Instituto Superior de Economia e Gestão
UNIVERSIDADE TÉCNICA DE LISBOA
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MATEMÁTICA FINANCEIRA

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DISSERTAÇÃO

INTEGRO-DIFFERENTIAL EQUATIONS FOR OPTION PRICING IN EXPONENTIAL LÉVY MODELS

JOSÉ MANUEL TEIXEIRA DOS SANTOS CRUZ

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Resumo

Este trabalho discute sob que condições se pode expressar a função que representa o preço de uma opção como solução de uma determinada equação integro-diferencial parcial num modelo exponencial de Lévy. A grande diferença entre o caso aqui considerado e o de Black-Scholes é que existe na equação um termo não local, o que faz com que a análise seja mais complexa. Também é discutido sob que condições se pode obter a fórmula de Feynman-Kač para o caso de um processo de saltos puros e sob que condições o preço de uma opção é solução clássica de uma equação integro-diferencial. Quando tais condições não são verificadas, considera-se o conceito de solução de viscosidade, que apenas exige que a função que representa o preço da opção seja contínua.

Para alguns tipos de processos de Lévy são apresentados resultados de continuidade para os preços de opções barreira. Para além disso demonstram-se os mesmos resultados para processos de variação finita e sem componente de difusão. Também são apresentados alguns exemplos em que a função que representa o preço da opção é descontínua. É apresentado um esquema numérico que permite obter o preço de uma opção de venda Europeia para o caso do processo "Variance Gamma". Este esquema de diferenças finitas foi proposto inicialmente por Cont e Voltchkova para resolver numericamente a equação integro-diferencial parcial associada.

Palavras-Chave:Processos de Lévy,Fórmula de Feynman-Kač,Equação integro-diferencial parcial, Valorização de opções.

Abstract

This dissertation discusses under which conditions we can express the function that represents the option price as the solution of a certain partial integro-differential equation (PIDE) in a exponential Lévy model. The main difference between this case and the Black-Scholes case is that there is a non-local term in the equation, which makes the analysis more complicated. Also, we discuss under which conditions we can obtain a Feynman-Kač formula for the case of a pure jump process and discuss the conditions under which option prices are classical solutions of the PIDEs. When such conditions are not verified, we consider the concept of viscosity solutions which only requires that the function representing the option price is continuous.

Continuity results for option prices of barrier options are presented for some types of Lévy processes. In addition, we show the same continuity results for processes of finite variation and with no diffusion component. Also, we present some examples in which the function that represents the option price is discontinuous. Moreover, we present a numerical scheme that gives the price of an European put option for the Variance Gamma process. This finite difference scheme was initially proposed by Cont and Voltchkova, to solve numerically the associated PIDE.

Keywords:Lévy Processes,Feynman-Kač formula,PIDEs,Option Pricing.

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Contents

Chapter 1

Introduction

One of the main reasons for using the Black-Scholes model is the existence of an analytical formula to price European options. However, evidence from the stock market suggests that this model is not the most realistic one. It is well known that the sample paths of a Brownian motion are continuous, but the stock price of a typical company suffers sudden jumps on an intraday scale, making the price trajectories discontinuous. In the classical Black-Scholes model the logarithm of the price process has normal distribution. However the empirical distribution of stock returns exhibits fat tails. Finally, when we calibrate the theoretical prices to the market prices, we realize that the implied volatility is not constant as a function of strike neither as a function of time to maturity, contradicting this way the prediction of the Black-Scholes model. Several alternatives have been proposed in the literature for the generalization of this model. The models with jumps can, at least in part, solve the problems inherent to the Black-Scholes model. The jump models have also an important role in the options market. While in the Black-Scholes model the market is complete, implying that every payoff can be exactly replicated, in jump models there is no perfect hedge and this way the options are not redundant.

The objective of this thesis is to study under which conditions we can obtain the function that represents the option price as a solution of a certain partial integro-differential equation. Moreover, we will discuss some examples where the price function is not regular enough in order to be a classical solution of this partial integro-differential equation.

The prices of options such as European options and barrier options can be caracterized in terms of solutions of a partial integro differential equation with some boundary conditions depending on the type of option considered. Conversely, if we have a solution of a certain partial integro-differential equation (PIDE) satisfying some conditions, then it is possible to arrive at a stochastic representation of the Feynman-Kač kind, analogous to the Black-Scholes case. The main difference between a model with jumps and the Black-Scholes case is a non-local term that appears in the equation, because now the price process possesses jumps, and the option price can be discontinuous. This non-local term makes PIDEs less easy to solve than partial differential equations. However, one of the numerical schemes used in the literature is presented to solve such equations. In analytical terms, if the price is not a classical (smooth) solution of the PIDE, the notion of viscosity solution can be used.

In Section 2.1 we present the definition of a Lévy process and some notation related to Lévy processes. In Section 2.2 we introduce the Lévy exponential models for financial assets. In Section 2.3 we give some examples of financial models. In Section 3.1 we present the definition of a price of an European option as a discounted expected value of the terminal payoff and a simple derivation of the integro-differential equation whose solution is the discounted expected value of the terminal payoff. In Section 3.2 we present a result shown by Nualart and Schoutens [NS01] that allows a probabilistic representation of solutions of PIDE's through the use of a Feynman-Kač formula. Section 3.3 is dedicated to present in detail the relation between the price of European options (Subsection 3.3.1) and barrier options (Subsection 3.3.2), and the solutions of the associated integro-differential equations. Also, in Subsection 3.3.2 some continuity results are presented for barrier options. In Section 3.3.3 we present the numerical scheme proposed by Cont and Voltchkova [CV05a] to solve a partial integro-differential equation and also present some numerical results. In the begining of Chapter 4, we introduce the notion of viscosity solution and its rigorous definition is given in full detail in Section 4.1. In Section 4.2 we present some results concerning the uniqueness and existence of a viscosity solution. In the appendix we present the proofs of Propositions 3.3.1, 3.3.2 and 3.3.5. Also in the appendix, we presente the numerical code (Mathemathica Code) used to compute the value of a binary option using the Monte Carlo method and the code to compute the value of an European option under the Variance Gamma process using a finite difference scheme proposed in [CV05a].

Chapter 2

Financial Modelling with Lévy Processes

2.1 Lévy Process: definitions

Let us start with the definition of a Lévy process.

Definition 2.1.1 *Consider a fixed probability space* $(\Omega, \mathcal{F}, \mathbb{Q})$. *A stochastic process* X_t *such that* $X_0 = 0$ *is called a Lévy process if:*

- X_t *has independent increments: for every* $t_0 < t_1 < .. < t_n$*, the random variables* $X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, ..., X_{t_n} - X_{t_{n-1}}$ *are independent.*
- X_t *has stationary increments: the law of* $X_{t+h} - X_t$ *does not depend on* t*;*
- X_t *is stochastically continuous ,i.e. for all* $a > 0$ *and* $s > 0$*:*

$$\lim_{t \to s} \mathbb{P}[|X_t - X_s| > a] = 0$$

We only consider a right continuous with limits to the left (cádlag) version of X_t and will denote $\Delta X_t = X_t - X_{t^-}$, the jump of X at time t.

The characteristic function of X_t has the following Lévy-Khintchine representation ([Sat99],[CT04],[App04]):

$$\mathbb{E}\left[e^{izX_t}\right] = e^{t\phi(z)}, \phi(z) = -\frac{\sigma^2 z^2}{2} + i\gamma z + \int_{-\infty}^{+\infty} \left(e^{izx} - 1 - izx1_{|x|\leq 1}\right) \nu(dx).$$

where $\sigma \geq 0$ and $\gamma \in \mathbb{R}$ and ν is a positive Radon measure on $\mathbb{R} \setminus \{0\}$ verifying:

$$\int_{-1}^{1} x^2 \nu(dx) < \infty. \tag{2.1}$$

and

$$\int_{|x|>1} \nu(dx) < \infty. \tag{2.2}$$

The measure ν is defined by:

$$\nu(A) = \mathbb{E}\left[\#\{t \in [0,1] : \Delta X_t \in A\}\right] = \frac{1}{T}\mathbb{E}\left[\#\{t \in [0,T] : \Delta X_t \in A\}\right], A \in \mathcal{B}(\mathbb{R}), \tag{2.3}$$

and is called the Lévy measure of X. It gives the mean number, per unit of time, of jumps whose amplitude belongs to A.

The Lévy-Itô decomposition gives a representation where X is interpreted as a combination of a Brownian motion with drift and a infinite sum of independent compensated Poisson processes with several jump sizes x (see [CT04])

$$X_t = \gamma t + \sigma W_t + \int_0^t \int_{|x|\geq 1} x J_X\left(ds, dx\right) + \int_0^t \int_{|x|<1} x \widetilde{J}_X\left(ds, dx\right), \tag{2.4}$$

where J_X is the Poisson random measure defined in the following way:

$$J_X\left([0,t]\times A\right) = \#\left\{s \in [0,t] : \Delta X_s \in A\right\}. \tag{2.5}$$

The compensated Poisson measure is defined by:

$$\widetilde{J}_X\left([0,t]\times A\right) = J_X\left([0,t]\times A\right) - t\nu\left(A\right). \tag{2.6}$$

A Lévy process is a strong Markov process, the associated semigroup is a convolution semigroup and its infinitesimal generator $L : f \to Lf$ is an integro-differential operator given by (see [App04]):

$$\begin{aligned} Lf(x) &= \lim_{t\to 0} \frac{\mathbb{E}\left[f\left(x+X_t\right)\right] - f\left(x\right)}{t} && (2.7)\\ &= \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} + \gamma\frac{\partial f}{\partial x} + \int_{\mathbb{R}} \left[f\left(x+y\right) - f\left(x\right) - y 1_{|x|\leq 1}\frac{\partial f}{\partial x}\left(x\right)\right]\nu\left(\mathrm{d}y\right), && (2.8) \end{aligned}$$

which is well defined for $f \in C^2\left(\mathbb{R}\right)$ with compact support.

2.2 Exponential Lévy models

Let S_t be a stochastic process representing the price of a financial asset under a filtered probability space $\left(\Omega, \mathcal{F}, \left\{\mathcal{F}_t\right\}, \mathbb{P}\right)$. The filtration $\left\{\mathcal{F}_t\right\}$ represents the price history up to time t. If the market is arbitrage-free, then there is a measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ under which the discounted prices of all traded financial assets are $\mathbb{Q}-$ martingales. This result is known as the fundamental theorem of asset pricing (see [CT04]). The measure $\mathbb{Q}$ is also known as the risk neutral measure. We consider here the exponential Lévy model in which the risk-neutral dynamics of S_t under $\mathbb{Q}$ is given by $S_t = e^{rt+X_t}$, where X_t is a Lévy process under $\mathbb{Q}$ with characteristic triplet (σ, γ, ν). Then the arbitrage-free market hypothesis imposes that $\widehat{S}_t = S_t e^{-rt} = e^{X_t}$ is a martingale, which is equivalent to the following conditions imposed on the triplet (σ, γ, ν)

$$\int_{|x|>1} e^y \nu\left(\mathrm{d}y\right) < \infty, \gamma = -\frac{\sigma^2}{2} - \int_{-\infty}^{+\infty} \left(e^y - 1 - y 1_{|y|\leq 1}\right)\nu\left(dy\right). \tag{2.9}$$

Then the infinitesimal generator (2.8) becomes

$$Lf(x) = -\frac{\sigma^2}{2}\frac{\partial f}{\partial x}\left(x\right) + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}\left(x\right) + \int_{\mathbb{R}} \left[f\left(x+y\right) - f\left(x\right) - \left(e^y - 1\right)\frac{\partial f}{\partial x}\left(x\right)\right]\nu\left(\mathrm{d}y\right) \tag{2.10}$$

The risk-neutral dynamics of S_t under $\mathbb{Q}$ is given by

$$S_t = S_0 + \int_0^t r S_{u^-}\,\mathrm{d}u + \int_0^t \sigma S_{u^-}\,\mathrm{d}W_u + \int_0^t \int_{\mathbb{R}} \left(e^x - 1\right) S_{u^-} \widetilde{J}_X\left(du, dx\right). \tag{2.11}$$

The price process S_t is also a Markov process with state space $(0,\infty)$ and infinitesimal generator (see [CT04])

$$L^S f(x) = \lim_{h\to 0} \frac{\mathbb{E}[f(xe^{X_h})] - f(x)}{h} \tag{2.12}$$
$$= rx\frac{\partial f}{\partial x}(x) + \frac{\sigma^2 x^2}{2}\frac{\partial^2 f}{\partial x^2}(x) + \int_{\mathbb{R}} \left[f(xe^y) - f(x) - x(e^y - 1)\frac{\partial f}{\partial x}(x) \right] \nu(\mathrm{d}y) \tag{2.13}$$

2.3 Examples of Lévy processes in finance

The exponential Lévy models considered in the financial literature are of two types. The first type of models are called jump-diffusion models where we represent the log-price as a Lévy process with a non zero diffusion part ($\sigma > 0$) and with a jump process with finite activity (i.e $\nu(\mathbb{R}) < \infty$). The second type of models are called infinite activity pure jump models in which case there is no diffusion part and only a jump process with infinite activity (i.e $\nu(\mathbb{R}) = \infty$).

There are a variety of exponential Lévy models proposed in the financial modelling literature that differ from each other only in the choice of the Lévy measure. In this section we present some examples of models used.

2.3.1 Jump-Diffusion models

A Lévy process of jump-diffusion type is of the following form:

$$X_t = \gamma t + \sigma W_t + \sum_{i=1}^{N_t} Y_i$$

where $\sigma > 0, N_t$ is a Poisson process with intensity λ that counts the jumps of X_t and $Y_i, i = 1,2,3...$ are independent and identically distributed random variables with distribution given by μ. The Lévy measure ν is given by $\lambda\mu$ and the drift γ is equal to

$$-\frac{\sigma^2}{2} - \int_{\mathbb{R}} \left(e^y - 1 - y1_{|y|\le 1}\right) \nu(\mathrm{d}y).$$

2.3.1.1 Merton's model

This model was introduced by Merton [Mer76] and was the first jump-diffusion model proposed in the financial literature. The random variables $Y_i, i = 1,2,3...$ are normally distributed with mean μ and variance δ. Its Lévy density is given by:

$$\nu(x) = \lambda \frac{1}{\delta\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\delta^2}} \tag{2.14}$$

Then it's possible to obtain the probability density of X_t as a series that converges rapidly (see [CT04]):

$$p_t(x) = \sum_{j=0}^{\infty} e^{-\lambda t}(\lambda t)^j \frac{e^{-\frac{(x-\gamma t - jm)^2}{2(\sigma^2 t + j\delta^2)}}}{j!\sqrt{2\pi(\sigma^2 t + j\delta^2)}}. \tag{2.15}$$

Thus, we can express the price of an European call option as a weighted sum of Black-Scholes prices:

$$C_{Merton}(S_0, K, T, \sigma, r) = e^{-rT} \sum_{j=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!} e^{r_j T} C_{BS}(S_0 e^{\frac{j\delta^2}{T}}, K, T, \sigma_j, r_j), \tag{2.16}$$

where $r_j = r - \lambda(e^{m+\frac{\delta^2}{2}} - 1) + \frac{jm}{T}$, $\sigma_j = \sqrt{\sigma^2 + \frac{j\delta^2}{T}}$ and $C_{BS}(S, K, T, \sigma, r)$ is the well known Black-Scholes formula.

2.3.2 Infinite activity pure jump models

The Variance Gammma and Normal Inverse Gaussian (NIG) processes are obtained by a subordination of a Brownian motion and a tempered α-stable process: the Variance Gamma process correspond to $\alpha = 0$ and the NIG process corresponds to $\alpha = 1/2$. These models are popular in the literature beacuse the probability density of the subordinator is known in a closed form for those values of α (see [CT04]).

2.3.2.1 Variance Gamma Process

The Variance Gamma process is a pure discontinuous process of infinite activity and finite variation ($\int_{|x|\leq 1} |x|\nu(\,dx) < \infty$) that is widely used in the financial modelling. Its Lévy measure is given by

$$\nu\left(x\right) = \frac{1}{\kappa\left|x\right|} e^{Ax - B|x|} \text{ with } A = \frac{\theta}{\sigma^2} \text{ and } B = \frac{\sqrt{\theta^2 + 2\frac{\sigma^2}{\kappa}}}{\sigma^2},$$

where σ and θ are parameters related with the volatility and drift of the Brownian motion with drift and κ is the parameter related with the variance of the subordinator, in this case the Gamma process (see [CT04]). The probability density is given by

$$p_t(x) = Ce^{Ax}|x|^{\frac{t}{k}} K_{\frac{t}{k}-\frac{1}{2}}(|x|),$$

where K is the modified Bessel Function of second kind.

The characteristic function of $X_t + \gamma t$ is equal to :

$$\Phi_t\left(u\right) = e^{itu\gamma}\phi_t\left(u\right) = e^{itu\gamma}\left(1 + \frac{\sigma^2 u^2 \kappa}{2} - i\theta\kappa u\right)^{-t/\kappa},$$

where γ is determined by the martingale condition and $\phi_t\left(u\right)$ is the characteristic function of X_t. In fact, we must have

$$\mathbb{E}[e^{-rT} S_T | \mathbb{F}_t] = e^{-rt} S_t, \tag{2.17}$$

where

$$S_t = S_0 e^{rt+\gamma t + X_t} \tag{2.18}$$

is the risk-neutral process introduced in [MCC98, MM91]. Therefore, $\gamma = \frac{1}{\kappa} log(1 - \frac{\sigma^2\kappa}{2} - \theta\kappa)$.

2.3.2.2 Normal Inverse Gaussian model

The NIG process is a process of infinite activity and infinite variation without any Brownian component. Its Lévy measure is given by (see [CT04])

$$\nu(x) = \frac{C}{|x|} e^{Ax} K_1 (B|x|)$$

and

$$C = \frac{\sqrt{\theta^2 + \frac{\sigma^2}{\kappa}}}{2\pi\sigma\sqrt{\kappa}}, A = \frac{\theta}{\sigma^2}, B = \frac{\sqrt{\theta^2 + \frac{\sigma^2}{\kappa}}}{\sigma^2},$$

where θ,σ and κ have the same meaning as in the Variance Gamma process. The probability density is:

$$p_t(x) = Ce^{Ax} \frac{K_1(B\sqrt{x^2 + \frac{t^2\sigma^2}{\kappa}})}{\sqrt{x^2 + \frac{t^2\sigma^2}{\kappa}}}$$

where K is the modified Bessel Function of second kind. The characteristic function is given by

$$\Phi_t(u) = e^{\frac{t}{\kappa} - \frac{t}{\kappa}\sqrt{1+u^2\sigma^2\kappa - 2iu\theta\kappa}}. \tag{2.19}$$

2.3.2.3 Generalized Hyperbolic model

The Generalized Hyperbolic model is a process of infinite variation whithout gaussian part. Its characteristic function is given by (see [CT04])

$$\phi_t(u) = e^{i\mu u} \left(\frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + iu)^2}\right)^{\frac{t}{2\kappa}} \frac{K_{\frac{t}{\kappa}}(\delta\sqrt{\lambda^2 - (\beta + iu)^2})}{K_{\frac{t}{\kappa}}(\delta\sqrt{\alpha^2 - \beta^2})}, \tag{2.20}$$

where δ is a scale parameter, μ is the shift parameter and κ has the same meaning that in the Variance Gamma process. The parameters λ, α and β determine the shape of the distribution. The density function

$$p_t(x) = C(\sqrt{\delta^2 + (x-\mu)^2})^{\frac{t}{k} - \frac{1}{2}} K_{\frac{t}{\kappa} - \frac{1}{2}}(\alpha\sqrt{\delta^2 - (x-\mu)^2}) e^{\beta(x-\mu)},$$

where K is the modified bessel function and

$$C = \frac{(\sqrt{\alpha^2 - \beta^2})^{\frac{t}{k}}}{\sqrt{2\pi}\alpha^{\frac{t}{\kappa} - \frac{1}{2}} \delta^{\frac{t}{\kappa}} K_{\frac{t}{\kappa}}(\delta\sqrt{\alpha^2 - \beta^2})}.$$

The Variance Gamma process is obtained for $\mu = 0$ and $\delta = 0$. The Normal Inverse Gaussian process corresponds to $\lambda = -\frac{1}{2}$.

Chapter 3

Integro-differential equations for option pricing

3.1 Definitions

The value of a European option is defined as the discounted conditional expectation of the terminal payoff $H(S_T)$ under the risk neutral probability $\mathbb{Q}$:

$$\begin{aligned} C_t = C(t, S_t) &= \mathbb{E}\left[e^{-r(T-t)} H(S_T) | \mathcal{F}_t\right] \\ &= \mathbb{E}\left[e^{-r(T-t)} H(S_T) | S_t = S\right] = e^{-r(T-t)} \mathbb{E}[H(Se^{r(T-t)+X_{T-t}})], \end{aligned}$$

because of the Markov property and the fact that X_t is a Lévy process.

