

MESTRADO
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DISSERTAÇÃO

**STABILITY OF NON-TRIVIAL SOLUTIONS OF
STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY
THE FRACTIONAL BROWNIAN MOTION**

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OUTUBRO-2019

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Abstract

This dissertation aims to generalize a result on the exponential stability of trivial solutions of stochastic differential equations driven by the fractional Brownian motion by Garrido-Atienza *et al.* to non-trivial solutions in the scalar case. Notions on fractional calculus are presented, as well as the definition and main properties of the fractional Brownian motion. Subsequently the framework for SDEs driven by fractional Brownian motion with a pathwise approach is characterized along with some existence and uniqueness results. The result on stability is then applied to the fractional Vasicek model for interest rates.

Keywords: fractional calculus, fractional Brownian motion, generalized Riemann-Stieltjes integral, exponential stability.

Contents

1	Introduction	1
2	Fractional Brownian Motion	3
2.1	Preliminaries - Fractional Calculus	3
2.1.1	Riemann-Liouville Fractional Calculus - Continuous Approach . .	4
2.1.2	Grünwald- Letnikov Fractional Calculus - Discrete Approach . .	6
2.2	Definitions and Important Properties	7
2.2.1	Fractional Brownian Motion	7
2.2.2	Increments and Long-range Dependence	8
2.2.3	Self-similarity	10
2.2.4	Mandelbrot-van Ness' Integral Representation	10
2.2.5	Hölder Continuity	11
2.2.6	Loss of Semi-martingale Property	12
3	Stochastic Differential Equations driven by Fractional Brownian Motion, $H > 1/2$	14
3.1	Pathwise Integrals with respect to FBM	14
3.1.1	Riemann-Stieltjes integral	14
3.1.2	Generalized Riemann-Stieltjes integral	15
3.2	Existence and Uniqueness Results	16
3.3	Stability	18
4	Application to Fractional Vasicek Interest Rate Model	22
5	Conclusion	25

Chapter 1

Introduction

Since the Black and Scholes model was developed by Black, Scholes and Merton in the 1970's, it has been broadly applied in financial markets. However the Brownian motion, random source in the the Black and Scholes model, does not fit the empirical financial data. Indeed Brownian motion's increments are independent, characteristic which is contrary to the "long range dependency" or "long memory" financial instruments exhibit when statistically analyzed. To answer to this problem, in the late 1990's (for example [NR02], [Zäh98]) researchers started to study the consequences of replacing the Brownian motion with one of its generalizations: the fractional Brownian motion. This more general stochastic process bears some fundamental common properties when compared to the Brownian motion, however its main advantage is that for a suitable choice of Hurst index (parameter that fully characterizes the fractional Brownian motion) this new source of randomness features dependent increments thereby aligning with the long memory that financial instruments display. Some criticism arose regarding pricing models that used the fractional Brownian motion, either because financial interpretation was lost ([BH05]) or because arbitrage issues were encountered ([Rog97]), consequently slowing down the research that applied the fractional Brownian motion in finance. However during the past decade stochastic volatility ([IS10], [Fuk17]) and interest rate models ([HLW14]) with fractional Brownian motion appeared in several papers.

This dissertation is mainly motivated by the work of Garrido-Atienza *et al.* in [GANS18]. This paper consists on establishing conditions that stochastic differential equations driven by a stochastic process with Hölder continuous paths (as it is the case for fractional Brownian motions) must satisfy in order to assure local exponential stability of a trivial, *i.e.* zero, solution. The original result of this dissertation, delineated in theorem (3.6), consists on generalizing [GANS18]'s result to non-trivial solutions in the scalar case. To arrive at this result, as the fractional Brownian motion loses the semi-martingale property that the standard Brownian motion had, a different kind of stochastic calculus has

to be formally established. Indeed one cannot use Itô's lemma when dealing with the fractional Brownian motion, instead a pathwise approach will be adopted: the Riemann-Stieltjes integral defined by Young in [You36] and its generalization by Zähle in [Zäh98]. As this later integral is defined with the help fractional derivatives, *i.e.* derivatives of non-integer order, a review on fractional calculus will be our starting point.

This work is structured in three chapters. Notions and definitions of fractional calculus are introduced in the first chapter. Immediately followed by the definition and main properties of the fractional Brownian motion, pointing out the correlations of its increments and its "long-range" dependence structure, the maintenance of the process' self-similarity but the loss of semi-martingale property, and finally the Hölder regularity of its sample paths. In the next chapter, a pathwise concept to define integrals with respect to fractional Brownian motion is outlined, before describing stochastic differential equations driven by fractional Brownian motion and some existence and uniqueness results. At the end of this chapter, the stability matter is tackled where we present existing results and finish with this dissertation's original result and proof. Finally, in the last chapter we will apply the previously established results to a fractional Vasicek interest rate model used by Hao *et al.*, [HLW14]), to price credit default swaps.

Chapter 2

Fractional Brownian Motion

2.1 Preliminaries - Fractional Calculus

Fractional calculus allows one to compute integrals and derivatives of a non-integer order. The term *fractional* can be misleading as the order of integration must not be a *fraction*, *i.e.* a rational number, indeed it can take any real value. In order to introduce smoothly this notion, let us first recall the idea of the multiple integral of order n .

$$I^n f(x) := \int_a^x \int_a^{x_0} \int_a^{x_1} \dots \int_a^{x_{n-2}} f(x_{n-1}) dx_{n-1} dx_{n-2} \dots dx_0, \text{ for all } x \in]a, b[\quad (2.1)$$

where $n \in \mathbb{N}$, $a \in \mathbb{R}$, $b \leq \infty$ and the function f is locally integrable in the interval $[a, b]$. With a simple proof by induction, one can prove that,

$$I^n f(x) = \frac{1}{(n-1)!} \int_a^x (x-\xi)^{n-1} f(\xi) d\xi \quad (2.2)$$

The idea behind the generalization of these integrals to a non-integer order is the use of the Gamma function, Γ^{-1} , instead of the factorial term.

Following the same structure as in [MG00]’s lecture notes, we will define two conceptualizations of fractional integrals, a continuous and a discrete. The continuous formulation is based on integral operators, as for the discrete, it uses sums and finite differences schemes and can be very useful when one needs numerical approaches to problems based on fractional calculus.

¹The Gamma function, Γ , is defined, for all $z \in \mathbb{C}$ such that $\text{Re}(z) > 0$, by: $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$. The fact that $\Gamma(z+1) = z\Gamma(z)$ (proved by integration by parts), implies $\forall n \in \mathbb{N}, \Gamma(n+1) = n!$.

2.1.1 Riemann-Liouville Fractional Calculus - Continuous Approach

Definition 2.1. Let $\alpha \in \mathbb{R}^{+*}$, $a \in \mathbb{R}$, $b \leq \infty$,

Let f a real valued function in $L^1([a, b[)$,

for $x \in]a, b[$, we define the Abel-Riemann fractional integrals of order α ,

$$I_{a+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x - \xi)^{\alpha-1} f(\xi) d\xi \quad (2.3)$$

$$I_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (\xi - x)^{\alpha-1} f(\xi) d\xi \quad (2.4)$$

Furthermore, we define the set $I_{a+}^{\alpha}(L_p)$ as the class of functions that can be represented as an integral of order α of some function g in L_p (i.e. such that $(\int_a^b |g(t)|^p dt)^{1/p} < \infty$).

