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MATHEMATICAL FINANCE

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INTERNSHIP REPORT

A QUANTITATIVE INVESTIGATION INTO THE DETERMINANTS OF RISK CAPACITY

HAYDN LLEWELLYN HERBERT MARTIN

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SUPERVISION:

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Abstract

For financial advisers, the Risk Profile is a crucial component of delivering the best possible experience to the client. This Risk Profile is composed of Risk Capacity, which relates to the socio-economic situation that the investor finds themselves in, and Risk Tolerance, which is associated with the psychological composition of the investor. Risk Tolerance is vague and of questionable use to the adviser in terms of determining the Risk Profile. Risk Capacity, conversely, can be measured objectively using data that is easy to obtain and process. Risk Capacity then, rather than Risk Tolerance, should be both the focus of academic research and the foundation of the Risk Profile. However, this is not true in reality. This project attempts to correct this misallocation of attention by quantitatively assessing the determinants of Risk Capacity. It measures the effect that investment horizon, goals, net income, and net assets have on the ability of the investor to take risks using simulations via Monte Carlo methodology, mathematical derivation utilising probability theory, and logical analysis. The conclusions of this project are that investors with a long investment horizon, small and flexible goals, small and stable expenses, and large and liquid net assets are able to take more risk. These findings have varied implications for advisers and supply the framework from which a model of Risk Capacity could be based on.

Keywords: Risk Profile; Risk Tolerance; Risk Capacity; Terminal Value of the Portfolio; Probability of Shortfall, Probability of Withdrawal.

Acknowledgements

I bestow thanks to those around me in aiding the completion of this project primarily for the act of staying out of the way. Not that I didn't receive any help, far from it, but I work best when left on my own to figure things out independently. These last 6 months have been the most isolated of my life, largely due to circumstances outside of my control. Whilst most people may find this depressing - it was at times - this allowed me to truly focus on this project and I was able to think deeply about the subject matter. Hopefully, this is reflected in the pages that follow.

Firstly, I would like to thank everyone at Advicefront, but mostly Jose. Jose, you allowed me to work independently, to follow my intuitions, and to explore areas that might not be useful. Thank you. Without this approach to management, I would not have been able to generate the quality of research that I did whilst working for Advicefront.

Secondly, I would like to thank my family. I completed this project almost entirely in a basement of my family home, locked down with the rest of my immediate family - all 5 of them - for months. Although there were some minor arguments I only wanted to tear my hair out (if I had any, that is) on two or three occasions, a good result considering the circumstances. I don't think I was the easiest person to live with at the time (irritable and hermit-like for much of the period), so the fact that they put up with me for the whole of quarantine and didn't bother me too much is a minor miracle. Thank you, Mum, Dad, Rhys, Carys and Ieuan. Love you all.

My friends deserve no thanks whatsoever. They did nothing. In fact, I would say they had an overall negative influence on my productivity by inconveniently doing things like making me laugh when my project was driving me insane, inviting me to BBQs, hosting weekend trips, going for a drink and a chin-wag, and doing other things that good friends do. Shame on you guys.

Finally, I must pay hommage to the ideas that this project was based on. To my lecturers at ISEG, thank you. I was initially sceptical as to the usefulness of the material that we were taught over the last two years but I found it extremely useful during the completion of this project. I especially thank Nuno for guiding me and answering some doubts I had. I would be remiss not to express my gratitude to the great thinkers in the field of probability. Without the use of probability theory, quantitative analysis in this report would not be possible. So, thank you Pascal, Fermat, Bayes, Gauss, Bernoulli, Laplace, Chebyshev, Markov, Kolmogorov, and others. I also thank thinkers who have significantly influenced my approach to several matters contained within this project: Taleb, Mandlebrot, Peters, Keynes, Hume, Knight, Arrow, Hayek and Mises are those who initially come to mind.

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Glossary

E/I Expenses/Income. v, 20, 21, 23, 27

GBM Geometric Brownian Motion. v, 8, 11, 18, 20, 23, 34

IH Investment Horizon. v, 8–13, 20, 32, 37

POS Probability of Shortfall. v, 8, 9, 11–13, 15

RC Risk Capacity. v, 1–7, 15, 20, 31–33, 37

RP Risk Profile. v, 1, 37

RT Risk Tolerance. v, 1, 2, 4, 5, 37

RTVP Real Terminal Value of the Portfolio. v, 17–19

TVP Terminal Value of the Portfolio. v, 8, 18, 19, 21, 23–25, 31

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1 Introduction

This report forms part of my master's degree in Mathematical Finance at the Lisbon School of Economics and Management. As part of this degree, I participated in a four-month internship at Advicefront, a FinTech start-up that provides a platform in which advisers can manage the relationship they have with their clients. The following report details my work for Advicefront, which was largely focused on risk. Specifically, how advisers can represent risk to clients and how they could assess the risk-taking abilities of these clients. In my work on risk representation, I outlined an explanation of the risks associated with holding financial assets that advisers could use to educate clients. This included my "Principles of Risk", as well as graphical representations of the concepts contained within this explanatory piece.

Before a description of my work associated with the risk-taking abilities of clients, some terms must first be defined, as this has been an area in which there is no agreed-upon terminology. In fact, there has been significant differences surrounding the terms used which has inevitably stunted progress (Nobre and Grable, 2015). The categorisation outlined below was used because it allows for the splitting of risk-considerations into subjective and objective elements. Doing so allows for a quantitative investigation into the objective elements and provides a clear framework from which an analysis of the area in it's totality is possible. However, these terms are not used universally in comparative literature on this topic. As such, I ask that this is taken into consideration when reading other works in this sphere.

When assessing the risk-taking abilities of the client, one must build a Risk Profile (RP) of the client. The RP represents the catalogue of all the measured information regarding how able and willing an investor is to take risks with their portfolio. This profile has traditionally been composed of Risk Tolerance (RT) and Risk Capacity (RC) (Klement, 2015, pg. 3). RT is a measure of how **willing** an investor is to accept the possibility of negative outcomes, knowing that this typically is accompanied by the possibility of experiencing positive outcomes. It is based on the psychological composition of the investor. RC, by contrast, is a measure of the **capacity** of the investor to take risks. It is based on the socio-economic situation that the investor finds themselves in. As there is a distinct difference between the two (Ananthan et al., 2017), RT should be separated from RC (Hanna et al., 2011).

RT has been analysed extensively in the academic literature (see Hanna et al., 2001; Roszkowski et al., 2005; Grable and Lytton, 1999); much ink has been spilled on how to develop supposedly "scientific" methods of measuring RT. However, no strong evidence to support its usage currently exists.

The most widely accepted method of assessing RT is generally acknowledged to be psychometric testing (Roszkowski et al., 2005). Although these tests are reliable (Grable and Lytton, 1999; Hallahan et al., 2004; Callan and Johnson, 2002), their predictive validity (how RT scores correlate to actual behaviour) remains questionable. One of the most comprehensive attempts to prove the predictive validity of these models was presented by Kuzniak et al. (2015), who looked at the model developed by Grable and Lytton (1999). They regressed several variables against the equity ownership percentage of the participants in their sample, including an RT score. They found that RT had the largest effect on equity ownership, with a statistically-significant coefficient of 0.25. This seems to prove that RT has predictive validity. However, examining this figure more closely it is understood that this implies that a 1-point increase in the RT score leads to a 0.25 percentage point increase in equity holding, *ceteris paribus*. This means that moving from the minimum of the RT scores observed, 13, to the maximum, 47, would only result in an 8.5 percentage point increase in equity holding. One potential explanation for this is that this effect isn't large only because equity holding is an imperfect proxy for the willingness of investors to take risk. This seems like a reasonable conclusion as the R-squared for the regression was only 0.31. Accepting this reasoning implies that this study provides only weak evidence in support of the predictive validity of RT. An alternative conclusion would be that their measurement of RT doesn't have a substantial correlation with the willingness to take risk.

The determinants of RC can, unlike RT, be quantitatively examined in an objective manner, allowing one to arrive at robust conclusions. This is what this project seeks to do.

If one wants to examine the capacity of investors to take risk, it is important to first arrive at a definition of risk. In the context of financial planning, the definition provided by Robert Jeffrey can be used:

> The real risk in holding a portfolio is that it might not provide its owner, either during the interim or at some terminal date or both, with the cash he requires to make essential outlays.
>
> In: Jeffrey (1984)

This definition is useful in its' correctness but not in its applicability. How does one know if an asset or portfolio is risky under this definition? It's hard to say. Under this definition, risk is determined by what I call *fluctuation risk* and *bankruptcy risk*. Fluctuation risk is the risk that the value of the asset or portfolio changes so that when it comes to liquidation it may not provide the investor with adequate “cash he requires to make essential outlays”. Bankruptcy risk is the risk of the asset or portfolio losing all of its value. Under this scenario, 0% of the cash the investor requires will be available to him. Both of these types of risk have to be taken into account of and, during risk representation, the client should be made aware of both of these distinct types of risk to ensure a holistic understanding of the risk environment as it pertains to capital markets.

The report focuses exclusively on fluctuation risk: when different levels of risk are discussed this refers exclusively to different levels of volatility of the portfolio. This is done for analytical purposes; the conclusions of the project would remain broadly the same if bankruptcy risk was considered. Using fluctuation risk as a proxy for risk, the impact that different factors have on the investor’s ability to take risk were assessed. Specifically, investment horizon, goals, net income, and net assets were considered in order to quantitatively gauge whether or not these factors influence RC.

The project is split into 8 chapters. Chapter 1 - “Introduction”, that you have just read, outlines my work for Advicefront and gives a brief overview of the project. Chapter 2 - “Setting the Scene” outlines the current state of affairs pertaining to RC. Chapters 3, 4, 5 and 6 are devoted to each of the four determinants of RC (investment horizon, goals, net income, and net assets) that were analysed. Chapter 7 - "Responding to Criticism: A Pre-emptive Strike”, outlines some of the potential criticisms of the project as well as my response to these criticisms. Chapter 8 - "Conclusion" summarises the project and discusses some of its implications.

2 Setting the Scene

2.1 RC in more detail

RC defines how much risk an investor is able to take, given their present socio-economic situation and what it is likely to look like in the future (Klement, 2015, pg. 3). Given their situation, how able are they to bear the risk of receiving a reduced income and/or valuation from/of their portfolio? Having a high RC is favourable for investors because, historically in capital markets, taking more risk has led to higher returns .

2.2 Academic approaches

The volume of scientific research as it pertains to RC pales in comparison to that of RT. Although the reason for this isn't immediately obvious, one suspects it is associated with the fact that RT is a more complex, elusive concept. RC is easier to define and far more concrete and therefore neglected in the academic community. It is difficult to find academic material associated with RC (aside from brief mentions in papers on RT), but it does exist.

Some researchers have attempted to *describe* what levels of risk are observed in populations differentiated by demographic and socio-economic factors. Cavezzali and Rigoni (2012) allude to this general concept, although they differentiate their ideas in two main ways. Firstly, they observe that the ability of an investor to take risk seems to affect the amount of cash they hold, rather than their unique mix of risky assets. Secondly, they emphasise the influence that socio-economic factors have on this cash holding, rather than demographic factors.

A few have offered ideas as to what could contribute to determining RC, but mostly in an informal way in the midst of a paper about RT. Cordell (2001) gave one of the only formalised lists that exists in the literature, stating that the following factors influence RC:

- Age
- Portfolio goals and constraints
- Income
- Expenses
- Balance sheet

- Financial obligations
- Insurance coverages

This is a fairly comprehensive list. Although he didn't outline a model, he did give hints as to how one could be developed, proposing the use of a method similar to credit rating except with an outcome on a continuum rather than pass/fail.

Some researchers have suggested ways to actual model RC, although most have taken predominantly qualitative approaches. Bosner and Lakehal-Ayat (2008), in their study of RT and RC amongst college students, used a questionnaire based on factors such as age, job security, living situation, number of years until retirement, etc. to determine RC. Cordell (2002), building on his work from the previous year, suggested advisers judge the RC of their clients by primarily considering "the amount and stability of income relative to fixed and discretionary expenses.". He also stated the need to incorporate other factors that he listed in his 2001 paper. The adviser should assess all of these factors to arrive at an RC score.

Grable (2008) measured RC in a similar way, albeit with a slightly higher level of quantification. He asked a group of advisers what they thought contributed to RC. These answers were then ranked from one to ten on a scale of importance by the advisers and the top six factors were taken (positive net worth, positive cash flow, emergency fund ratio, savings ratio, adequate life insurance and current ratio). These were then used to create questions with binary answers that would give the investor an RC score out of five.

These qualitative approaches are good for a 'quick and dirty' analysis: they give a good initial approximation of the risks an investor is able to take. However, this method of determining RC has several flaws which make it a non-viable long-term solution. Firstly, these methods often miss a vital variable. For example, the score Grable (2008) suggests does not take goals into account. Secondly, these models are often imprecise. It's better to be roughly right than precisely wrong, but this doesn't mean that one should use blunt instruments. To paraphrase Einstein[1], one should be as precise as possible, but no more. There is scope for more precision than these qualitative models allow for.

Thirdly, these approaches are usually heavily reliant on adviser judgement. Whilst practitioner judgement often trumps methods proposed by academics, adviser judgement has been shown to be systematically flawed (Roszkowski et al., 2005). One is also subject to the judgement of one's specific adviser under this regime. The practitioners would make the defence that "that is what they get paid for". This may be true, but wouldn't it

[1]Einstein supposedly said something like "Everything should be as simple as it can be, but not simpler." (Sessions, 1950)

be better to have a systematic way to determine RC, rather than being vulnerable to the specific judgemental capacity of each individual adviser? Overall, qualitative methods rely on commonly-held beliefs by advisers, in either the factors that determine RC or the actual assessment of RC itself. Again, it should be emphasised that using the experience of practitioners isn't inherently wrong. It's very likely to be right, in fact. However, currently-proposed factors require quantitative investigation to determine their efficacy.