If H is in the domain of the infinitesimal generator L^S, then if we differentiate $C(t, S_t)$ with respect to t ,we obtain the following integro-differential equation

$$\frac{\partial C}{\partial t}(t, S) + L^S C(t, S) - rC(t, S) = 0; C(T, S) = H(S), \tag{3.1}$$

where L^S is defined by (2.13).

Defining $\tau = T - t, x = ln\left(\frac{S}{S_0}\right), h(x) = H(S_0 e^x)$ and $f(\tau, x) = e^{r\tau} C(T - t, S_0 e^x)$ we get

$$f(\tau, x) = \mathbb{E}\left[H\left(Se^{r\tau + X_\tau}\right)\right] = \mathbb{E}\left[H\left(S_0 e^{x + r\tau + X_\tau}\right)\right] = \mathbb{E}\left[h(x + r\tau + X_\tau)\right]. \tag{3.2}$$

The associated infinitesimal generator is given by (2.10). Then, similarly to the previous case, differentiating (3.2) with respect to τ we obtain the integro-differential equation

$$\frac{\partial f}{\partial \tau} = Lf + r\frac{\partial f}{\partial x}, (\tau, x) \in (0, T] \times \mathbb{R}; \tag{3.3}$$

$$f(0, x) = h(x), x \in \mathbb{R}. \tag{3.4}$$

Indeed, by the definition of the associated infinitesimal generator we get

$$\begin{aligned}
Lf(x) &= \lim_{k\to 0} \frac{\mathbb{E}[f(\tau, x+X_k)] - f(\tau,x)}{k} \\
&= \lim_{k\to 0} \frac{\mathbb{E}[h(x+r\tau+X_{k+\tau})] - \mathbb{E}[h(x+r\tau+X_\tau)]}{k} \\
&= \lim_{k\to 0} \frac{\mathbb{E}[h(x+r(\tau+k)+X_{k+\tau})] - \mathbb{E}[h(x+r\tau+X_\tau)]}{k} \\
&\quad - \lim_{k\to 0} \frac{\mathbb{E}[h(x+r(\tau+k)+X_{k+\tau})] - \mathbb{E}[h(x+r\tau+X_{k+\tau})]}{k} \\
&= \frac{\partial f}{\partial \tau} - \lim_{k\to 0} \frac{\mathbb{E}[h(x+r(\tau+k)+X_{k+\tau})] - \mathbb{E}[h(x+r\tau+X_{k+\tau})]}{k} \\
&= \frac{\partial f}{\partial \tau} - \lim_{z\to 0} r \frac{\mathbb{E}[h(x+z+r\tau+X_{\frac{z}{r}+\tau})] - \mathbb{E}[h(x+r\tau+X_{\frac{z}{r}+\tau})]}{z} = \frac{\partial f}{\partial \tau} - r\frac{\partial f}{\partial x}.
\end{aligned}$$

3.2 Feynman-Kač formula for PIDEs

For $t \geq 0$, let $\mathcal{F}_t$ denote the $\sigma-$algebra generated by the random variables $\{X_s, 0 \leq s \leq t\}$ and

$$H_T^2 = \left\{ \phi_t, t \in [0,T] : \|\phi\|^2 = \mathbb{E}[\int_0^T |\phi_t|^2 \,\mathrm{d}t] < \infty \right\}$$

M_T^2 is the subspace of H_T^2 that contains predictable processes. Let $H_T^2(l^2)$ and $M_T^2(l^2)$ denote the corresponding spaces of l^2-valued processes equipped with the norm:

$$\|\phi\|^2 = \mathbb{E}[\int_0^T \sum_{i=1}^{\infty} |\phi_t^{(i)}|^2 \,\mathrm{d}t].$$

Finally set $\mathcal{H}_T^2 = H_T^2 \times M_T^2(l^2)$.

Following Nualart and Schoutens [NS00], they define the power-jump processes for every $i = 1, 2, 3, ...$, $\left\{X_t^{(i)}, t \geq 0\right\}$ and the compensated power-jump processes or Teugel martingales $\left\{Y_t^i = X_t^i - \mathbb{E}\left[X_t^{(i)}\right], t \geq 0\right\}$, in the following way:

$$X_t^{(1)} = X_t,$$

$$X_t^{(i)} = \sum_{0<s\leq t} (\Delta X_s)^i \, , i = 2, 3, 4, ...,$$

$$Y_t^{(i)} = X_t^{(i)} - t\mathbb{E}\left[X_t^{(i)}\right], i \geq 1,$$

Then applying a orthonormalization procedure to the martingales $Y^{(i)}$ we obtain a set of pairwise strongly orthonormal martingales $\left\{H^{(i)}, t \geq 0\right\}, i = 1, 2, ...$ such that each $H^{(i)}$ is a linear combination of the $Y^{(j)}, j = 1, 2, ...i$:

$$H^{(i)} = c_{i,i}Y^{(i)} + ... + c_{i,1}Y^{(1)},$$

where

$$c_{1,1} = [\int_{\mathbb{R}} y^2 \nu(dy)]^{-1/2}$$

and

$$\mathbb{E}[X_1] = a + \int_{|x|\geq 1} x\nu(dx).$$

The constants $c_{i,j}$ are the orthonormalization coefficients of the polynomials $\{1, x, x^2, x^3, ...\}$ with respect to the measure $\mu(dx) = x^2\nu(dx) + \sigma^2\delta_0(dx)$ and the polynomials we want to find are of the form

$$q_{i-1}(x) = c_{i,1} + c_{i,2}x + c_{i,3}x^2 + ... + c_{i,i-1}x^{i-2} + c_{i,i}x^{i-1}, i = 1, 2, 3....$$

Then, we just have to multiply by x to get the desired pairwise strongly orthonormal martingales:

$$p_i(x) = c_{i,1}x + c_{i,2}x^2 + c_{i,3}x^3 + ... + c_{i,i-1}x^{i-1} + c_{i,i}x^i, i = 1, 2, 3....$$

We now see that:

$$H_t^{(i)} = p_i(Y^{(i)}).$$

An important result in Nualart and Schoutens [NS00] is the predictable representation property:

Theorem 3.2.1 *Let $F \in L^2(\Omega, \mathbb{F}_T, \mathbb{P})$. Then F has a representation of the form:*

$$F = \mathbb{E}[F] + \sum_{j=1}^{\infty} \int_0^T \phi_t^{(j)}\, \mathrm{d}H_t^{(j)} \;\; \textit{where } \phi_t^{(j)} \tag{3.5}$$

are predictable processes such that

$$\mathbb{E}[\int_0^T \sum_{j=1}^{\infty} |\phi_t^{(j)}|^2\, \mathrm{d}t] < \infty. \tag{3.6}$$

Consider the Backward Stochastic Differential Equation (BSDE):

$$-dY_t = b(t, Y_{t^-}, Z_t)\, \mathrm{d}t - \sum_{i=0}^{\infty} Z_t^{(i)} dH_t^{(i)}, Y_T = \xi \tag{3.7}$$

where $H_t^{(i)}$ is the orthonormalized Teugel martingale of order i associated with the Lévy process X, $b : \Omega \times [0, T] \times \mathbb{R} \times M_T^2(l^2) \to \mathbb{R}$ is a measurable function and uniformly Lipschitz in the first two components and $\xi \in L_T^2$.

Consider the particular case of a BSDE:

$$dY_t = \sum_{i=0}^{\infty} Z_t^{(i)} dH_t^{(i)}, Y_T = h\,(X_T) \tag{3.8}$$

Let $f(\tau, x)$ be the solution of the following PIDE:

$$\frac{\partial f}{\partial \tau} = c\frac{\partial f}{\partial x} + \int_{\mathbb{R}} \left[f\,(\tau, x+y) - f\,(\tau, x) - y\frac{\partial f}{\partial x} \right] \nu\,(\,\mathrm{d}y)\,,$$

$$f\,(0, x) = h\,(x) \tag{3.9}$$

where $c = r + \gamma + \int_{|y|\geq 1} y\nu\,(\,\mathrm{d}y) = a + \int_{|y|\geq 1} y\nu\,(\,\mathrm{d}y)\,.$

Defining $g(t,x) = f(\tau,x)$, where $\tau = T-t$, we obtain from (3.9):

$$\frac{\partial g}{\partial t} + c\frac{\partial g}{\partial x} + \int_{\mathbb{R}} \left[g(t,x+y) - g(t,x) - y\frac{\partial g}{\partial x} \right] \nu(\mathrm{d}y) = 0,$$
$$g(T,x) = h(x) \tag{3.10}$$

If g is sufficiently smooth, then by applying the Itô formula to $g(t,X_t)$ we obtain the following probabilistic representation for the case of a Lévy process given by $X_t = (r+\gamma)t + J_t = a + J_t$,where J_t is a pure jump process. For a detailed proof of this proposition see [NS01].

Proposition 3.2.1 *Assume* $\sigma = 0$ *and* $\exists\lambda > 0$ *such that*

$$\int_{|x|>1} e^{\lambda|x|}\nu(dx) < \infty.$$

If $g \in C^{1,2}$ *is a classical solution of* (3.10) *and* $\frac{\partial g}{\partial x}$ *and* $\frac{\partial^2 g}{\partial x^2}$ *are bounded by a polynomial function of* x*, uniformly in* t*, then the unique adapted solution of* (3.8) *is given by*

$$Y_t = g(t,X_t),$$

where

$$Z_t^1 = \int_{\mathbb{R}} \left[g(t,X_{t^-}+y) - g(t,X_{t^-}) - y\frac{\partial g}{\partial x}(t,X_{t^-}) \right] p_1(y)\nu(\mathrm{d}y) + \frac{\partial g}{\partial x}(t,X_{t^-}) \left(\int_{\mathbb{R}} y^2 \nu(\mathrm{d}y) \right)^{1/2},$$

$$Z_t^i = \int_{\mathbb{R}} \left[g(t,X_{t^-}+y) - g(t,X_{t^-}) - y\frac{\partial g}{\partial x}(t,X_{t^-}) \right] p_i(y)\nu(\mathrm{d}y), i \geq 2$$

and $g(t,x) = \mathbb{E}[h(X_T)|X_t = x]$.

The probabilistic representation

$$g(t,x) = \mathbb{E}[h(X_T)|X_t = x] \tag{3.11}$$

obtained in the previous proposition is a Feynman-Kač formula for the solution of the PIDE (3.10).

Sketch of the proof. We can apply Itô's formula for processes with jumps, presented in Proposition 8.19 of [CT04], to $g(s,X_s)$ from $s=t$ to $s=T$:

$$\begin{aligned} g(T,X_T) = g(t,X_t) &+ \int_t^T \frac{\partial g}{\partial t}(s,X_{s^-})\,\mathrm{d}s + \int_t^T \frac{\partial g}{\partial x}(s,X_{s^-})\,\mathrm{d}X_s \\ &+ \sum_{t<s\leq T} [g(s,X_s) - g(s,X_{s^-}) - \frac{\partial g}{\partial x}(s,X_{s^-})\Delta X_s]. \end{aligned} \tag{3.12}$$

Making use of Lemma 5 in [NS00] and applying it to $h(s,y) = g(s,X_s) - g(s,X_{s^-}+y) - \frac{\partial g}{\partial x}(s,X_{s^-})y$ we get,

$$\begin{aligned} g(T,X_T) = g(t,X_t) &+ \int_t^T \frac{\partial g}{\partial t}(s,X_{s^-})\,\mathrm{d}s + \int_t^T \frac{\partial g}{\partial x}(s,X_{s^-})\,\mathrm{d}X_s \\ &+ \sum_{i=1}^{\infty} \int_t^T \int_{\mathbb{R}} (g(s,X_s) - g(s,X_{s^-}+y) - \frac{\partial g}{\partial x}(s,X_{s^-})y) p_i(y)\nu(\mathrm{d}y)\,\mathrm{d}H_s^{(i)} \\ &+ \int_t^T (\int_{\mathbb{R}} g(s,X_s) - g(s,X_{s^-}+y) - \frac{\partial g}{\partial x}(s,X_{s^-})y\nu(\mathrm{d}y))\,\mathrm{d}s. \end{aligned} \tag{3.13}$$

But $X_t = Y_t^{(1)} + t\mathbb{E}[X_1] = (\int_{\mathbb{R}} y^2\nu(\,\mathrm{d}y))^{1/2} H_t^{(1)} + t(a + \int_{|y|\geq 1} y\nu(\,\mathrm{d}y))$, so

$$\begin{aligned} h(X_T) = g(t,X_t) &+ \int_t^T [\frac{\partial g}{\partial t}(s,X_{s^-}) + \int_{\mathbb{R}} (g(s,X_s) - g(s,X_{s^-}+y) - \frac{\partial g}{\partial x}(s,X_{s^-})y)\nu(\,\mathrm{d}y) \\ &+ (a + \int_{|y|\geq 1} y\nu(\,\mathrm{d}y))\frac{\partial g}{\partial x}(s,X_{s^-})]\,\mathrm{d}s + \int_t^T \frac{\partial g}{\partial x}(s,X_{s^-})(\int_{\mathbb{R}} y^2\nu(\,\mathrm{d}y))^{1/2}\,\mathrm{d}H_s^{(1)} \\ &+ \sum_{i=1}^{\infty}\int_t^T (\int_{\mathbb{R}} (g(s,X_s) - g(s,X_{s^-}+y) - \frac{\partial g}{\partial x}(s,X_{s^-})y)p_i(y)\nu(\,\mathrm{d}y))\,\mathrm{d}H_s^{(i)}. \end{aligned} \tag{3.14}$$

Then because $g(t,x)$ solves (3.9) and taking expectations in (3.14), we get:

$$g\,(t,x) = \mathbb{E}\left[h\,(X_T)\,|X_t = x\right].$$

■

The next example shows how to perform the orthonormalization procedure described above and presents the Feynman-Kač formula for a pure jump process.

Example 3.2.2 *Consider the case where we have the sum of two compensated Poisson processes, $X_t = N_t^1 + N_t^2$ where $N_t^1 = N_t - \lambda_1 t$ and $N_t^2 = N_t - \lambda_2 t$, with Lévy measure $\nu(dx) = (\lambda_1+\lambda_2)\delta_1(x)\,\mathrm{d}x$. Then performing a orthonormalization procedure we get*

$$\begin{aligned} \psi_0 = 1 \Rightarrow q_0 &= \frac{1}{(\int_{\mathbb{R}} x^2\nu(dx))^{1/2}} = \frac{1}{(\int_{\mathbb{R}} x^2(\lambda_1+\lambda_2)\delta_1(x)\,\mathrm{d}x)^{1/2}} = \frac{1}{(\lambda_1+\lambda_2)^{1/2}} \\ \psi_1 = x + a_{1,0}q_0 &\Rightarrow \psi_1 = x - \langle x, q_0\rangle\, q_0 = x - \int_{\mathbb{R}} x\frac{1}{\sqrt{\lambda_1+\lambda_2}}(\lambda_1+\lambda_2)x^2\delta_1(x)\frac{1}{\sqrt{\lambda_1+\lambda_2}}\,\mathrm{d}x \\ &= x - 1 \Rightarrow \psi_1 = 0 \Rightarrow q_1 = 0. \end{aligned}$$

By recurrence we get that $q_i = 0, i = 1,2,3,\ldots$. Then in terms of previous notation $p_1(x) = \frac{x}{\sqrt{\lambda_1+\lambda_2}}$ and $p_i(x) = 0, i = 2,3,\ldots$, which implies

$$H_t^{(1)} = \frac{1}{\sqrt{\lambda_1+\lambda_2}}X_t \text{ and } H_t^{(i)} = 0, i = 2,3,\ldots. \tag{3.15}$$

Then, by Proposition (3.2.1)

$$Y_t = h\,(X_T) - \int_t^T Z_s^{(1)} dH_s^{(1)},$$

where

$$Z_t^{(1)} = [g(t,x+1) - g(t,x) - \frac{\partial g}{\partial x}]\sqrt{\lambda_1+\lambda_2} + \frac{\partial g}{\partial x}\sqrt{\lambda_1+\lambda_2} = [g(t,x+1) - g(t,x)]\sqrt{\lambda_1+\lambda_2}.$$

Then,

$$\begin{aligned} Y_t &= h\,(X_T) - \int_t^T [g(t,X_{s^-}+1) - g(t,X_{s^-})]\sqrt{\lambda_1+\lambda_2}\frac{1}{\sqrt{\lambda_1+\lambda_2}}\,\mathrm{d}X_s \Leftrightarrow \\ Y_t &= h\,(X_T) - \int_t^T [g(t,X_{s^-}+1) - g(t,X_{s^-})]\,\mathrm{d}X_s. \end{aligned}$$

Moreover,

$$g(t,x) = \mathbb{E}[h\,(X_T)\,|X_t = x]. \tag{3.16}$$

Notice that in Proposition 3.2.1, g is assumed to be smooth and its derivatives have to be bounded by a polynomial function of x, uniformly in t. However, these conditions are rarely satisfied in applications.

Example 3.2.3 *Consider an European call option with payoff function $H(x) = (x-1)^+$ and strike price $K = 1$. We see that the first derivative of the payoff function has a discontinuity at $x = 1$:*

$$H^{'}(x) = \begin{cases} 1 & \text{if } x > 1, \\ 0 & \text{if } x < 1 \end{cases}$$

Then, we see that the second derivative diverges at $x = 1$. So, when t tends to T and if the option is at the money $(S = K)$ the second derivative of the price function tends to the second derivative of the payoff function that diverges when $S = K$. This means that the gamma of the call option is not uniformly bounded in time.

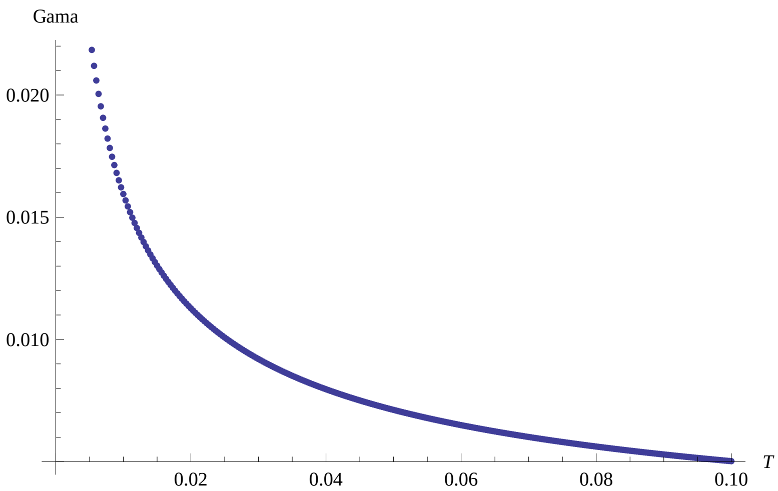


Figure 3.1: As T tends to zero the gamma of the option tends to infinity.

3.3 Option prices as classical solutions of PIDEs

3.3.1 European Options

Consider an European option with maturity T and payoff $H(S_T)$.Assume that the payoff function is a Lipschitz function

$$|H(x) - H(y)| \leq c\,|x-y| \tag{3.17}$$

for some $c > 0$. As we already know, the value of that option at time t is :

$$C(t, S_t) = \mathbb{E}\left[e^{-r(T-t)}H(S_T)\,|S_t = S\right] = e^{-r(T-t)}\mathbb{E}\left[H\left(Se^{r(T-t)+X_{T-t}}\right)\right]. \tag{3.18}$$

We will assume that $\widehat{S}_t = e^{X_t}$ is a square integrable martingale

$$\int_{|x|>1} e^{2y}\nu(dy) < \infty. \tag{3.19}$$

Then the dynamics of $\widehat{S}_t$ is given by:

$$\frac{\mathrm{d}\widehat{S}_t}{\widehat{S}_{t^-}} = \sigma\,\mathrm{d}W_t + \int_{\mathbb{R}} (e^x - 1)\,\widetilde{J}_X\,(dt, dx)\,,\ \sup_{t\in[0,T]} \mathbb{E}\left[\widehat{S}_t^2\right] < \infty. \tag{3.20}$$

The proofs of the following propositions are presented in [Vol05] and are shown in greater detail in the appendix. These propositions will be needed to prove the Proposition 3.3.3.

Proposition 3.3.1 *Let the payoff function H satisfy the Lipschitz condition* (3.17)*. Then the forward value of an European option defined by* (3.2)*, $f(\tau, x) = \mathbb{E}[H(Se^{x+r\tau+X_\tau})]$, is continuous on $[0, T] \times \mathbb{R}$.*

Proposition 3.3.2 *Let h be a measurable function with polynomial growth at infinity: $\exists p > 0, |h(x)| \le C(1 + x^p)$. If*

$$\sigma > 0 \quad \text{or} \quad \exists\beta \in (0,2) \quad \text{such} \quad \text{that} \quad \liminf_{\epsilon\to 0} \frac{1}{\epsilon^{2-\beta}} \int_{-\epsilon}^{\epsilon} |x|^2\,\nu\,(\,\mathrm{d}x) > 0 \tag{3.21}$$

and

$$\forall n \ge 0, \int_{|y|>1} |y|^n \nu(\,\mathrm{d}y) < \infty, \tag{3.22}$$

Then, $f(\tau, x) = \mathbb{E}[h(x + r\tau + X_\tau)]$ belongs to $C^\infty((0, T] \times \mathbb{R})$.

