LISBOA SCHOOL OF ECONOMICS & MANAGEMENT

Master in Actuarial Science

Masters Final Work

Dissertation

Premiums and Reserves in Life Insurance Policies: The Worst-case Scenario and Solvency II

Eunice Alexandra Madeira Balau

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Supervisor:
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Resumo

As reservas de capital representam um instrumento fundamental no processo de gestão de risco das empresas de seguros, sendo utilizadas no cálculo do capital económico e regulamentar. Como o valor das reservas e dos prémios é fortemente influenciado pelos pressupostos atuariais utilizados, a escolha adequada das bases técnicas é um dos temas de principal interesse para as Companhias de Seguros e para as Entidades Reguladoras.

O principal objetivo deste trabalho é o estudo de um método de construção de cenários biométricos para o cálculo de reservas e prémios, adotando uma posição conservadora em relação às bases técnicas de segunda ordem, seguindo a orientação de dois trabalhos fundamentais neste domínio, Christiansen (2010) e Milbrodt and Stracke (1997). Este cenário é determinado através da resolução de um problema de maximização da reserva prospetiva que nos permite definir as bases biométricas de primeira ordem que representam o pior caso do ponto de vista do Segurador. As apólices do ramo vida são descritas pelo modelo Markoviano de estados múltiplos, sendo as reservas prospetivas calculadas recorrendo à equação de Thiele.

O novo regime de solvência da União Europeia, Solvência II, também recorre à noção de piores cenários, por forma a quantificar os requisitos de capitais no ramo vida, embora com uma definição diferente. Assim, um objetivo adicional, e também importante, deste trabalho é procurar integrar o método estudado no enquadramento estabelecido pelo projeto Solvência II.

O novo método, bem como as propostas anteriores existentes na literatura, serão objeto de apresentação e discussão, recorrendo nomeadamente a dois Casos de Estudo, que permitirão observar a sua praticabilidade no cálculo dos prémios e das reservas, enquanto se avalia uma possível aplicação no enquadramento estabelecido em Solvência II. Por forma a fazê-lo, os casos analisados pelo autor serão estendidos a outros produtos que, embora não sendo comuns no mercado Português, pela sua complexidade, nos permitem mostrar toda a versatilidade inerente ao modelo e tirar importantes conclusões.

Palavras-Chave: Seguros Vida, Prémios, Reserva, Pior Cenário, Sum-at-Risk, Solvência II

Abstract

Reserves are a fundamental tool in insurance risk management since they are used to determine the economic or regulatory capital required for insurers to remain solvent. As the values of reserves and premiums are strongly dependent on the actuarial assumptions used, the choice of the adequate elements of the technical basis is a major concern of both regulators and insurance companies.

The main purpose of this work is to study a method for the construction of biometric worst-case scenarios that allow premiums and reserves to be on the safe side with respect to given confidence bands for the biometric second-order basis, following the essential works of Christiansen (2010) and Milbrodt and Stracke (1997). This scenario is obtained by solving a maximization problem for the prospective reserve that allows one to find the worst-case biometric valuation basis from the insurer's point of view. In life insurance, policies are often described by the multi-state Markov model of life contingencies and the prospective reserves computed using Thiele's equation.

The new solvency regime of the European Union, Solvency II, also uses worst-case scenarios, although constructed in a different way, in order to quantify the solvency capital requirements for life insurance business. Thus, a further important purpose of this thesis is to integrate the method in study under the Solvency II framework.

The new method, as well as the previous approaches offered in the literature, will be presented and discussed with two Case Studies, demonstrating the usefulness for the calculations of premiums and reserves, while a possible application in the calculation of solvency reserves in Solvency II are introducing. In order to do so, the examples discussed by the author are extended to products, which although not common in the Portuguese market, are complex situations that allow us to show the versatility of the model in study and to derive significant conclusions.

Keywords: Life insurance, Premiums, Reserve, Worst-case Scenario, Sum-at-Risk, Solvency II

Contents

List of Figures

List of Tables

There is nothing more practical than a good theory.
Lewin, K. (1952)

Chapter 1

Introduction

Since the early 18th century that actuaries applied scientific principles and techniques to life insurance problems involving risk, uncertainty and finance. Although the basic risks insured have not changed, contracts have become more complex in recent years as well as the techniques needed to manage them.

The insurer should use those techniques to maintain the total liabilities plus the company's net value equal to its total assets. Assets are mainly investments such as bonds, equities and property, resulting from the collection of premiums from the policyholders and from earnings on the investments. On the other hand, the liabilities of life insurers primarily comprise the reserves hold to back its obligations to policyholders and beneficiaries. As a matter of fact, the calculation of the sum of the reserves for all policies in force and the value of all the company's investments, at the valuation time, is an important element in the financial control of an insurance company.