Fractional integrals do not always have a closed formula, as it is not always possible to find an anti-derivative to the function $\xi \mapsto (x - \xi)^{\alpha-1} f(\xi)$. Let us, nonetheless see a simple example.

Example 2.1. Integral of order α of the power function of order k .

$$\begin{aligned} I^{\alpha} t^k &= \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \tau^k d\tau \\ &= \frac{t^{\alpha+k}}{\Gamma(\alpha)} \int_0^1 (1 - \tau)^{\alpha-1} \tau^k d\tau \\ &= \frac{t^{\alpha+k}}{\Gamma(\alpha)} B(k+1, \alpha) \\ &= \frac{t^{\alpha+k}}{\Gamma(\alpha)} \frac{\Gamma(k+1)\Gamma(\alpha)}{\Gamma(\alpha+k+1)} \\ &= \frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)} t^{\alpha+k} \end{aligned}$$

In futures sections of our work it will be important to use the fact that these fractional integral operators commute and that there is a generalization of the law of exponents known for integer order integral operators.

Proposition 2.1. Let a real valued function $f \in L^1([a, b[)$,

If $\alpha > 0$ and $\beta > 0$, then:

$$I_{a+}^{\alpha} I_{a+}^{\beta} f(x) = I_{a+}^{\alpha+\beta} f(x), \text{ for all } x \in]a, b[\quad (2.5)$$

Proof. By definition, for $x \in]a, b[$,

$$\begin{aligned}
 I_{a+}^{\alpha} I_{a+}^{\beta} f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x - \xi)^{\alpha-1} \left(I_{a+}^{\beta} f(\xi) \right) d\xi \\
 &= \frac{1}{\Gamma(\alpha)} \int_a^x (x - \xi)^{\alpha-1} \frac{1}{\Gamma(\beta)} \int_a^{\xi} (\xi - \psi)^{\beta-1} f(\psi) d\psi d\xi \\
 &= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \int_a^x f(\psi) \left[\int_{\psi}^x (x - \xi)^{\alpha-1} (\xi - \psi)^{\beta-1} d\xi \right] d\psi \\
 &= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \int_a^x f(\psi) \left[\int_0^1 (x - \psi - u(x - \psi))^{\alpha-1} (u(x - \psi))^{\beta-1} (x - \psi) du \right] d\psi \\
 &= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \int_a^x f(\psi) \left[\int_0^1 (x - \psi)^{\alpha-1} (1 - u)^{\alpha-1} (u(x - \psi))^{\beta-1} (x - \psi) du \right] d\psi \\
 &= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \int_a^x f(\psi) \left[(x - \psi)^{\alpha+\beta-1} \int_0^1 (1 - u)^{\alpha-1} u^{\beta-1} du \right] d\psi \\
 &= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \int_a^x f(\psi) \left[(x - \psi)^{\alpha+\beta-1} B(\alpha, \beta) \right] d\psi^2 \\
 &= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \int_a^x f(\psi) \left[(x - \psi)^{\alpha+\beta-1} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} \right] d\psi \\
 &= \frac{1}{\Gamma(\alpha + \beta)} \int_a^x (x - \psi)^{\alpha+\beta-1} f(\psi) d\psi \\
 I_{a+}^{\alpha} I_{a+}^{\beta} f(x) &= I_{a+}^{\alpha+\beta} f(x)
 \end{aligned}$$

□

Remark 2.1. Using definition 2.1, one can define the fractional derivative of order α operator D^{α} , for $\alpha > 0$ and $n \in \mathbb{N}$ such that $n - 1 < \alpha \leq n$,

$$D_{a+}^{\alpha} = D^n I_{a+}^{n-\alpha}, \text{ for } x \in]a, b[$$

$$D_{b-}^{\alpha} = (-1)^n D^n I_{a+}^{n-\alpha}, \text{ for } x \in]a, b[$$

Where the operator D^n denotes the usual derivative of order n , $D^n f(x) := \frac{d^n}{dx^n} f(x)$ for all $x \in]a, b[$.

One cannot however generalize proposition 2.1 to the fractional derivative operator, as it is proved in [Pod99] section 2.3.6.

A similar definition for fractional integrals in an interval with infinite bounds can also

²B stands for Beta function defined for all $(\alpha, \beta) \in (\mathbb{R}_+^*)^2$ by $B(\alpha, \beta) = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx$. Considering the Laplace transform of B , one we can show, cf [Pod99], that for all $(\alpha, \beta) \in (\mathbb{R}_+^*)^2$, $B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

be useful in some financial applications where there is no time constraint. Such approach was proposed by Liouville and Weyl.

Definition 2.2. Let $\alpha \in \mathbb{R}^{+*}$, $a \in \mathbb{R}$, $b \leq \infty$,

Let f a real valued function in $L^1(]-\infty, b])$ and well behaved in $]-\infty, b]$, for $x \in]-\infty, b]$, we define the Liouville fractional integral of order α ,

$$I_+^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x (x - \xi)^{\alpha-1} f(\xi) d\xi \quad (2.6)$$

Let g a real valued function in $L^1(]a, +\infty[)$ and well behaved in $]a, +\infty[$, for $x \in]a, +\infty[$, we define the Weyl fractional integral of order α ,

$$I_-^\alpha g(x) := \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} (\xi - x)^{\alpha-1} f(\xi) d\xi \quad (2.7)$$

2.1.2 Grünwald- Letnikov Fractional Calculus - Discrete Approach

The Grunwald-Letnikov's approach is based on a generalization of the definition of derivative.

Recalling that, for a real valued differentiable function f ,

$$\mathbf{D}_t^1 f(t) := \frac{df}{dt}(t) = \lim_{h \rightarrow 0} \frac{f(t) - f(t-h)}{h}$$

One can apply this formula twice, 3 times, ..., n times, and:

$$\mathbf{D}_t^n f(t) := \frac{d^n f}{dt^n}(t) = \lim_{h \rightarrow 0} \frac{\sum_{r=0}^n (-1)^r \binom{n}{r} f(t-rh)}{h^n}$$

Generalizing it to a non integer order, we have:

Definition 2.3. For $\alpha \in \mathbb{R}$, the Grunwald-Letnikov fractional derivative/integral of order α of a function f is defined by:

$$\mathbf{D}_t^\alpha f(t) = \lim_{h \rightarrow 0} \frac{\sum_{r=0}^m (-1)^r \binom{\alpha}{r} f(t-rh)}{h^\alpha} \quad (2.8)$$

where m is such that $mh = t$, and the generalized binomial coefficients are, for $\alpha \in \mathbb{R}$ and $r \in \mathbb{N}$, defined by:

$$\binom{\alpha}{r} = \frac{\alpha(\alpha-1)\dots(\alpha-r+1)}{r(r-1)\dots 1}$$

Under certain conditions on the function being differentiated, the definitions proposed by Riemann-Liouville and Grunwald-Letnikov coincide. Podlubny, [Pod99], proved the following theorem in section 2.3.7.

Theorem 2.1. *If a function f is $n - 1$ times continuously differentiable in an interval $[0, T]$ and $f^{(n-1)}$ is integrable in the same interval, then for $0 < \alpha < n$ the Grunwald-Letnikov and Riemann-Liouville fractional derivative coincide.*

We have:

$$D_t^\alpha f(t) = \mathbf{D}_t^\alpha f(t) = \sum_{j=0}^{m-1} \frac{f^{(j)}(0)t^{j-p}}{\Gamma(1+j-p)} + \frac{1}{\Gamma(m-p)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{p-m+1}} d\tau \quad (2.9)$$

where $m \in \mathbb{N}$ such that $m - 1 \leq p < m$.