The proof of this proposition, following the proof given in Voltchkova [CV05b], is presented here in greater detail.

Proposition 3.3.3 *Consider the exponential Lévy model $S_t = S_0e^{rt+X_t}$ where the Lévy process X verifies* (3.19) *and* (3.21)*. Then the value of a European option with terminal payoff $H(S_T)$ (satisfying* (3.17)*) given by*

$$C : [0,T] \times (0,\infty) \to \mathbb{R}, (t,S) \to C\,(t,S) = \mathbb{E}\left[e^{-r(T-t)}H\,(S_T)\,|S_t = S\right] \tag{3.23}$$

is continuous on $[0,T] \times (0,\infty)$, C^∞ on $(0,T) \times (0,\infty)$ and satisfies the integro-differential equation:

$$\begin{aligned} &\frac{\partial C}{\partial t}\,(t,S) + rS\frac{\partial C}{\partial S}\,(t,S) + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}\,(t,S) - rC\,(t,S) + \\ &+ \int \left[C\,(t,Se^y) - C\,(t,S) - S\,(e^y - 1)\,\frac{\partial C}{\partial S}\,(t,S)\right]\nu\,(\,\mathrm{d}y) = 0; \end{aligned} \tag{3.24}$$

on $[0,T) \times (0,\infty)$ with the terminal condition:

$$C\,(T,S) = H\,(S)\ \ ,\forall S > 0 \tag{3.25}$$

Proof. By Proposition 3.3.1 we know that $C\,(t,S) = e^{r\tau}f(\tau,x)$ is continuous on $[0,T] \times \mathbb{R}$ and by Proposition 3.3.2, $C(t,S_t) \in C^\infty((0,T) \times (0,\infty))$.

It remains to prove that $C\,(t,S)$ satisfies (3.24).

The risk neutral dynamics of S_t under $\mathbb{Q}$ is given by

$$\mathrm{d}S_t = rS_{t^-}\,\mathrm{d}t + \sigma S_{t^-}\,\mathrm{d}W_t + \int_{\mathbb{R}} (e^x - 1)S_{t^-}\widetilde{J}_X\,(dt,dx)\,.$$

Applying the Itô formula to $\widehat{C}_t = e^{-rt}C(t,S_t)$ where $S_t = e^{rt+X_t}$ we get (see Proposition 8.18 of [CT04])

$$\begin{aligned}
\mathrm{d}(e^{-rt}C(t,S_t)) = e^{-rt}(-rC(t,S_{t^-})\,\mathrm{d}t + \frac{\partial C}{\partial t}(t,S_{t^-})\,\mathrm{d}t + \frac{\sigma^2}{2}S_{t^-}^2\frac{\partial^2 C}{\partial S^2}(t,S_{t^-})\,\mathrm{d}t \\
+ \frac{\partial C}{\partial S}(t,S_{t^-})\,\mathrm{d}S_t + \int_{\mathbb{R}}(C(t,y+S_{t^-}) - C(t,S_{t^-}) - y\frac{\partial C}{\partial S}(t,S_{t^-}))\widetilde{J}_S\,(dt,dy)).
\end{aligned}$$

simplifying and plugging in the dynamics for S_t we get:

$$\begin{aligned}
\mathrm{d}\widehat{C}_t &= e^{-rt}\frac{\partial C}{\partial S}(t,S_{t^-})\sigma S_{t^-}\,\mathrm{d}W_t + e^{-rt}\int_{\mathbb{R}}(C(t,S_{t^-}e^x) - C(t,S_{t^-}))\widetilde{J}_X\,(dt,dx) \\
&\quad + e^{-rt}(-rC(t,S_{t^-}) + \frac{\partial C}{\partial t}(t,S_{t^-}) + \frac{\sigma^2}{2}e^{-rt}S_{t^-}^2\frac{\partial^2 C}{\partial S^2}(t,S_{t^-}) + rS_{t^-}\frac{\partial C}{\partial S}(t,S_{t^-}) \\
&\quad + \int_{\mathbb{R}}(C(t,S_{t^-}e^x) - C(t,S_{t^-}) - S_{t^-}(e^x-1)\frac{\partial C}{\partial S}(t,S_{t^-}))\nu(\,\mathrm{d}x))\ \mathrm{d}t \\
&= b(t)\,\mathrm{d}t + \,\mathrm{d}M_t.
\end{aligned}$$

where

$$\begin{aligned}
b(t) &= -rC(t,S_{t^-}) + \frac{\partial C}{\partial t}(t,S_{t^-}) + \frac{\sigma^2}{2}e^{-rt}S_{t^-}^2\frac{\partial^2 C}{\partial S^2}(t,S_{t^-}) + rS_{t^-}\frac{\partial C}{\partial S}(t,S_{t^-}) \\
&\quad + \int_{\mathbb{R}}(C(t,S_{t^-}e^x) - C(t,S_{t^-}) - S_{t^-}(e^x-1)\frac{\partial C}{\partial S}(t,S_{t^-}))\nu(\,\mathrm{d}x), \\
M_t &= \int_0^T e^{-rt}\frac{\partial C}{\partial S}(t,S_{t^-})\sigma S_{t^-}\,\mathrm{d}W_t + \int_0^T e^{-rt}\int_{\mathbb{R}}(C(t,S_{t^-}e^x) - C(t,S_{t^-}))\widetilde{J}_X\,(dt,dx)\,.
\end{aligned}$$

It remains to prove that M_t is a martingale, because by proposition 8.9 of [CT04], if $\widehat{C}_t - M_t = \int_0^t b(s)\,\mathrm{d}s$ is a continuous martingale with finite variation paths then $\int_0^t b(s)\,\mathrm{d}s = X_0$ a.s, which implies that b(t)=0 a.s.

In order for $\int_0^t e^{-rt}\int_{\mathbb{R}}(C(t,S_{t^-}e^x) - C(t,S_{t^-}))\widetilde{J}_X\,(dt,dx)$ to be a martingale we have to show that:

$$\mathbb{E}[\int_0^T e^{-2rt}\int_{\mathbb{R}}(C(t,S_{t^-}e^x) - C(t,S_{t^-}))^2\nu(\,\mathrm{d}x)\,\mathrm{d}t] < \infty.$$

Then, by the Lipschitz condition

$$\mathbb{E}[\int_0^T e^{-2rt}\int_{\mathbb{R}}(C(t,S_{t^-}e^x) - C(t,S_{t^-}))^2\nu(\,\mathrm{d}x)\,\mathrm{d}t] \le \mathbb{E}[\int_0^T e^{-2rt}\int_{\mathbb{R}}c^2S_{t^-}^2(e^x-1)^2\nu(\,\mathrm{d}x)\,\mathrm{d}t].$$

Moreover,

$$\begin{aligned}
c^2\int_{\mathbb{R}}(e^x-1)^2\nu(\,\mathrm{d}x) &= c^2\int_{|x|\le 1}(e^x-1)^2\nu(\,\mathrm{d}x) + c^2\int_{|x|>1}(e^x-1)^2\nu(\,\mathrm{d}x) \\
&\le \widetilde{k}^2\int_{|x|\le 1}|x|^2\nu(\,\mathrm{d}x) + c^2\int_{|x|>1}(e^x-1)^2\nu(\,\mathrm{d}x) \\
&= \widetilde{k}^2\int_{|x|\le 1}|x|^2\nu(\,\mathrm{d}x) + c^2\int_{|x|>1}(e^{2x}+1-2e^x)\nu(\,\mathrm{d}x) \\
&\le \widetilde{k}^2\int_{|x|\le 1}|x|^2\nu(\,\mathrm{d}x) + \widetilde{k}^2\int_{|x|>1}(e^{2x}+1)\nu(\,\mathrm{d}x)
\end{aligned}$$

for some $\widetilde{k}$ sufficiently big.

Then,

$$\begin{aligned}
&\mathbb{E}[\int_0^T e^{-2rt}\int_{\mathbb{R}} c^2 S_{t^-}^2\,(e^x-1)^2\,\nu(\,\mathrm{d}x)\,\mathrm{d}t] \\
&\leq \mathbb{E}[\int_0^T S_{t^-}^2 e^{-2rt}\left(\widetilde{k}^2\int_{|x|\leq 1}|x|^2\nu(\,\mathrm{d}x)+\widetilde{k}^2\int_{|x|>1}(e^{2x}+1)\nu(\,\mathrm{d}x)\right)\,\mathrm{d}t] \\
&= \widetilde{k}^2\left(\int_{|x|\leq 1}|x|^2\nu(\,\mathrm{d}x)+\int_{|x|>1}(e^{2x}+1)\nu(\,\mathrm{d}x)\right)\mathbb{E}[\int_0^T S_{t^-}^2 e^{-2rt}\,\mathrm{d}t] \\
&= \widetilde{k}^2\left(\int_{\mathbb{R}} 1\wedge|x|^2\nu(\,\mathrm{d}x)+\int_{|x|>1}e^{2x}\nu(\,\mathrm{d}x)\right)\int_0^T \mathbb{E}[S_{t^-}^2]e^{-2rt}\,\mathrm{d}t<\infty.
\end{aligned}$$

Then $\int_0^t e^{-rt}\int_{\mathbb{R}}(C(t,S_{t^-}e^x)-C(t,S_{t^-}))\widetilde{J}_X\,(dt,dx)$ is a square integrable martingale.

It remains to prove that $\int_0^T e^{-rt}\frac{\partial C}{\partial S}(t,S_{t^-})\sigma S_{t^-}\,\mathrm{d}W_t$ is also a martingale, such that M_t is a martingale.

$$\begin{aligned}
\mathbb{E}[\int_0^T e^{-2rt}(\frac{\partial C}{\partial S}(t,S_{t^-})\sigma S_{t^-})^2\,\mathrm{d}t] &\leq \mathbb{E}[\int_0^T e^{-2rt}\left\|\frac{\partial C}{\partial S}(t,S_{t^-})\right\|_{L^\infty}^2\sigma^2 S_{t^-}^2\,\mathrm{d}t] \\
&\leq c^2\sigma^2\int_0^T e^{-2rt}\mathbb{E}[S_{t^-}^2]\,\mathrm{d}t<\infty,
\end{aligned}$$

because if C is Lipschitz, then $\frac{\partial C}{\partial S}(t,S_{t^-})\in L^\infty$. ■

The condition (3.21) holds for all jump-diffusion models with Brownian component or for processes with Lévy densities with behavior near zero as $\nu(x)\sim\frac{c}{x^{1+\beta}}$ with $\beta>0$. This condition is not satisfied for the Generalized Hyperbolic model or in particular for the Variance Gamma model. The next example shows that if we do not impose any conditions on a given Lévy triplet, then the function that represents the price of a binary option is not smooth.

Example 3.3.1 *Consider the Generalized Hyperbolic model and for simplicity assume $\delta=0$. Then the density function becomes*

$$p_t(x)=C|x-\mu|^{\frac{t}{\kappa}-\frac{1}{2}}K_{|\frac{t}{\kappa}-\frac{1}{2}|}(\alpha|x-\mu|)e^{\beta(x-\mu)}.$$

Notice that, when

$$z\to 0\ ,\ K_v(z)\sim\frac{1}{2}\Gamma(v)(\frac{z}{2})^{-v}\Rightarrow$$

$$\begin{aligned}
\lim_{x\to\mu}p(t,x) &= \lim_{x\to\mu}C|x-\mu|^{\frac{t}{\kappa}-\frac{1}{2}}K_{|\frac{t}{k}-\frac{1}{2}|}(\alpha|x-\mu|)e^{\beta(x-\mu)} \\
&= \lim_{x\to\mu}C|x-\mu|^{\frac{t}{\kappa}-\frac{1}{2}-|\frac{t}{\kappa}-\frac{1}{2}|}\frac{1}{2}\Gamma(\frac{t}{\kappa}-\frac{1}{2})e^{\beta(x-\mu)}=\infty
\end{aligned}$$

if and only if

$$\frac{t}{\kappa}-\frac{1}{2}-|\frac{t}{\kappa}-\frac{1}{2}|<0\Leftrightarrow 2(\frac{t}{\kappa}-\frac{1}{2})<0.$$

Then we conclude that $p(t,x)$ is locally unbounded at $x=\mu$ if $t<\frac{\kappa}{2}$.

If $0<\frac{t}{\kappa}-\frac{1}{2}-|\frac{t}{\kappa}-\frac{1}{2}|<1$ *, then* $p(t,x)\in C^0$ *but not in* C^1*.*

If $0<\frac{t}{\kappa}-\frac{1}{2}-|\frac{t}{\kappa}-\frac{1}{2}|<1$ *or* $1<2\frac{t}{\kappa}<2$ *, then* $p(t,x)\in C^0$ *but not in* C^1*.*

If $1<\frac{t}{\kappa}-\frac{1}{2}-|\frac{t}{\kappa}-\frac{1}{2}|<2$ *or* $2<2\frac{t}{\kappa}<3$ *, then* $p(t,x)\in C^1$ *but not in* C^2*.*

So by recurrence we conclude that

$$\text{if } p-1 < \frac{t}{\kappa} - \frac{1}{2} - \left|\frac{t}{\kappa} - \frac{1}{2}\right| < p \text{ or } p < 2\frac{t}{\kappa} < p+1 \text{ , then } p(t,x) \in C^{p-1} \text{ but not in } C^p.$$

So if $t \in (p\frac{\kappa}{2}, (p+1)\frac{\kappa}{2})$ *then* $p(t,x)$ *belongs to* $C^{p-1}(\mathbb{R})$ *but not in* $C^p(\mathbb{R})$ *and for* $t < \frac{\kappa}{2}$, $p(t,.)$ *is locally unbounded.*

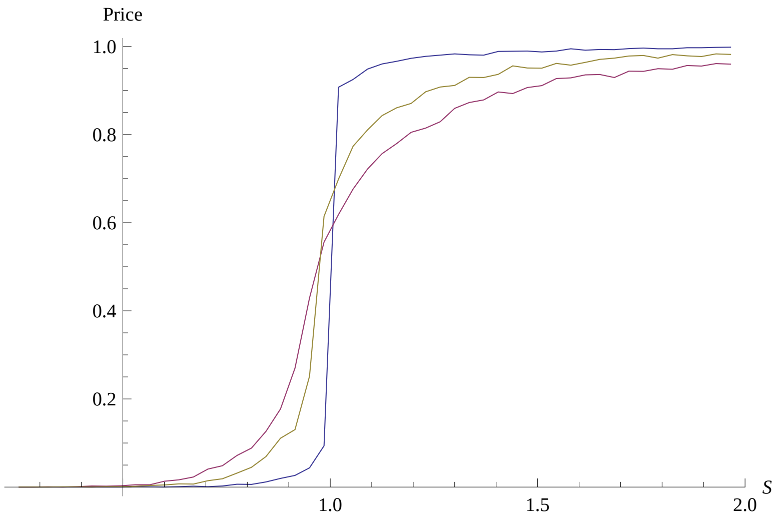


Figure 3.2: The price of a binary option is not differentiable at the money, using the Monte Carlo method, when $\mu = 0$ with $\kappa = 2, r = 0, \sigma = 0.25, \theta = -0.1$. Blue: $T = 0.1$, red: $T = 0.5$, yellow: $T = 1$.

Consider a binary option whose payoff function is given by $h(x) = 1_{x \geq l_0}$. *Its price is given by*

$$\begin{aligned} C(t,S) &= e^{-r(T-t)} E[H(S_T)|S_t = S] = e^{-r(T-t)} E[h(x + r(T-t) + X_{T-t})] \\ &= e^{-r(T-t)} E[1_{x+r(T-t)+X_{T-t} \geq l_0}] = e^{-r(T-t)} \mathbb{Q}[x + r(T-t) + X_{T-t} \geq l_0] \\ &= e^{-r(T-t)} \mathbb{Q}[X_{T-t} \geq l_0 - r(T-t) - x] = \int_d^\infty p(t,x)\, \mathrm{d}x. \end{aligned}$$

where $d = l_0 - r(T-t) - x$. *Then for* $t < \frac{k}{2}$ *the binary option is continuous but is not differentiable.*

3.3.2 Barrier Options

We now present the result without proof, analogous to Proposition 3.3.3. It tells us that the price function of a barrier option is smooth enough if and only if it satisfies a PIDE. For a full detailed proof of this proposition see [Vol05].

Proposition 3.3.4 *Consider $S_t = S_0 e^{rt+X_t}$ where the Lévy process X verifies (3.19). Let $\theta_t = \inf\{s \geq t | S_t \notin (L,U)\}$ where $0 \leq L < U \leq \infty$ and suppose that $H \geq 0$ and $\exists N > 0 : H(S) \leq N(1+S)$. Define*

$$C_b(t,S) = e^{-r(T-t)} \mathbb{E}[H(S_T) 1_{T<\theta_t} | S_t = S], \tag{3.26}$$

as the value of a knock-out option, where $C_b(t,S) \in C^{1,2}([0,T) \times (L,U))$. Then it satisfies the integro-differential equation:

$$\frac{\partial C_b}{\partial t}(t,S) + rS\frac{\partial C_b}{\partial S}(t,S) + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C_b}{\partial S^2}(t,S) - rC_b(t,S) +$$

$$+\int \left[C_b(t, Se^y) - C_b(t,S) - S(e^y - 1)\frac{\partial C_b}{\partial S}(t,S) \right] \nu(\,\mathrm{d}y) = 0; \tag{3.27}$$

on $[0,T) \times (L,U)$ with the conditions:

$$C_b(T,S) = H(S) \ \forall S \in (L,U), \tag{3.28}$$

$$C_b(t,S) = 0 \ \forall S \notin (L,U). \tag{3.29}$$

Conversely, every solution of (3.27) – (3.29) belonging to $C^{1,2}([0,T) \times (L,U))$ has the stochastic representation given by (3.26).

Before we study the continuity of barrier option prices we will need the definition of first passage process: Let $\{Y_t\}$ be a Lévy process defined by $Y_t = rt + X_t$. Finally set $M_t = \sup_{0 \leq s \leq t} Y_s$. Following the notation of Sato [Sat99], we define

$$R_x = \inf\{s \geq 0 | Y_s > x\}, R_x^- = \inf\{s \geq 0 | -Y_s > x\},$$

$$R_x^{''} = \inf\{s \geq 0 | Y_s \vee Y_{s^-} \geq x\}.$$

We know that $\{R_x, x \geq 0\}$ is a process with non-decreasing paths, so we can define $R_{x^-}(\omega) = \lim_{\epsilon \to 0} R_{x-\epsilon}(\omega)$. As for the right continuity, since Y_t is right-continuous, R_x is also right-continuous. Following the terminology of Voltchkova:

Definition 3.3.2 *Consider a Lévy process Y_t with triplet (σ, γ, ν).*

If $\sigma = 0$ and $\nu(\mathbb{R}) < \infty$, then Y_t is of type A (Compound Poisson).

If $\sigma = 0$, $\nu(\mathbb{R}) = \infty$ and $\int_{|x|\leq 1} |x|\nu(\,\mathrm{d}x) < \infty$, then Y_t is of type B (finite variation,infinite activity).

If $\sigma > 0$ or $\int_{|x|\leq 1} |x|\nu(\,\mathrm{d}x) = \infty$, then Y_t is of type C (infinite variation) .

In order to prove the continuity of barrier option prices we need some properties of the process $\{R_x\}$.

The first lemma is an extension of the Lemma 3.5.3 presented in [Vol05], in the sense that also applies to Lévy processes of type A. The second and third lemmas are presented in [Vol05].

Lemma 3.3.3 *If $\{Y_t\}$ is of type B or C or A with $\gamma \neq 0$ then:*

$$\forall x > 0, \mathbb{Q}[R_{x^-} = R_x] = 1. \tag{3.30}$$

Proof.

Introducing

$$\Omega_1 = \{\omega \in \Omega : R_{x^-} < R_x\}, \Omega_2 = \left\{\omega \in \Omega : R_x^{''} = R_x\right\}.$$

Define $R_{x'} = \inf\{s \geq 0 | Y_s \geq x\}$ By Lemma 49.6 of [Sat99] we have that, for any $x > 0, \mathbb{P}[R_x = R_{x'} = R_x^{''}] = 1$, because Y_t is non zero and is not Compound Poisson process, which means that is of type B or C or A with $\gamma \neq 0$. Then $\mathbb{Q}[\Omega_2] = 1$. In order that, $\mathbb{Q}[R_{x^-} = R_x] = 1$ we must have $\mathbb{Q}[\Omega_1] = 0$, because we always have $R_{x^-} \leq R_x$. So we have to prove that $\mathbb{Q}[R_{x^-} < R_x] = \mathbb{Q}[R_{x^-} < R_x^{''}] = \mathbb{Q}[\Omega_1 \cap \Omega_2] = 0$.