The legal framework for the insurance activity in member states of the European Union (EU) is based upon common rules. In Portugal, the Portuguese Insurance and Pension Funds Supervisory Authority (ISP) ensures that insurance companies carry out their business in accordance with the European law, supervising the prudent management. In order to do so, ISP requires each company to maintain its reserves at a level that will assure payment of all policy obligations as they fall due. In fact, reserves are not only used to determine the profit or loss for the company over any time period, but also to determine the economic or regulatory capital needed to remain solvent, being a fundamental tool in insurance risk management.

When a life insurance company enters into an insurance contract it does not know the precise moment when the benefits and expenses will occur or how much they will be. The life insurance company underwrites a set of risks and accepts the premiums, entering into a long term contractual commitment to pay certain benefits. Thus, life insurance calculations are performed on technical basis, which is the set of estimates and assumptions that are used to project future cash flows under a life insurance contract and discount them to the present. They are, for instance, mortality probabilities, disability probabilities, reactivation probabilities and interest rates. Following Macdonald (2004), technical bases have three main purposes: pricing, i.e. setting premiums, valuation for solvency and valuation in order to measure and distribute surplus. In many countries, the basis used to calculate premiums, called premium basis, differ from the assumptions used to calculate the reserves, that is, the valuation basis. Another distinction is between first-order basis and second-order basis. While first-order basis include a safety margin, making conservative or safe-side assumptions about the future, the second-order basis, in principle, do not contain any margin and consists in the best estimate with respect to the insured population, being often called experience basis.

The past has shown that the variation in the actuarial assumptions can be wide within a contract period, for instance, the recent increase in life expectancies in many developed countries. The influence such changes can have on premiums and reserves is an important issue and the choice of the technical basis is a major concern of regulators and insurance companies. As a matter of fact, premiums and reserves should be set on the safe-side, in the sense that they should be high enough to cover the benefits in all possible scenarios. In order to do so, it is a common method to choose first-order technical basis that represents some worst-case scenario for the insurer.

Literature offers three ways for the construction of these scenarios. There is a first method based on the sum-at-risk which was developed by Lidstone (1905), Norberg (1985), Hoem (1988), Ramlau-Hansen (1988) and Linnemann (1993). Verbally, it studies the financial effect at any point in time resulting from the transition of the policyholder from one state to another. The authors showed that, for a given first-order basis with corresponding sums-at-risk for the different states of the policy, the premiums and reserves are on the safe side if: the second-order basis is smaller than the first-order basis when the first-order sums-at-risk are positive; or the second-order basis is greater than the first-order basis when the first-order sums-at-risk are negative.

The two other methods are based on derivatives. The second one was presented by Dienst (1995), Bowers et al. (1997), Christiansen and Helwich (2008) and Christiansen (2008a,b) and approximates the relevant functions with local linearisations by using first-order derivatives, given that worst-case scenarios can be found much more easily for linear mappings. The third one, developed by Kalashnikov and Norberg (2003), differentiates the reserve and premium with respect to one arbitrary real parameter and, using the assumption of normality, obtains confidence bands for them in an one step-approach.

However, as we will see later, none of these three methods is an exact method and all of them have problems: the first method does not tell us how to find the first-order basis; the second method works only for narrow confidence bands and yields only approximate results; the third method runs counter to the traditional rules of insurance regulation in many countries. Trying to fill this gap in the literature, Christiansen (2010) presents a new method.

The method allows one to construct the biometric worst-case scenarios that let premiums and reserves be always on the safe side with respect to given confidence bands for the biometric second-order basis. Thus, one should choose the first-order valuation basis that maximizes the reserve with respect to all biometric scenarios within the bounds imposed for our actuarial assumptions. The method allows to construct such scenarios for homogeneous portfolios of single life insurance policies and the results are especially interesting for life insurance policies with a mixed character, i.e., survival and occurrence character.

There are two main purposes in this thesis:

1. To explore the method introduced by Christiansen (2010) and adapting it to the Portuguese case (at this point, use was made of the practical knowledge the author acquired while performing her professional tasks);

2. Attention should be paid to the fact that the new solvency regime of the EU, Solvency II (Directive 2009/138/EC), uses worst-case scenarios for the calculation of the solvency capital requirement (SCR) in life insurance business. Thus, a possible application of this method in the Solvency II framework is also discussed.