2.2 Definitions and Important Properties

Whenever needed we will consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with an increasing filtration.

2.2.1 Fractional Brownian Motion

In this section we introduce the fractional Brownian motion, a generalization of the standard Brownian motion, as well as some basic properties that will be useful when defining and manipulating stochastic integrals and differential equations with respect to a fractional Brownian motion. We refer [BHØZ08] for further details to any curious reader.

Definition 2.4. *Let $H \in]0, 1[$,*

The stochastic process $(B_t^H)_{t \geq 0}$ is a continuous, Gaussian process of zero mean, with the following covariance function:

$$E[B_t^H B_s^H] = \frac{1}{2} \left(t^{2H} + s^{2H} - |t - s|^{2H} \right), \quad t, s \in \mathbb{R}^+ \quad (2.10)$$

$(B_t^H)_{t \geq 0}$ is called Fractional Brownian Motion of Hurst index H , throughout this dissertation we shall use the abbreviation FBM (and if needed FBM_H).

Remark 2.2.

- $B_0^H = 0$ for all $0 < H < 1$,
- $E[B_t^H] = 0$ for all $0 < H < 1$ and $t \geq 0$.
- When $H = \frac{1}{2}$, the $FBM_{\frac{1}{2}}$ is simply the standard Brownian motion.

2.2.2 Increments and Long-range Dependence

One of the main reasons to introduce the FBM to financial models, replacing the standard Brownian motion, is the fact that FBM exhibits some sort of long-term memory when $H > 1/2$, which is indeed a characteristic we find when statistically analyzing realizations of financial instruments.

Proposition 2.2. *Here is a non exhaustive list of results regarding FBM's increments.*

1. A FBM has stationary increments, i.e. for $s > 0$, $B_{t+s}^H - B_s^H \stackrel{d}{\sim} B_t^H$.

2. $(B_t^{\frac{1}{2}})_{t \geq 0}$ has independent increments.

3. When $H \neq \frac{1}{2}$, $(B_t^H)_{t \geq 0}$ its increments have the following covariance function:

$$\rho(n) := E[(B_{t+h}^H - B_t^H)(B_{s+h}^H - B_s^H)] = \frac{1}{2}h^{2H}((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}), \quad (2.11)$$

where n is such that: $t - s = nh$.

(a) For $H > \frac{1}{2}$, $(B_t^H)_{t \geq 0}$ has positively correlated increments.

(b) For $H < \frac{1}{2}$, $(B_t^H)_{t \geq 0}$ has negatively correlated increments.

Proof.

- Let $s > 0$, to simplify notations let the process X , $X_t = B_{t+s}^H - B_s^H$ for all $t \geq 0$. As B^H is a Gaussian process, so is X . Furthermore, $E(X_t) = E(B_{t+s}^H - B_s^H) = 0$ since $E(B_t^H) = 0$. To conclude, we only need to show that covariance function of process X can be reduced to formula (2.10).

$$\begin{aligned} E(X_t X_u) &= E\left((B_{t+s}^H - B_s^H)(B_{u+s}^H - B_s^H)\right) \\ &= E(B_{t+s}^H B_{u+s}^H) - E(B_{t+s}^H B_s^H) - E(B_{u+s}^H B_s^H) + E(B_s^H B_s^H) \\ &= \frac{1}{2}\left((t+s)^{2H} + (u+s)^{2H} - |t-u|^{2H}\right. \\ &\quad \left.- [(t+s)^{2H} + s^{2H} - t^{2H} + (u+s)^{2H} + s^{2H} - u^{2H}] + 2s^{2H}\right) \\ &= \frac{1}{2}\left(t^{2H} + u^{2H} - |t-u|^{2H}\right) \\ E(X_t X_u) &= E(B_t^H B_u^H) \end{aligned}$$

2. and 3. Let $n \in \mathbb{N}$ and $h \in \mathbb{R}$ such that $t - s = nh$, and suppose $t > s$,

$$\begin{aligned}
 \rho(n) &= E[(B_{t+h}^H - B_t^H)(B_{s+h}^H - B_s^H)] \\
 &= \frac{1}{2} \left((t+h)^{2H} + (s+h)^{2H} - |t-s|^{2H} \right. \\
 &\quad \left. - [(t+h)^{2H} + s^{2H} - |t+h-s|^{2H} + (s+h)^{2H} + t^{2H} - |t-s-h|^{2H}] \right. \\
 &\quad \left. + t^{2H} + s^{2H} - |t-s|^{2H} \right) \\
 &= \frac{1}{2} \left(-2(nh)^{2H} + (nh+h)^{2H} + (nh-h)^{2H} \right) \\
 \rho(n) &= \frac{1}{2} h^{2H} ((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}).
 \end{aligned}$$

The function ρ is the null iff $H = 1/2$, thus proving 2.

Furthermore 3.(a) and 3.(b) follow from the fact $\lim_{n \rightarrow \infty} \frac{\rho(n)}{h^{2H} H(2H-1)n^{2H-2}} = 1$.

□

When it comes to long-range dependence, there are different definitions, we will follow both Biagini *et al.* (2000) and Taqqu (2003)'s main approach in [BHØZ08] and [Taq13].

Definition 2.5. Let $(X_n)_{n \in \mathbb{N}}$ a stationary stochastic process,

$(X_n)_{n \in \mathbb{N}}$ is said to have long-range dependence if its covariance function, $\rho(n)$, is such that:

$$\exists c \in \mathbb{R}, \exists a \in]0, 1[: \lim_{n \rightarrow \infty} \frac{\rho(n)}{cn^{-a}} = 1$$

From this definition of long-range dependence, we can deduce that a process that has long-range dependence verifies:

$$\lim_{n \rightarrow 0} \rho(n) = 0, \text{ with however } \sum_{n=1}^{\infty} \rho(n) = \infty.$$

Let us now see what we can say about FBM's increments, when its Hurst index is bigger than 1/2.

Proposition 2.3. If $H > \frac{1}{2}$, then $(B_t^H)_{t \geq 0}$'s increments have long-range dependence.

Proof. Denoting $X_n = B_n^H - B_{n-1}^H$, $(X_n)_{n \in \mathbb{N}}$ is a stationary process and we have:

$$\rho(n) = \frac{1}{2} ((n+1)^{2H} + (n-1)^{2H} - 2n^{2H})$$

Therefore,

$$\lim_{n \rightarrow \infty} \frac{\rho(n)}{H(2H-1)n^{2H-2}} = 1$$

2.2. Definitions and Important Properties

As $H > \frac{1}{2}$, the term $2H - 2$ will be bigger than -1 , implying that the series of general term $\rho(n)$ diverges, even if the covariance function's limit is zero. □

2.2.3 Self-similarity

Unlike many properties that are lost when the Hurst index of the FBM is different from $\frac{1}{2}$, $(B_t^H)_{t \geq 0}$ keeps the well-known self-similarity property of the standard Brownian motion. Let us first recall its definition and some vocabulary.

Definition 2.6. A process $(X_t) \in (\mathbb{R}^n)^{\mathbb{R}^+}$, $n \in \mathbb{N}$ is said to be self-similar if:

$$\forall a > 0, \exists b > 0 : (X_{at})_{t \geq 0} \stackrel{d}{\sim} (bX_t)_{t \geq 0}$$

Vocabulary 2.1. Following the notations of the definition above,

- H such that $b = a^H$ is called Hurst index, X is said to be self-similar of index H .
- $D = \frac{1}{H}$ is called statistical fractal dimension of the process X .