By contradiction, suppose that $\exists \omega \in \Omega_1 \cap \Omega_2 \Rightarrow \omega \in \Omega_1, \omega \in \Omega_2$.Then,

$$\exists u \geq 0, R_{x^-} = u \tag{3.31}$$

$$\exists u < t, t = R_x^{''}. \tag{3.32}$$

By definition of $R_{x^-} = \lim_{\epsilon \to 0}, R_{x-\epsilon}$ and because $R_{x^-} = u$ we get,

$$\forall_{\delta>0}, \exists_{\eta>0} : |\epsilon| < \eta \Rightarrow u - \delta < R_{x-\epsilon} < u + \delta.$$

$$\forall_{\epsilon>0}, \forall_{\delta>0}, \exists s < u + \delta : Y_s > x - \epsilon.$$

Now, choose $\epsilon_n = \delta_n = \frac{1}{n} \to 0$.Then,

$$\exists s_n \forall n : s_n < u + \frac{1}{n}, Y_{s_n} > x - \frac{1}{n}.$$

Because $\{s_n\}$ is bounded, there is a convergent subsequence $s_{n_k} \uparrow s_0$ with $s_0 \leq u < t$. This means that,

$$Y_{s_n} > x - \frac{1}{n} \Rightarrow Y_{s_0^-} \geq x.$$

and if $s_{n_k} \downarrow s_0$ with $s_0 \leq u < t$,then

$$Y_{s_n} > x - \frac{1}{n} \Rightarrow Y_{s_0} \geq x.$$

Then, $Y_{s_0^-} \vee Y_{s_0} \geq x$. But this contradicts (3.32) because it implies that $\forall s < t$, $Y_{s^-} \vee Y_s < x$. Then $\Omega_1 \cap \Omega_2 = \emptyset$. ■

Lemma 3.3.4 *If $\{Y_t\}$ is of type B with $R_0 = 0$ a.s or of type C, then:*

$$\forall x > 0, \forall t \geq 0, \mathbb{Q}[R_x = t] = 0. \tag{3.33}$$

Lemma 3.3.5 *If $\{Y_t\}$ is of type B or C, then $\forall x > 0, \forall t \geq 0$:*

$$\mathbb{Q}[R_x \leq t < R_{x+\epsilon}] \to 0, \tag{3.34}$$

$$\mathbb{Q}[R_{x-\epsilon} \leq t < R_x] \to 0, \tag{3.35}$$

when $\epsilon \to 0$.If we have also $R_0 = 0$ a.s, then (3.34) is satisfied for $x = 0, t > 0$.

The next proposition shows that the up-and-out option is continuous. The sketched proof of this proposition is shown in [Vol05] and a more detailed version is shown in the appendix.

Proposition 3.3.5 *Let Y_t be of type B or C with $R_0 = 0$ a.s. Suppose that $H : (0, U) \to [0, \infty)$ is Lipschitz:*

$$\forall S_1, S_2 \in (0, U), |H(S_1) - H(S_2)| \leq k|S_1 - S_2|, \tag{3.36}$$

for some $k > 0$ and let $u = log(\frac{U}{S_0})$. Then the function $f_U(\tau, x)$ defined by

$$f_U(\tau, x) = \begin{cases} E\left[H\left(S_0 e^{x+Y_\tau}\right) 1_{\tau < R_{u-x}}\right] & \text{if } x < u, \\ 0, & \text{if } x \geq u \end{cases} \tag{3.37}$$

is continuous on $(0, T] \times \mathbb{R}$.

The following proposition gives the continuity result for the case of a down-and-out option. The proof of this proposition, similar to the previous one, can be found in [Vol05].

Proposition 3.3.6 *Let Y_t be of type B or C with $R_{0^-} = 0$ a.s. Suppose that $H : (L, \infty) \to [0, \infty)$ is Lipschitz:*

$$\forall S_1, S_2 \in (0, U), |H(S_1) - H(S_2)| \leq k|S_1 - S_2|, \tag{3.38}$$

with $L < S_0$ and let $l = log(\frac{L}{S_0})$. Then the function $f_L(\tau, x)$ defined by

$$f_L(\tau, x) = \begin{cases} E\left[H\left(S_0 e^{x+Y_\tau}\right) 1_{\tau < R^-_{x-l}}\right] & \text{if } x > l, \\ 0, & \text{if } x \leq l \end{cases} \tag{3.39}$$

is continuous on $(0, T] \times \mathbb{R}$.

Finally the continuity result of a double-barrier option with payoff $H(S_T)1_{T < inf\{t \geq 0, S_t \in (L,U)\}}$, where $L < S_0 < U$, $u = log(\frac{U}{S_0})$ and $l = log(\frac{L}{S_0})$ is presented here without proof and can be found in [Vol05].

Proposition 3.3.7 *Let Y_t be of type B or C with $R_{0^-} = 0$ and $R_0 = 0$ a.s. Suppose that $H : (L, \infty) \to [0, \infty)$ is Lipschitz:*

$$\forall S_1, S_2 \in (0, U), |H(S_1) - H(S_2)| \leq k|S_1 - S_2|, \tag{3.40}$$

with $k > 0$.Then the function $f_D(\tau, x)$ defined by

$$f_D(\tau, x) = \begin{cases} E\left[H\left(S_0 e^{x+Y_\tau}\right) 1_{\tau < R^-_{x-l} \cap R_{u-x}}\right] & \text{if } x \in (l, u), \\ 0, & \text{if } x \notin (l, u) \end{cases} \tag{3.41}$$

is continuous on $(0, T] \times \mathbb{R}$.

The results for the continuity of a up-and-out option and a down-and-out option are proven here when the Lévy process is of type A.

Proposition 3.3.8 *Suppose $\{Y_t\}$ is a Lévy process of type A with $\gamma \neq 0$, $R_0 = 0$ a.s and $\mathbb{Q}[R_x = t] = 0, \forall x \geq 0, t \geq 0, (t, x) \neq (0, 0)$. Suppose that H:$(0, U) \to (0, \infty)$ is Lipschitz.Then for every τ sufficiently small $f_u(\tau, x)$ defined by*

$$f_u(\tau, x) = \begin{cases} E\left[H\left(S_0 e^{x+Y_\tau}\right) 1_{\tau < R_{u-x}}\right] & \text{if } x < u, \\ 0 & \text{if } x \geq u \end{cases}$$

is continuous.

Proof. Considering $\tau > 0$ and $x = u$, we have by definition,

$$|f_u(\tau, u-\epsilon) - f_u(\tau, u)| = E\left[H\left(S_0 e^{u-\epsilon+Y_\tau}\right) 1_{\tau < R_\epsilon}\right] \le ME\left[1_{\tau<R_\epsilon}\right] = M\mathbb{Q}\left[\tau < R_\epsilon\right]$$

Let $\{\epsilon_n\} \to 0$ and $\Omega_n = \{\omega \in \Omega : \tau < R_{\epsilon_n}\}$, then $\{\Omega_n\}$ is a decreasing sequence: $\Omega_1 \supset \Omega_2 \supset \ldots\Omega_n \supset \ldots$, because R_x is an increasing process. Then

$$\lim_{n\to\infty} \mathbb{Q}\left[\tau < R_{\epsilon_n}\right] = \mathbb{Q}\left[\cap_{n=1}^{\infty}\Omega_n\right] = \mathbb{Q}\left[\tau < R_0\right] = 0$$

because $R_0 = 0$ a.s.

For $\tau > 0$ and $x < u$:

$$\begin{aligned}|f_u(\tau, x+\epsilon) - f_u(\tau, x)| &= \left|E\left[H\left(S_0 e^{x+\epsilon+Y_\tau}\right) 1_{\tau<R_{u-x-\epsilon}}\right] - \mathbb{E}\left[H\left(S_0 e^{x+Y_\tau}\right) 1_{\tau<R_{u-x}}\right]\right| \\ &= |E[H(S_0 e^{x+\epsilon+Y_\tau}) - H(S_0 e^{x+Y_\tau}))1_{\tau<R_{u-x-\epsilon}} \\ &\quad + H(S_0 e^{x+Y_\tau})(1_{\tau<R_{u-x-\epsilon}} - 1_{\tau<R_{u-x}})]| \\ &\le cS_0 e^{x+r\tau}\mathbb{E}[e^{Y_\tau}]|e^\epsilon - 1| + M\mathbb{Q}[R_{u-x-\epsilon} \le \tau < R_{u-x}]\end{aligned}$$

But, $|e^\epsilon - 1| \to 0$ as $\epsilon \to 0$ and

$$\mathbb{Q}[R_{u-x-\epsilon} \le \tau < R_{u-x}] \to \mathbb{Q}[R_{(u-x)^-} \le \tau < R_{u-x}] \le \mathbb{Q}[R_{(u-x)^-} \ne R_{u-x}] = 0 \tag{3.42}$$

because of lemma (3.3.3). In a similar way:

$$\begin{aligned}|f_u(\tau, x-\epsilon) - f_u(\tau, x)| &= \left|E\left[H\left(S_0 e^{x-\epsilon+Y_\tau}\right) 1_{\tau<R_{u-x+\epsilon}}\right] - \mathbb{E}\left[H\left(S_0 e^{x+Y_\tau}\right) 1_{\tau<R_{u-x}}\right]\right| \\ &\le cS_0 e^{x+r\tau}|1-e^{-\epsilon}| + M\mathbb{Q}[R_{u-x} \le \tau < R_{u-x+\epsilon}] \to 0,\end{aligned}$$

because $|1-e^{-\epsilon}| \to 0$ as $\epsilon \to 0$, and

$$\mathbb{Q}[R_{u-x} \le \tau < R_{u-x+\epsilon}] \to \mathbb{Q}[R_{u-x} = \tau] = 0. \tag{3.43}$$

The continuity in time is proven in the same way as in the proof of Proposition 3.3.5 (see the Appendix). Finally, using the triangular inequality we can prove continuity for all $(\tau, x) \in [0, T] \times (-\infty, u)$. ■

Proposition 3.3.9 *Suppose $\{Y_t\}$ is a Lévy process of type A with $\gamma \ne 0$, $R_0^- = 0$ a.s and $\mathbb{Q}[R_x^- = t] = 0, \forall x \ge 0, t \ge 0, (t,x) \ne (0,0)$.Suppose that H:$(L,\infty) \to (0,\infty)$ is Lipschitz. Then, for every τ sufficiently small $f_l(\tau, x)$ defined by*

$$f_l(\tau, x) = \begin{cases} E\left[H\left(S_0 e^{x+Y_\tau}\right) 1_{\tau<R^-_{x-l}}\right] & \text{if } x > l, \\ 0 & \text{if } x \le l\end{cases}$$

is continuous.

Proof. Considering $\tau > 0$ and by definition of the price of a down-and-out option,

$$|f_l(\tau, l+\epsilon) - f_l(\tau, l)| = E\left[H\left(S_0 e^{l+\epsilon+Y_\tau}\right) 1_{\tau<R_\epsilon^-}\right] \le CE\left[\left(1 + S_0 e^{l+\epsilon+Y_\tau}\right) 1_{\tau<R_\epsilon^-}\right] \tag{3.44}$$

$$= C\mathbb{Q}\left[\tau < R_\epsilon^-\right] + CS_0 e^{l+\epsilon} E\left[e^{Y_\tau} 1_{\tau<R_\epsilon^-}\right]. \tag{3.45}$$

Let $\{\epsilon_n\} \to 0$ and $\Omega_n = \left\{\omega \in \Omega : \tau < R_{\epsilon_n}^-\right\}$, then $\{\Omega_n\}$ is a decreasing sequence: $\Omega_1 \supset \Omega_2 \supset \ldots\Omega_n \supset \ldots$, because R_x^- is an increasing process. Then,

$$\lim_{n\to\infty} \mathbb{Q}\left[\tau < R^-_{\epsilon_n}\right] = \mathbb{Q}\left[\cap_{n=1}^{\infty}\Omega_n\right] = \mathbb{Q}\left[\tau < R^-_0\right] = 0$$

because $R^-_0 = 0$ a.s. Then, $\mathbb{Q}\left[\tau < R^-_\epsilon\right] \to 0$ as $\epsilon \to 0$.
The quantity $e^{Y_\tau} 1_{\tau < R^-_\epsilon}$ is bounded by an integrable variable e^{Y_τ} that is, $e^{Y_\tau} 1_{\tau < R^-_\epsilon} \le e^{Y_\tau}$ and converges in probability to zero because, $\forall\ \delta > 0$,

$$\mathbb{Q}\left[e^{Y_\tau} 1_{\tau < R^-_\epsilon} > \delta\right] \le \mathbb{Q}\left[\tau < R^-_\epsilon\right].$$

Then, by dominated convergence theorem,

$$\lim_{\epsilon\to 0} E\left[e^{Y_\tau} 1_{\tau < R^-_\epsilon}\right] = \lim_{\epsilon\to 0} \int_\Omega e^{Y_\tau} 1_{\tau < R^-_\epsilon}\, \mathrm{d}\mathbb{Q} = 0. \tag{3.46}$$

Then, $E\left[e^{Y_\tau} 1_{\tau < R^-_\epsilon}\right] \to 0$ as $\epsilon \to 0$. This means that $f_l(\tau, x)$ is continuous in $x = l$. The proof for all (τ, x), follows the same steps of the proof of Proposition 3.5.9 in [Vol05]. ■

The next example shows that if we do not impose any restriction on the Lévy process, then the value of a knock-out option is discontinuous for every t:

Example 3.3.6 *Let us consider the following Lévy process $X_t = N^1_t - N^2_t$ where N^1_t and N^2_t are independent Poisson processes with jump intensities λ_1 and λ_2. Assuming $r = 0$, we have $\lambda_2 = e\lambda_1$ in order for $S_t = S_0 e^{X_t}$ to be a martingale. Consider now a knock-out option with a payoff function defined by $H_T = 1_{T<\theta(S_0)}$, where $\theta(S) = inf\left\{t \ge 0 : S_0 e^{X_t} \le L\right\}$ is the first exit time if the process starts from S. We will show that the initial option value $C(0, S) = \mathbb{E}\left[1_{T<\theta(S_0)} | S_0 = S\right] = \mathbb{E}\left[1_{T<\theta(S)}\right]$ is not continuous at $S^* = Le$. Let $0 < \epsilon < S^* - L$, so that $L = L - S^* + S^* < S^* - \epsilon < S^* < S^* + \epsilon$.*

$$|C(0, S^* + \epsilon) - C(0, S^* - \epsilon)| = \left|\mathbb{E}\left[1_{\theta(S^*-\epsilon) \le T < \theta(S^*+\epsilon)}\right]\right| = \mathbb{Q}\left[\theta(S^* - \epsilon) \le T < \theta(S^* + \epsilon)\right].$$

Consider the following event $\left\{N^1_T = 0, N^2_T = 1\right\}$ of non-zero probability. Then, if S_t starts from $S^* - \epsilon$, then $S_T = (S^* - \epsilon)\, e^{-1} = (Le - \epsilon)\, e^{-1} = L - \epsilon e^{-1} < L$, which means that $T \ge \theta(S^* - \epsilon)$. On the other hand, if S_t starts from $S^* + \epsilon$, then $S_T = (S^* + \epsilon)\, e^{-1} = (Le + \epsilon)\, e^{-1} = L + \epsilon e^{-1} > L$, implying $T < \theta(S^* + \epsilon)$. Because $\left\{N^1_T = 0, N^2_T = 1\right\}$ is a possible realization of the trajectory of X_t, we have:
$\left\{\omega \in \Omega : N^1_T = 0, N^2_T = 1\right\} \subset \left\{\omega \in \Omega : \theta(S^* - \epsilon) \le T < \theta(S^* + \epsilon)\right\}$. *Then,*

$$\begin{aligned} |C(0, S^* + \epsilon) - C(0, S^* - \epsilon)| &= \mathbb{Q}\left[\theta(S^* - \epsilon) \le T < \theta(S^* + \epsilon)\right] \\ &\ge \mathbb{Q}\left[N^1_T = 0, N^2_T = 1\right] = e^{-T\lambda_1} T\lambda_2 e^{-T\lambda_2} \\ &= e^{-T\lambda_1(1+e)} T\lambda_2 > 0. \end{aligned}$$

Thus, $C(0, S)$ is discontinuous at $S = S^*$.

3.3.3 Numerical procedure

This section is dedicated to present the finite difference scheme proposed by [Vol05], used to solve numerically the PIDE whose solution allows to obtain the price of an European option when the Lévy process is the Variance Gamma process. The problem that has to be solved is:

$$\frac{\partial f}{\partial \tau} - Lf = 0, \ (\tau, x) \in [0,T] \times \mathbb{R} \tag{3.47}$$

$$f(0,x) = h(x), \ x \in \mathbb{R}, \tag{3.48}$$

where

$$Lf = -\left(\frac{\sigma^2}{2} - r\right)\frac{\partial f}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} + \int_{-\infty}^{+\infty}\left[f(\tau, x+y) - f(\tau,x) - (e^y - 1)\frac{\partial f}{\partial x}\right]\nu(\,\mathrm{d}y). \tag{3.49}$$

In order to solve this equation numerically, the domain of integration of the integral term needs to be truncated by a bounded interval and because the Variance Gamma process is a jump process of infinite activity, the small jumps of the initial Lévy process need to be approximated by a process of finite activity, namely the Brownian Motion. The Lévy process obtained has a new characteristic triplet given by $(\gamma(\epsilon), \sigma^2(\epsilon), \nu 1_{|x|>\epsilon})$,where $\sigma^2(\epsilon) = \int_{-\epsilon}^{\epsilon} y^2 \nu(\,\mathrm{d}y)$ and the drift is given by the associated martingale condition. The scheme proposed in [Vol05] is the explicit-implicit finite difference scheme. The idea is to separate Lf into two parts, the differential part Df and the integral part Jf. The operator then becomes in this case:

$$Lf(\tau,x) = Df(\tau,x) + Jf(\tau,x), \tag{3.50}$$

where

$$Df(\tau,x) = -\left(\frac{\sigma^2(\epsilon)}{2} - r + \alpha\right)\frac{\partial f}{\partial x}(\tau,x) + \frac{\sigma^2(\epsilon)}{2}\frac{\partial^2 f}{\partial x^2}(\tau,x) - \lambda f(\tau,x), \tag{3.51}$$

$$Jf(\tau,x) = \int_{B_l}^{B_r} f(\tau, x+y)\,\nu(\,\mathrm{d}y)\,1_{|y|>\epsilon}, \tag{3.52}$$

$\alpha = \int_{B_l}^{B_r}(e^y - 1)\nu(\,\mathrm{d}y)1_{|y|>\epsilon}$ and $\lambda = \int_{B_l}^{B_r}\nu(\,\mathrm{d}y)1_{|y|>\epsilon}$. The localized problem becomes:

$$\frac{\partial f}{\partial \tau} - Lf = 0, \ (\tau,x) \in [0,T] \times (-A,A) \tag{3.53}$$

$$f(0,x) = h(x), \ x \in (-A,A), \tag{3.54}$$

$$f(\tau,x) = g(\tau,x), \ x \notin (-A,A). \tag{3.55}$$

In [Vol05], it is shown that the best choice for $g(\tau,x)$ is $h(x + r\tau)$. Let $\{f_i^n\}$ be the numerical solution of the scheme proposed and define a uniform grid:

$Q_{\Delta t,\Delta x} = \left\{(\tau_n, x_i) : \tau_n = n\Delta t, n = 0,1,...M, x_i = -A + i\Delta x, i \in \mathbb{Z}, \Delta t = \frac{T}{M}, \Delta x = \frac{2A}{N}\right\}$ and choose K_l, K_r such that $[B_l, B_r] \subset [(K_l - 1/2)\Delta x, (K_r + 1/2)\Delta x]$.