Hanna and Chen (1997) considered what effect RC (which they called "objective risk tolerance") would have on optimal portfolios, using investment horizon and the ratio of financial assets as a proportion of wealth as a proxy for RC. They simulated across different rates of return, levels of risk aversion and financial assets as a proportion of wealth over different time horizons and calculated the expected utility of each outcome. Their conclusion was simple: with a short time horizon and a large ratio of financial assets to wealth, one cannot afford to take risks.

MES (2012) documented one of the most sophisticated quantitative approaches available. Using Monte Carlo analysis, they simulated portfolio performance over multiple investors with different characteristics and noted important outputs from each simulated path. For example, they noted the number of negative periods (in the case of a shortage scenario before death, the number periods the investor experienced after they ran out of money was counted). Using this information they formulated the following measure:

New Metric = 100 − MINIMUM((% Negative Periods*(100 − Average % of Target When Negative)*AVERAGE(Value of the $ based on weighting)/100),100)

Both these pieces of research are valuable and give a useful hint at quantitative methods. However, there is still room for improvement. The problem with the Hanna and Chen (1997) paper is obvious: they fail to take into account important factors that affect RC. The MES (2012) paper, meanwhile, doesn't tell the investor what their RC is *today* based on objective measures of their present situation. That is to say, they fail to translate their findings into implications for the risk-taking capacity of investors based on socio-economic and demographic factors.

2.3 My thoughts

As MES (2012) acknowledged, the current approach to RC is largely anecdotal. An objective measure of RC is required, based on the *current* situation of the investor. The future is fundamentally unknowable and for the measure to remain objective, it can only take facts about the current situation, both socio-economic and demographic, into account.

To lay the foundations for the development of a measure for RC, one must first analyse the determinants of RC. By assessing the body of research associated with RC discussed in the previous section, as well as via interaction with practitioners, a list of determinants of RC to quantitatively examine was determined. These factors encapsulate essentially everything that is currently thought to determine RC. Although interrelated they are sufficiently separate, allowing one to isolate specifically how each factor influences RC. Analysing only four factors will allow for clear conclusions and will make later work in this area easier.

RC essentially depends on four things:

1. Investment Horizon
2. Goals
3. Net Income
4. Net Assets

These four factors will determine how much risk an investor is able to take. All other possible contributors are either incorporated into one or more of these factors or do not significantly contribute to RC. The analysis, therefore, will focus on these four factors.

3 Investment Horizon

Investment Horizon (IH) is defined as the length of time between the present date and the goal date. Two tests were devised in an attempt to illustrate the effects that IH has on one's ability to take risks. In the first, shortfalls that an investor with a set portfolio risk experienced with different IHs and different goals were analysed. In the second, different risk levels were introduced and IH was examined more granularly.

3.1 Constant risk

Geometric Brownian Motion (GBM), which was outlined by Black and Scholes (1973) in their model for option pricing (based on work from Samuelson (1965)), was used to simulate the returns of the portfolio, with μ (a value of 0.05 was assigned) being the rate of return and σ (a value of 0.1 was assigned) being the volatility of the portfolio (see Appendix B.3 for R code used and B.2 and B.1 for proof of its validity). The evolution of the portfolio price can be characterised by the following equation:

$$dX_t = \mu X_t dt + \sigma X_t dW_t \tag{1}$$

With price at any time t being given by:

$$X_t = X_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t} \tag{2}$$

Where X_t is portfolio value at time t and W_t refers to a Weiner process at time t where $W_t \sim N(0, t)$.

Figure 1 shows the output of 100,000 observations of the Terminal Value of the Portfolio (TVP) for a portfolio with the stated statistical characteristics over a 1, 5, 10, 20 and 30 year period. Comparing a 1-year IH to a 30-year IH it is clear that under the longer IH an investor is more likely to achieve more favourable investment outcomes.

One must compare these generated returns to goals that the investor has to make the benefits of IH clearer. To do this, the TVP for each simulation was compared to different goals. Specifically, goals of £100,000, £200,000, £500,000 and £1,000,000 were examined, paths for each IH were simulated and the Probability of Shortfall (POS) for each goal was considered (Figure 2 - see Appendix B.4 for R code).

By looking at Figure 2, it's immediately obvious that the POS decreases as IH in-

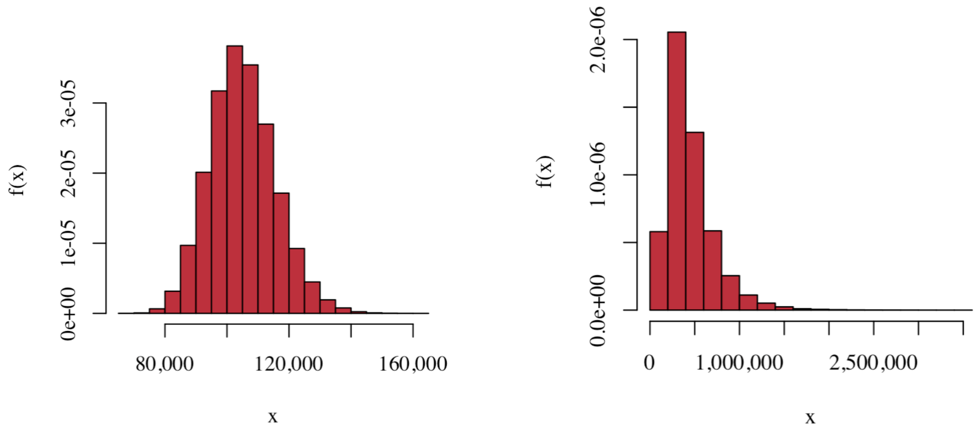


FIGURE 1: Comparison of TVPs 1-year vs. 30-year IH.

Figure 1 Note: Histogram of 100,000 simulations of the Terminal Value of the Portfolio (TVP) generated using Geometric Brownian Motion for 1-year (left) and 30-Year (right) Investment Horizons (IHs).

creases. At higher goal levels, the goal is simply not attainable at short IHs. What's more, the relationship appears to be concave: the more the time horizon increases, the greater the rate of decrease in the POS at higher goal levels. For goal two (doubling the £100,000 portfolio to £200,000), the drop in the POS is dramatic and roughly linear. Here, arguably the most realistic goal scenario, changing the IH from 10 to 20 years drastically decreases the POS. Turning now towards the goal of maintaining portfolio value (goal one), a convex relationship seems to be emerging. This means that the largest benefits of IH occur over shorter time frames (<10 years) if one simply wants to maintain one's investment. Note how the POS drops to near 0 in this scenario as the time horizon is extended.

This shows that shortfalls are less likely at longer IHs. So, investors with these longer IHs are able to take more risk with the knowledge that their goal will probably still be hit anyway. Take goal one, for example. In this scenario, an investor with a 30-year IH will be able to take more risk than one with a 10-year IH. This is because an investor with a 10-year IH still has a reasonable chance of not hitting their goal at their current level of risk, so cannot really afford to take on extra risk. However, because the chance of not hitting their goal is so low for the investor with the 30-year IH, they can afford to take more risk, safe in the knowledge that they will probably still hit their goal.

It is also possible to estimate some descriptive statistics for the shortfalls of each combination of IH and goal size for 100,000 observations (Table I, II and III - see Appendix B.5 and B.6 for R code). For the maximum shortfalls, the picture remains largely the same, regardless of IH: the worst-case scenario is roughly the same for all IHs. When looking at the means and medians, the average shortfall decreases as IH increases for goal two, three and four. However, the average shortfall seems to increase as IH increases for

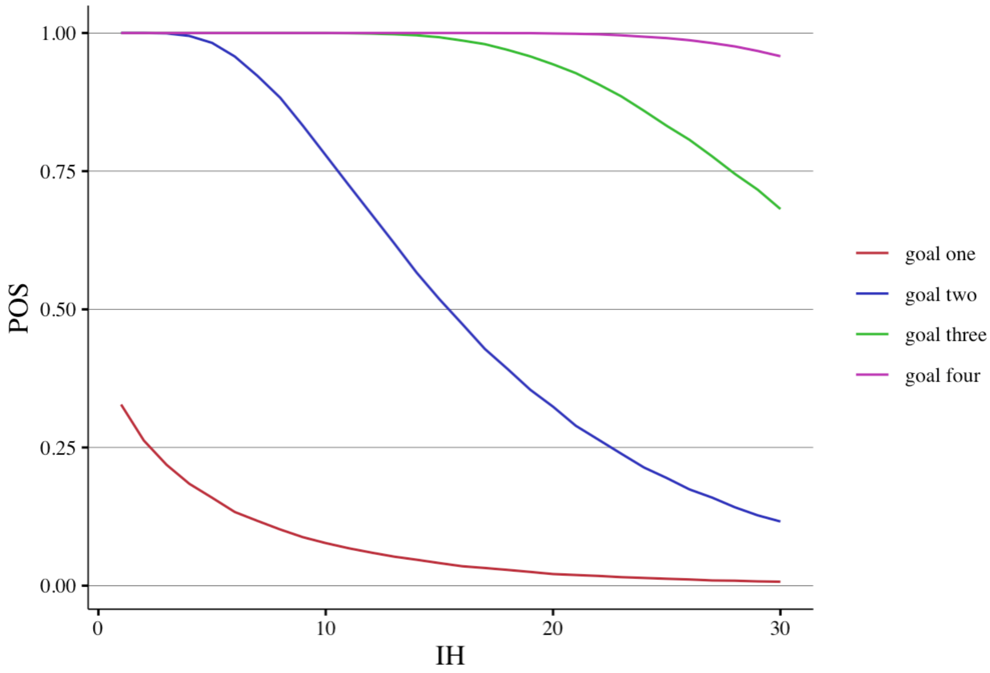


FIGURE 2: POS for different goals and different IHs.

Figure 2 Note: Estimates the Probability of Shortfall (POS) generated by 100,000 observations of the Terminal Value of the Portfolio of a portfolio with fixed characteristics at different Investment Horizons (IHs) and different goal sizes.

goal one (but there is not much acceleration after 5 years). This is due to the stochastic nature of the model: at longer time horizons there are more possible price paths. This means that when shortfalls are exclusively analysed, as they are here, the mean shortfall is likely to be slightly greater at longer IHs, but it is important to bear in mind that the probability of this shortfall is very low (see Figure 2).

	£100,000	£200,000	£500,000	£1,000,000
1-Year	33,300	135,200	432,100	932,800
5-Year	50,600	150,700	456,900	951,000
10-Year	59,700	160,500	456,100	959,500
20-Year	65,700	166,700	462,100	971,000
30-Year	57,400	163,900	461,200	970,200

TABLE I: MAXIMUM SHORTFALL OBSERVED

Table I Note: Maximum shortfall observed over 100,00 simulations for the 4 goal sizes and 5 Investment Horizons displayed. Shortfall (£) amounts are listed to the nearest £100.

Even in the case of a shortfall (less likely in longer time horizons) the size of the shortfall that one experiences is likely to be smaller under longer IHs in most scenarios.

	£100,000	£200,000	£500,000	£1,000,000
1-Year	6,300	94,900	394,900	894,900
5-Year	10,600	73,300	371,400	871,600
10-Year	12,500	57,100	335,000	835,200
20-Year	14,700	46,500	248,100	729,800
30-Year	15,400	43,100	190,400	587,600

TABLE II: MEAN SHORTFALL OBSERVED

Table II Note: Mean shortfall observed over 100,00 simulations for the 4 goal sizes and 5 Investment Horizons displayed. Shortfall (£) amounts are listed to the nearest £100.

	£100,000	£200,000	£500,000	£1,000,000
1-Year	5,100	95,400	395,300	895,400
5-Year	8,800	75,500	374,500	874,800
10-Year	10,100	56,100	343,300	843,200
20-Year	12,000	41,700	261,300	754,700
30-Year	12,200	37,300	192,900	625,100

TABLE III: MEDIAN SHORTFALL OBSERVED

Table III Note: Median shortfall observed over 100,00 simulations for the 4 goal sizes and 5 Investment Horizons displayed. Shortfall (£) amounts are listed to the nearest £100.

Investors are therefore more able to take more risks under longer IHs because they know that both the POS and the size of shortfall are likely to be smaller under these scenarios.

3.2 Varied risk

Now consider IH under three different portfolio scenarios with three different risk levels. Still using the GBM model outlined above, three new portfolios were constructed:

1. No Risk: Portfolio One (P1) with μ of 0.01 and σ of 0
2. Some Risk: Portfolio Two (P2) with μ of 0.05 and σ of 0.1
3. High Risk: Portfolio Three (P3) with μ of 0.1 and σ of 0.2

One can compare the minimum of these portfolios obtained over 1,000,000 observations against the maximum over different time horizons[2] (Figure 3 and 4 - see Appendix

[2]Obviously, for P1 the simulated maximum and minimum would be the same as it is a deterministic process.

B.7 for R code). The benefits of riskier portfolios are obvious, particularly at long IHs; the difference between minimum portfolios is relatively very small when compared to the potential maximums. Also note how after 10 years or so, the simulated minimum for each risk level seems to flatten out whereas the maximum continues to grow.