Proposition 2.4. Let $0 < H < 1$, $(B_t^H)_{t \geq 0}$ is self-similar with Hurst index H .

Proof. Let $a > 0, t, s \geq 0$,

$$\begin{aligned} E[B_{at}^H B_{as}^H] &= \frac{1}{2} \left((at)^{2H} + (as)^{2H} - |at - as|^{2H} \right) \\ &= \frac{1}{2} a^{2H} \left(t^{2H} + s^{2H} - |t - s|^{2H} \right) \\ &= a^{2H} E[B_t^H B_s^H] \\ &= E[(a^H B_t^H)(a^H B_s^H)] \end{aligned}$$

Furthermore, $(B_{at}^H)_{t \geq 0}$ is Gaussian and $E[B_{at}^H] = 0$. Therefore $(B_{at}^H)_{t \geq 0} \stackrel{d}{\sim} (bB_t^H)_{t \geq 0}$ where $b = a^H$. □

2.2.4 Mandelbrot-van Ness' Integral Representation

An important characteristic the FBM has, that can also be seen as a alternative way of defining it, is the fact it can be represented through a stochastic integral *w.r.t.* the standard Brownian motion B_t . This was first exploited by Mandelbrot and van Ness in [MvN68], considered to be pioneers in doing research concerning the fractional Brownian motion.

Theorem 2.2. *The stochastic process*

$$\left(\frac{1}{C(H)} \int_{\mathbb{R}} ((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}) dB_s \right)_{t \in \mathbb{R}} \quad (2.12)$$

with the constant $C(H)$ depending only on H , $C(H) := \sqrt{\int_0^\infty ((1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}})^2 ds} + \frac{1}{2H}$, is the fractional Brownian motion with Hurst index H .

An elegant and simple proof for this theorem can be found in [Nua06b]. Nualart starts by giving arguments on the regularity of the integrated function in (2.12), thus justifying the existence of this Itô integral implying (2.12) is a Gaussian process. Then, the author computes the expected value of (2.12) squared and its increment squared to arrive at a formula for a covariance function and compares the result with the fractional Brownian motion, concluding (2.12) is in fact FBM with Hurst index H .

2.2.5 Hölder Continuity

Even if FBM's sample paths are not differentiable (as proved in [BHØZ08] proposition 1.7.1), one of the characteristics of the FBM, that will prove to be fundamental when dealing with stochastic differential equations driven by a FBM, is the fact that the FBM has a version whose paths are Hölder continuous with probability one.

Let us first recall what it means to be Hölder continuous.

Definition 2.7. *Let $\alpha \in]0, 1]$, $T \geq 0$.*

A function $f : [0, T] \rightarrow \mathbb{R}$ is said to be α -Hölder continuous, or Hölder continuous of order α , if:

$$\exists M > 0, \forall s, t \in [0, T], |f(t) - f(s)| \leq M|t - s|^\alpha$$

We denote C^α the space of α -Hölder continuous functions, equipped with the following norm,

$$\|f\|_\alpha = \sup_{0 \leq t \leq T} |f(t)| + \sup_{0 \leq s < t \leq T} \frac{|f(t) - f(s)|}{(t - s)^\alpha}$$

Theorem 2.3. *Let $0 < H < 1$, FBM_H has a version with α -Hölder continuous paths, for all $\alpha < H$.*

Proof. Let us first recall the Kolmogorov criterion:

Considering the process $(X_t)_{t \geq 0}$, if there are M, a and b positive constants such that:

$E(|X_t - X_s|^a) \leq M|t - s|^{1+b}$, then X has a version with d -Hölder continuous trajectories as long as $d < \frac{b}{a}$.

2.2. Definitions and Important Properties

Since B^H is self-similar and its increments are stationary, we have:

$$E(|B_t^H - B_s^H|^\alpha) = E(|B_{|t-s|}^H|^\alpha) = E(|B_1^H|^\alpha)|t-s|^{\alpha H},$$

therefore from the Kolmogorov criterion FBM_H has a version with d -Hölder continuous paths, for all $d < \frac{\alpha H}{\alpha} = H$. \square

2.2.6 Loss of Semi-martingale Property

In order to establish results regarding the fact that FBM is no longer a semi-martingale when $H \neq 1/2$, let us first introduce the definition of p -variation in the sense of [Gue08].

Definition 2.8. Let a stochastic process $(u_t)_{t \geq 0}$, consider the interval $[0, T]$ and the set of uniform partitions of the type $\{0 = t_0^n < t_1^n < \dots < t_n^n = T\}$ for $n \in \mathbb{N}$, with $t_i^n = \frac{i}{n}T$ for all $i = 0, \dots, n$. The p -variation of u in the interval $[0, T]$, with respect to this set of partitions, is defined by:

$$V^p(u) := \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} |u_{t_{i+1}^n} - u_{t_i^n}|^p \quad (2.13)$$

In [Che01], Cheridito proves that when $p = \frac{1}{H}$, $V^{1/H}(B^H) = E[|B_1^H|^{1/H}]$. This result will prove to be useful when considering the case $p > \frac{1}{H}$.

Theorem 2.4. The p -variation in the sense of the previous definition of a FBM in the interval $[0, T]$ is: $V^p(B^H) = \begin{cases} \infty, & p < 1/H \\ 0, & p > 1/H \end{cases}$

Proof.

– Case $p < 1/H$:

Let us consider the interval $[0, T]$ with $T > 0$, and let $n \in \mathbb{N}$,

$$\begin{aligned} \sum_{i=0}^{n-1} \left(B_{\frac{i+1}{n}T}^H - B_{\frac{i}{n}T}^H \right)^p &\stackrel{d}{\sim} \sum_{i=0}^{n-1} \left(\frac{T}{n} \right)^{pH} \left(B_{i+1}^H - B_i^H \right)^p, \text{ as the FBM is a self-similar process} \\ &\stackrel{d}{\sim} \left(\frac{T}{n} \right)^{pH} \sum_{i=0}^{n-1} \left(B_1^H \right)^p, \text{ as FBM's increments are stationary} \\ &= \frac{T^{pH}}{n^{pH-1}} \left(B_1^H \right)^p \end{aligned}$$

As $p < 1/H$ this will diverge in L^1 .

– Case $p > 1/H$:

Let $\epsilon > 0$, as $p > 1/H$ we denote $p = 1/H + \epsilon$.

$$\begin{aligned}
 V^p(B^H) &= \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} \left(|B_{\frac{i+1}{n}T}^H - B_{\frac{i}{n}T}^H| \right)^p \\
 &= \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} \left(|B_{\frac{i+1}{n}T}^H - B_{\frac{i}{n}T}^H| \right)^{1/H+\epsilon} \\
 &= \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} \left(|B_{\frac{i+1}{n}T}^H - B_{\frac{i}{n}T}^H| \right)^{1/H} \left(|B_{\frac{i+1}{n}T}^H - B_{\frac{i}{n}T}^H| \right)^\epsilon \\
 &\leq \lim_{n \rightarrow \infty} \underbrace{\sup_{i=\{0, \dots, n\}} \left(|B_{\frac{i+1}{n}T}^H - B_{\frac{i}{n}T}^H| \right)^\epsilon}_{\rightarrow 0, \text{ FBM sample paths are continuous}} \underbrace{\sum_{i=0}^{n-1} \left(|B_{\frac{i+1}{n}T}^H - B_{\frac{i}{n}T}^H| \right)^{1/H}}_{\rightarrow E[|B_1^H|^{1/H}]} \\
 V^p(B^H) &= 0
 \end{aligned}$$

□

Remark 2.3. The previous result on FBM's p -variation can be applied to its quadratic

variation:
$$V^2(B^H) = \begin{cases} \infty, & H < 1/2 \\ 0, & H > 1/2 \end{cases}$$

Theorem 2.5. The FBM is a semi-martingale only in the case $H = \frac{1}{2}$.