Then,

$$\alpha \approx \hat{\alpha} = \sum_{j=K_l}^{K_r}(e^{y_j} - 1)\nu_j 1_{|y_j|>\epsilon}, \lambda \approx \hat{\lambda} = \sum_{j=K_l}^{K_r}\nu_j 1_{|y_j|>\epsilon}, \tag{3.56}$$

$$\int_{B_l}^{B_r} f(\tau, x_i + y)\,\nu(\,\mathrm{d}y)\,1_{|y|>\epsilon} \approx \sum_{j=K_l}^{K_r}\nu_j f_{i+j} 1_{|y_j|>\epsilon}, \tag{3.57}$$

where

$$\nu_j \approx \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x}\nu(\,\mathrm{d}y) \approx 0.5\Delta x(\nu((j-1/2)\Delta x) + \nu((j+1/2)\Delta x)) \tag{3.58}$$

$$(\frac{\partial f}{\partial x})_i \approx \begin{cases} \frac{f_{i+1}-f_i}{\Delta x} & \text{if } (\frac{\sigma^2(\epsilon)}{2} - r + \hat{\alpha}) < 0, \\ \frac{f_i - f_{i-1}}{\Delta x} & \text{if } (\frac{\sigma^2(\epsilon)}{2} - r + \hat{\alpha}) \geq 0 \end{cases} \tag{3.59}$$

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_i \approx \frac{f_{i+1} - 2f_i + f_{i-1}}{(\Delta x)^2}, \tag{3.60}$$

$$\left(\frac{\partial f}{\partial \tau}\right)_i \approx \frac{f_i^{n+1} - f_i^n}{\Delta t}. \tag{3.61}$$

Then replacing all these quantities in the problem (3.53)-(3.55) the algorithm becomes:

Initialization:

$$f_i^0 = h(x_i), i \in \{0, 1..., N\}, \tag{3.62}$$
$$f_i^0 = g(0, x_i), i \notin \{0, 1..., N\}. \tag{3.63}$$

For n=0,....M-1:

$$\frac{f_i^{n+1} - f_i^n}{\Delta t} = (D_\Delta f^{n+1})_i + (J_\Delta f^n)_i, i \in \{0, 1..., N\}, \tag{3.64}$$
$$f_i^{n+1} = g((n+1)\Delta t, x_i), i \notin \{0, 1..., N\}. \tag{3.65}$$

where

$$(D_\Delta f^{n+1}) = -\left(\frac{\sigma^2(\epsilon)}{2} - r + \hat{\alpha}\right) \frac{f_{i+1}^{n+1} - f_i^{n+1}}{\Delta x} + \frac{\sigma^2(\epsilon)}{2} \frac{f_{i+1}^{n+1} - 2f_i^{n+1} + f_{i-1}^{n+1}}{(\Delta x)^2} - \hat{\lambda} f_i^{n+1}, \tag{3.66}$$

$$(J_\Delta f^n)_i = \sum_{j=K_l}^{K_r} \nu_j f_{i+j}^n 1_{|y_j|>\epsilon}. \tag{3.67}$$

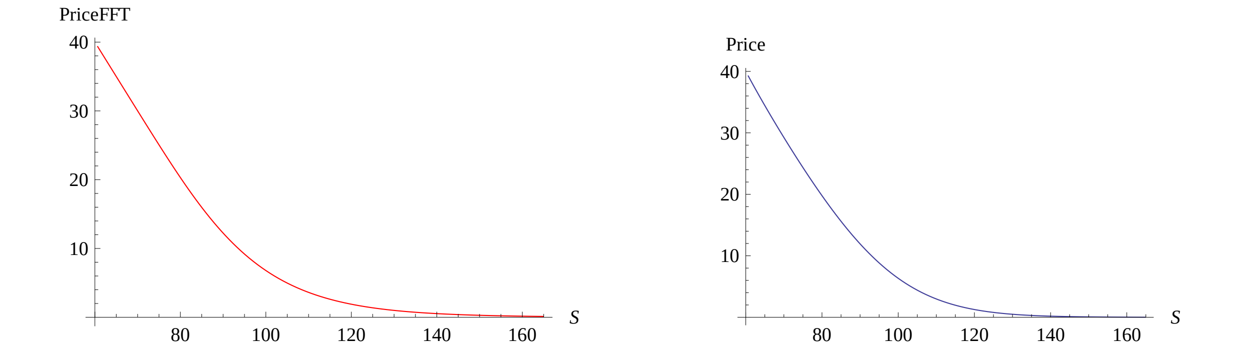


Figure 3.3: Left: The price of an European put option using Fast Fourier Transform. Right: The price of an European put option using the finite difference scheme.

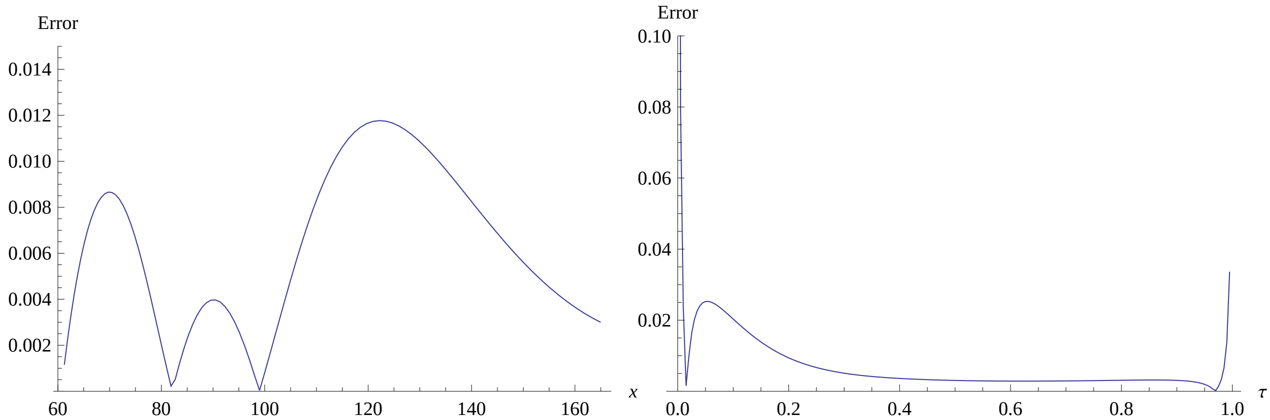


Figure 3.4: Left: The error metric in % as a function of x ,when $\tau = T$. Right: The error in % as a function of τ, when $x = 0$.

3.3.3.1 Numerical Results

In this section we briefly present a numerical experiment using the finite difference scheme described above and compare it with the widely known Fast Fourier transform method that can be used in this case for option pricing because the characteristic function of the Variance Gamma process is known analytically. The process considered was the Variance Gamma with $\theta = -0.33, \sigma = 0.12, \kappa = 0.16, r = 0, \Delta x = 0.01, \Delta t = 0.005, K = 100, A = 1, \epsilon = 0.6, T = 1$ for the case of an European put option. The price as a function of the underlying for the two methods is shown in Figure (3.3). We see that the function that represents the European put option price using the finite difference scheme is very similar to the price of an European put option using the Fast Fourier Transform. The error between the two methods for two cases is displayed in Figure (3.4) and as we can see is relatively small. In order to compare the two methods the error metric used (as in [CV05a]) was the difference in absolute value between the implied volatilities under the two methods:

$$\epsilon(\tau, x) = |\Sigma^{PIDE}(\tau, x) - \Sigma^{FFT}(\tau, x)| \text{ in } \%. \tag{3.68}$$

Chapter 4

Option prices as viscosity solutions of PIDEs

In this chapter we introduce the notion of viscosity solution in the PIDEs frame, that was first introduced by Crandall and Lions ([CL92]). We saw in Section 3.1 that if some conditions of smoothness are satisfied, then the option price function is a classical solution of the associated partial integro-differential equation. But, as we saw in the examples of pure jump processes in Section 3.2, the option price functions were not smooth. This motivates us to consider the notion of viscosity solution. As we will see in Section 4.2, if we consider more general conditions, we can express option prices , such as barrier or European options, as viscosity solutions of certain PIDEs. In this introduction we give an idea of such concept. Consider a regular solution f of the equation:

$$\frac{\partial f}{\partial \tau} - Lf - r\frac{\partial f}{\partial x} = 0 \tag{4.1}$$

where Lf is given by (2.10).

Then, if φ is a regular function such that $f - \varphi$ has a global maximum at (τ, x), we have:

$$\frac{\partial \varphi}{\partial \tau} - L\varphi - r\frac{\partial \varphi}{\partial x} \leq 0 \tag{4.2}$$

In fact, because (τ, x) is a global maximum, we have:

$$\frac{\partial (f-\varphi)}{\partial \tau}(\tau, x) = \frac{\partial (f-\varphi)}{\partial x}(\tau, x) = 0, \tag{4.3}$$

$$\frac{\partial^2 (f-\varphi)}{\partial x^2}(\tau, x) \leq 0. \tag{4.4}$$

and also $\forall(\tau, y)$,

$$f(\tau, x) - \varphi(\tau, x) \geq f(\tau, y) - \varphi(\tau, y) \Leftrightarrow f(\tau, y) - f(\tau, x) \leq \varphi(\tau, y) - \varphi(\tau, x). \tag{4.5}$$

Then,

$$0 = \frac{\partial f}{\partial \tau} - Lf - r\frac{\partial f}{\partial x} \tag{4.6}$$

$$= \frac{\partial f}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} + \left(\frac{\sigma^2}{2} - r\right)\frac{\partial f}{\partial x}(\tau, x) - \int_{\mathbb{R}}[f(\tau, x+y) - f(\tau, x) - (e^y - 1)\frac{\partial f}{\partial x}]\nu(\,\mathrm{d}y) \tag{4.7}$$

$$\geq \frac{\partial \varphi}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 \varphi}{\partial x^2} + \left(\frac{\sigma^2}{2} - r\right)\frac{\partial \varphi}{\partial x} - \int_{\mathbb{R}}[\varphi(\tau, x+y) - \varphi(\tau, x) - (e^y - 1)\frac{\partial \varphi}{\partial x}]\nu(\,\mathrm{d}y) \tag{4.8}$$

$$= \frac{\partial \varphi}{\partial \tau} - L\varphi - r\frac{\partial \varphi}{\partial x}. \tag{4.9}$$

On the other hand if φ is a regular function and (τ, x) is a globlal minimum of $f - \varphi$ we have

$$\frac{\partial \varphi}{\partial \tau} - L\varphi - r\frac{\partial \varphi}{\partial x} \geq 0. \tag{4.10}$$

We are going to see that if f satisfies (4.9) and (4.10), then f is called a viscosity solution, which is nothing but a generalization of solution. This way f doesn't need to belong to $C^{1,2}$. If f is a viscosity solution and also a function of class $C^{1,2}$, then f is also a solution in the classical sense. In fact, we could set $\varphi = f$, then $f - \varphi = 0$ and all (τ, x) is a global minimum and maximum. Then by (4.9) and (4.10) we have:

$$\frac{\partial f}{\partial \tau} - Lf - r\frac{\partial f}{\partial x} = \frac{\partial \varphi}{\partial \tau} - L\varphi - r\frac{\partial \varphi}{\partial x} = 0. \tag{4.11}$$

4.1 Definitions

Consider the following definitions that we will need to formalize the concept of viscosity solution. Let

$$USC = \{v : [0, T) \times \mathbb{R} \to \mathbb{R} : \ v \text{ is an upper semicontinuous function } \}, \tag{4.12}$$

and

$$LSC = \{v : [0, T) \times \mathbb{R} \to \mathbb{R} : \ v \text{ is a lower semicontinuous function } \}. \tag{4.13}$$

Also,

$$\mathcal{M} = \{\phi : \phi \text{ is a measurable function } \}$$

$$C_p^+([0, T] \times \mathbb{R}) = \{\phi : \phi \in \mathcal{M} \text{ and } \exists C > 0, |\phi(t, x)| \leq C(1 + |x|^p 1_{x>0})\}.$$

This way, $L\varphi$ can be defined for $\varphi \in C_p^+([0, T] \times \mathbb{R}) \cap C^2([0, T] \times \mathbb{R})$:

$$L\varphi(x) = \gamma\frac{\partial f}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} + \int_{|y|\le 1}\nu(\,\mathrm{d}y)[\varphi(x+y) - \varphi(x) - y\frac{\partial \varphi}{\partial x}(x)]$$
$$+ \int_{|y|>1}\nu(\,\mathrm{d}y)[\varphi(x+y) - \varphi(x)]. \tag{4.14}$$

The terms in (4.14) are well defined because for $\varphi \in C^2([0,T]\times\mathbb{R})$:

$$|\varphi(\tau, x+y) - \varphi(\tau, x) - y\frac{\partial\varphi}{\partial x}(\tau, x)| \le y^2 \sup_{|x|\le 1}|\varphi''(\tau,.)| \text{ for } |y|\le 1. \tag{4.15}$$

and for $\varphi \in C_p^+([0,T]\times\mathbb{R})$:

$$\int_{y>1} y^p \nu(\,\mathrm{d}y) < \infty. \tag{4.16}$$

Let $O = (a,b) \subseteq \mathbb{R}$ be an open interval, $\partial O = \{a,b\}$ the boundary of O. Consider the following initial boundary value problem on $[0,T]\times\mathbb{R}$:

$$\frac{\partial f}{\partial \tau} = Lf + r\frac{\partial f}{\partial x}, (0,T]\times O, \tag{4.17}$$
$$f(0,x) = h(x), x\in O; \tag{4.18}$$
$$f(\tau,x) = g(\tau,x), x\notin O \tag{4.19}$$

We now present the definition of viscosity solution:

Definition 4.1.1 *A function $v \in USC$ is a viscosity subsolution of* (4.17)−(4.19) *if* $\forall\varphi \in C^2([0,T]\times\mathbb{R})\cap C_p^+([0,T]\times\mathbb{R})$ *and for all global maximum point $(\tau,x)\in[0,T]\times\mathbb{R}$ of $v-\varphi$ the following properties are verified:*

$$\frac{\partial\varphi}{\partial\tau} - L\varphi - r\frac{\partial\varphi}{\partial x} \le 0, \;\; \textit{if } (\tau,x)\in(0,T]\times O \tag{4.20}$$
$$\min\left\{\frac{\partial\varphi}{\partial\tau} - L\varphi - r\frac{\partial\varphi}{\partial x}, v(\tau,x) - h(x)\right\} \le 0, \;\; \textit{if } \tau = 0, x\in\overline{O}, \tag{4.21}$$
$$\min\left\{\frac{\partial\varphi}{\partial\tau} - L\varphi - r\frac{\partial\varphi}{\partial x}, v(\tau,x) - g(\tau,x)\right\} \le 0, \;\; \textit{if } \tau\in(0,T], x\in\partial O, \tag{4.22}$$
$$v(\tau,x) \le g(\tau,x), \;\; \textit{if } x\notin\overline{O} \tag{4.23}$$

A function $v \in LSC$ is a viscosity supersolution of (4.17)−(4.19) *if* $\forall\varphi\in C^2([0,T]\times\mathbb{R})\cap C_p^+([0,T]\times\mathbb{R})$ *and for all global minimum point $(\tau,x)\in[0,T]\times\mathbb{R}$ of $v-\varphi$ the following properties are verified:*

$$\frac{\partial\varphi}{\partial\tau} - L\varphi - r\frac{\partial\varphi}{\partial x} \ge 0, \;\; \textit{if } (\tau,x)\in(0,T]\times O \tag{4.24}$$
$$\max\left\{\frac{\partial\varphi}{\partial\tau} - L\varphi - r\frac{\partial\varphi}{\partial x}, v(\tau,x) - h(x)\right\} \ge 0, \;\; \textit{if } \tau = 0, x\in\overline{O}, \tag{4.25}$$
$$\max\left\{\frac{\partial\varphi}{\partial\tau} - L\varphi - r\frac{\partial\varphi}{\partial x}, v(\tau,x) - g(\tau,x)\right\} \ge 0, \;\; \textit{if } \tau\in(0,T], x\in\partial O, \tag{4.26}$$
$$v(\tau,x) \ge g(\tau,x), \;\; \textit{if } x\notin\overline{O} \tag{4.27}$$

A function $v\in C_p^+([0,T]\times\mathbb{R})$ is called a viscosity solution of (4.17)−(4.19) *if it is simultaneously a subsolution and a supersolution. Then v is continuous on* $(0,T]\times\mathbb{R}$.

Note that the definition of viscosity solution requires that the test functions φ have second-order derivatives and the viscosity solution only has to be continuous on $[0,T]\times\mathbb{R}$.

4.2 Option prices as viscosity solutions of PIDEs

The main tool for showing uniqueness of viscosity solutions is the comparison principle presented in [Vol05].

Proposition 4.2.1 *Let $u \in USC$ and $v \in LSC$ with polynomial growth. If u is a subsolution and v is a supersolution of* (4.17) − (4.19) *with $O = \mathbb{R}$ and the function h is continuous, then $u \leq v$ on $[0, T] \times \mathbb{R}$.*

The following proposition indicates that the values of European and barrier options in a exponential Lévy model can be expressed as the viscosity solutions of (4.17) − (4.19).

Proposition 4.2.2 *Let the payoff function H verify the lipschitz condition* (3.17) *and let $h(x) = H(S_0 e^x)$ verify the condition of polynomial growth at infinity:*

$$|h(x)| \leq C(1 + |x|^p 1_{x>0}). \tag{4.28}$$

Then:

1) The forward value $f_e(\tau, x)$ of an European option defined by $f(\tau, x) = \mathbb{E}[h(x + r\tau + X_\tau)]$ is a viscosity solution of the Cauchy problem (4.17)− (4.19) *with $O = \mathbb{R}$.*

2) The forward value $f_b(\tau, x)$ of a knockout barrier option defined by (3.37), (3.39) *or* (3.41) *is a viscosity solution of the Cauchy problem* (4.17)− (4.19) *with $g = 0$.*

Notice that we do not make any kind of assumptions on the Lévy triplet of the Lévy process as in Propositions 3.3.3 and 3.3.4. The conditions imposed on the payoff function imply that the option price belongs to C_p^+, which is a requirement in the definition of a viscosity solution. The proof that this viscosity solution satisfies (4.17)− (4.19) can be found in [Vol05].

Chapter 5

Conclusions

In this dissertation we showed that if the payoff function and the Lévy process satisfy some conditions, then we can obtain the option price as a solution of a certain partial integro-differential equation. Also, if a solution of a certain PIDE is smooth enough and if the Lévy process satisfies a exponential moment condition, then we can apply the Feynman-Kač formula for option pricing in a Lévy market. In Chapter 3 we present this formula for the case of a pure jump process. Two of the possible methods that can be used to compute the option price numerically are the Fast Fourier technique and the finite difference method. In this dissertation we present the latter in the form proposed by Cont and Voltchkova [CV05a]. The numerical results showed that the finite difference scheme is a good approximation to compute the price of an European option under the Variance Gamma process. Although the Fast Fourier transform method requires less operations than the finite difference method, the latter can be applied to cases in which the characteristic function is not known in closed form and can be used to compute the price of barrier options. We could see that the price function of a binary option was not smooth when we used the Generalized Hyperbolic process. So we can not apply the results of Chapter 3, instead we use the notion of viscosity solution that allows us to consider cases in which the function that represents the option price is not a smooth function. To find such solutions, we need to find test functions that bound the solution of the problem. A similar result for the case of classical solutions is presented in Section 4.2, and shows that the only requirement is that the payoff function has polynomial growth at infinity. In this dissertation we present also a proof for the continuity of an up-and-out and down-and-out options when the Lévy process is of type A, besides the cases of type B and C presented in [Vol05] and [CV05b].

It would be interesting to study, in a future research, the continuity of barrier options, when the Lévy process is of type B with infinite activity, which is not contemplated in the propositions 3.3.5,3.3.6 and 3.3.7. It would be also interesting to study alternative numerical methods for PIDEs such as the Analytic method of lines, finite element methods and Wavelet-Galerkin methods, because they allow to compute the price of American options, unlike the finite difference methods. One of the reasons to use numerical methods for partial integro-differential equations is that they are computationally efficient in the case of single-asset options. However, in the case of three or more assets these methods become inefficient and the most used method to price American or barrier options is the Monte carlo method. So it would be also interesting to study the case when there are three or more underlying assets. The potential theory could also be an interesting topic for future research because it explores the deep connection between partial integro-differential operators and Markov processes with jumps. Another issue that could be interesting to study in the future is the hedging in incomplete markets.

Appendix A

Proofs of propositions and Numerical Code

A.1 Proof of Proposition 3.3.1.

Proof. First, we need to prove the continuity with respect to x.

$$\begin{aligned}|f(\tau,x+\Delta x)-f(\tau,x)| &= |\mathbb{E}[H(S_0e^{x+\Delta x+r\tau+X_\tau})-H(S_0e^{x+r\tau+X_\tau})]| \\ &\leq \mathbb{E}[c|S_0e^{x+\Delta x+r\tau+X_\tau})-S_0e^{x+r\tau+X_\tau}|] = c\mathbb{E}[S_0e^{x+r\tau+X_\tau}|e^{\Delta x}-1|] \\ &= cS_0e^{x+r\tau}\mathbb{E}[e^{X_\tau}]|e^{\Delta x}-1| = cS_0e^{x+r\tau}|e^{\Delta x}-1|\end{aligned}$$

because $\mathbb{E}[e^{X_\tau}]=1$ since e^{X_τ} is a martingale.

Then,

$$\lim_{\Delta x\to 0} f(\tau,x+\Delta x)-f(\tau,x)\leq \lim_{\Delta x\to 0} cS_0e^{x+r\tau}|e^{\Delta x}-1| = 0,$$

which means that $f(\tau,x)$ is continuous in x.