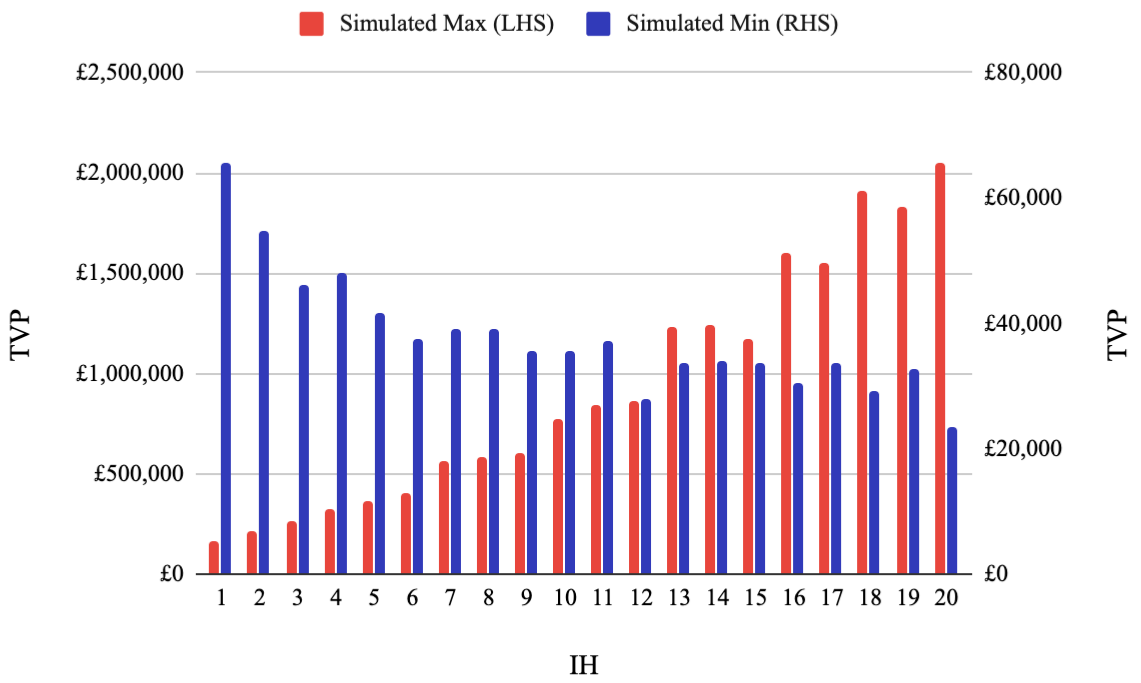


FIGURE 3: Min vs. Max TVP: Some Risk.

Figure 3 Note: The maximum (left) and minimum (right) value observed of the Terminal Value of the Portfolio (TVP) over 1,000,000 observations of the "Some Risk" portfolio (Geometric Brownian Motion with $\mu = 0.05$ and $\sigma = 0.1$).

With a longer IH, the annual rate of return required to achieve the same total return is lower than with a short IH. For example, say two investors both need to return 100% in some period. An investor with a 10-year IH will need to return roughly 7.18% annually to achieve this goal. However, an investor with a 30-year IH will only need to return 2.34%. This investor can invest in higher-risk assets because they only **require** a low return. However, the investor with the 10-year horizon **requires** a higher level of return and cannot afford to invest in riskier assets because these are more volatile and might not give the investor the return they require.

Using these portfolios consider again the POS, comparing against the same four goals (Figure 5 - see Appendix B.8 for R code). The conclusion from the low-risk portfolio is clear: if the goal exceeds £100,000 by a certain amount, invest in riskier assets otherwise the investor won't reach it. If, however, it is below some set threshold, one can divert some funds to riskier assets, safe in the knowledge that one's goal will be achieved with

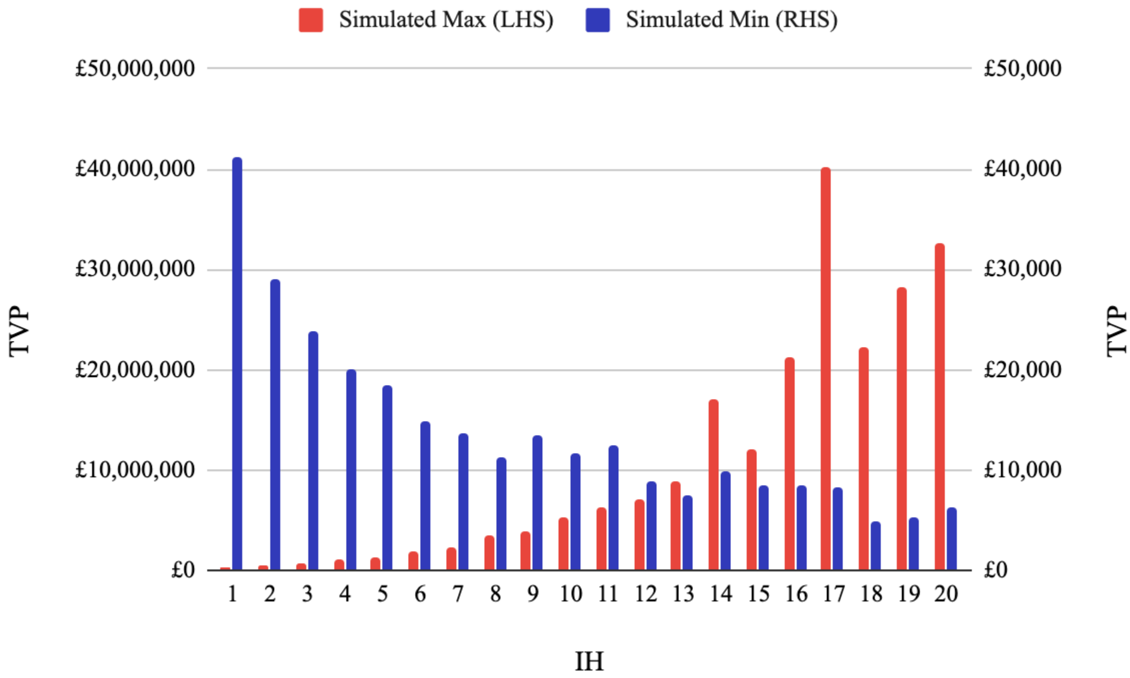


FIGURE 4: Min vs. Max TVP: High Risk.

Figure 4 Note: The maximum (left) and minimum (right) value observed of the Terminal Value of the Portfolio (TVP) over 1,000,000 observations of the "High Risk" portfolio (Geometric Brownian Motion with $\mu = 0.1$ and $\sigma = 0.2$).

(theoretical) probability 1, enjoying the upside that riskier assets are accompanied by.

This threshold increases as the IH increases. So, investors with low goals are able to take more risks when their IH is longer. For the riskier assets, note that increasing the IH decreases the POS. This effect is more pronounced with the high-risk asset. So, to decrease the POS over longer IHs for goals exceeding some certain level, it is actually less risky, in terms of POS, to invest in risky assets. Again, this shows how investors with longer IHs are more able - if not compelled - to invest in riskier assets.

	P1	P2	P3
1-Year	1.00	0.326	0.345
5-Year	1.00	0.263	0.285
10-Year	1.00	0.218	0.245
20-Year	1.00	0.184	0.212
30-Year	1.00	0.157	0.185

(a) Goal 1 (£100,000)

	P1	P2	P3
1-Year	0.00	1.00	0.999
5-Year	0.00	1.00	0.970
10-Year	0.00	0.999	0.905
20-Year	0.00	0.995	0.824
30-Year	0.00	0.982	0.744

(b) Goal 2 (£200,000)

	P1	P2	P3
1-Year	0.00	1.00	1.00
5-Year	0.00	1.00	1.00
10-Year	0.00	1.00	1.00
20-Year	0.00	1.00	0.999
30-Year	0.00	1.00	0.997

(c) Goal 3 (£500,000)

	P1	P2	P3
1-Year	0.00	1.00	1.00
5-Year	0.00	1.00	1.00
10-Year	0.00	1.00	1.00
20-Year	0.00	1.00	1.00
30-Year	0.00	1.00	1.00

(d) Goal 4 (£1,000,000)

FIGURE 5: POS for each of the 4 goals for each risk level.

Figure 5 Note: Estimated Probability of Shortfall (POS) for each of the 4 goal sizes and 5 Investment Horizons using 100,000 simulations. P1 represents a portfolio based on a Geometric Brownian Motion with $\mu = 0.01$ and $\sigma = 0$, P2 has $\mu = 0.05$ and $\sigma = 0.1$ and P3 has $\mu = 0.1$ and $\sigma = 0.2$. The Terminal Value of the Portfolio generated was compared against the goal size and the frequency of shortfall over 100,000 observations was used to generate estimated probabilities.

4 Goals

Most people invest with some kind of goal in mind. This could be funding retirement, saving for a holiday, or building a portfolio for future generations. Goals as a concept in its totality was deconstructed into size and flexibility in order to isolate the effect that each would have individually on RC.

4.1 Size

Looking again at Figure 5, one can clearly see that the larger the goal, the higher the probability of not reaching it. This effect is also observable by looking at a fixed time horizon of 10 years and using, once again, three different portfolios (except this time all three are stochastic):

1. Low Risk: Portfolio One (P1) with μ of 0.025 and σ of 0.05
2. Medium Risk: Portfolio Two (P2) with μ of 0.05 and σ of 0.1
3. High Risk: Portfolio Three (P3) with μ of 0.1 and σ of 0.2

Returning to an examination of the POS in Figure 6, one can clearly see the effect that increasing the goal size has (see Appendix C.1 for R code). The most striking feature is the severity of the relationship; the lines are very steep and in the case of P1 almost vertical. This shows that small changes in goal size can affect the POS dramatically. The effect is also apparently concave (although increasingly linear as risk increases): at high goal sizes, reducing goals by the same amount will not have as significant an effect as at low goal sizes.

As one moves from very high goals to smaller goals, the benefit of investing in higher-risk assets grows in the form of reduced POS. So as goal size reduces, the investor is better off, in terms of POS, investing in riskier assets. At very low goal size, the POS is very similar for all levels of risk. This illustrates how an investor with a low goal size can take more risk, safe in the knowledge that their goal will probably still be achieved anyway, even if they experiences a negative outcome. Investing in lower-risk assets doesn't really benefit the investor, in terms of POS. So it would seem they would be better served investing in higher-risk assets, given that these are accompanied by higher upside.

One can also demonstrate the effect of size mathematically by insisting that the goal of an investor has to be achieved using safe assets. Assume the investor can invest in two

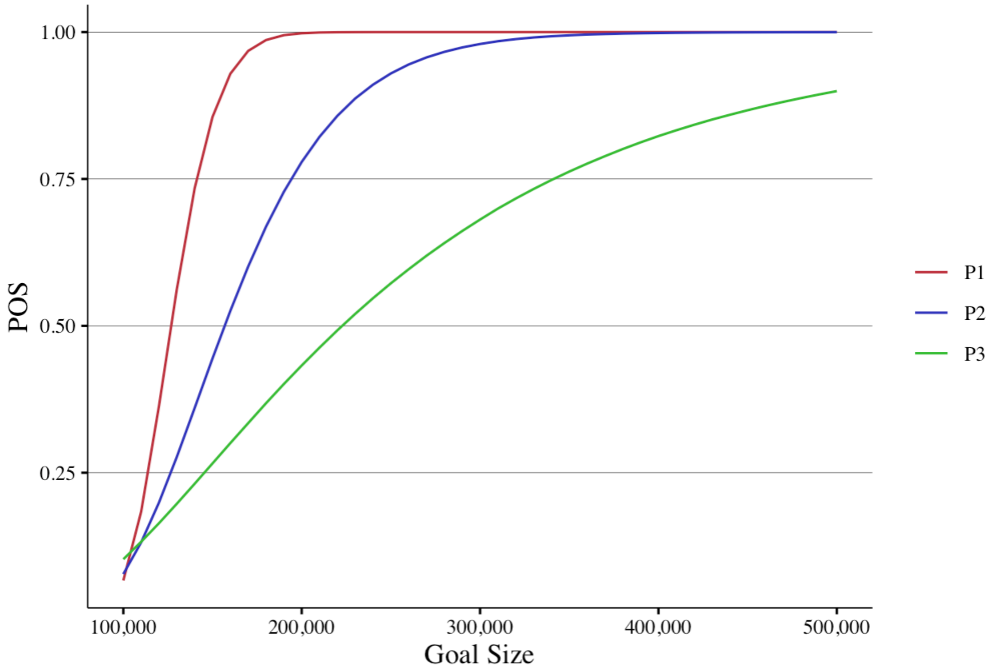


FIGURE 6: POS for different risk levels as goals increase in size.

Figure 6 Note: Terminal Value of the Portfolio for each portfolio type was calculated. This was done using a Geometric Brownian Motion with $\mu = 0.025$ and $\sigma = 0.05$ for Portfolio 1 (P1), $\mu = 0.05$ and $\sigma = 0.1$ for Portfolio 2 (P2), and $\mu = 0.1$ and $\sigma = 0.2$ for Portfolio 3 (P3). These TVPs were then compared to the goal sizes (x-axis) for 1,000,000 observations to estimate the Probability of Shortfall (POS).

assets: asset r, a risky asset, and asset s, a risk-free asset. Let X_t denote portfolio value at time t. Π_n is the proportion invested in asset n ($\Pi_r + \Pi_s = 1$) and Γ_n is the annual return of asset n ($\Gamma_r > \Gamma_s$).

So, at year t,

$$X_t = X_0 \left[(1 + \Gamma_r)^t \, \Pi_r + (1 + \Gamma_s)^t \, \Pi_s \right] \tag{3}$$

Suppose that the investor can use safe assets **only** to meet a goal G_t and invests the rest in risky assets:

$$G_t = (1 + \Gamma_s)^t \, \Pi_s \tag{4}$$

Imagine two different scenarios (1 and 2) in which the investor has two possible goal sizes where $G_t^1 < G_t^2$. So,

$$\frac{X_t^1}{X_0} = (1 + \Gamma_r)^t \, \Pi_r^1 + (1 + \Gamma_s)^t \, \Pi_s^1 \tag{5}$$

$$\frac{X_t^2}{X_0} = (1+\Gamma_r)^t\,\Pi_r^2 + (1+\Gamma_s)^t\,\Pi_s^2 \tag{6}$$

Note that, because $G_t^1 < G_t^2$, one must necessarily have $\Pi_s^1 < \Pi_s^2$. Additionally, because $\Pi_r + \Pi_s = 1$, $\Pi_r = 1 - \Pi_s$. So, (5) - (6) is equal to:

$$\begin{aligned}&(1+\Gamma_r)^t\left(1 - \Pi_s^1 - 1 + \Pi_s^2\right) + (1+\Gamma_s)^t\left(\Pi_s^1 - \Pi_s^2\right)\\ &= (1+\Gamma_r)^t\left(\Pi_s^2 - \Pi_s^1\right) + (1+\Gamma_s)^t\left(\Pi_s^1 - \Pi_s^2\right)\end{aligned} \tag{7}$$

This equation is greater than 0 because:

1. $(\Pi_s^2 - \Pi_s^1) = -(\Pi_s^1 - \Pi_s^2)$
2. $\Gamma_r > \Gamma_s \Leftrightarrow (1+\Gamma_r)^t > (1+\Gamma_s)^t \quad (\text{for } t > 0)$

Therefore, (5) > (6) and $X_t^1 > X_t^2$.