Proof. - Sketch -

It is a well know result that the classical Brownian Motion is a semi-martingale. One method to prove this theorem, that can be found in [Sot03], is to consider two cases, $H < 1/2$ and $H > 1/2$.

- For $H < 1/2$, the previous proposition states that the FBM has infinite quadratic variation. Therefore it can not be a semi-martingale.
- For $H > 1/2$, it is assumed B^H is a semi-martingale to later arrive at a contradiction.

□

Chapter 3

Stochastic Differential Equations driven by Fractional Brownian Motion, $H > 1/2$

3.1 Pathwise Integrals with respect to FBM

In the last section of chapter 2, two important results were established: FBM is not a semi-martingale when $H \neq 1/2$ which means that classical Itô calculus can not be used when defining stochastic integrals *w.r.t.* FBM with a Hurst index different from $1/2$; however FBM does have, for every Hurst index, a version whose paths are α -Hölder continuous, for every $\alpha < H$. This last property will be fundamental when defining the Riemann-Stieltjes stochastic integral and later the generalized Riemann-Stieltjes stochastic integral developed by Zähle, [Zäh98].

3.1.1 Riemann-Stieltjes integral

In 1936 [You36], Young defined the Riemann-Stieltjes integral.

Definition 3.1. Let $[a, b]$ an interval, and $\pi = \{t_0 = a < t_1 < \dots < t_i < t_{i+1} < \dots < t_n = b\}$ a partition of this interval. The Riemann-Stieltjes integral is defined by :

$$(RS) \int_a^b f(t) dg(t) := \lim_{|\pi| \rightarrow 0} \sum_{i=0}^{n-1} f(t_i) (g(t_{i+1}) - g(t_i)) \quad (3.1)$$

In the same paper, Young gives conditions on the functions f and g for the existence of the Riemann-Stieltjes integral.

Theorem 3.1. With the notations of the previous definition, the Riemann-Stieltjes integral

$$(RS) \int_a^b f(t) dg(t)$$

exist when $V_p(f) < \infty$, $V_q(g) < \infty$ and g is continuous, with $\frac{1}{p} + \frac{1}{q} > 1$.

The following corollary consists in applying the previous theorem to the FBM.

Corollary 3.1. *Let $(u_t)_{t \geq 0}$ a stochastic process, If the process u has α -Hölder continuous trajectories, such that $\alpha > 1 - H$, Then, the Riemann-Stieltjes integral w.r.t. B^H ,*

$$(RS) \int_0^t u_s dB_s^H \quad (3.2)$$

is well defined for every $t > 0$.

Proof. As B^H has a version whose trajectories are λ -Hölder continuous for all $\lambda < H$, $V_{1/\lambda}(B^H) < \infty$. The same happens with u , we have $V_{1/\alpha}(u) < \infty$.

To finish we just need to verify that $\alpha + \lambda > 1$.

Indeed, the differences $H - \lambda$ and $\alpha - (1 - H)$ are both positive by assumption, we just need to choose a λ close enough to H such that $\alpha - (1 - H) > H - \lambda$ which will directly imply what is needed : $\alpha + \lambda > 1$. \square

3.1.2 Generalized Riemann-Stieltjes integral

Inspired by the fact that (3.1) becomes $\int_a^b f(t)g'(t)dt$ when function g is C^1 , Zähle ([Zäh98]) wanted to generalize this idea for a less regular g in order to still have a closed-formula to compute or estimate these integrals. If g' cannot exist but for some positive α , $D^{1-\alpha}g$ is well defined in the sense of chapter 2. Asking "less" from g as to be, in some sense, compensated by a stronger regularity of function f .

Definition 3.2. *Let $f \in I_{a+}^\alpha(L_p)$, $g \in I_{b-}^{1-\alpha}(L_q)$, with $0 \leq \alpha \leq 1$ and such that $\frac{1}{p} + \frac{1}{q} \leq 1$, the generalized Riemann-Stieltjes is defined as :*

$$(Z) \int_a^b f(t)dg(t) = (-1)^\alpha \int_a^b D_{a+}^\alpha f_{a+}(t) D_{b-}^{1-\alpha} g_{b-}(t) dt + f(a+)(g(b-) - g(a+)) \quad (3.3)$$

where $f(a+) = \lim_{\epsilon \rightarrow 0+} f(a + \epsilon)$, $g(b-) = \lim_{\epsilon \rightarrow 0+} g(b - \epsilon)$,

and $f_{a+}(t) := \chi_{(a,b)}(t)(f(t) - f(a+))$, $g_{b-}(t) := \chi_{(a,b)}(t)(g(t) - g(b-))$.

Zähle, [Zäh98], proves this definition is both correct and independent of the choice of α in the interval $[0, 1]$. The proof consists in taking the definition with an order $\alpha' = \alpha + \beta$ and arriving to the definition with the order α , mainly thanks to the composition formula for fractional integrals presented in the second chapter (proposition 2.1).

After having applied this integral to several types of functions f , starting with the simple case of an indicator function, then to step functions, as it is usual in the construction of an integral; Zähle proves the following theorem, connecting this integral to Young's Riemann-Stieltjes integral.

Theorem 3.2. *If $f \in C^\lambda$, $g \in C^\mu$ with $\lambda + \mu > 1$,*

Then,

$$(Z) \int_a^b f(t)dg(t) = (RS) \int_a^b f(t)dg(t) \quad (3.4)$$

The final important result from [Zäh98], that will prove to be crucial in the final section of chapter 3, gives us a change of variable formula for (Z)-type integrals. Its proof can be found in p350.

Theorem 3.3. *Let $f \in C^\lambda$ and $F \in C^1(\mathbb{R}, (a, b))$, such that $F'_1(f(\cdot), \cdot) \in C^\mu$ with $\lambda + \mu > 1$.*

$$F(f(t), t) - F(f(a), a) = \int_a^t F'_1(f(s), s)df(s) + \int_a^t F'_2(f(s), s)ds \quad (3.5)$$

where F'_1 is the partial derivative of F wrt its first variable, and F'_2 to its second variable.

3.2 Existence and Uniqueness Results

In several financial applications, instruments are modeled with the help of some differential equation that describes the evolution of either a price, an interest rate, a volatility, etc. In this section we will introduce stochastic differential equations whose randomness will come from the FBM, contrary to what we might be used to where the randomness comes from the standard Brownian motion. We shall start with a definition of such equation as well as what it means to be a solution.