Second,we need to prove the continuity in τ.

$$\begin{aligned}|f(\tau+\Delta\tau,x)-f(\tau,x)| &= |\mathbb{E}[H(S_0e^{x+r(\tau+\Delta)\tau+X_{\tau+\Delta\tau}})-H(S_0e^{x+r\tau+X_\tau})]| \\ &\leq \mathbb{E}[c|S_0e^{x+r(\tau+\Delta)+X_{\tau+\Delta}})-S_0e^{x+r\tau+X_\tau}|] \\ &= cS_0e^{x+r\tau}\mathbb{E}[e^{X_\tau}]\mathbb{E}|e^{r\Delta\tau+X_{\Delta\tau}}-1|\end{aligned}$$

But,

$$\begin{aligned}\mathbb{E}|e^{r\Delta\tau+X_{\Delta\tau}}-1| &= \begin{cases}\mathbb{E}[e^{r\Delta\tau+X_{\Delta\tau}}-1] & \text{if } e^{r\Delta\tau+X_{\Delta\tau}}-1>0,\\ \mathbb{E}[1-e^{r\Delta\tau+X_{\Delta\tau}}] & \text{if } e^{r\Delta\tau+X_{\Delta\tau}}-1<0\end{cases} \\ &= \begin{cases}e^{r\Delta\tau}-1 & \text{if } e^{r\Delta\tau+X_{\Delta\tau}}-1>0,\\ 1-e^{r\Delta\tau} & \text{if } e^{r\Delta\tau+X_{\Delta\tau}}-1<0\end{cases} \\ &= e^{r\Delta\tau}-1+2\begin{cases}\mathbb{E}[1-e^{r\Delta\tau+X_{\Delta\tau}}] & \text{if } 1-e^{r\Delta\tau+X_{\Delta\tau}}>0,\\ 0 & \text{if } 1-e^{r\Delta\tau+X_{\Delta\tau}}<0\end{cases} \\ &= e^{r\Delta\tau}-1+2\mathbb{E}[(1-e^{r\Delta\tau+X_{\Delta\tau}})^+]\end{aligned}$$

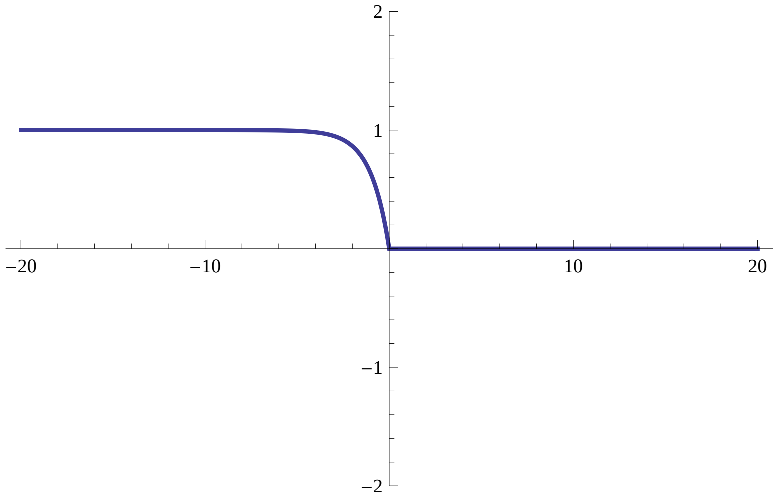
Then, because $e^{r\Delta\tau}-1\to 0$ when $\Delta\tau\to 0$, we only have to prove that:

$$\mathbb{E}[(1-e^{r\Delta\tau+X_{\Delta\tau}})^+]\to 0, \text{ when } \Delta\tau\to 0.$$

Let $C_0(\mathbb{R}) = \{h : h$ is continuous and vanishing at infinity$\}$. The Feller property tells us that,for any $h \in C_0(\mathbb{R})$ we have:

$$P_\tau h(0) = \mathbb{E}[h(r\tau + X_\tau)] \to h(0) \text{ as } \tau \to 0$$

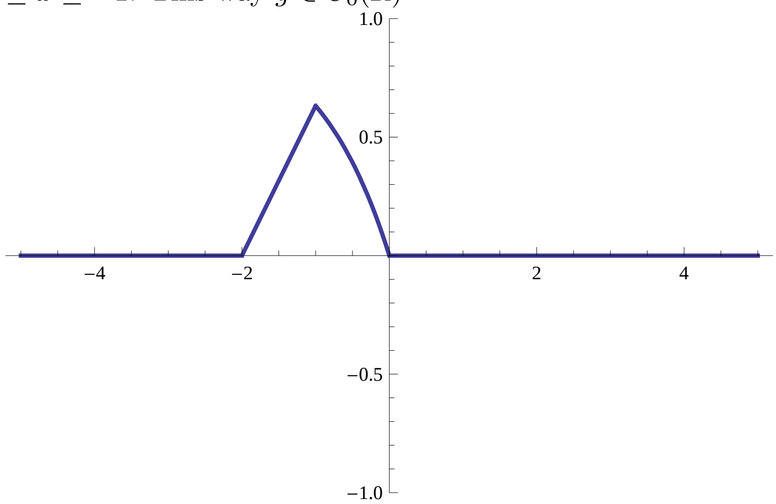
But, in this case $h(x) = (1 - e^x)^+$ does not belong to $C_0(\mathbb{R})$.



Then we try to approximate h with the function $g(x)$ such that:

$$g(x) = h(x), \text{ if } x \geq -1, g(x) = 0, \text{ if } x \leq -2, 0 \leq g(x) \leq h(x)$$

and $g(x)$ is continuously interpolated between -2 and -1, that is $g(x) = h(-1)x + 2h(-1)$ for $-2 \leq x \leq -1$. This way $g \in C_0(\mathbb{R})$



$$\begin{aligned}
\mathbb{E}[(1 - e^{r\Delta\tau + X_{\Delta\tau}})^+] &= |P_\tau h(0)| = |P_\tau h(0) - P_\tau g(0) + P_\tau g(0)| \leq |P_\tau h(0) - P_\tau g(0)| + |P_\tau g(0)| \\
&= |\mathbb{E}[(h(r\Delta\tau + X_{\Delta\tau}) - g(r\Delta\tau + X_{\Delta\tau}))1_{r\Delta\tau + X_{\Delta\tau} < -1}]| + |P_\tau g(0)| \\
&\leq \mathbb{E}[1_{r\Delta\tau + X_{\Delta\tau} < -1}] + |P_\tau g(0)| = \mathbb{Q}[r\Delta\tau + X_{\Delta\tau} < -1] + |P_\tau g(0)| \\
&\leq \mathbb{Q}[X_{\Delta\tau} \leq -1] + |P_\tau g(0)|,
\end{aligned}$$

because $g = h$ when $r\Delta\tau + X_{\Delta\tau} \geq -1$ and $h(x) \leq 1$, $g(x) \geq 0$ by construction.

Since $|P_\tau g(0)| \to g(0) = 0$ as $\Delta\tau \to 0$, we only have to prove that: $\mathbb{Q}[X_{\Delta\tau} \leq -1] \to 0$ as $\Delta\tau \to 0$.

Defining $M_\tau^- = \sup_{0\le s\le\tau}(-X_s)$ we have $\mathbb{Q}[X_{\Delta\tau} \le -1] = \mathbb{Q}[(-X_{\Delta\tau}) \ge 1] \le \mathbb{Q}[M_\tau^- \ge 1]$.

Consider $\tau_n \downarrow 0$ and define $\Omega_n = \left\{\omega \in \Omega : M_{\tau_n}^-(\omega) \ge 1\right\}$. This way, the sequence Ω_n is decreasing. Therefore,

$$\begin{aligned}\lim_{n\to\infty} \mathbb{Q}[\Omega_n] &= \lim_{n\to\infty} \mathbb{Q}[M_{\tau_n}^- \ge 1] = \mathbb{Q}[\bigcap_{n=1}^{\infty} \left\{\omega \in \Omega : M_{\tau_n}^-(\omega) \ge 1\right\}] \\ &= \mathbb{Q}[M_0^-(\omega) \ge 1] = 0,\end{aligned}$$

since $M_0^- = -X_0$ and $X_0 = 0$ a.s . Then $\mathbb{Q}[M_\tau^- \ge 1] \to 0$ since τ_n is arbitrary. Therefore $\mathbb{Q}[X_{\Delta\tau} \le -1] \to 0$.

In order to show continuity for any $(\tau, x) \in [0, T] \times \mathbb{R}$, we use the triangular inequality:

$$\begin{aligned}|f(\tau + \Delta\tau, x + \Delta x) - f(\tau, x)| &\le |f(\tau + \Delta\tau, x + \Delta x) - f(\tau + \Delta\tau, x)| + |f(\tau + \Delta\tau, x) - f(\tau, x)| \\ &\le cS_0 e^{x+r(\tau+\Delta\tau)}|e^{\Delta x} - 1| + cS_0 e^{x+r\tau}\mathbb{E}[|e^{r\Delta\tau+X_{\Delta\tau}} - 1|] \to 0,\end{aligned}$$

Then $f(\tau, x)$ is continuous on $[0, T] \times \mathbb{R}$. ■

A.2 Proof of Proposition 3.3.2.

First step: We prove that the density function of $r\tau + X_\tau$, $p_\tau(x) \in C^\infty$

The condition

$$\liminf_{\epsilon\to 0} \epsilon^{-\beta} \int_{-\epsilon}^{\epsilon} |x|^2 \nu(\,\mathrm{d}x) > 0$$

implies that

$$\exists_{c_1>0} \int_{-\epsilon}^{\epsilon} |x|^2 \nu(\,\mathrm{d}x) \ge c_1 \epsilon^\beta$$

for small ϵ. Following the notation on [Vol05]: Let $p_\tau(x)$, be the density function of the Lévy process $r\tau + X_\tau$ with characteristic function:

$$\psi_{r\tau+X_\tau}(z) = e^{\tau\phi_{r+X_1}(z)}$$

with

$$\phi_{r+X_1}(z) = -\frac{\sigma^2 z^2}{2} + i\gamma z + \int_{-\infty}^{+\infty} \left(e^{izx} - 1 - izx1_{|x|\le 1}\right) \nu(dx).$$

Then,

$$\begin{aligned}|\psi_{r\tau+X_\tau}(z)| &= \left| e^{\tau\left(-\frac{\sigma^2 z^2}{2} + i\gamma z + \int_{-\infty}^{+\infty}(e^{izx} - 1 - izx1_{|x|\le 1})\nu(dx)\right)} \right| \\ &= \left| e^{\tau(i(\gamma z + \int_{-\infty}^{+\infty}(\sin(zx) - zx1_{|x|\le 1})\nu(dx))} e^{-\frac{\sigma^2 z^2}{2} + \int_{-\infty}^{+\infty}(\cos(zx)-1)\nu(dx))} \right| \le e^{\int_{-\infty}^{+\infty}(\cos(zx)-1)\nu(dx))}.\end{aligned}$$

Notice that, $1 - \cos(u) = 1 - \cos(\frac{u}{2} + \frac{u}{2}) = 1 - (\cos^2(\frac{u}{2}) - \sin^2(\frac{u}{2})) = 2\sin^2(\frac{u}{2}) \ge 2(\frac{u}{\pi})^2$ for $\frac{|u|}{\pi} \le 1$. Then,

$$\begin{aligned}|\psi_{r\tau+X_\tau}(z)| &\le e^{\int_{-\infty}^{+\infty}(\cos(zx)-1)\nu(dx))} \le e^{\int_{|x|\le\frac{\pi}{|z|}} -2(\frac{zx}{\pi})^2\nu(dx))} \\ &= e^{-\frac{2}{\pi^2}z^2\int_{|x|\le\frac{\pi}{|z|}} x^2\nu(dx))} = e^{-Kz^2\int_{|x|\le\frac{\pi}{|z|}} x^2\nu(dx)}.\end{aligned}$$

But $\int_{-\epsilon}^{\epsilon}|x|^2\nu(\,\mathrm{d}x) \geq c_1\epsilon^\beta$, so by choosing $\epsilon = \frac{\pi}{|z|}$, we get $-\int_{|x|\leq\frac{\pi}{|z|}} x^2\nu(dx) \leq -C(\frac{\pi}{|z|})^\beta$. Then,

$$|\psi_{r\tau+X_\tau}(z)| \leq e^{-Kz^2C(\frac{\pi}{|z|})^\beta} = e^{-c|z|^{2-\beta}} = e^{-c|z|^\alpha} \text{ with } c = KC\pi^\beta \text{ and } \alpha = 2-\beta.$$

Also,

$$\int_{\mathbb{R}} |\psi_{r\tau+X_\tau}(z)|z^n\,\mathrm{d}z \leq \int_{\mathbb{R}} e^{-c|z|^\alpha}z^n\,\mathrm{d}z < \infty \tag{A.1}$$

and by inversion formula of the Fourier transform,

$$p_\tau(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ixz}\psi_{r\tau+X_\tau}(z)\,\mathrm{d}z.$$

Then, the right hand-side is n times differentiable with respect to x and differentiation is possible under the integral sign because of (A.1). In fact,

$$\begin{aligned}\frac{\partial^n p_\tau(x)}{\partial x^n} &= \frac{1}{2\pi}\int_{\mathbb{R}}(-iz)^n e^{-izx}\psi_{r\tau+X_\tau}(z)\,\mathrm{d}z = \frac{1}{2\pi}\int_{\mathbb{R}}|z|^n e^{(\frac{3}{2}n\pi-zx)i}\psi_{r\tau+X_\tau}(z)\,\mathrm{d}z\\ &\leq \frac{1}{2\pi}\int_{\mathbb{R}}|z|^n\psi_{r\tau+X_\tau}(z)\,\mathrm{d}z < \infty.\end{aligned}$$

Then, by proposition 28.1 of [Sat99], the process $r\tau + X_\tau$ has density function $p_\tau(x)$ of class C_∞.

Second step:Let us prove that $f(\tau,x) = \mathbb{E}[h(x+r\tau+X_\tau)] \in C^\infty((0,T)\times\mathbb{R})$.

Defining,$\widetilde{p_\tau}(x) = p_\tau(-x)$, we have

$$\begin{aligned}f(\tau,x) = \mathbb{E}[h(x+r\tau+X_\tau)] = h(x)*\widetilde{p_\tau}(x) &= \int_{\mathbb{R}} h(x-z)\widetilde{p_\tau}(z)\,\mathrm{d}z\\ = \int_{\mathbb{R}} h(x-z)p_\tau(-z)\,\mathrm{d}z &= \int_{\mathbb{R}} h(x+w)p_\tau(w)\,\mathrm{d}w,\end{aligned}$$

by making the substitution $w = -z$.

So we have to show that $h(x)*\widetilde{p_\tau}(x)$ belongs to C^∞ and for that to happen, $\frac{\partial^n\widetilde{p_\tau}(x)}{\partial x^n}$ has to decrease sufficiently fast at the infinity so that

$$\frac{\partial^n f(\tau,x)}{\partial x^n} = h(x)*\frac{\partial^n\widetilde{p_\tau}(x)}{\partial x^n} = \int_{\mathbb{R}} h(x-y)\frac{\partial^n\widetilde{p_\tau}(y)}{\partial x^n}\,\mathrm{d}y$$

makes sense.

We have

$$\begin{aligned}\phi'_{r+X_1}(z) &= -\sigma^2 z + i\gamma + \int_{\mathbb{R}} iy(e^{iyz} - 1_{|y|\leq 1})\nu(\,\mathrm{d}y),\\ \phi''_{r+X_1}(z) &= -\sigma^2 + \int_{\mathbb{R}}(iy)^2 e^{iyz}\nu(\,\mathrm{d}y),\\ \phi^{(k)}_{r+X_1}(z) &= \int_{\mathbb{R}}(iy)^k e^{iyz}\nu(\,\mathrm{d}y), \forall k\geq 3.\end{aligned}$$

Therefore,

$$\begin{aligned}|\phi'_{r+X_1}(z)| &= \left|-\sigma^2 z + i\gamma + \int_{\mathbb{R}} iy(e^{iyz}-1_{|y|\leq 1})\nu(\,\mathrm{d}y)\right| \leq \sigma^2|z| + |\gamma| + \left|\int_{\mathbb{R}} iy(e^{iyz}-1_{|y|\leq 1})\nu(\,\mathrm{d}y)\right|\\ &\leq \sigma^2|z| + |\gamma| + \int_{\mathbb{R}}\left|iy(e^{iyz}-1_{|y|\leq 1})\right|\nu(\,\mathrm{d}y) \leq \sigma^2|z| + |\gamma| + \int_{\mathbb{R}}|y||e^{iyz}|\nu(\,\mathrm{d}y)\\ &= \sigma^2|z| + |\gamma| + \int_{\mathbb{R}}|y|\nu(\,\mathrm{d}y) = \sigma^2|z| + |\gamma| + \int_{|y|\leq 1}|y|\nu(\,\mathrm{d}y) + \int_{|y|>1}|y|\nu(\,\mathrm{d}y) < \infty,\end{aligned}$$

because of (3.22).

$$|\phi''_{r+X_1}(z)| = \left|-\sigma^2 + \int_{\mathbb{R}} (iy)^2 e^{iyz} \nu(\,\mathrm{d}y)\right| \leq \sigma^2 + \left|\int_{\mathbb{R}} (iy)^2 e^{iyz} \nu(\,\mathrm{d}y)\right|$$
$$\leq \sigma^2 + \int_{\mathbb{R}} |(iy)^2||e^{iyz}|\nu(\,\mathrm{d}y) = \sigma^2 + \int_{\mathbb{R}} |y|^2 \nu(\,\mathrm{d}y) < \infty,$$

also by (3.22) and (2.1).

$$|\phi^{(k)}_{r+X_1}(z)| = \left|\int_{\mathbb{R}} (iy)^k e^{iyz} \nu(\,\mathrm{d}y)\right| \leq \int_{\mathbb{R}} |(iy)^k e^{iyz}|\nu(\,\mathrm{d}y) = \int_{\mathbb{R}} |y|^k |e^{iyz}|\nu(\,\mathrm{d}y)$$
$$= \int_{\mathbb{R}} |y|^k \nu(\,\mathrm{d}y) = \int_{|y|\leq 1} |y|^k \nu(\,\mathrm{d}y) + \int_{|y|>1} |y|^k \nu(\,\mathrm{d}y) < \infty, \forall k \geq 3.$$

Then $\phi_{r+X_1}(z) \in C^\infty$ which implies that $\psi_{r\tau+X_\tau}(z) = e^{\tau\phi_{r+X_1}(z)} \in C^\infty$.

Next,we conclude that,

$$|\phi'_{r+X_1}(z)| \leq |\sigma^2|z| + |\gamma| + \int_{\mathbb{R}} |y|\nu(\,\mathrm{d}y)| \leq A_1(1+|z|)$$
$$|\phi''_{r+X_1}(z)| \leq |\sigma^2 + \int_{\mathbb{R}} |y|^2 \nu(\,\mathrm{d}y)| \leq A_2$$
$$|\phi^{(k)}_{r+X_1}(z)| \leq |\int_{\mathbb{R}} |y|^k \nu(\,\mathrm{d}y)| \leq A_k, \forall k \geq 3$$

and also that

$$|\frac{\partial \psi_{r\tau+X_\tau}(z)}{\partial z}| = \tau|\phi'_{r+X_1}(z)|e^{\tau\phi_{r+X_1}(z)} \leq K(1+|z|)\psi_{r\tau+X_\tau}(z) \leq K(1+|z|)e^{-c|z|^\alpha}$$
$$|\frac{\partial^2 \psi_{r\tau+X_\tau}(z)}{\partial z^2}| = \left|\tau\phi''_{r+X_1}(z)e^{\tau\phi_{r+X_1}(z)} + \tau^2(\phi'_{r+X_1}(z))^2 e^{\tau\phi_{r+X_1}(z)}\right|$$
$$\leq \tau A_2 e^{\tau\phi_{r+X_1}(z)} + \tau^2 A_1(1+|z|)^2 e^{\tau\phi_{r+X_1}(z)}$$
$$\leq \tau A_2 e^{-c|z|^\alpha} + \tau A_1(1+|z|)^2 e^{-c|z|^\alpha}$$
$$\leq K(1+|z|^2)e^{-c|z|^\alpha}.$$

So, by recurrence, we get

$$|\frac{\partial^k \psi_{r\tau+X_\tau}(z)}{\partial z^k}| \leq K(1+|z|^k)\psi_{r\tau+X_\tau}(z) \leq K(1+|z|^k)e^{-c|z|^\alpha}, \forall k \geq 0$$

Also,

- $$\left|\frac{d^k}{dz^k}\int_{\mathbb{R}} e^{izx}\frac{\partial}{\partial x}\widetilde{p}_\tau(x)\,\mathrm{d}x\right| = \left|\frac{d^k}{dz^k}([e^{izx}\widetilde{p}_\tau(x)]_{-\infty}^{\infty} - \int_{\mathbb{R}} (iz)e^{izx}\widetilde{p}_\tau(x)\,\mathrm{d}x)\right|$$
$$= \left|\frac{d^k}{dz^k}(-iz)\int_{\mathbb{R}} e^{izx}\widetilde{p}_\tau(x)\,\mathrm{d}x\right| \leq K|z|^{1+k}e^{-c|z|^\alpha}.$$

- $$\left|\frac{d^k}{dz^k}\int_{\mathbb{R}} e^{izx}\frac{\partial^2}{\partial x^2}\widetilde{p}_\tau(x)\,\mathrm{d}x\right| = \left|\frac{d^k}{dz^k}([e^{izx}\frac{\partial}{\partial x}\widetilde{p}_\tau(x)]_{-\infty}^{\infty} - \int_{\mathbb{R}} (iz)e^{izx}\frac{\partial}{\partial x}\widetilde{p}_\tau(x)\,\mathrm{d}x)\right|$$
$$= |\frac{d^k}{dz^k}([e^{izx}\frac{\partial}{\partial x}\widetilde{p}_\tau(x)]_{-\infty}^{\infty} + [(-iz)e^{izx}\widetilde{p}_\tau(x)]_{-\infty}^{\infty}$$
$$- \int_{\mathbb{R}} (iz)^2 e^{izx}\widetilde{p}_\tau(x)\,\mathrm{d}x)| = \left|\frac{d^k}{dz^k}(-iz)^2\int_{\mathbb{R}} e^{izx}\widetilde{p}_\tau(x)\,\mathrm{d}x\right|$$
$$\leq K|z|^{2+k}e^{-c|z|^\alpha}, \forall k \geq 0,$$

because by proposition 28.1 of [Sat99] the partial derivatives of $\widetilde{p}_\tau$ of orders 0,..,n tend to zero as $|x| \to \infty$. Once again by recurrence,

- $$\left|\frac{d^k}{dz^k}\int_{\mathbb{R}} e^{izx}\frac{\partial^n}{\partial x^n}\widetilde{p}_\tau(x)\,\mathrm{d}x\right| = \left|\frac{d^k}{dz^k}(-iz)^n\int_{\mathbb{R}} e^{izx}\widetilde{p}_\tau(x)\,\mathrm{d}x\right| \leq K|z|^{n+k}e^{-c|z|^\alpha},$$

  for all $n, k \geq 0$.