4.2 Flexibility

In reality, some goals matter more than others. For example, one would think that investors care more about whether they save enough for planned retirement in 10 years time than if they go on a trip to Hawaii in the summer. Different goals have different *real* values; not achieving certain goals will be more painful than not achieving others. All goals have an *absolute* size (monetary value) and a *real* size (how much they are worth to the investor).

Let X_t represent the absolute value of the portfolio and Z_t represent real value (at time t). τ is the importance that the investor places on goal G_t ($\tau \in [0,1]$).

$$Z_t = \begin{cases} \frac{1}{1+\tau}\cdot X_t & \text{if } X_t \le G_t \\ X_t & \text{if } X_t > G_t \end{cases} \tag{8}$$

What is the expectation of the Real Terminal Value of the Portfolio (RTVP) where period

t is the last period observed? Well, because of the Law of Total Expectation,

$$\begin{aligned} E[Z_t] &= P(X_t \le G_t) \cdot E[Z_t] + P(X_t > G_t) \cdot E[Z_t] \\ &= P(X_t \le G_t) \cdot E\left[\frac{X_t}{1+\tau}\right] + P(X_t > G_t) \cdot E[X_t] \\ &= E[X_t] \cdot \left[\frac{P(X_t \le G_t) + (1+\tau) \cdot P(X_t > G_t)}{1+\tau}\right] \\ &= E[X_t] \cdot \left[\frac{1 + \tau \cdot P(X_t > G_t)}{1+\tau}\right] \end{aligned} \tag{9}$$

This expression gets smaller as τ increases because $P(X_t > G_t) < 1$. In fact, $Z_t \to X_t$ as $\tau \to 0$. That is to say, RTVP approaches TVP as the goal of the investor becomes less and less important.

One can use this formula to analyse the effect that changing τ has on the RTVP of a portfolio with set characteristics. Using a GBM model for the portfolio with μ of 0.05 and σ of 0.1, values of the RTVP for goals of £100,000, £200,000, £300,000 and £400,000 were simulated whilst changing τ incrementally (Figure 7 - see Appendix C.2 for R code).

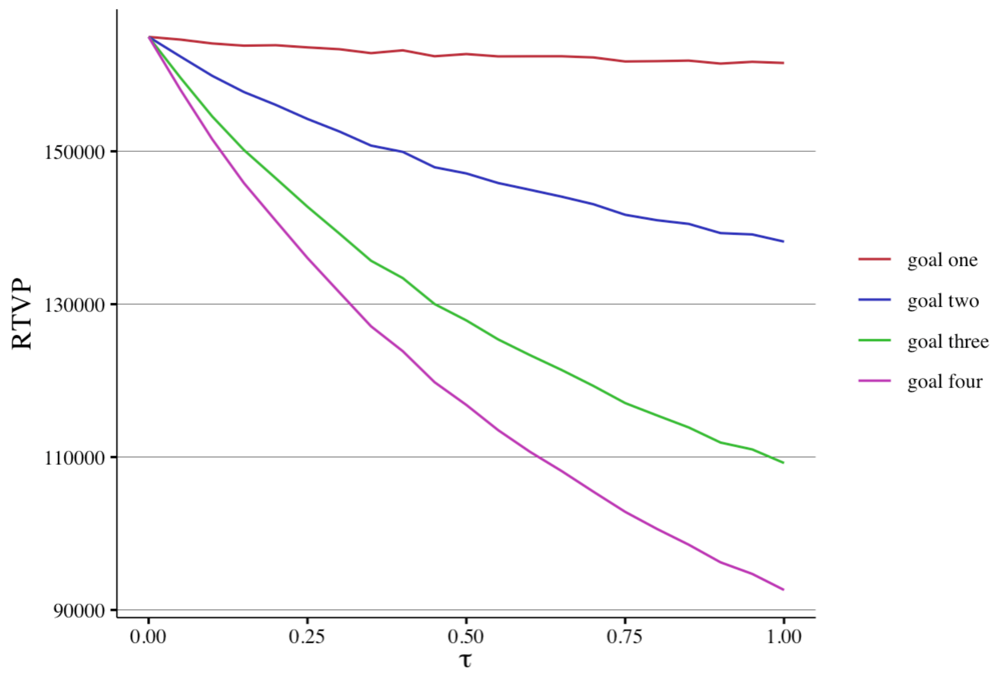


FIGURE 7: POS for different risk levels as goals increase in size.

Figure 7 Note: The Real Terminal Value of the Portfolio (RTVP) was determined using Equation (8), a Geometric Brownian Motion with $\mu = 0.05$ and $\sigma = 0.1$, and different values of the importance the investor places on achieving their goals (τ). This calculated for 4 goals (goal one = £100,000, goal two = £200,000, goal three = £300,000 and goal four = £400,000). The average of 100,000 observations was calculated to generate an estimate for the RTVP.

Clearly, as τ increases and goals become more important to the investor, the simulated RTVP decreases across all goal sizes. This happens in a fairly linear way although there is some convexity present, indicating that absolute differences in τ are more impactful at lower values of τ. At lower values of τ the RTVP moves closer and closer to the TVP: achieving the goal becomes less and less important and the value of the portfolio becomes more important.

When there is a high degree of flexibility in the goal, investors care more about the return of the portfolio and less about hitting the goal. So, investors are less concerned with worst-case scenarios of their portfolio because they are not strongly negatively impacted by not hitting their goal. They can focus more on the upside and are able to invest in riskier assets as a result.

5 Net Income

Net income can be defined as income minus expenses. So, both income and expenses influence net income. However, one can focus exclusively on expenses because income and expenses are two sides of the same coin (net income). Any analysis applied to one can be applied to the other (but the conclusions will be the opposite). For example, if one were to come to the conclusion that high expenses negatively impact RC, then low income would also negatively impact RC. If it is discovered that the number of income sources positively influences RC, then the number of expense sources will negatively influence RC. Additionally, it doesn't really make sense to look at either income or expenditure in isolation. How much, exactly, is a high income? Income and expenses need to be compared to something. The best way to gauge either is via comparison with the other.

5.1 Size

Firstly, the size of expenses (characterised by the fraction Expenses/Income (E/I)) was examined. A simulation of a portfolio using the GBM model with μ equal to 0.05 and σ equal to 0.1 was generated, given a 10-year IH. 10 shocks were subsequently generated by taking numbers from a uniform distribution with lower bound of £1,000 and an upper bound of £5,000; the location of these shocks were generated by taking observations from a uniform distribution with an upper bound of 120 and a lower bound of one (here representing month numbers of the 10-year period, see Appendix D.1 for the shock values and locations). Holding income constant at £50,000 per year, expenses were varied to ensure that the E/I took three values: 1/4, 1/2 and 3/4. These fractions were assigned to three investors: investor one (I_1) had an E/I of 1/4, investor two (I_2) had an E/I of 1/2 and investor three (I_3) had an E/Iof 3/4 (Table IV).

	I_1	I_2	I_3
Monthly Earnings	£4,166.67	£4,166.67	£4,166.67
Monthly Expenses	£1,041.67	£2,083.33	£3,125.00
Net Income	£3,125.00	£2,083.33	£1,041.67
E/I	0.25	0.50	0.75

TABLE IV: NET INCOME OF INVESTORS WITH DIFFERENT E/IS

Table IV Note: A comparison of the characteristics for the three different investors I_1, I_2 and I_3. The Expenses/Income (E/I) ratios is used to generate a figure for Net Income for all 3. All figures correct to 2 decimal places.

For each period, if an expense shock occurred it would either be able to be absorbed by the investor or the investor would have to liquidate some of their portfolio to cover the expense shock. For example, for the first shock of £2,331, I_1 would not have to withdraw from their investment account (because 3,125 > 2,331). However, I_2 would have to withdraw £247.67 and I_3 would have to withdraw £1,289.33. Over 10 years, this can have a significant effect on the TVP. In Table V a typical price path of the simulated portfolio is examined - significant differences in the TVPs are seen. Of particular note here is the gap between the withdrawal amounts and the difference between the original portfolio and the TVP for each investor.

	TVP	Difference with original portfolio	Number of Withdrawals	Withdrawal Total
Original Portfolio	£139,825	£0	0	£0
Low E/I	£133,692	£-6,133	5	£5,219
Med E/I	£126,997	£-12,828	7	£10,982
High E/I	£115,042	£-24,782	10	£20,672

TABLE V: EFFECT OF DIFFERENT E/IS ON RANDOM PRICE PATH

Table V Note: Price path generated using Geometric Brownian Motion with $\mu = 0.05$ and $\sigma = 0.1$. Withdrawal occurs if total expenses for each period (indluding the random expense shocks - see Appendix D.1) exceed income for that period. This is observed for 3 investors with different Expenses/Income (E/I) ratios - Low E/I who has an E/I of 0.25, Med E/I who has an E/I of 0.50 and High E/I who has an E/I of 0.75. The Terminal Value of the Portfolio (TVP) seen is the result that these withdrawals have on the portfolio. All figures correct to the nearest whole number.

These same expense shocks were then applied to many different simulations of the portfolio, utilising Monte Carlo methodology. After 1,000,000 simulations one can roughly determine some descriptive statistics of the TVPs (Table VI - see Appendix D.2 for R code). Clearly, as the E/I fraction increases the minimum TVP and the average TVP decrease. One can also detect a hint of non-linearity in the relationship: the jump between the returns that I_1 and I_2 might receive is smaller than that between I_2 and I_3.

In Figure 8 the same random shocks were applied to different values of E/I (0 to 1 incremented by 0.05) and the effect this change of E/I had on the TVP for the same random price path observed earlier in Table V was examined (see Appendix D.3 for R code). As expected, this line is downward-sloping: as E/I increases, the TVP decreases. The non-linearity hinted at earlier is clearly visible here: the relationship between E/I and TVP is concave. This means that the higher the value of E/I, the more the same absolute changes affect the TVP. So, it is more important to reduce one's E/I fraction at higher absolute levels of E/I to prevent unplanned withdrawal. Note that the relationship becomes linear at very high levels of E/I, when the investor is withdrawing from their

	Minimum	Mean	Median
No E	35,008.86	164,884.20	156,811.90
Low E/I	30,940.70	158,075.80	150,116.90
Med E/I	26,376.32	150,713.80	142,874.20
High E/I	19,353.44	137,735.00	130,163.90

TABLE VI: DESCRIPTIVE STATISTICS OF TVPS FOR DIFFERENT E/IS

Table VI Note: Terminal Value of the Portfolio (TVP) generated using Geometric Brownian Motion with $\mu = 0.05$ and $\sigma = 0.1$. Withdrawal occurs if total expenses for each period (indluding the random expense shocks - see Appendix D.1) exceed income for that period. This is observed for 3 investors with different Expenses/Income (E/I) ratios - Low E/I who has an E/I of 0.25, Med E/I who has an E/I of 0.50 and High E/I who has an E/I of 0.75. Simulation occurred 1,000,000 times to generate the figures seen in the table. All figures in £ and estimated to 2 decimal places.

portfolio with every expense shock.

5.2 *Volatility*

In reality, expenses vary. No-one spends the same amount each month, no matter how financially organised and careful one is. Some people's expenses vary more than others: investors experience different levels of volatility in their monthly expenses. To investigate this theoretical framework, expenses were modelled using the log-normal distribution. The logic behind this being that most expenses can't go below 0, cluster around some low mean and have the potential to be significantly higher than the mean. Think of an investor with typical expenses of £1,000 who goes on a £2,000 holiday, purchases a £5,000 watch, places a £30,000 deposit down on a house, or has a particularly expensive trip to the casino. The log-normal distribution will generate mostly low observation with increasingly-low probability of higher observations. This is similar to expenses: for most months they are roughly the same but something significant can happen with low probability that can increase the total expenses for that month significantly. The log-normal distribution probably doesn't reflect expenses perfectly (it practically doesn't allow for very large expenses, like the purchase of a house outright, for example) but it will serve perfectly adequately for the purposes of this analysis.

Let monthly income be denoted by a constant I.

Withdrawal occurs if $E_t \sim lognormal\left(\mu_n, \sigma_n^2\right) > I$ for any given month. So, the probability of withdrawal from any month $t = P\left(E_t > I\right)$.