Definition 3.3. *Let $H > 1/2$, consider the interval $[0, T]$ with $T > 0$. The following stochastic differential equation:*

$$du_t = b(t, u_t)dt + \sigma(t, u_t)dB_t^H \quad (3.6)$$

admits a pathwise solution $(u_t)_{0 \leq t \leq T}$ given that for almost every trajectory both b and σ are C^0 in $[0, T]$, the generalized Riemann-Stieltjes integral $\int_0^t \sigma(s, u_s)dB_s^H$ is well defined for $t \in]0, T]$, and we have for every $s \leq t$ in $[0, T]$:

$$u_t = u_s + \int_s^t b(r, u_r)dr + \int_s^t \sigma(r, u_r)dB_r^H$$

In this section we shall consider the following integral equation,

$$u_t = u_0 + \int_0^t b(s, u_s) ds + \int_0^t \sigma(s, u_s) dB_s^H \quad (3.7)$$

Before presenting, Nualart and Răşcanu's existence and uniqueness theorem, developed in [NR02], let us recall that:

- u_0 is a random variable,
- B^H 's Hurst index H is bigger than $1/2$,
- $b, \sigma : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ are measurable functions,
- We define, for $\alpha \in]1 - H, 1/2[$, the Sobolev space $W_0^{\alpha, \infty}$ the space of measurable functions $f : [0, T] \rightarrow \mathbb{R}$ such that

$$\|f\|_{\alpha, \infty} := \sup_{t \in [0, T]} \left(|f(t)| + \int_0^t \frac{|f(t) - f(s)|}{(t-s)^{\alpha+1}} ds \right) < \infty$$

Remark 3.1. In [NR02], it is proved that for $\epsilon > 0$, $C^{\alpha+\epsilon} \subset W_0^{\alpha, \infty} \subset C^{\alpha-\epsilon}$

The following existence and uniqueness theorem, as other theorems of this kind, has some expected hypothesis on the regularity of the coefficients σ and b , such as local or global Hölder continuity and Lipschitz conditions in both time and space variables.

Theorem 3.4. *If the following conditions are satisfied, for almost all trajectories in Ω ,*

1. *Function $\sigma : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is differentiable in space, such that:*

- (a) $\exists M > 0, \forall u, v \in \mathbb{R}, \forall t \in [0, T],$
 $|\sigma(t, u) - \sigma(t, v)| \leq M|u - v|$
- (b) $\exists d \in]\frac{1}{H} - 1, 1], \forall U \in \mathbb{R}^+, \exists M_U > 0, \forall t \in [0, T],$
 $(|u| \leq U \wedge |v| \leq U) \implies \left| \frac{\partial \sigma}{\partial u}(t, u) - \frac{\partial \sigma}{\partial u}(t, v) \right| \leq M_U |u - v|^d$
- (c) $\exists a > 1 - H, \exists N > 0, \forall u \in \mathbb{R}, \forall t, s \in [0, T],$
 $|\sigma(t, u) - \sigma(s, u)| + \left| \frac{\partial \sigma}{\partial u}(t, u) - \frac{\partial \sigma}{\partial u}(s, u) \right| \leq N|t - s|^a$

2. *Function $b : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is such that:*

- (a) $\forall U \in \mathbb{R}^+, \exists M_U > 0, \forall t \in [0, T],$
 $(|u| \leq U \wedge |v| \leq U) \implies |b(t, u) - b(t, v)| \leq M_U |u - v|$

3.3. Stability

$$(b) \exists p \geq 2, \exists b_0 : [0, T] \rightarrow \mathbb{R} \in L^p, \exists A > 0, \forall u \in \mathbb{R}, \forall t \in [0, T], \\ |b(t, u)| \leq A|u| + b_0(t)$$

Then, denoting $a_0 = \min(\frac{1}{2}, a, \frac{d}{1+d})$, there exists a unique measurable function $u \in W_0^{\alpha, \infty}$ that verifies equation 3.7, where $\alpha \in]1 - H, a_0[$ and $p \geq 1/\alpha$. Furthermore the solution of equation 3.7 is $(1 - \alpha)$ -Hölder continuous.

Nualart and Răşcanu, [NR02], prove this theorem with the help of a result regarding an analogous deterministic differential equation (theorem 5.1 in [NR02]) based on technical existence lemmas outside of the scope of this master thesis.

As the conditions presented in the previous theorem are rather complex and can sometimes be hard to verify, we shall present a less general version of theorem 3.4 that can be used in practical examples in a quicker and simpler way.

Corollary 3.2. Consider equation 3.7,

If:

- Function $\sigma : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is C^2 in both variables and $\partial_u \sigma$, $\partial_u^2 \sigma$ and $\partial_t \partial_u \sigma$ are bounded.
- Function $b : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is C^1 in the space variable, and

$$\exists p \geq 2, \exists b_0 : [0, T] \rightarrow \mathbb{R} \in L^p, \exists A > 0, \forall u \in \mathbb{R}, \forall t \in [0, T], |b(t, u)| \leq A|u| + b_0(t)$$

Then, there exists a unique measurable function $u \in W_0^{\alpha, \infty}$ that verifies equation 3.7, where $\alpha \in]1 - H, 1/2[$. This solution will also be $(1 - \alpha)$ -Hölder continuous.

Proof. The facts that $\partial_u \sigma$, $\partial_u^2 \sigma$ and $\partial_t \partial_u \sigma$ are bounded respectively imply hypothesis 1.(a), 1.(b) and 1.(c) in theorem 3.4 hold, with constants d and a both equal to 1. As for conditions 2., they come directly from what is assumed regarding function b . \square

3.3 Stability

Let us consider the following stochastic differential equation,

$$dy_t = F(y_t)dt + G(y_t)dB_t^H \tag{3.8}$$

A result on global exponential stability of the trivial (zero) solution of equation (3.8) was developed by Garrido-Atienza *et al.* in [GANS18]. The authors suppose that the function G is linear and that F verifies for all x : $F(x) = -\lambda.x + f(x)$, where $\lambda > 0$ and such that it exists constant $\delta < \lambda$ with $|f(x)| \leq \delta|x|$.

In this same paper, the authors proved under what conditions equation (3.8)'s solution is locally exponentially zero stable and with which rate. Let us first recall the definition of such stability to later present Garrido-Atienza *et al.*'s main result (proved in [GANS18] in the multidimensional case but only presented here in the scalar case).

Definition 3.4. A solution, x_t , of equation (3.8) is said to be locally exponentially zero stable with rate r , if there is a neighborhood, V , of zero where:

$$x_0 \in V \implies \lim_{t \rightarrow \infty} e^{rt} |x_t| = 0.$$

Theorem 3.5. Consider equation (3.8),

Suppose:

- $F, G \in C^2$ with bounded derivatives in a neighborhood of zero,
- $F(0) = 0$ and $G(0) = 0$,
- $G'(0) = 0$,
- There is $\lambda > 0$ such that $F'(0) < -\lambda$.

Then, the trivial solution of equation 3.8 is locally exponentially zero stable with rate r for every r verifying $r < -\ln(\epsilon + e^{-\lambda})$ for every $\epsilon \in]0, 1 - e^{-\lambda}[$.

We wish now to build an original result, inspired by [GANS18] results displayed above, that gives conditions on functions F and G that will ensure local exponential stability but of a non-trivial solution of equation (3.8).

We consider y_t a non-trivial solution of this equation, our goal is to introduce a small perturbation to the solution y , here denoted u , and to understand what conditions the functions F and G must satisfy to establish local exponential stability of y_t .

Let us first define such form of stability.

Definition 3.5. A solution, y_t , of equation (3.8) is said to be locally exponentially stable, if for any other solution x_t :

$$\sup_{t \geq 0} |y_t - x_t| < 1 \implies \lim_{t \rightarrow \infty} \frac{1}{t} \ln |y_t - x_t| < 0$$

Theorem 3.6. Consider equation (3.8), and y_t a non-trivial solution.