Then,

$$\forall k, n \geq 0, \int_{\mathbb{R}}\left(\frac{d^k}{dz^k}\int_{\mathbb{R}} e^{izx}\frac{\partial^n}{\partial x^n}\widetilde{p}_\tau(x)\,\mathrm{d}x\right)^2\,\mathrm{d}z \leq \int_{\mathbb{R}}\left(K|z|^{n+k}e^{-c|z|^\alpha}\right)^2\,\mathrm{d}z < \infty,$$

which means that $\frac{d^k}{dz^k}\int_{\mathbb{R}} e^{izx}\frac{\partial^n}{\partial x^n}\widetilde{p}_\tau(x)\,\mathrm{d}x \in L^2(\mathbb{R})$. But this implies that

$$\int_{\mathbb{R}}\left(|x|^k\frac{\partial^n}{\partial x^n}\widetilde{p}_\tau(x)\right)^2\,\mathrm{d}x < \infty$$

or that $|x|^k\frac{\partial^n}{\partial x^n}\widetilde{p}_\tau(x) \in L^2(\mathbb{R})$, and this in turn implies that:

$$\begin{aligned}\int_{\mathbb{R}}|\frac{\partial^n\widetilde{p}_\tau(x)}{\partial x^n}(1+|x|^k)\,\mathrm{d}x| &\leq C\int_{\mathbb{R}}\frac{1}{1+|x|}(1+|x|^{k+1})\frac{\partial^n\widetilde{p}_\tau(x)}{\partial x^n}\,\mathrm{d}x\\ &\leq C\left(\int_{\mathbb{R}}\left(\frac{1}{1+|x|}\right)^2\mathrm{d}x\right)^{1/2}\left(\int_{\mathbb{R}}\left((1+|x|^{k+1}\right)\frac{\partial^n\widetilde{p}_\tau(x)}{\partial x^n}\right)^2\mathrm{d}x\right)^{1/2}\\ &= C\left\|\frac{1}{1+|x|}\right\|_{L^2}\left\|(1+|x|^{k+1})\frac{\partial^n\widetilde{p}_\tau(x)}{\partial x^n}\right\|_{L^2} < \infty.\end{aligned}$$

Then,

$$\begin{aligned}\left|\frac{\partial^n f}{\partial x^n}(\tau,x)\right| &= \left|h(x) * \frac{\partial^n\widetilde{p}_\tau(x)}{\partial x^n}\right| = \left|\int_{\mathbb{R}} h(x-z)\frac{\partial^n\widetilde{p}_\tau(z)}{\partial x^n}\,\mathrm{d}z\right|\\ &\leq C\int_{\mathbb{R}}(1+|x-z|^p)\left|\frac{\partial^n\widetilde{p}_\tau(z)}{\partial x^n}\right|\,\mathrm{d}z \leq C(1+|x|^p)\int_{\mathbb{R}}(1+|z|^p)\left|\frac{\partial^n\widetilde{p}_\tau(z)}{\partial x^n}\right|\,\mathrm{d}z\\ &\leq C(1+|x|^p)K\left\|\frac{1}{1+|x|}\right\|_{L^2}\left\|(1+|x|^{k+1})\frac{\partial^n\widetilde{p}_\tau(x)}{\partial x^n}\right\|_{L^2} = D(1+|x|^p).\end{aligned}$$

Then $\frac{\partial^n f}{\partial x^n}(\tau,x)$ is continuous and finite, which means that f is regular with respect to x. To prove the regularity in time we notice that:

$$|\phi_{r+X_1}(z)| = \left|-\frac{\sigma^2 z^2}{2} + i\gamma z + \int_{-\infty}^{+\infty}(e^{izx} - 1 - izx1_{|x|\leq 1})\nu(dx)\right| \leq C(1+|z|^2)$$

and verify by recurrence that,

$$\begin{aligned}\left|\frac{d^k}{dz^k}\int_{\mathbb{R}} e^{izx}\frac{\partial^m}{\partial\tau^m}\widetilde{p}_\tau(x)\,\mathrm{d}x\right| &= \left|\frac{d^k}{dz^k}\frac{\partial^m}{\partial\tau^m}\int_{\mathbb{R}} e^{izx}\widetilde{p}_\tau(x)\,\mathrm{d}x\right|\\ &= \left|\frac{d^k}{dz^k}[\phi_{r+X_1}(z)]^m e^{\tau\phi_{r+X_1}(z)}\right| \leq C|z|^{2m+k}e^{-c|z|^\alpha}.\end{aligned}$$

Then, $\frac{d^k}{dz^k}\int_{\mathbb{R}} e^{izx}\frac{\partial^m}{\partial\tau^m}\widetilde{p}_\tau(x)\,\mathrm{d}x \in L^2(\mathbb{R})$,which implies that $\frac{\partial^m\widetilde{p}_\tau(x)}{\partial\tau^m}(1+|x|^k)\in L^1(\mathbb{R})$.
Therefore,

$$\begin{aligned}\left|\frac{\partial^m f}{\partial\tau^m}(\tau,x)\right| = \left|h(x) * \frac{\partial^m\widetilde{p}_\tau(x)}{\partial\tau^m}\right| = \left|\int_{\mathbb{R}} h(x-z)\frac{\partial^m\widetilde{p}_\tau(z)}{\partial\tau^m}\,\mathrm{d}z\right| &\le C\int_{\mathbb{R}}(1+|x-z|^p)\left|\frac{\partial^m\widetilde{p}_\tau(z)}{\partial\tau^m}\right|\mathrm{d}z\\ &\le C(1+|x|^p)\int_{\mathbb{R}}(1+|z|^p)\left|\frac{\partial^m\widetilde{p}_\tau(z)}{\partial\tau^m}\right|\mathrm{d}z\\ &\le C(1+|x|^p)K\left\|\frac{1}{1+|x|}\right\|_{L^2}\left\|(1+|x|^{k+1})\frac{\partial^m\widetilde{p}_\tau(x)}{\partial\tau^m}\right\|_{L^2}\\ &= D(1+|x|^p),\end{aligned}$$

which means that $\frac{\partial^n f}{\partial\tau^m}(\tau,x)$ is continuous and finite.

In the same way we conclude that:

$$\begin{aligned}\left|\frac{d^k}{dz^k}\int_{\mathbb{R}} e^{izx}\frac{\partial^{n+m}}{\partial x^n\partial\tau^m}\widetilde{p}_\tau(x)\,\mathrm{d}x\right| &= \left|\frac{d^k}{dz^k}(-iz)^n[\phi_{r+X_1}(z)]^m e^{\tau\phi_{r+X_1}(z)}\int_{\mathbb{R}} e^{izx}\widetilde{p}_\tau(x)\,\mathrm{d}x\right|\\ &\le C|z|^{2m+n+k}e^{-c|z|^\alpha}.\\ \left|\frac{\partial^{n+m} f}{\partial x^n\partial\tau^m}(\tau,x)\right| &= \left|h(x) * \frac{\partial^{n+m}\widetilde{p}_\tau(x)}{\partial x^n\partial\tau^m}\right| = \left|\int_{\mathbb{R}} h(x-z)\frac{\partial^{n+m}\widetilde{p}_\tau(z)}{\partial x^n\partial\tau^m}\,\mathrm{d}z\right|\\ &\le C\int_{\mathbb{R}}(1+|x-z|^p)\left|\frac{\partial^{n+m}\widetilde{p}_\tau(z)}{\partial x^n\partial\tau^m}\right|\mathrm{d}z\\ &\le C(1+|x|^p)\int_{\mathbb{R}}(1+|z|^p)\left|\frac{\partial^{n+m}\widetilde{p}_\tau(z)}{\partial x^n\partial\tau^m}\right|\mathrm{d}z\\ &\le C(1+|x|^p)K\left\|\frac{1}{1+|x|}\right\|_{L^2}\left\|(1+|x|^{k+1})\frac{\partial^{n+m}\widetilde{p}_\tau(x)}{\partial x^n\partial\tau^m}\right\|_{L^2}\\ &= D(1+|x|^p).\end{aligned}$$

Then $f(\tau,x)\in C^\infty((0,T],\mathbb{R})$.

A.3 Proof of Proposition 3.3.5.

Proof. Define $M=\sup_{S\in(0,U)}H(S)$. We can do this because H is bounded due to the fact that it is Lipschitz. We will prove first the continuity in x and τ and finally prove the continuity using the triangular inequality.

First step: Prove continuity in x for all $\tau>0$ and $x<u$.Choosing $\delta\in(0,u-x)$ we get:

$$\begin{aligned}|f_U(\tau,x+\delta)-f_U(\tau,x)| &= |\mathbb{E}[H(S_0e^{x+\delta+Y_\tau})1_{\tau<R_{u-x-\delta}} - H(S_0e^{x+Y_\tau})1_{\tau<R_{u-x}}]|\\ &\le \mathbb{E}[|H(S_0e^{x+\delta+Y_\tau})1_{\tau<R_{u-x-\delta}} - H(S_0e^{x+Y_\tau})1_{\tau<R_{u-x}}|]\\ &= \mathbb{E}[|(H(S_0e^{x+\delta+Y_\tau}) - H(S_0e^{x+Y_\tau}))1_{\tau<R_{u-x-\delta}}\\ &\quad + H(S_0e^{x+Y_\tau})(1_{\tau<R_{u-x-\delta}} - 1_{\tau<R_{u-x}})|]\\ &\le \mathbb{E}[k|(S_0e^{x+\delta+Y_\tau} - S_0e^{x+Y_\tau})|1_{\tau<R_{u-x-\delta}}] + M\mathbb{E}[1_{R_{u-x-\delta}<\tau<R_{u-x}}]\\ &\le ke^{x+r\tau}\mathbb{E}[S_0e^{X_\tau}]|e^\delta-1| + M\mathbb{Q}[R_{u-x-\delta}<\tau<R_{u-x}]\\ &\le kS_0e^{x+r\tau}|e^\delta-1| + M\mathbb{Q}[R_{u-x-\delta}<\tau<R_{u-x}],\end{aligned}$$

because by the martingale condition $\mathbb{E}[e^{X_\tau}]=\mathbb{E}[e^{X_0}]=1$.

Then,

$$\lim_{\delta\to 0} |f_U(\tau, x+\delta) - f_U((\tau, x))| \leq \lim_{\delta\to 0} kS_0 e^{x+r\tau}|e^{\delta} - 1| + M\mathbb{Q}[R_{u-x-\delta} < \tau < R_{u-x}] = 0.$$

because $|e^{\delta} - 1| \to 0$ and $\mathbb{Q}[R_{u-x-\delta} < \tau < R_{u-x}] \to 0$ when $\delta \to 0$ by Lemma 3.3.5.

Similarly we prove for $x < u$:

$$\lim_{\delta\to 0} |f_U(\tau, x-\delta) - f_U((\tau, x))| \leq \lim_{\delta\to 0} kS_0 e^{x+r\tau}|e^{-\delta} - 1| + M\mathbb{Q}[R_{u-x} < \tau < R_{u-x+\delta}] = 0.$$

also by Lemma 3.3.5 and by the martingale condition.

As for $x = u$ the right continuity of $f_U(\tau, x)$ is proven easily so:

$$|f_U(\tau, u-\delta) - f_U((\tau, u))| = |\mathbb{E}[H(S_0 e^{u-\delta+Y_\tau})1_{\tau<R_\delta}]| \leq M\mathbb{Q}[\tau < R_\delta].$$

Considering $\delta_n \to 0$ we have:

$$\mathbb{Q}[\tau < R_\delta] \to \mathbb{Q}[\cap_{n=1}^{\infty} \{\omega \in \Omega | R_{\delta_n} > \tau\}] = \mathbb{Q}[\tau \leq R_0] = 0,$$

because $R_0 = 0$ $a.s.$ Therefore, we proved the continuity of $f_U(\tau, x)$ for all $x \in \mathbb{R}$.

Second step: Let us prove continuity in time. For $x < u$ and $0 \leq s \leq t$:

$$\begin{aligned}
|f_U(t, x) - f_U(s, x)| &= |\mathbb{E}[H(S_0 e^{x+Y_t})1_{t<R_{u-x}} - H(S_0 e^{x+Y_s})1_{s<R_{u-x}}]| \\
&\leq \mathbb{E}[|H(S_0 e^{x+Y_t}) - H(S_0 e^{x+Y_s})|1_{t<R_{u-x}} + |H(S_0 e^{x+Y_s})|1_{s\leq R_{u-x}} < t] \\
&\leq kS_0 e^{x+rs}\mathbb{E}[|e^{Y_{t-s}} - 1|] + M\mathbb{Q}[s \leq R_{u-x} < t] \\
\lim_{t\to s} |f_U(t, x) - f_U(s, x)| &\leq \lim_{t\to s} kS_0 e^{x+rs}\mathbb{E}[|e^{Y_{t-s}} - 1|] + M\mathbb{Q}[s \leq R_{u-x} < t] = 0
\end{aligned}$$

because we know that, by the proof of the Proposition 3.3.1, $\mathbb{E}[|e^{Y_{t-s}} - 1|] \to 0$ when $t \to s$ and considering a decreasing set $\Omega_n = \{\omega \in \Omega | s \leq R_{u-x}(\omega) < t_n\}$, $t_n \to s$:

$$\lim_{n\to\infty} \mathbb{Q}[s \leq R_{u-x}(\omega) < t_n] = \mathbb{Q}[\cap_{n=1}^{\infty}\Omega_n] = \mathbb{Q}[\emptyset] = 0.$$

Third step: Use the triangular inequality. Let $(\tau, x) \in [0, T] \times (-\infty, u)$ and $(\Delta\tau, \Delta x) \in \mathbb{R}^2$

$$\begin{aligned}
|f_U(\tau + \Delta\tau, x + \Delta x) - f_U(\tau, x)| \leq\ & |f_U(\tau + \Delta\tau, x + \Delta x) - f_U(\tau, x + \Delta x)| \\
&+ |f_U(\tau, x + \Delta x) - f_U(\tau, x)|.
\end{aligned}$$

- First term.

  Defining $y = x + \Delta x$ and $t = \tau + \Delta\tau$ with $\Delta\tau > 0$ we obtain:

$$\begin{aligned}
|f_U(t, y) - f_U(\tau, y)| &= |\mathbb{E}[H(S_0 e^{y+Y_t})1_{t<R_{u-y}} - H(S_0 e^{y+Y_\tau})1_{\tau<R_{u-y}}]| \\
&= |\mathbb{E}[(H(S_0 e^{y+Y_t}) - H(S_0 e^{y+Y_\tau}))1_{t<R_{u-y}} \\
&\quad + H(S_0 e^{y+Y_\tau})(1_{t<R_{u-y}} - 1_{\tau<R_{u-y}})]| \\
&\leq \mathbb{E}[|H(S_0 e^{y+Y_t}) - H(S_0 e^{y+Y_\tau})|1_{t<R_{u-y}} + |H(S_0 e^{y+Y_\tau})|1_{\tau<R_{u-y}<t}] \\
&\leq k\mathbb{E}[|S_0 e^{y+Y_t} - S_0 e^{y+Y_\tau}|1_{t<R_{u-y}}] + M\mathbb{Q}[\tau < R_{u-y} < t] \\
&\leq kS_0 e^{y}\mathbb{E}[|e^{Y_t} - e^{Y_\tau}|] + M\mathbb{Q}[\tau < R_{u-y} < t] \text{ but } Y_t - Y_\tau \stackrel{d}{=} Y_{\Delta\tau} \\
&\leq kS_0 e^{y}\mathbb{E}[|e^{Y_\tau}|e^{Y_{\Delta\tau}} - 1|] + M\mathbb{Q}[\tau < R_{u-y} < t] \\
&= kS_0 e^{y}\mathbb{E}[e^{Y_\tau}]\mathbb{E}[|e^{Y_{\Delta\tau}} - 1|] + M\mathbb{Q}[\tau < R_{u-y} < t] \\
&= kS_0 e^{y+r\tau}\mathbb{E}[|e^{Y_{\Delta\tau}} - 1|] + M\mathbb{Q}[\tau < R_{u-y} < t]
\end{aligned}$$

Similarly for the case $\Delta\tau < 0$, we get:

$$\begin{aligned}
|f_U(t,y) - f_U(\tau,y)| &= |f_U(\tau,y) - f_U(t,y)| \\
&= |\mathbb{E}[H(S_0e^{y+Y_\tau})1_{\tau<R_{u-y}} - H(S_0e^{y+Y_t})1_{t<R_{u-y}}]| \\
&= |\mathbb{E}[(H(S_0e^{y+Y_\tau}) - H(S_0e^{y+Y_t}))1_{\tau<R_{u-y}} \\
&\quad + H(S_0e^{y+Y_t})(1_{\tau<R_{u-y}} - 1_{t<R_{u-y}})]| \\
&\leq \mathbb{E}[|H(S_0e^{y+Y_\tau}) - H(S_0e^{y+Y_t})|1_{\tau<R_{u-y}} + |H(S_0e^{y+Y_t})|1_{t<R_{u-y}<\tau}] \\
&\leq k\mathbb{E}[|S_0e^{y+Y_\tau} - S_0e^{y+Y_t}|1_{\tau<R_{u-y}}] + M\mathbb{Q}[t < R_{u-y} < \tau] \\
&\leq kS_0e^y\mathbb{E}[|e^{Y_\tau} - e^{Y_t}|] + M\mathbb{Q}[t < R_{u-y} < \tau] \text{ but } , Y_\tau \stackrel{d}{=} Y_{\tau+\Delta\tau} - Y_{\Delta\tau} \\
&\leq kS_0e^y\mathbb{E}[|e^{Y_t}|e^{-Y_{\Delta\tau}} - 1|] + M\mathbb{Q}[t < R_{u-y} < \tau] \\
&= kS_0e^y\mathbb{E}[e^{Y_\tau}]\mathbb{E}[|e^{Y_{-\Delta\tau}} - 1|] + M\mathbb{Q}[t < R_{u-y} < \tau] \\
&= kS_0e^{y+r\tau+\Delta\tau}\mathbb{E}[|e^{Y_{-\Delta\tau}} - 1|] + M\mathbb{Q}[t < R_{u-y} < \tau]
\end{aligned}$$

So for $\Delta x \in \mathbb{R}$,

$$\begin{aligned}
&|f_U(t,y) - f_U(\tau,y)| \\
&\quad \leq kS_0e^y(e^{r\tau}1_{\Delta\tau\geq 0} + e^{rt}1_{\Delta\tau<0})\mathbb{E}[|e^{Y_{|\Delta\tau|}} - 1|] + M(\mathbb{Q}[\tau < R_{u-y} \leq t]1_{\Delta\tau\geq 0} \\
&\qquad + \mathbb{Q}[t < R_{u-y} \leq \tau]1_{\Delta\tau<0}) \\
&\quad = kS_0e^{y+r\tau}(1_{\Delta\tau\geq 0} + e^{r\Delta\tau}1_{\Delta\tau<0})\mathbb{E}[|e^{Y_{|\Delta\tau|}} - 1|] + M(\mathbb{Q}[t < R_{u-y} \leq \tau]1_{\Delta\tau<0} \\
&\qquad + \mathbb{Q}[\tau < R_{u-y} \leq t]1_{\Delta\tau\geq 0}) \\
&\quad \leq kS_0e^{y+r\tau}\mathbb{E}[|e^{Y_{|\Delta\tau|}} - 1|] + M(\mathbb{Q}[t < R_{u-y} \leq \tau]1_{\Delta\tau<0} \\
&\qquad + \mathbb{Q}[\tau < R_{u-y} \leq t]1_{\Delta\tau\geq 0}) \\
&\quad = kS_0e^{y+r\tau}\mathbb{E}[|e^{Y_{|\Delta\tau|}} - 1|] + M(\mathbb{Q}[\Delta\tau < R_{u-y} - \tau \leq 0]1_{\Delta\tau<0} \\
&\qquad + \mathbb{Q}[0 < R_{u-y} - \tau \leq \Delta\tau]1_{\Delta\tau\geq 0})] \\
&\quad = kS_0e^{y+r\tau}\mathbb{E}[|e^{Y_{|\Delta\tau|}} - 1|] + M(\mathbb{Q}[-\Delta\tau < R_{u-y} - \tau \leq 0] \\
&\qquad + \mathbb{Q}[0 < R_{u-y} - \tau \leq \Delta\tau])1_{\Delta\tau\geq 0} \\
&\quad \leq kS_0e^{y+r\tau}\mathbb{E}[|e^{Y_{|\Delta\tau|}} - 1|] + M\mathbb{Q}[|R_{u-y} - \tau| \leq \Delta\tau].
\end{aligned}$$

We would like to apply Lemma 3.3.5, but we can't, because we still have a bound that depends on $\Delta\tau$ and Δx. However, note that $\forall_{\epsilon>0}, \forall_{\Delta x} - \epsilon \leq \Delta x \leq \epsilon, R_{u-x-\epsilon} \leq R_{u-x-\Delta x} \leq R_{u-x+\epsilon}$.