One can use this simple estimation, as well as R and the knowledge of the relationship between the mean and variance of a normal distribution with the log-normal distribution

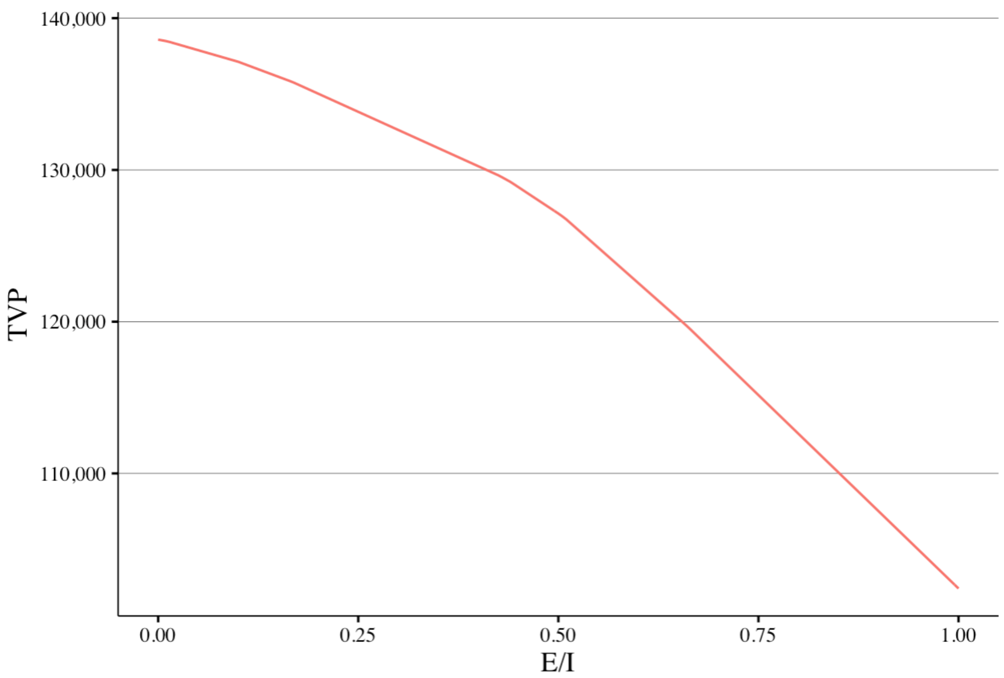


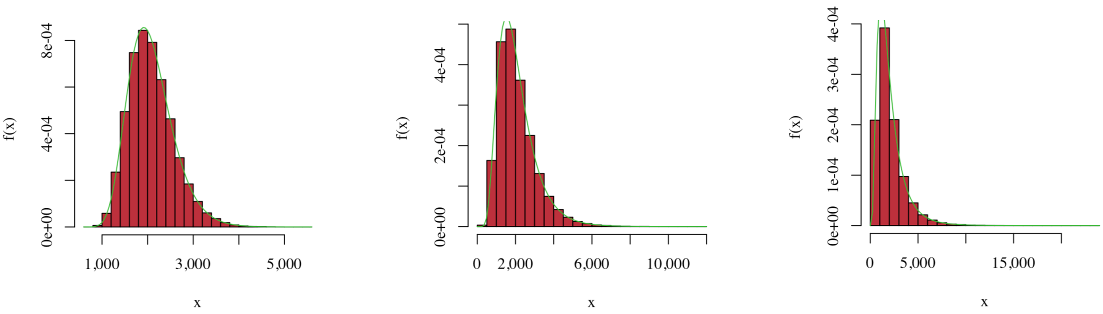
FIGURE 8: Changes to TVP of random price path as E/I changes.

Table 8 Note: Terminal Value of the Portfolio (TVP) generated using the same price path in Table V: Geometric Brownian Motion with $\mu = 0.05$ and $\sigma = 0.1$ with different values of Expenses/Income (E/I) ratios. Withdrawal occurs if total expenses for each period (indluding the random expense shocks - see Appendix D.1) exceed income for that period.

proved in Appendix D.4, to asses the impact that volatility has on the TVP. E/I was set at 1/2, with monthly income to £4,166.67 (annually £50,000) and expenses to £2,083.33 (annually £25,000). Three hypothetical investors were separated by the volatility of their expenses, the first experiencing a standard deviation of £500, the second of £1,000 and the third of £1,500. By looking at Figure 9, it is obvious that this has a significant impact on the distribution of their respective expenses.

Looking now at their descriptive statistics in Table VII, it is clear how the expenses of the high-volatility investor have the potential to go very high but also very low (see Appendix D.5 for R code). Reducing volatility has the pleasant effect of effectively eliminating the probability of a very expensive month: the low-volatility investor has a practically 0 probability of a month of over £10,000 in expenses. Contrast this with the high-volatility investor, who still has a chance, albeit very small, of a month of over £20,000 in expenses. Finally, one can see that the probability of withdrawal increases as volatility increases.

These three investors were used to see what affect these different expense volatilities might have on the investment account. Again using the GBM model, a portfolio was simulated over 10 years with $\mu = 0.05$ and $\sigma = 0.1$. As before, when the investor's expenses exceed their income for a certain month, they must withdraw from the portfolio



(a) Low Volatility (b) Medium Volatility (c) High Volatility

FIGURE 9: Histograms for the 3 levels of expense volatilites.

Figure 9 Note: Histogram of 100,000 observations for investors with 3 different expense structures - the Low Volatility has expenses with $\sigma = 500$, the Medium Volatility has expenses with $\sigma = 1,000$ and the High Volatility investor has expenses with $\sigma = 1,500$. All have the same $\mu = 2,083.33$.

	σ	Estimated Minimum	Estimated Maximum	$P(E_t > I)$	$P(E_t > 10,000)$	$P(E_t > 20,000)$
Low Volatility	500	£719.60	£5,730.50	0.00104	0.00000	0.00000
Medium Volatility	1000	£256.82	£13,945.02	0.04006	0.00012	0.00000
High Volatility	1500	£101.08	£29,260.15	0.08076	0.00298	0.00007

TABLE VII: STATISTICS FOR EXPENSES

Table VII Note: All 3 investors have the same $\mu = 2,083.33$. The estimated minima and maxima are the average of 10,000 observations of the min and max of trials of 100,000 expense occurrences. Figures accurate to 2 decimal places. Probabilities are calculated using cumulative distribution function for the log-normal distribution and Z-tables. Figures accurate to 5 decimal places.

to cover this shortfall. This leads to differing price paths, withdrawal amounts and TVPs (Figure 10, Table VIII and Table IX respectively - see Appendix D.6 for R code).

Not only do the lower-volatility investors withdraw less on average but their maximum withdrawal (in the worst-case scenario) is likely to be much less. Now looking at the TVPs, the statistics are broadly similar for the low-volatility investor and the medium-volatility investor. This is a trivial conclusion of the fact that if one's expenses don't exceed one's income very often, one won't make many withdrawals from one's portfolio and one's TVP won't be too adversely affected. There seems to be some kind of limit to the volatility of expenses that the investor can add without significant consequences. Beyond this point, extra volatility seems to start to severely negatively impact the TVP. This also means that if the volatility of expenses is low, the investor can afford to increase

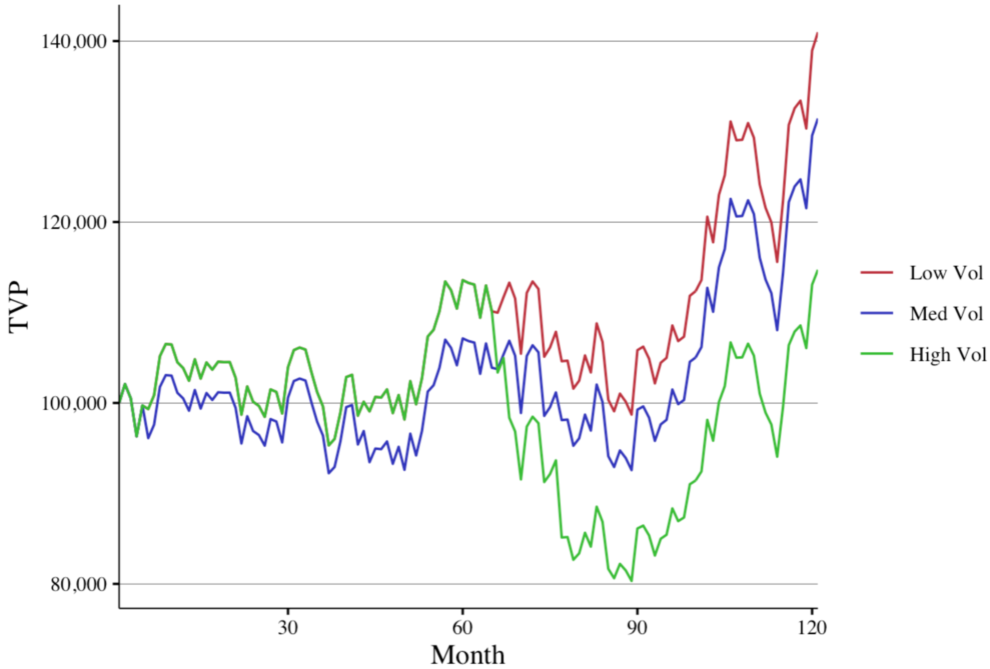


FIGURE 10: Example price path for the 3 different volatility levels.

Figure 10 Note: Example of price path generated using Geometric Brownian Motion with $\mu = 0.05$ and $\sigma = 0.1$ where 3 different investors have expenses are drawn from a log-normal distribution where Low Vol has expenses with $\sigma = 500$, the Med Vol has expenses with $\sigma = 1,000$ and the High Vol has expenses with $\sigma = 1,500$. All have the same $\mu = 2,083.33$. If any observation exceeds their income for that period (£4,166.67) then withdrawal from the portfolio occurs.

it without many negative consequences. The high-volatility investor has been crushed by their expenses. Both the mean and the maximum of the TVP that they are likely to enjoy are significantly lower. The simulated value of the minimum portfolio has turned negative. This is obviously not possible in real life and would simply result in their portfolio evaporating as they withdraw more and more money to cover pricey month after pricey month. This is the mechanism via which successively expensive months can cripple one's returns.

Granulating sigma to a greater extent, Figure 11, one can derive more insight into its exact relationship to the TVP (see Appendix D.7 for R code). The hunch from earlier appears to be correct: volatility does not appear to affect expenses before a σ of 0.2. Beyond this point, volatility does has an accelerating negative effect on the TVP, becoming fairly aggressive after σ 0.4. The effect is concave: the damage inflicted by more volatility accelerates beyond σ 0.2, but seems to become linear after roughly σ 0.6. The reason for this isn't clear but it could be associated with the non-linear relationship that volatility has with withdrawals. At some point, increasing volatility more and more may have less and less of an effect on withdrawal probability and amount.

	Mean	Max	Min
Low Vol	40.91	3,561.63	0.00
Med Vol	4,400.33	31,454.94	0.00
High Vol	15,992.71	100,682.00	0.00

TABLE VIII: WITHDRAWALS FOR EACH VOLATILITY LEVEL

Table VIII Note: 1,000,000 price paths generated using Geometric Brownian Motion with $\mu = 0.05$ and $\sigma = 0.1$ where 3 different investors have expenses are drawn from a log-normal distribution where Low Vol has expenses with $\sigma = 500$, the Med Vol has expenses with $\sigma = 1,000$ and the High Vol has expenses with $\sigma = 1,500$. All have the same $\mu = 2,083.33$. If any observation exceeds their income for that period (£4,166.67) then withdrawal from the portfolio occurs. Withdrawal amounts (£) accurate to 2 decimal places.

	Mean	Max	Min
Low Vol	164,797.00	682,818.50	35,176.30
Med Vol	159,153.30	660,098.90	28,466.71
High Vol	91,435.04	543,745.30	-73,636.65

TABLE IX: TVPS FOR EACH VOLATILITY LEVEL

Table IX Note: 1,000,000 price paths generated using Geometric Brownian Motion with $\mu = 0.05$ and $\sigma = 0.1$ where 3 different investors have expenses are drawn from a log-normal distribution where Low Vol has expenses with $\sigma = 500$, the Med Vol has expenses with $\sigma = 1,000$ and the High Vol has expenses with $\sigma = 1,500$. All have the same $\mu = 2,083.33$. If any observation exceeds their income for that period (£4,166.67) then withdrawal from the portfolio occurs. The Terminal Value of the Portfolio (TVP) was calculated and analysed. TVPs (£) accurate to 2 decimal places.

5.3 Fixed costs

A different way to approach the problem is to look not just at the total of these expenses but also at the composition. Expenses can be split into two categories: fixed costs and variable costs. Fixed costs are those that are the same, or roughly the same, each month. Examples include rent, electricity/gas/water, internet connection, food (not including restaurants), etc. Variable costs are those which are subject to change and vary, sometimes dramatically, from month to month. Examples include expenditure in restaurants and bars, going to the theatre, trips abroad, etc. Fixed costs are much harder, both psychologically and logistically, to change and, for the most part, remain the same month to month. Variable costs, however, can and do change relatively easily.

Fixed costs are important because they cannot be altered in the face of an expense shock. Variable costs can. For example, say one's boiler breaks down and a repair is required. One cannot simply reduce one's rent payment for that month to provide one with

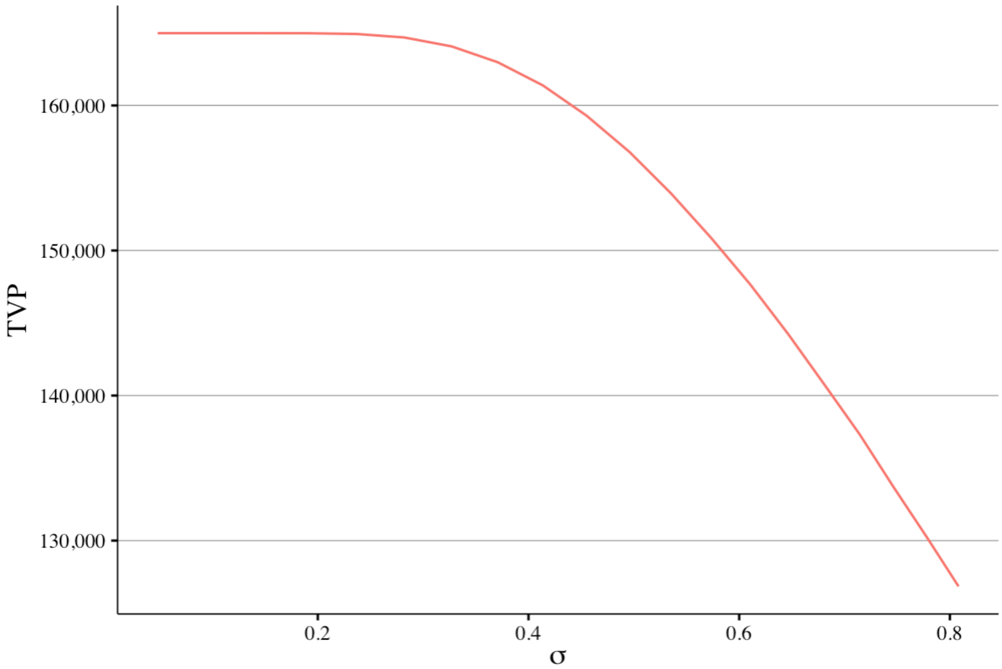


FIGURE 11: Effect of σ on TVP.