Suppose:

- F is twice differentiable and its second derivative is limited. Let $\delta := \|F''\|_{C^0}$ ¹.

¹ $\|f\|_{C^0} = \sup_x |f(x)|$

3.3. Stability

– G is an affine function.

If $\delta < \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \lambda(s) ds$, where the function λ is such that $\lambda(t) = -F'(y_t)$, then the solution y_t is locally exponentially stable.

Proof. Let $x_t = y_t + u_t$, suppose x_t is also a solution of (3.8). We have,

$$dx_t = dy_t + du_t = F(y_t)dt + G(y_t)dB_t^H + du_t,$$

and

$$dx_t = F(y_t + u_t)dt + G(y_t + u_t)dB_t^H.$$

So, denoting $G(x) = \gamma x + c$ for all x , u_t satisfies the following,

$$du_t = [F(y_t + u_t) - F(y_t)]dt + \gamma u_t dB_t^H$$

Let us now focus our attention on the drift coefficient.

$$\begin{aligned} F(y_t + u_t) - F(y_t) &= \int_0^1 F'(y_t + su_t) ds \cdot u_t, \text{ fundamental theorem of calculus,} \\ &= \int_0^1 F'(y_t + su_t) ds \cdot u_t + (F'(y_t) - F'(y_t))u_t \\ &= \underbrace{F'(y_t)}_{=-\lambda(t)} u_t + \int_0^1 (F'(y_t + su_t) - F'(y_t)) ds \cdot u_t \\ &= -\lambda(t)u_t + \int_0^1 \int_0^1 F''(y_t + \eta su_t) d\eta \cdot su_t ds \cdot u_t \\ &= -\lambda(t)u_t + \underbrace{\int_0^1 \int_0^1 F''(y_t + \eta su_t) d\eta \cdot s ds \cdot u_t^2}_{:=f(u_t, t)}. \end{aligned}$$

So, $du_t = [-\lambda(t)u_t + f(u_t, t)]dt + \gamma u_t dB_t^H$.

From the condition on F'' , we can deduce that $|f(u_t, t)| \leq \frac{1}{2}\delta|u_t|^2$. Furthermore, as the perturbation u_t is supposed to be small we have $|u_t|^2 \leq |u_t|$. Therefore:

$$|f(u_t, t)| \leq \frac{1}{2}\delta|u_t|, \text{ for every } t.$$

Let us now consider the following change of variable: $v_t = e^{\int^t \lambda(s) ds} u_t := g(u_t, t)$. From theorem 3.3 we have that $g(u_b, b) - g(u_a, a) = \int_a^b g'_u(u_s, s) du_s + \int_a^b g'_t(u_s, s) ds$.

Here $g'_u(u_t, t) = e^{\int^t \lambda(s) ds}$ and $g'_t(u_t, t) = \lambda(t)e^{\int^t \lambda(s) ds} u_t$.

So,

$$\begin{aligned} dv_t &= g'_t(u_t, t)dt + g'_u(u_t, t)du_t \\ &= \lambda(t)e^{\int^t \lambda(s)ds}u_t dt + e^{\int^t \lambda(s)ds} \left([-\lambda(t)u_t + f(u_t, t)]dt + \gamma u_t dB_t^H \right) \\ dv_t &= e^{\int^t \lambda(s)ds} f(e^{-\int^t \lambda(s)ds}v_t, t)dt + \gamma v_t dB_t^H \end{aligned}$$

Denoting $b(v_t, t) := e^{\int^t \lambda(s)ds} f(e^{-\int^t \lambda(s)ds}v_t, t)$, v_t follows the differential equation:

$$dv_t = b(v_t, t)dt + \gamma v_t dB_t^H \quad (3.9)$$

We can now perform the Doss transform, [Dos77], to the process v_t . It can be written as

$$v_t = h(D_t, B_t^H), \text{ where } h \text{ and } D \text{ are such that: } \begin{cases} \frac{\partial}{\partial y} h(x, y) = \gamma h(x, y); h(x, 0) = x \\ dD_t = e^{-\gamma B_t^H} b(D_t e^{\gamma B_t^H}, t)dt; D_0 = u_0 \end{cases}$$

This implies that $h(x, y) = x e^{\gamma y}$, so $v_t = D_t e^{\gamma B_t^H}$.

Let us now focus our attention in D_t . We shall denote $r_t := |D_t|^2$. As D_t verifies $dD_t = e^{-\gamma B_t^H} b(D_t e^{\gamma B_t^H}, t)dt = e^{-\gamma B_t^H + \int^t \lambda(s)ds} f(e^{-\int^t \lambda(s)ds + \gamma B_t^H} D_t, t)dt$, we have:

$$\begin{aligned} dr_t &= 2D_t e^{-\gamma B_t^H + \int^t \lambda(s)ds} f(e^{-\int^t \lambda(s)ds + \gamma B_t^H} D_t, t)dt \\ &\leq 2D_t e^{-\gamma B_t^H + \int^t \lambda(s)ds} \frac{1}{2} \delta |e^{-\int^t \lambda(s)ds + \gamma B_t^H} D_t| dt \\ &\leq \delta |D_t|^2 dt \\ dr_t &\leq \delta r_t dt \end{aligned}$$

By Grönwald's lemma, $dr_t \leq \delta r_t dt$ implies that $r_t \leq r_0 e^{\delta t}$. Therefore, since

$u_t = e^{-\int^t \lambda(s)ds}v_t$ and $v_t = D_t e^{\gamma B_t^H}$ we can deduce that:

$$|u_t| \leq |u_0| e^{\gamma B_t^H + \delta t - \int_0^t \lambda(s)ds} \quad (3.10)$$

From [Mao97], we know the process (u_t) is exponentially stable if $\limsup_{t \rightarrow \infty} \frac{1}{t} \ln |u_t| < 0$. Here:

$$\frac{1}{t} \ln |u_t| \leq \frac{1}{t} \left(\gamma B_t^H + \delta t - \int_0^t \lambda(s)ds + \ln |u_0| \right)$$

In [Nua06a], Nualart shows, using the Borel-Cantelli Lemma, that $\lim_{t \rightarrow \infty} \frac{|B_t^H|}{t} = 0$ for almost all trajectories. Therefore in order to assure exponential stability we must have:

$$\delta < \frac{1}{t} \limsup_{t \rightarrow \infty} \int_0^t \lambda(s)ds$$

□

Chapter 4

Application to Fractional Vasicek Interest Rate Model

In [HLW14], Hao *et al.* develop a model to price credit default swaps, also known as CDS, based on a fractional Vasicek model for the interest rate.

Since their conception in 1994 by JP Morgan Chase investment bank, CDS have been enormously popular in financial markets. A credit default swap is a financial derivative used to hedge credit risk, its buyer transfers the risk of a loan or bond default to the seller in exchange for periodical premia. After the subprime financial crisis in 2008, where CDS played an important role, a need for better and more realistic models to price these swaps was clear.

Since credit default swaps are highly sensitive to interest rate fluctuations, an accurate model to describe interest rates' evolution will have a positive impact in CDS' pricing model. Hao *et al.* introduced the fractional Vasicek model to describe interest rate evolution in time.

$$dr_t = k(\rho - r_t)dt + \sigma dB_t^H \quad (4.1)$$

Where:

- $(r_t)_{t \geq 0}$ is the stochastic process that represents the interest rate,
- ρ is the long-term interest rate,
- k is the speed at which the interest rate reverses to the long-term value, $0 < k < 1$,
- σ is the interest rate volatility,
- B^H is a fractional Brownian motion, with $1/2 < H < 1$.