Then,

$$\begin{aligned}
\lim_{\Delta\tau,\Delta x)\to 0} \mathbb{Q}[|R_{u-y} - \tau| \leq \Delta\tau] &\leq \lim_{(\Delta\tau,\Delta x)\to 0} (\mathbb{Q}[|R_{u-x-\epsilon} - \tau| \leq \Delta\tau] + \mathbb{Q}[|R_{u-x+\epsilon} - \tau| \leq \Delta\tau] \\
&\quad + \mathbb{Q}[R_{u-x-\epsilon} \leq \tau \leq R_{u-x+\epsilon}]) \\
&= \mathbb{Q}[R_{u-x-\epsilon} = \tau] + \mathbb{Q}[R_{u-x+\epsilon} = \tau] \\
&\quad + \mathbb{Q}[R_{u-x-\epsilon} \leq \tau \leq R_{u-x+\epsilon}] \\
&= \mathbb{Q}[R_{u-x-\epsilon} \leq \tau \leq R_{u-x+\epsilon}]
\end{aligned}$$

after Lemma 3.3.4.

- Second term

$$
\begin{aligned}
|f_U(\tau, y) - f_U(\tau, x)| &= |\mathbb{E}[H(S_0 e^{y+Y_\tau}) 1_{\tau < R_{u-y}} - H(S_0 e^{x+Y_\tau}) 1_{\tau < R_{u-x}}]| \\
&\leq k S_0 e^{x + r\tau} |e^{\Delta x} - 1| + M(\mathbb{Q}[R_{u-y} \leq \tau < R_{u-x}] 1_{\Delta x \geq 0} \\
&\quad + \mathbb{Q}[R_{u-x} \leq \tau < R_{u-y}] 1_{\Delta x < 0})
\end{aligned}
$$

As already demonstrated this expression tends to zero when $\Delta x \to 0$.

Then,

$$
\begin{aligned}
\lim_{(\Delta\tau, \Delta x) \to 0} |f_U(\tau + \Delta\tau, x + \Delta x) - f_U(\tau, x)| \leq & \lim_{(\Delta\tau, \Delta x) \to 0} (M\mathbb{Q}[R_{u-x-\epsilon} \leq \tau \leq R_{u-x+\epsilon}] \\
& + k S_0 e^{x + r\tau} |e^{\Delta x} - 1| \\
& + M(\mathbb{Q}[R_{u-y} \leq \tau < R_{u-x}] 1_{\Delta x \geq 0} \\
& + \mathbb{Q}[R_{u-x} \leq \tau < R_{u-y}] 1_{\Delta x < 0})) \\
= & M\mathbb{Q}[R_{u-y-\epsilon} \leq \tau \leq R_{u-y+\epsilon}].
\end{aligned}
$$

So it remains to prove that when $\epsilon \to 0$, which implies $\Delta x \to 0$, that

$$\mathbb{Q}[R_{u-x-\epsilon} \leq \tau \leq R_{u-x+\epsilon}] \to 0.$$

But once again taking $\epsilon_n \to 0$ and if

$$A_n = \{\omega \in \Omega | R_{u-x-\epsilon_n} \leq \tau \leq R_{u-x+\epsilon_n}\},$$

then

$$\lim_{n=\infty} \mathbb{Q}[R_{u-x-\epsilon_n} \leq \tau \leq R_{u-x+\epsilon_n}] = \mathbb{Q}[\cap_{n=1}^{\infty} A_n] = \mathbb{Q}[R_{u-x} = \tau] = 0,$$

after Lemma 3.3.4.

Fourth Step:Let us show continuity in $x = u$.

$$
\begin{aligned}
|f_U(\tau + \Delta\tau, u + \Delta x) - f_U(\tau, u)| &= |f_U(\tau + \Delta\tau, u + \Delta x) 1_{\Delta x < 0}| \\
&= |\mathbb{E}[H(S_0 e^{u + \Delta x + Y_{\tau + \Delta\tau}}) 1_{\tau + \Delta\tau < R_{-\Delta x}} 1_{\Delta x < 0}| \\
&\leq M\mathbb{Q}[\tau + \Delta\tau < R_{-\Delta x}] 1_{\Delta x < 0} = M\mathbb{Q}[\tau + \Delta\tau < R_{|\Delta x|}].
\end{aligned}
$$

But, for all $\xi > 0$ such that $|\Delta\tau| \leq \xi$, implies:

$$\left\{\omega \in \Omega | \tau + \Delta\tau < R_{|\Delta x|}\right\} \subset \left\{\omega \in \Omega | \tau - \xi < R_{|\Delta x|}\right\}.$$

which in turn implies:

$$\mathbb{Q}[\tau + \Delta\tau < R_{|\Delta x|}] \leq \mathbb{Q}[\tau - \xi < R_{|\Delta x|}] \to 0,$$

when $\Delta x \to 0$, because it only depends on Δx. ∎

A.4 Numerical Solution of PIDE

```
VGDensity[k_, sigma_, theta_, x_] := Module[{A, B, v},
  A = theta / sigma^2;
  B = Sqrt[theta^2 + (2 / k) * sigma^2] / sigma^2;
  v = (1 / (k * Abs[x])) * Exp[A * x - B * Abs[x]]
 ]
PutOptVGFFFinal[S_, K_, vgdensity_, r_, k_, sigma_, theta_,
  x_, W_, A_, Al_, Ar_, T_, N_, M_, epsilon_, Kl_, payoff_, xx_] :=
 Module[{B, F, E, G, P, w, q, t, h, ν, aux1, aux2, μ, μfourier, υ, υfourier, d,
   vg, g, deltax, deltat, deltaep, y, alpha, alphae, lambdae, alpha1, alpha2,
   alpha3, lambda, u, sigmaepsilon, sigmaepsilone, L, R, dtdx2, dtdx, zeros},
  g[u_, v_, y_, z_] := vgdensity /. {k → u, sigma → v, theta → y, x → z};
  h[w_] := payoff /. {xx → w};
  ν[c_] := 0.5 * deltax * (g[k, sigma, theta, (c - 0.5) * deltax] +
      g[k, sigma, theta, (c + 0.5) * deltax]) * If[Abs[c * deltax] >= epsilon, 1, 0];
  deltax = (A - W) / (N); deltat = T / M; deltaep = 2 * epsilon / Kl;
  dtdx2 = deltat / (deltax^2); dtdx = deltat / deltax; R = N / 2; L = -R;
  alphae = Sum[(Exp [j * deltax] - 1) * ν[j], {j, L, R}];
  lambdae = Sum[ν[j], {j, L, R}];
  sigmaepsilone = Sum[ν[j] * (j * deltax)^2, {j, L, R}];
  alpha = NIntegrate[(Exp [y] - 1) * g[k, sigma, theta, y],
     {y, -Kl, -epsilon}, Method → {Automatic, "SymbolicProcessing" → 0}] +
    NIntegrate[(Exp [y] - 1) * g[k, sigma, theta, y], {y, epsilon, Kl},
     Method → {Automatic, "SymbolicProcessing" → 0}];
  lambda = NIntegrate[g[k, sigma, theta, y], {y, -Kl, -epsilon}] +
    NIntegrate[g[k, sigma, theta, y], {y, epsilon, Kl}];
  sigmaepsilon = NIntegrate[g[k, sigma, theta, y] * y^2, {y, -epsilon, -0.00001}] +
    NIntegrate[g[k, sigma, theta, y] * y^2, {y, 0.00001, epsilon}];
  y = Table[h[Al + (i) * deltax + r * j * deltat], {j, 0, M}, {i, 0, N - L + R}];
  zeros = Table[0, {i, 0, N - 1}];
  μ = Join[Reverse[Table[ν[i], {i, L, R}]], zeros]; μfourier = Fourier[μ];
  υ = Table[0, {j, 0, M}, {i, 0, N + R - L}]; u = Table[0, {j, 0, M}, {i, 0, N - L + R}];
  aux1 = Table[0, {j, 0, M}, {i, 0, N + R - L}];
  aux2 = Table[0, {j, 0, M}, {i, 0, N + R - L}];
  alpha1 = -(sigmaepsilon) * 0.5 * (dtdx2) +
    0.5 * (0.5 * (sigmaepsilon) - r + alpha) * (dtdx);
  alpha2 = 1 + (dtdx2) * (sigmaepsilon) -
    0.5 * (0.5 * (sigmaepsilon) - r + alpha) * (dtdx) + lambda * deltat;
  alpha3 = -0.5 * (sigmaepsilon) * (dtdx2);
  F = DiagonalMatrix[Table[alpha2, {i, 0, N}]];
  E = DiagonalMatrix[Table[alpha1, {i, 0, N - 1}], 1];
  G = DiagonalMatrix[Table[alpha3, {i, 0, N - 1}], -1];
  B = F + E + G; P = Inverse[B];
  For[i = 1, i ≤ N + 1 - L + R, i++,
   u[[1, i]] = y[[1, i]];
  ];
  Do[
   For[i = L, i ≤ R, i++,
    aux1[[n - 1, i - L + 1]] = u[[n - 1, i - L + 1]];
   ];
   For[i = R + 1, i ≤ N + R, i++,
    aux2[[n - 1, i - R]] = u[[n - 1, i - L + 1]];
```

```
    ];
    For[i = 1, i ≤ N, i++,
     υ[[n - 1, i]] = aux2[[n - 1, i]];
    ];
    For[i = N + 1, i ≤ N + R - L + 1, i++,
     υ[[n - 1, i]] = aux1[[n - 1, i - N]];
    ];
    υfourier = Fourier[Table[υ[[n - 1, i]], {i, 1, N + R - L + 1}]];
    d = InverseFourier[Table[μfourier[[i]] * υfourier[[i]], {i, 1, N + R - L + 1}]];
    Do[u[[n, i]] = y[[n - 1, i]], {i, 1, -L}];
    Do[u[[n, i]] = y[[n - 1, i]], {i, N - L + 2, N + 1 - L + R}];
    Do[
     u[[n, i]] = P[[i + L, 1]] *
         (u[[n - 1, -L + 1]] + deltat * d[[N + R - L + 1]] - alpha3 * u[[n, -L]]) +
        Sum[P[[i + L, m + L]] * (u[[n - 1, m]] + deltat * d[[m + L - 1]]),
         {m, -L + 2, N - L}] + P[[i + L, N + 1]] *
         (u[[n - 1, N - L + 1]] + deltat * d[[N]] - alpha1 * u[[n, N - L + 2]]);
     , {i, -L + 1, N - L + 1}];
    , {n, 2, M + 1}];
   Table[{S * Exp[Al + (i - 1) * deltax], Exp[-r (j - 1) * deltat] * u[[j, i]]},
    {j, 1, M + 1}, {i, -L + 1, N - L + 1}]
  ]
PriceputVGPIDE =
 N[PutOptVGFFFinal[100, 100, VGDensity[0.16, 0.12, -0.33, x], 0, 0.16,
    0.12, -0.33, x, -0.5, 0.5, -1, 1, 1, 100, 200,
    0.3, 200, Max[100 - 100 * Exp[xx], 0], xx]][[201]]
ListLinePlot[{PriceputVGPIDE, FPut}, PlotRange → All, AxesLabel → {S, Price}]
```

Computation of the price of an european option using Fast Fourier Transform.

```
    VGCF[u_, s_, σ_, r_, q_, θ_, ν_, t_] :=
  With[{w = (1 / ν) Log[-θ ν - ν σ^2/2 + 1]}, (1/2 ν u^2 σ^2 - ⅈ ν u θ + 1)^(-t/ν) ⅇ^(I u (t (r+w)+Log[s]))];
VGEuropeanCall[K_, s_, σ_, r_, q_, θ_, ν_, t_] :=
  With[{k = Log[K], a = 1}, Exp[- a k]/(2 Pi)
    Re[NIntegrate[Exp[- I v k] Exp[-r t] VGCF[v - (a + 1) I, s, σ, r, q, θ, ν, t]/(a^2 + a - v^2 + I (2 a + 1) v),
      {v, -10, 10}, Method → {Automatic, "SymbolicProcessing" → 0}]]];
VGEuropeanPut[k_, s_, σ_, r_, q_, θ_, ν_, t_] :=
  VGEuropeanCall[k, s, σ, r, q, θ, ν, t] + k Exp[-r t] - s;
PutOptionVGFFTx[N_, A_, S_] := Module[{deltax},
    deltax = 2 * A / N;
    Table[{S * Exp[-A + (i - 1) * deltax], VGEuropeanPut[100,
       S * Exp[-A + (i - 1) * deltax], 0.12, 0, 0, -0.33, 0.16, 1]}, {i, 1, N + 1}]
   ];
```

Code to compute the errors.

```
    PutOptionVGFFTt[M_, T_] := Module[{deltat},
      deltat = T / M;
      Table[{(j - 1) * deltat, VGEuropeanPut[100,
```

```
        100, 0.12, 0, 0, -0.33, 0.16, (j - 1) * deltat]}, {j, 1, M + 1}]
     ];
FPut = PutOptionVGFFTx[200, 0.5, 100]
ImpliedVolatilityx[price_, x_, K_, T_, r_, S_] := Module[{p, Ix, It, o},
   p[o_] := price /. {o → x};
   Ix = Table[{p[Length[price]][[j, 1]],
      FinancialDerivative[{"European", "Put"}, {"StrikePrice" → K,
        "Expiration" → T, "Value" → p[Length[price]][[j, 2]]},
       {"InterestRate" → r, "CurrentPrice" → S, "Dividend" → 0},
       "ImpliedVolatility"]}, {j, 1, Length[price]}]
  ];
ImpliedFFTx =
  ImpliedVolatilityx[PutOptionVGFFTx[200, 0.5, 100], x, 100, 1, 0, 100];
ImpliedPIDEx = ImpliedVolatilityx[PriceputVGPIDE, x, 100, 1, 0, 100];
Err[W_, A_, U_] := Module[{deltax, error},
   deltax = (A - W) / U;
   error =
    Table[Abs[ImpliedFFTx[[j, 2]] - ImpliedPIDEx[[j, 2]]] * 100, {j, 1, U}];
   ListLinePlot[Table[{-A + i * deltax, error[[i]]}, {i, 1, U}],
    PlotRange → {0, 50}]
  ];
Err[-0.5, 0.5, 100]
PutOptionVGFFTt[200, 1];
priceputVGFFFinalt =
  Transpose[N[PutOptVGFFFinal[100, 100, VGDensity[0.16, 0.12, -0.33, x],
      0, 0.16, 0.12, -0.33, x, -0.5, 0.5, -1, 1, 1, 100, 200,
      0.6, 200, Max[100 - 100 * Exp[xx], 0], xx]]][[100 / 2 + 1]];
ImpliedVolatilityt[price_, x_, M_, K_, T_, r_, S_] :=
  Module[{p, Ix, It, o, m, deltat},
   p[o_] := price /. {o → x};
   deltat = T / M;
   Ix = Table[{(j - 1) * deltat,
      FinancialDerivative[{"European", "Put"}, {"StrikePrice" → K,
        "Expiration" → (j - 1) * deltat, "Value" → p[Length[price]][[j, 2]]},
       {"InterestRate" → r, "CurrentPrice" → S, "Dividend" → 0},
       "ImpliedVolatility"]}, {j, 1, Length[price]}]
  ];
ImpliedFFTt = ImpliedVolatilityt[PutOptionVGFFTt[200, 1], x, 100, 100, 1, 0, 100]
ImpliedPIDEt =
 Reverse[ImpliedVolatilityt[priceputVGFFFinalt, x, 200, 100, 1, 0, 100]]
Err[T_, M_] := Module[{deltat, error},
   deltat = T / M;
   error =
    Table[(Abs[ImpliedFFTt[[j, 2]] - ImpliedPIDEt[[j, 2]]]), {j, 1, M + 1}];
   ListLinePlot[Table[{(i - 1) * deltat, error[[i]]}, {i, 1, M}],
    PlotRange → {0, 1}]
  ];
Err[1, 100]
```

A.5 Value of a Knock-out option using Monte Carlo

```
Simulation of prices under the Variance Gamma Process.
    path[s_, r_, sigma_, k_, theta_, n_, T_] :=
  Module[{deltat, deltax, normal, gama, sum, m, g, prices, distGama, distND},
   deltat = T / n;
   g = (1 / k) * Log[1 - theta * k - 0.5 * k * sigma^2];
   m = Table[r * i * deltat + i * deltat * g, {i, 1, n}];
   distND = NormalDistribution
     [0, 1];
   distGama = GammaDistribution[deltat / k, 1];
   normal = Table[RandomReal[distND], {i, 1, n}];
   gama = Table[k * RandomReal[distGama], {i, 1, n}];
   deltax = Table[0, {i, 1, n}];
   For[i = 1, i ≤ n, i++,
    deltax[[i]] = sigma * normal[[i]] * Sqrt[gama[[i]]] + theta * gama[[i]];
   ];
   sum = Table[0, {i, 1, n}];
   sum[[1]] = deltax[[1]];
   For[j = 2, j ≤ n, j++,
    sum[[j]] = sum[[j - 1]] + deltax[[j]];
   ];
   prices = Table[0, {i, 1, n}];
   For[i = 1, i ≤ n, i++,
    prices[[i]] = s * Exp[sum[[i]] + m[[i]]];
   ];
   Last[prices]
  ];
Computation of a binary option using Monte Carlo.
    mcprice[n_, nt_, s_, K_, T_, r_, sigma_, k_, theta_] :=
  Module[{simul, soma, preço, y, S, mat},
   simul = n; S = s; mat = T; soma = 0;
   y = Table[path[S, r, sigma, k, theta, nt, mat], {j, 1, simul}];
   For[i = 1, i ≤ simul, i++,
    If[y[[i]] ≥ K, soma = soma + 1, soma = soma];
   ];
   preço = Exp[-r * T] * (1 / simul) * soma
  ];

knockout[sim_, n_, nt_, s_, K_, T_, r_, sigma_, k_, theta_] := Module[{kprice},
   kprice = (Sum[mcprice[n, nt, s, K, T, r, sigma, k, theta], {j, 1, sim}]) / sim
  ];
kp = Table[{0.25 + (2 - 0.25) / (50) * (j - 1), knockout[25, 100, 100,
    0.25 + (2 - 0.25) / (50) * (j - 1), 1, 0.1, 0, 0.25, 2, -0.1]}, {j, 1, 50}]
kp01 = Table[{0.25 + (2 - 0.25) / (50) * (j - 1), knockout[25, 100, 100,
     0.25 + (2 - 0.25) / (50) * (j - 1), 1, 0.5, 0, 0.25, 2, -0.1]}, {j, 1, 50}];
kp05 = Table[{0.25 + (2 - 0.25) / (50) * (j - 1), knockout[25, 100, 100,
     0.25 + (2 - 0.25) / (50) * (j - 1), 1, 1, 0, 0.25, 2, -0.1]}, {j, 1, 50}];
ListLinePlot[{kp, kp05, kp01}, DataRange → {0, 2}, AxesLabel → {S, Price}]
```

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