Figure 11 Note: 100,000 price paths generated using Geometric Brownian Motion with $\mu = 0.05$ and $\sigma = 0.1$ where the volatility of expenses varies with σ. All have the same $\mu = 2,083.33$. If any observation exceeds their income for that period (£4,166.67) then withdrawal from the portfolio occurs. The Terminal Value of the Portfolio (TVP) was estimated based on an average over the 100,000 observations for each σ.

the extra money required to cover this unexpected expense. One can, however, not go to the bar on Friday and not go out to eat on Saturday. Hence, when one posits that the size of expenses matter what one should really posit is that the size of **fixed** expenses matter. These are the expenses that cannot be changed and result in unplanned withdrawals from the portfolio.

This can be illustrated by looking at the following three investors with equal E/I ratios but different fixed costs. The "Low FC" investor's fraction of total expenses as fixed costs is 1/4, for the "Med FC" investor it's 1/3, and for the "High FC" investor it's 1/2. Note that each investor has the same monthly expenses - £2,083.33 - and income - £4,166.67, it's only the composition of those expenses that is different. Looking at Figure 12 it is clear that, even though E/I remains constant, withdrawal amounts can differ significantly for different expense shocks, depending on the fraction of total expenses that are fixed.

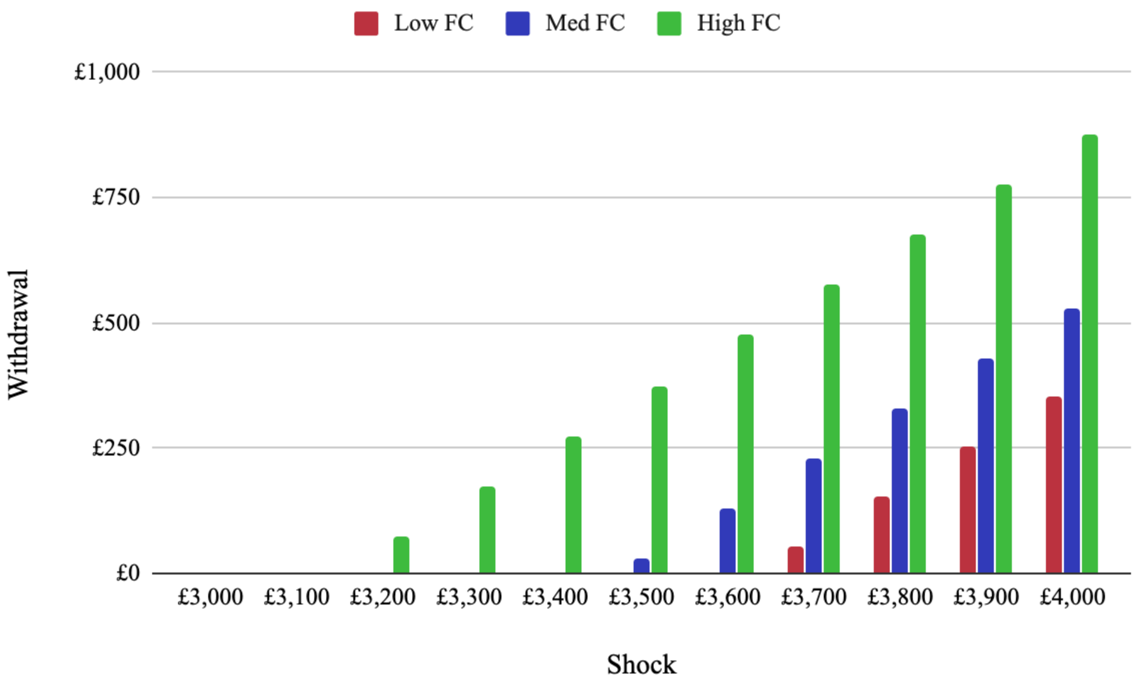


FIGURE 12: The effect of different fixed costs on withdrawal.

Figure 12 Note: Each investor has the same Expenses/Income ratio (income is £4,166,67 and expenses are £2,083.33) but a different fraction of expenses as fixed costs: Low FC has a fraction of 1/4, Med FC has a fraction of 1/3 and High FC has a fraction of 1/2. If the expense for each period exceeds non-variable costs (fixed costs and expense shocks) for a period then withdrawal occurs.

One can also investigate this relationship mathematically. Let income be a constant I s.t. $I_1 = I_2 = ... = I_t = I$. Expenses in each (non-expense-shock) period t are the aggregate of two separate components: fixed costs (C_t^F) and variable costs (C_t^V). Total costs $C_t = C_t^F + C_t^V$. Fixed costs are constant $C_1^F = C_2^F = ... = C_t^F = C^F$. Variable costs are also constant *except* from periods in which the investor experiences expense shocks S_i in periods $j \neq k \neq ... \neq n$:

$$S = \begin{bmatrix} S_j \\ S_k \\ \vdots \\ S_n \end{bmatrix} \tag{10}$$

In periods in which there is are no expense shock, variable costs are equal to a constant:

$$C_1^V = C_2^V = ... = C_t^V = C^V \quad \text{if } t \neq j, k, ..., n \tag{11}$$

Note that $C^F + C^V < I$. This means that in non-shock periods, the investor does not need to withdraw and there is no uncertainty associated with the situation in this context. Here, periods in which the investor experiences an expense shock will be exclusively focused on, a far more uncertain proposition. In shock periods, withdrawal may be necessary to cover expenses for that month. To prevent this, the investor can alter their variable costs in these periods and/or withdraw from their portfolio. Note that even if there is a shock, it may be absorbed by net income, after variable costs have been altered. Variable costs are only reduced to prevent, or limit, withdrawals from the investment account. A full classification of variable costs in shock periods can therefore be represented by the following:

$$C_t^{VS} = \begin{cases} C^V & \text{if } I - C^F - C^V \geqslant S_t \\ I - C^F - S_t & \text{if } I - C^F - C^V < S_t \leqslant I - C^F \\ 0 & \text{if } I - C^F < S_t \end{cases} \tag{12}$$

In shock periods, $C_t = C^F + C_t^{VS} + S_t$. A withdrawal occurs if total expenses, C_t, are greater than income in that period. Now let X_t be a binary random variable, defining whether or not the investor is forced to withdraw from their investment account in shock period t:

$$X_t = \begin{cases} 1 & \text{if } C_t > I \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

How often can the investor expect to withdraw in shock periods? In other words, what is $E[X]$? Via the Law of Total Expectation,

$$\begin{aligned} E[X] &= E[X \mid C > I] \cdot P(C > I) + E[X \mid C \leqslant I] \cdot P(C \leqslant I) \\ &= P(C > I) \\ &= P\left(C^F + C^{VS} + S > I\right) \end{aligned} \tag{14}$$

Now utilising the Law of Total Probability (as well as (12)),

$$\begin{aligned} &E[X] = P\left(C^F + C^{VS} + S > I \mid I - C^F - C^V \geqslant S\right) \cdot P\left(I - C^F - C^V \geqslant S\right) + \\ &P\left(C^F + C^{VS} + S > I \mid I - C^F - C^V < S \leqslant I - C^F\right) \cdot P\left(I - C^F - C^V < S \leqslant I - C^F\right) \\ &+ P\left(C^F + C^{VS} + S > I \mid I - C^F < S]\right) \cdot P\left(I - C^F < S\right) \end{aligned} \tag{15}$$

To simplify this equation one can analyse the three different situations in which S takes three different values:

$\mathbf{I - C^F - C^V \geqslant S \Rightarrow C^{VS} = C^V}$
$\Rightarrow$

$$\begin{gathered} P\left(C^F + C^{VS} + S > I \mid I - C^F - C^V \geqslant S\right) \cdot P\left(I - C^F - C^V \geqslant S\right) = \\ P\left(C^F + C^V + S > I \mid C^F + C^V + S \leqslant I\right) \cdot P\left(I - C^F - C^V \geqslant S\right) = 0 \end{gathered} \tag{16}$$

$\mathbf{I - C^F - C^V < S \leqslant I - C^F \Rightarrow C^{VS} = I - C^F - S}$
$\Rightarrow$

$$P\left(C^F + C^{VS} + S > I \mid I - C^F - C^V < S \leqslant I - C^F\right) \cdot P\left(I - C^F - C^V < S \leqslant I - C^F\right)$$
$$= P\left(C^F + I - C^F - S + S > I \mid S \leqslant C^V\right) \cdot P\left(S \leqslant C^V\right) = 0 \tag{17}$$

$\mathbf{I - C^F < S \Rightarrow C^{VS} = 0}$
$\Rightarrow$

$$\begin{aligned} & P\left(C^F + C^{VS} + S > I \mid I - C^F < S]\right) \cdot P\left(I - C^F < S\right) \\ & = P\left(C^F + S > I \mid C^F + S > I]\right) \cdot P\left(I - C^F < S\right) \\ & = P\left(I - C^F < S\right) \end{aligned} \tag{18}$$

Using (16), (17) and (18) in (15),

$$E\left[X\right] = P\left(S > I - C^F\right) \tag{19}$$

The probability of withdrawal is dependent on the expense shocks and the size of both income and fixed costs. As the expense shocks and fixed costs become larger, and income becomes smaller, the probability of withdrawal increases. Of particular note is the absence of variable costs in this equation.

6 Net Assets

Net assets are defined as the total value of assets minus the total value of liabilities. Note that assets will mostly be discussed. Similarly to the reasons for only focusing on expenses in Chapter 5, assets and liabilities are two sides of the same coin. What is true for assets as it pertains to RC will be true, but in reverse, for liabilities. Hence, one only needs to focus on one (here it is assets). Note also that when net assets are discussed, net assets **other than assets contained within the portfolio** are what are *really* being discussed.

Net assets affect RC via the effects they have on other factors already analysed. They have an indirect effect by impacting the other factors that contribute to RC.

6.1 Assets as a buffer

Assets, subject to liquidity, act as a buffer to expense shocks. Typically, when an investor experiences monthly expenses above their income, they must withdraw from their investment account to cover this expense. However, if they have a buffer in the form of assets (such as an emergency fund), they may not have to withdraw after all. As previously demonstrated in the analysis of net income, unplanned withdrawal can have disastrous consequences for the investor and inhibit the investor's ability to take risk. Considering the same example as mentioned in Chapter 5 (as used in table V), it is obvious to see that having a buffer can dramatically reduce the amount an investor withdraws from their portfolio. Liquid assets of £10,000 would have prevented the majority of withdrawals for all 3 investors in the example discussed. The presence of a buffer increases the TVP by decreasing both the size and likelihood of withdrawals, allowing the investor to take more risk.

6.2 Reducing goal size

The second way that assets influence RC is by effectively reducing the goal size that investors have. If an investor has a goal size of £1,000,000 but already has £500,000 of assets, this goal has been effectively reduced to one of £500,000. For example, if an investor wants to move to Australia in 10 years time and calculates they need £500,000 to do it then without any assets they would need to accumulate £500,000. However, if they own a house with a value of £250,000, they can liquidate this asset to be left with only £250,000 left to accumulate. They have effectively reduced their goal size! Take this to the extreme: if the investor has assets that they are happy to liquidate that have value greater than all their goals combined, they can take all the risk they want, safe in the knowledge that their goals will still be achieved no matter what happens in the capital markets. Goal size has a significant impact on RC, as was discussed in Chapter C. Hence, net assets also have a significant impact on RC, by influencing effective goal size.

6.3 Extending the IH

Assets act as a realised investment path that has just been liquidated and can be invested again. For example, an investor could have a 20-year IH with an original investment of £100,000. At the 10-year mark, their portfolio is valued at £150,000. So, they have total assets (including their portfolio) of £150,000 with 10 years remaining on their IH. Consider now an investor with a 10-year IH with an original investment of £100,000. However, this investor also has liquid assets worth £50,000. So, they have total assets (including their portfolio) of £150,000 with 10 years remaining on their IH. Exactly the same position that the first investor found themselves in.

This point may seem purely philosophical, and maybe it is, but there is no accounting difference between the two investors. Assets can be viewed as realised returns: they can signify an already-realised investment returns path from the past. This is how net assets can be seen to be equivalent to extending the time horizon of the investor, which influences RC as has already been demonstrated.

7 Responding to Criticism: A Pre-emptive Strike

7.1 Other factors

Although the main contributors to RC have already been investigated, there are other factors that have not been mentioned that could have some kind of influence. In this section, these ideas will be given some attention, before the reasons for their exclusion from the analysis are briefly outlined.

Income and debt

The most obvious factors that are not present in the analysis are income and debt. These two factors are included in the list Cordell (2001) provided. However, the reasons for not including these factors in detail in their respective sections has already been discussed earlier in the project and will not be repeated here.

Special factors

Others would argue for the inclusion of what I call *specific factors*. These factors are incorporated into already-existing factors but receive special attention in both the academic literature and in the minds of practitioners. The most prominent of these special factors are the existence of insurance, the size of one's emergency fund, and the number of dependants and the costs of those dependants (Cordell, 2001). These concerns can be quickly and easily dismissed by stating that they are already incorporated into one of the four examined factors.

Ratios

There are those who favour the use of ratios, rather than absolute figures. For example, Hanna and Chen (1997) use the ratio of financial assets to total wealth in their analysis. The reasons for this are clear: it provides a relative measure for factors that influence RC, making the assessment of RC easier. This is simply a different, equally valid way of approaching the problem, although more applicable when constructing a model for RC, rather than assessing the determinants of RC.