The authors present a closed-form solution for equation (4.1):

$$r_t = \rho + (r_0 - \rho)e^{-kt} + \sigma e^{-kt} \int_0^t e^{ks} dB_s^H \quad (4.2)$$

However, since the integral *w.r.t.* the FBM in [HLW14] is defined in the Wick sense and not pathwise, we must show caution and verify if (4.2) is still a solution of (4.1) when considering $\int_0^t e^{ks} dB_s^H$ as a generalized Riemann-Stieltjes integral.

Proof.

$$\begin{aligned} dr_t &= d\left(\rho + (r_0 - \rho)e^{-kt} + \sigma e^{-kt} \int_0^t e^{ks} dB_s^H\right) \\ &= \left(-k(r_0 - \rho)e^{-kt} - k\sigma e^{-kt} e^{kt} \int_0^t e^{ks} dB_s^H\right) dt + \sigma e^{-kt} e^{kt} dB_t^H \\ &= \left(-k(r_0 - \rho)e^{-kt} - k\sigma \int_0^t e^{ks} dB_s^H\right) dt + \sigma dB_t^H \\ &= k\left(\rho - \underbrace{\left[\rho + (r_0 - \rho)e^{-kt} + \sigma \int_0^t e^{ks} dB_s^H\right]}_{=r_t}\right) dt + \sigma dB_t^H \\ dr_t &= k(\rho - r_t)dt + \sigma dB_t^H \end{aligned}$$

□

The stochastic process (r_t) given by (4.2) is a Gaussian process, let us compute its expected value and variation to fully characterize the interest rate process.

$$\begin{aligned} E[r_t] &= E\left[\rho + (r_0 - \rho)e^{-kt} + \sigma e^{-kt} \int_0^t e^{ks} dB_s^H\right] \\ &= \rho + (r_0 - \rho)e^{-kt} + \sigma e^{-kt} E\left[\int_0^t e^{ks} dB_s^H\right] \\ &= \rho + (r_0 - \rho)e^{-kt} + 0, \text{ because we can write this integral as in (3.1) and we know} \\ &\quad \text{the expected value of FBM increments is equal to zero.} \end{aligned}$$

$$E[r_t] = \rho + (r_0 - \rho)e^{-kt}$$

$$\begin{aligned}
\text{Var}(r_t) &= E[(r_t - E[r_t])^2] \\
&= (\sigma e^{-kt})^2 E\left[\left(\int_0^t e^{ks} dB_s^H\right)^2\right] \\
&= (\sigma e^{-kt})^2 E\left[\left(e^{kt} B_t^H - k \int_0^t e^{ks} B_s^H ds\right)^2\right], \text{ by formula (3.3)} \\
&= (\sigma e^{-kt})^2 E\left[e^{2kt} (B_t^H)^2 + k^2 \left(\int_0^t e^{ks} B_s^H ds\right)^2 - 2k e^{kt} B_t^H \int_0^t e^{ks} B_s^H ds\right] \\
&= (\sigma e^{-kt})^2 \left(e^{2kt} t^{2H} + k^2 E\left[\left(\int_0^t e^{ks} B_s^H ds\right)^2\right] - 2k e^{kt} \int_0^t e^{ks} E[B_t^H B_s^H] ds \right) \\
&= (\sigma e^{-kt})^2 \left(e^{2kt} t^{2H} + k^2 E\left[\left(\int_0^t e^{ks} B_s^H ds\right) \left(\int_0^t e^{kr} B_r^H dr\right)\right] - 2k e^{kt} \int_0^t e^{ks} E[B_t^H B_s^H] ds \right) \\
&= (\sigma e^{-kt})^2 \left(e^{2kt} t^{2H} + k^2 \left(\int_0^t \int_0^s e^{k(s+r)} E[B_s^H B_r^H] dr ds\right) - 2k e^{kt} \int_0^t e^{ks} E[B_t^H B_s^H] ds \right) \\
\text{Var}(r_t) &= (\sigma e^{-kt})^2 \left[e^{2kt} t^{2H} + \frac{1}{2} k^2 \left(\int_0^t \int_0^s e^{k(s+r)} (s^{2H} + r^{2H} - |s-r|^{2H}) dr ds\right) \right. \\
&\quad \left. - k e^{kt} \int_0^t e^{ks} (t^{2H} + s^{2H} - |t-s|^{2H}) ds \right]
\end{aligned}$$

The solution (4.2) is obviously a non-trivial one, and thanks to theorem 3.6 we can examine whether it is locally exponential stable. With the notations of theorem 3.6 we have:

$$\begin{aligned}
F(x) &= k(\rho - x), \quad x \in \mathbb{R} \\
\lambda(x) &= -F'(x) = k, \quad x \in \mathbb{R} \\
\delta &= F''(x) = 0, \quad x \in \mathbb{R}
\end{aligned}$$

As $\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \lambda(s) ds = k$ is positive by definition, we verify condition $\delta < \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \lambda(s) ds$.

Therefore solution (4.2) is locally exponentially stable.

The fact that this solution exhibits local exponential stability gives us more confidence in this model. Indeed interest rates can suffer small changes that are not explained by the model nor expected. Therefore, this result on stability ensures that even if with these unexpected perturbations, the solution will still behave correctly and will not "explode", instead it will converge to the original solution at an exponential rate.

Chapter 5

Conclusion

In this dissertation we generalized a result on the exponential stability of a trivial solution of a stochastic differential equation driven by fractional Brownian motion (in [GANS18]) to a non-trivial solution in the scalar case. In the last chapter we applied this result to the fractional Vaiscek model for interest rate and showed that its solution, which is non-trivial, is locally exponentially stable. In order to establish theorem (3.6), we first had to study the most important properties of the fractional Brownian motion. Even if the family of fractional Brownian motions kept the known self-similarity property of the standard Brownian motion, it lost the ability to be decomposed into a local martingale and adapted process. *i.e.* to be a semi-martingale, when $H \neq 1/2$. With this loss, the classical stochastic calculus based on the Itô lemma used for computing stochastic integrals *w.r.t.* the standard Brownian motion could no longer be applied to the fractional Brownian motion. Nevertheless, because FBM's sample paths are α -Hölder continuous for $\alpha < H$, we are able to define the pathwise Riemann-Stieltjes integral ([You36]) and the generalized Riemann-Stieltjes integral ([Zäh98]) *w.r.t.* the fractional Brownian motion. Having a clear and structured theoretical framework for stochastic differential equations driven by the FBM, we were able to present some existence and uniqueness results and to finally arrive at our main goal: the establishment of a result on the local exponential stability of non-trivial solutions. The importance of this result resides in the fact that in every phenomenon described by a mathematical model, or equation, there are always perturbations and changes that are not explained by the model nor expected, hence the stability of solutions ensures that the "perturbed" solution will converge to the original one. Moreover the fact that we can now apply this result to non-trivial solutions is highly meaningful, as in financial mathematical models we are not usually interested in studying a zero solution (or a constant solution), but a non-trivial one.

It is important to emphasize that our result on exponential stability of non-trivial solutions only considers the scalar case, therefore it would be interesting to focus future research in the multi-dimensional case.

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