$

After some calculus we obtain the final expression for θ_s:

$$\left(h \circ \psi_s \circ h^{-1}\right)(x) = \begin{cases} \frac{m[s(2x-1)-1]+m}{2m} & \text{if} \quad h^{-1}(x) \geq 0 \\ \frac{m[s(2x-1)+1]+m}{2m} & \text{if} \quad h^{-1}(x) < 0 \end{cases} = \begin{cases} s(x - 1/2) & \text{if} \quad x \geq 1/2 \\ s(x - 1/2) + 1 & \text{if} \quad x < 1/2 \end{cases}$$

□

A.12. Proof of Lemma 5.4

Proof. From the Definition 4.3 we know that $s \in \left(2^{\frac{1}{2^{n+1}}}, 2^{\frac{1}{2^n}}\right]$. Then, we may rewrite this condition based on the value n:

$$s \leq 2^{\frac{1}{2^n}} \Leftrightarrow n \leq \log_2\left(\frac{1}{\log_2 s}\right)$$

$$s > 2^{\frac{1}{2^{n+1}}} \Leftrightarrow n > \log_2\left(\frac{1}{\log_2 s}\right) - 1$$

Therefore:

$$n \in \left(\log_2\left(\frac{1}{\log_2 s}\right) - 1, \log_2\left(\frac{1}{\log_2 s}\right)\right]$$

□

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