Future considerations

Some define RC as the art of comparing future cash flows. Why, then, are considerations about the future missing from this analysis? Because the future is not objective. Models that approach the future in a quantitative manner are naive and can lead to misleading results. Models that use qualitative methods, such as asking the investor questions such as "What is your income likely to be in 10 years time?" are overly-optimistic about our ability to forecast. If one desires an objective measure, it cannot include considerations of the future. In this case, it is better to have inputs to the model that are definitely correct and have a simpler model than to have faulty inputs in a more complicated model.

7.2 The Geometric Brownian Motion model of asset prices is flawed

The GBM model used as a proxy for portfolio returns has been shown to be flawed. Primarily, the problem with the model is that it assumes that asset returns are normally distributed. This is demonstrably false, as has been illuminated by Mandelbrot (1997), amongst others. Furthermore, Peters (2011) showed that GBM is non-ergodic. This means that the ensemble average of an observation may be different from its time average, making analysis based on average returns (μ) difficult.

These criticisms are valid. GBM is flawed and is not an accurate representation of asset prices. However, for the purposes of this analysis, this doesn't really matter. Firstly, generic portfolios were modelled. No model would fit because each portfolio has different components. Secondly, the outcome from different portfolios were only used **for purposes of comparison**. How accurately these portfolios may reflect reality is irrelevant. The GBM model was only used to generate observations to compare different scenarios, not to model real-life asset returns. So, GBM does not need to reflect real asset returns for the analysis to be accurate.

7.3 Lack of empirical analysis

It is customary in these types of quantitative investigations to use empirical data. This has not been the case in this project, largely due to my scepticism around our ability to derive meaningful inductive conclusions in uncertain domains. Techniques developed in domains of risk may not be always applicable in these uncertain domains.

The distinction between domains in which probability, and therefore statistics, can be freely applied and domains in which one must be more careful is important. When conducting statistical analysis, the obstacle that must be overcome is the "tendency to impose on inductive thought the conventions and preconceptions appropriate only to deductive reasoning." (Fisher, 1956, pg. 109). The two must be kept distinct and separate. This is a distinction that Arrow ensures that he emphasises:

> With some inaccuracy, descriptions of uncertain consequences can be classified into two major categories, those which use exclusively the language of probability distri-

> butions and those which call for some other principle, either to replace or to supplement.
>
> In: Arrow, 1951, pg. 410

He states that the main difference between the two types is the ability to apply techniques developed evaluating games, gambling and insurance on general areas of uncertainty. Essentially, what applicability do techniques based in theoretical, fully-understood environments have on those that are not so well understood? Knight thought not much: "There is much question as to how far the world is intelligible at all...It is only in the very special and crucial cases that anything like a mathematical study can be made" (Knight, 1921, pg. 209). He had the similar insight to differentiate between risk, where the generating function is known, and uncertainty, where it is unknown. Probability, in his view, and therefore statistics, could only be applied to problems of risk; problems of repeated trials with independent and identically distributed observations. Keynes had similar ideas. In his *A Treatise on Probability* he argues that probabilities are not always calculable (Keynes, 1921, Part I). He was also somewhat critical of statistical inference, particularly the Law of Large Numbers (Keynes, 1921, Part V).

Taleb and Pilpel repeat the point: "Certainty, risk, and uncertainty differ not merely in the probabilities (or range of probabilities) one assigns to P, but in the strategies one must use to make a decision under these different conditions." (Taleb and Pilpel, 2004, pg. 5). They go further than others in providing insight into *when* exactly one may be operating in an uncertain domain. They come to the conclusion that only distributions that are bounded or exhibit fast convergence can be dealt with using a risk framework. Others must be considered under the lens of uncertainty. In these uncertain domains there are problems with statistical inferences and no solution may be actually possible. So, one must ensure that one is in a domain of risk, rather than uncertainty.

How, Taleb asks, can one know this? The problem of estimating the distribution of a random variable is that these are self-referential (Taleb, 2007). Naive estimation of these distributions can be harmful, especially in domains in which small probabilities can have large impacts because estimation errors are important when the consequences are severe, such as in the case of the returns of assets in the capital markets. Even if the distribution is somewhat accurately determined, knowing the distribution and the ability to make meaningful predictions are not the same thing. Additionally, this distribution may change over time because often we find ourselves in an open system, in an unbounded domain.

This is often the case in complex environments, such as asset prices (Mauboussin, 2002). This categorisation is accompanied by several problems for the ways that capital markets have been historically analysed and makes future analysis difficult because of the attributes of complex adaptive systems (CASs). As Mauboussin (2002) states, markets exhibit aggregation, adaptive decision rules, nonlinearity, feedback loops, and other characteristics of CASs. This possesses several problems for statisticians. Firstly, the link between risk and reward is non-linear and non-clear. Secondly, the market can't be defined as an entirely stochastic process. Cause and effect is present, just very difficult to decipher. This is something that Mandelbrot (1997) discovered too, observing that large price changes seemed to be able to be explained by causal relationships. He also alluded to the fact that price records are non-stationary, meaning that descriptive statistics such as the variance are effectively impossible to obtain.

8 Conclusion

The aim of this project was to quantitatively asses the determinants of RC. As a result of this investigation it was demonstrated that investors with a long investment horizon, small and flexible goals, small and stable expenses, and large and liquid net assets are able to take more risk.

As mentioned in the introduction, the purpose of this project was not to propose a measure for RC. However, the analysis outlined could provide the foundations for an objective model to be developed. The task for future researchers will be determining to what extent each of the outlined factors influence RC. After this has been established, a model can be easily constructed.

Ultimately, the purpose of the project was to help financial advisers, who want to retain clients and add new clients. The best way to do this is performing all the duties of an adviser in a highly-competent fashion. Although this includes generating acceptable returns, this is not the only thing that influences the adviser's ability to retain and attract clients. In fact, according to The Financial Adviser Client Experience Report published by Qualtrics (2017), the primary reason why surveyed investors originally selected their financial adviser was because they were "Trusted", not because of investment track record. Accurately assessing the risks clients can take is both a big part of returns and trust. Hence, an accurate measure of RC is vital to the success of advisers.

This is good news for advisers because RC is (should be) an objective measure which is theoretically easy to calculate. It also has the additional benefit that it is easy to change. Simply adjusting the goals of the client, their IH, the structure of their expenses, etc. can make a significant impact on their ability to take risks. In the broader picture, RC forms the cornerstone of the RP of the client. RT, difficult to define, conceptualise and assess, should be viewed with scepticism, regardless of how "scientific" the method of acquisition is. What advisers really need on the psychological side of the RP is some kind of measure of how much anxiety volatility/uncertainty gives to clients and how likely they are to panic and force the adviser to sell in difficult periods, deviating from the financial plan.

Representing risks to clients is also of vital importance. Clients need to be educated on the risks they are taking and the potential consequences they have for the portfolio. They also need some kind of way to monitor their progress in terms of risk, to incentivise risk-control in order to increase their capacity to take risks within their portfolio (or, put another way, to increase their RC).

Returning to the same questions asked by Qualtrics (2017), only **one** of the reasons given by the surveyed clients was associated with the investment track record. All other factors - such as fees, customer service, introductory offers - were associated with the business of advisery. These things are much easier to control than investment returns. Maybe, to attract and retain clients, advisers should first focus on these things and on managing the expectations of the clients (via accurate risk profiling and representation) and worry about out-sized returns later. A combination of excellent risk-related practices and services, quality business operations, and acceptable investment performance is sure to be a winning combination.

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A General

Please note that when working in R, unless stated otherwise or evidence to the contrary is clearly visible:

- 100,000 observations were used for the various Monte Carlo analyses.
- The seed was set to 336.

B Investment Horizon

B.1 GBM reformulation for use in R

$$X_t = X_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t} \tag{20}$$

Note that a Weiner process has normally-distributed increments with unit variance: $W_{t+1} - W_t \sim N(0,1)$. This also means that $W_T - W_0 \sim N(0,T)$. Let $W_T = \sqrt{T} \cdot Z$ where $Z \sim N(0,1)$. In this scenario, the expectation and variance of W_T are the following:

$$E\left[W_T\right] = E\left[\sqrt{T}\right] \cdot Z = \sqrt{T} \cdot E[Z] = 0 \tag{21}$$

$$Var\left(W_T\right) = Var\left(\sqrt{T}\right) \cdot Z = T \cdot Var(Z) = 1 \tag{22}$$

Hence, in terms of expectation and variance, $\sqrt{T} \cdot Z$ is equivalent to W_T. One can use this knowledge, as well as Equation (20), to formulate an expression for X_T.

$$X_T = X_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T} \cdot Z} \tag{23}$$

$$X_T = X_{t_0} e^{\left(\mu - \frac{\sigma^2}{2}\right)(T - t_0) + \sigma\sqrt{T - t_0} \cdot Z} \tag{24}$$

Now introduce N increments. This is the number of time steps that the portfolio will be modelled over. Now $\Delta t = \delta = \frac{T - t_0}{N}$.

$$X_{t_0 + \delta} = X_{t_0} e^{\left(\mu - \frac{\sigma^2}{2}\right)\delta + \sigma\sqrt{\delta} \cdot Z} \tag{25}$$

One can use this equation to model a portfolio in R when one needs to examine individual time steps.

B.2 GBM for individual time steps

```
x0 # Start value
mu # Expected return
sigma # Sigma value (SD of stock)
t0 # Starting time
t # End time
n # Number of periods observed (in days)
gbm_all_values <- function(x0, mu, sigma, t0, t, n){
  dt <- (t-t0)/n
  X <- vector(length = n+1)
  X[1] <- x0
  for(i in 2:(n+1)){
    X[i] <-  X[i-1] * exp((mu-sigma^2/2)*(dt) + sigma*sqrt(dt)*rnorm(1))}X}
```

FIGURE 13: Code used for generating a GBM model with all time steps.

B.3 GBM for terminal value

```
t_valu <- function(x0, mu, sigma, t){
  x0 * exp((mu-sigma^2/2)*(t) + sigma*sqrt(t)*rnorm(1))}
```

FIGURE 14: Code used for generating a GBM model with only the terminal value.

B.4 IH constant risk POS

```
prob_short <- function(t, goal){
count_g <- 0
for(i in 1:obs){t_val_obs <- t_valu(x0, mu, sigma, t)
  if(t_val_obs < goal){count_g <- count_g + 1}
  i <- i + 1}
prob_g <- count_g/obs
print(prob_g)}
goals <- matrix(c(100000, 200000, 500000, 1000000))
y <- 1
g <- 1
prob_outcome <- matrix(nrow = 30, ncol = 4)
for(g in 1:4){
for(y in 1:30){
  prob_outcome[y, g] <- prob_short(y, goals[g])}}
```

FIGURE 15: R code used for generating the POS for different IHs.

B.5 IH shortfall function

```
short_dist <- function(t, goal){
  s <- 1
  short <- matrix()
  for(i in 1:obs){t_val_obs <- t_valu(x0, mu, sigma, t)
    if(t_val_obs < goal){
      short[s] <- (goal - t_val_obs)
      s <- s + 1}
      i <- i + 1}
      short}
```

FIGURE 16: Function for generating shortfalls for different IHs.

B.6 IH shortfall tables generation

```
goals <- matrix(c(g_one, g_two, g_three, g_four))
years <- matrix(c(1, 5, 10, 20, 30))
y <- 1
g <- 1
short_mean <- matrix(nrow = 5, ncol = 4) # Mean
  for(g in 1:4){for(y in 1:5){short_mean[y, g] <- mean(short_dist(years[y], goals[g]))}}
  short_median <- matrix(nrow = 5, ncol = 4) # Median
  for(g in 1:4){for(y in 1:5){short_median[y, g] <- median(short_dist(years[y], goals[g]))}}
  short_max <- matrix(nrow = 5, ncol = 4) # Max
  for(g in 1:4){for(y in 1:5){short_max[y, g] <- max(short_dist(years[y], goals[g]))}}
```

FIGURE 17: Generating tables of shortfalls for different IHs.

B.7 Different risk levels max vs. min functions

```
min_over_t <- function(nt, obs_new){
  var_low_risk <- matrix(length(1:nt))
  var_med_risk <- matrix(length(1:nt))
  var_high_risk <- matrix(length(1:nt))
  med_risk_mat <- matrix(nrow = length(1:nt), ncol = length(1:obs_new))
  high_risk_mat <- matrix(nrow = length(1:nt), ncol = length(1:obs_new))
  var_matrix <- matrix(nrow = length(1:nt), ncol = 3)
    for (i in 1:nt){for(j in 1:obs_new){
        med_risk_mat [i, j] <- t_valu(x0, mu = 0.05, sigma = 0.1, t = i)
        high_risk_mat [i, j] <- t_valu(x0, mu = 0.1, sigma = 0.2, t = i)
        j <- j + 1}
      var_low_risk [i] <- t_valu(x0, mu = 0.01, sigma = 0, t = i)
      var_med_risk [i] <- min(med_risk_mat[i, ])
      var_high_risk [i] <- min(high_risk_mat[i, ])
      var_matrix [i, 1] <- var_low_risk [i]
      var_matrix [i, 2] <- var_med_risk [i]
      var_matrix [i, 3] <- var_high_risk [i]
      i <- i + 1}
    var_matrix <- as.data.frame(var_matrix)
    var_matrix}
```

(a) Function for generating the minima

```
max_over_t <- function(nt, obs_new){
  var_low_risk <- matrix(length(1:nt))
  var_med_risk <- matrix(length(1:nt))
  var_high_risk <- matrix(length(1:nt))
  med_risk_mat <- matrix(nrow = length(1:nt), ncol = length(1:obs_new))
  high_risk_mat <- matrix(nrow = length(1:nt), ncol = length(1:obs_new))
  var_matrix <- matrix(nrow = length(1:nt), ncol = 3)
    for (i in 1:nt){for(j in 1:obs_new){
      med_risk_mat [i, j] <- t_valu(x0, mu = 0.05, sigma = 0.1, t = i)
      high_risk_mat [i, j] <- t_valu(x0, mu = 0.1, sigma = 0.2, t = i)
      j <- j + 1}
    var_low_risk [i] <- t_valu(x0, mu = 0.01, sigma = 0, t = i)
    var_med_risk [i] <- max(med_risk_mat[i, ])
    var_high_risk [i] <- max(high_risk_mat[i, ])
    var_matrix [i, 1] <- var_low_risk [i]
    var_matrix [i, 2] <- var_med_risk [i]
    var_matrix [i, 3] <- var_high_risk [i]
    i <- i + 1}
  var_matrix <- as.data.frame(var_matrix)
  var_matrix}
```

(b) Function for generating the maxima

FIGURE 18: Functions for generating the different values for the minima and maxima.

B.8 IH varied risk POS function

```
prob_short_med_risk <- function(goal, y, obs_p){ # Some risk
  count_g <- 0
  for(i in 1:obs_p){t_val_obs <- t_valu(x0, mu = 0.05, sigma = 0.1, t = y)
    if(t_val_obs < goal){count_g <- count_g + 1}
    i <- i + 1}
  prob_g <- count_g/obs_p
  print(prob_g)}
prob_short_high_risk <- function(goal, y, obs_p){ # High risk
  count_g <- 0
  for(i in 1:obs_p){t_val_obs <- t_valu(x0, mu = 0.1, sigma = 0.2, t = y)
    if(t_val_obs < goal){count_g <- count_g + 1}
    i <- i + 1}
  prob_g <- count_g/obs_p
  print(prob_g)}
```

FIGURE 19: Functions for generating POSs for different IHs under the two different risk scenarios.

C Goals

C.1 POS for goal size analysis

```
g_size <- seq(100000, 500000, 10000)
low_count_g <- vector(length = length(g_size))
med_count_g <- vector(length = length(g_size))
high_count_g <- vector(length = length(g_size))
obs <- 1000000
  for (i in 1:obs) {
    t_val_low <- t_valu(x0, low_mu, low_sigma, t)
    t_val_med <- t_valu(x0, med_mu, med_sigma, t)
    t_val_high <- t_valu(x0, high_mu, high_sigma, t)
    for (j in 1:length(g_size)) {
      if(t_val_low < g_size[j]){low_count_g[j] <- low_count_g[j] + 1}
      if(t_val_med < g_size[j]){med_count_g[j] <- med_count_g[j] + 1}
      if(t_val_high < g_size[j]){high_count_g[j] <- high_count_g[j] + 1}}
      i <- i + 1}
  low_prob_g <- low_count_g/obs
  med_prob_g <- med_count_g/obs
  high_prob_g <- high_count_g/obs
```

FIGURE 20: Code for generating the POSs for the Goal Size analysis.

C.2 TVP for flexibility analysis

```
goals <- c(100000, 150000, 200000, 250000)
I <- seq(0 , 1, 0.05)
nsim <- 100000
term_vec_goals <- matrix(nrow = nsim, ncol = length(goals))
real_term_vals <- matrix(nrow = length(I), ncol = length(goals))
for (k in 1:length(I)) {for (i in 1:nsim) {
    term_vec <- vector(length = nsim)
    term_vec[i] <- t_valu(x0, mu, sigma, t)
    for (j in 1:length(goals)) {if (term_vec[i] <= goals[j]) {
      term_vec_goals[i, j] <- (1/(1 + I[k]))*term_vec[i]} else {
        term_vec_goals[i, j] <- term_vec[i]}}}
  for (j in 1:length(goals)) {
    real_term_vals[k, j] <- mean(term_vec_goals[, j])}}
```

FIGURE 21: Code for generating the TVPs for the Goal Flexibility analysis.

D Net Income

D.1 Expense shocks

Shock Month	Shock Amount (£)
118	3467
75	1410
52	2072
70	3750
65	4873
38	4637
7	2331
22	2042
45	2390
48	4117

TABLE X: EXPENSE SHOCKS

D.2 Statistics of TVPs code

```
nsim <- 1000000
nl <- n + 1
no_ei <- matrix(nrow = nl, ncol = nsim)
low_ei <- matrix(nrow = nl, ncol = nsim)
med_ei <- matrix(nrow = nl, ncol = nsim)
high_ei <- matrix(nrow = nl, ncol = nsim)
shock_month <- c(118, 75, 52, 70, 65, 38, 7, 22, 45, 48)
shock_amount <- c(3467, 1410, 2072, 3750, 4873, 4637, 2331, 2042, 2390, 4117)
G <- matrix(nrow = nl, ncol = nsim)
for (i in 1:nsim) {
  X <- vector(length = nl)
  X[1] <- x0
  G[1, i] <- 0
  no_ei[1, i] <- X[1]
  low_ei[1, i] <- X[1]
  med_ei[1, i] <- X[1]
  high_ei[1, i] <- X[1]
  for(l in 2:nl){
    X[l] <-  X[l-1] * exp((mu-sigma^2/2)*(dt) + sigma*sqrt(dt)*rnorm(1))
    G[l, i] <- (X[l] - X[l-1]) / X[l-1]
    no_ei[l, i] <- no_ei[l-1, i] * (1 + G[l, i])
    low_ei[l, i] <- low_ei[l-1, i] * (1 + G[l, i])
    med_ei[l, i] <- med_ei[l-1, i] * (1 + G[l, i])
    high_ei[l, i] <- high_ei[l-1, i] * (1 + G[l, i])
    for(j in 1:10){if(l == shock_month[j]){
        if(shock_amount[j] > 3125){low_ei[l, i] <- low_ei[l, i] - (shock_amount[j] - 3125)}
        if(shock_amount[j] > 2083.33){med_ei[l, i] <- med_ei[l, i] - (shock_amount[j] - 2083.33)}
        if(shock_amount[j] > 1041.67){high_ei[l, i] <- high_ei[l, i] - (shock_amount[j] - 1041.67)}
}}}}
```

FIGURE 22: Code for generating the descriptive statistics for the TVPs.

D.3 Incremental E/I code

```
ratios <- seq(from = 0, to = 1, by = 0.005)
income <- vector(length = length(ratios))
ratio_val <- matrix(nrow = nl, ncol = length(ratios))
for (i in 1:length(ratios)) {
  income[i] <- 4166.67 * (1 - ratios[i])
  CX <- rand_with [1:121, 2]
  GX <- vector(length = nl)
  GX[1] <- 0
  ratio_val[1, i] <- CX[1]
  for (l in 2:nl) {
  GX[l] <- (CX[l] - CX[l-1]) / CX[l-1]
  ratio_val[l, i] <- ratio_val[l-1, i] * (1 + GX[l])
    for(j in 1:10){if(l == shock_month[j]){if(shock_amount[j] > income[i]){
        ratio_val[l, i] <- ratio_val[l, i] - (shock_amount[j] - income[i])}}}}}
```

FIGURE 23: Code for generating the TVPs for incremental changes in E/I.

D.4 Sigma and mew proof

Let the mean of the log-normal distribution be denoted by μ and the variance σ^2. Let the mean of the normal distribution required to produce a log-normal distribution with mean μ be denoted by μ_n and the variance required to produce a log-normal distribution with variance σ^2 be denoted by ${\sigma_n}^2$.

It is known that,

$$\mu = e^{\mu_n + \frac{{\sigma_n}^2}{2}} \tag{26}$$

$$\sigma^2 = \left(e^{{\sigma_n}^2} - 1\right) e^{2\mu_n + {\sigma_n}^2} \tag{27}$$

From equation (26),

$$ln(\mu) = \mu_n + \frac{{\sigma_n}^2}{2} \tag{28}$$

$$\mu_n = ln(\mu) - \frac{{\sigma_n}^2}{2} \tag{29}$$

Substituting this into equation (27),

$$\sigma^2 = \left(e^{{\sigma_n}^2} - 1\right) e^{2\left(ln(\mu) - \frac{{\sigma_n}^2}{2}\right) + {\sigma_n}^2} \tag{30}$$

$$\sigma^2 = \left(e^{{\sigma_n}^2} - 1\right) e^{2ln(\mu)} \tag{31}$$

$$\sigma^2 = \left(e^{{\sigma_n}^2} - 1\right) \mu^2 \tag{32}$$

$$e^{{\sigma_n}^2} = \frac{\sigma^2}{\mu^2} + 1 \tag{33}$$

$${\sigma_n}^2 = ln\left(\frac{\sigma^2}{\mu^2} + 1\right) \tag{34}$$

Using this in equation (26),

$$\mu = e^{\mu_n + \frac{1}{2}\left(ln\left(\frac{\sigma^2}{\mu^2} + 1\right)\right)} \tag{35}$$

$$ln(\mu) = \mu_n + ln\left(\frac{\sqrt{\sigma^2 + \mu^2}}{\mu}\right) \tag{36}$$

$$\mu_n = ln\left(\frac{\mu^2}{\sqrt{\sigma^2 + \mu^2}}\right) \tag{37}$$

D.5 Code for descriptive statistics under different expense volatilities

```
n <- 100000
trials <- 10000
min_mat <- vector(length = trials)
max_mat <- vector(length = trials)
lrnom_vec <- vector(length = n)
for (i in 1:trials) {
  lrnom_vec <- rlnorm(n = n, meanlog = mew_500, sdlog = sigma_500)
  min_mat[i] <- min(lrnom_vec)
  max_mat[i] <- max(lrnom_vec)}
for (i in 1:trials) {
  lrnom_vec <- rlnorm(n = n, meanlog = mew_1000, sdlog = sigma_1000)
  min_mat[i] <- min(lrnom_vec)
  max_mat[i] <- max(lrnom_vec)}
for (i in 1:trials) {
  lrnom_vec <- rlnorm(n = n, meanlog = mew_1500, sdlog = sigma_1500)
  min_mat[i] <- min(lrnom_vec)
  max_mat[i] <- max(lrnom_vec)}
```

FIGURE 24: Code for the simulations used for generating descriptive statistics of expenses.

D.6 Price path code

```
nsim <- 1000000
nl <- n + 1
low_vol_exp <- matrix(nrow = nl, ncol = nsim)
med_vol_exp <- matrix(nrow = nl, ncol = nsim)
high_vol_exp <- matrix(nrow = nl, ncol = nsim)
low_vol_with <- matrix(nrow = nl, ncol = nsim)
med_vol_with <- matrix(nrow = nl, ncol = nsim)
high_vol_with <- matrix(nrow = nl, ncol = nsim)
low_vol_val <- matrix(nrow = nl, ncol = nsim)
med_vol_val <- matrix(nrow = nl, ncol = nsim)
high_vol_val <- matrix(nrow = nl, ncol = nsim)

for (i in 1:nsim) {
  dt <- (t-t0)/n
  X <- vector(length = nl)
  X[1] <- 1
  low_vol_val[1, i] <- x0
  med_vol_val[1, i] <- x0
  high_vol_val[1, i] <- x0
  low_vol_exp[, i] <- rlnorm(n = nl, meanlog = mew_500, sdlog = sigma_500)
  med_vol_exp[, i] <- rlnorm(n = nl, meanlog = mew_1000, sdlog = sigma_1000)
  high_vol_exp[, i] <- rlnorm(n = nl, meanlog = mew_1500, sdlog = sigma_1500)
  low_vol_with[1, i] <- 0
  med_vol_with[1, i] <- 0
  high_vol_with[1, i] <- 0

  for(l in 2:nl){
    X[l] <- exp((mu-sigma^2/2)*(dt) + sigma*sqrt(dt)*rnorm(1))

    low_vol_val[l, i] <- low_vol_val[l-1, i] * X[l]
    med_vol_val[l, i] <- med_vol_val[l-1, i] * X[l]
    high_vol_val[l, i] <- high_vol_val[l-1, i] * X[l]
```

(a) Price path code part one.

```
      if(low_vol_exp[l, i] > income){

        low_vol_with[l, i] <- low_vol_exp[l, i] - income

        low_vol_val[l, i] <- low_vol_val[l, i] - low_vol_with[l, i]

      } else {low_vol_with[l, i] <- 0}

      if(med_vol_exp[l, i] > income){

        med_vol_with[l, i] <- med_vol_exp[l, i] - income

        med_vol_val[l, i] <- med_vol_val[l, i] - med_vol_with[l, i]

      } else {med_vol_with[l, i] <- 0}

      if(high_vol_exp[l, i] > income){

        high_vol_with[l, i] <- high_vol_exp[l, i] - income

        high_vol_val[l, i] <- high_vol_val[l, i] - high_vol_exp[l, i]

      } else {high_vol_with[l, i] <- 0}


    }

  }
```

(b) Price path code part two.

FIGURE 25: Price path code.

D.7 Incremental sigma code

```
nsim <- 100000
nl <- n + 1
sigma_out <- seq(100, 2000, 100)
mew_in <- log((expenditure^2) / (sqrt(expenditure^2 + sigma_out^2)))
sigma_in <- sqrt(log(1 + (sigma_out^2) / (expenditure^2)))
expenses_mat <- matrix(nrow = nl, ncol = length(sigma_out))
investment_mat <- matrix(nrow = nl, ncol = length(sigma_out))
terminal_mat <- matrix(nrow = nsim, ncol = length(sigma_out))
for (i in 1:nsim) {for (j in 1:length(sigma_out)) {
    expenses_mat[, j] <- rlnorm(n = nl, meanlog = mew_in[j], sdlog = sigma_in[j])}
dt <- (t-t0)/n
X <- vector(length = nl)
X[1] <- 1
investment_mat[1, ] <- x0
for(l in 2:nl){X[l] <- exp((mu-sigma^2/2)*(dt) + sigma*sqrt(dt)*rnorm(1))
  for (k in 1:length(sigma_out)) {investment_mat[l, k] <- investment_mat[l-1, k] * X[l]
    if(expenses_mat[l, k] > income){
      investment_mat[l, k] <- investment_mat[l, k] - (expenses_mat[l, k] - income)}}}
terminal_mat[i, ] <- investment_mat[nl, ]}
```

FIGURE 26: Code for generating TVPs for incremental values of sigma.