The paper is organized as follows: In Chapter 2 we introduce the basic concepts of life insurance. In Chapter 3 we start with the general modelling framework for multi-state life policies based on Time-Continuous Markov Model for life insurance and on Thiele's Equations and then describe the methods offered in the literature to construct biometric worst-case scenarios. In Chapter 4, numerical applications are presented and discussed with the purpose of showing how this new approach is useful to obtain premiums and reserves on the safe side and in the calculations of biometric solvency reserves for Solvency II. The study is completed with some conclusions and suggestions for further research, in Chapter 5. Some results that would lead to an easier reading are presented in appendix because the size restrictions of the text.

Chapter 2

Life Insurance

The origins of the concept of life insurance go back to burial clubs in Rome, in 100 B.C., which were created to pay for the funeral of members. Norton (1852) refers that traditional life insurance were established in the early 18th century, in order to provide financial assistance to widows in the event of policyholder death. These policies remains very similar to the contracts written up to the 1980s, experienced enormous changes in the last three decades.

2.1 Types of contracts

In its simplest form, an insurance policy is a contract between two parties placing obligations on both of them. The policyholder agrees to pay an amount or a series of amounts to the insurer, called premiums, in return for a later payment or set of payments from the insurer, called benefit(s), if and when the event insured against occurs.

Life insurance policies exist in many forms, most of them providing considerable flexibility in premiums (amount, duration and frequency) and benefits (amount and the circumstances under which these will be paid). The benefits payable under simple life insurance contracts are of two main types: insurance or annuity. While the term insurance is usually used when the benefit is paid as a single lump sum, contingent on the death or on survival of the policyholder to a predetermined maturity date, an annuity is a benefit in the form of a regular series of payments, usually depends on the survival of the policyholder.

The traditional life insurance products are the whole life, term, endowment and pure endowment insurance and can be seen for instance in Bowers et al. (1997), Gerber (1997) and Dickson et al. (2012). The plainest life insurance contract is the whole life insurance which pays a benefit, called the sum assured, on the policyholder's death. If the sum assured is paid provided death occurs during a specified period, the term of the contract, it is called a term insurance. On the other hand, a pure endowment contract provides a sum assured at the end of a fixed term, if the policyholder is then alive. Finally, an endowment insurance is a combination of a term insurance and a pure endowment insurance which pays a sum assured either on death of the policyholder or at the end of a specified term, whichever occurs first.

Annuity contracts provides payments of amounts, which might be level or variable, at stated times, provided a life is still alive. There are many variants of annuity contracts, described e.g. in Garcia and Simões (2010). For instance, a whole life annuity provides payments until the death of the annuitant; if the payments are made for some maximum period, provided the annuitant survives that period, it is called a term annuity; and in the deferred annuity the start of payment is deferred for a given term.

There are other life contingent risks, depending on the state of health of the policyholder, described e.g. in Booth et al. (2005) as health insurance products: income protection insurance; critical illness insurance; long-term care insurance and private medical insurance. For instance, while an income protection insurance replaces some income to the insured whilst they are unable to work, by reason of illness or injury, a critical illness insurance pays a benefit on diagnosis of a severe condition, such as certain cancers or heart disease.

In recent years, in order to competing for policyholders' savings with other institutions, insurers have provided more flexible products that combine the death benefit coverage with a significant investment element, known as the modern insurance contracts (Dickson et al., 2012).

2.2 The loss random variable

The cash flows for a life insurance contract consist of the insurance and/or annuity benefit outgo (with associated expenses) and the premium income. All of the cash flows in a contract are uncertain, depending on the death, survival or possibly the state of health of a life, unless the contract is purchased by a single premium, in which case there is no uncertainty regarding the premium income. Therefore, the loss incurred by the insurer on a particular policy can be modelled with the loss random variable L.

Definition 2.2.1. Consider a policy which is still in force t years after it was issued. The random loss variable at time t, $L(t)$, is the difference between the present values of future outgo and of future income: $L(t) = PV\left[Future\,Outgo\right] - PV\left[Future\,Income\right]$.

Insurers wish to determine a distribution for $L(t)$ with the purpose of finding the adequate premium for a given benefit and compute reserves.

2.3 Premiums

Under an insurance policy the policyholder agrees to pay premiums to the insurance company. Regarding to the frequency and amount, the premium payment arrangement will commonly be:

- one single payment, known as a single premium;
- a regular series of m payments of a constant or varying amount, made every $1/m$ years, typically quarterly or monthly, known as regular premiums.

The key feature of any life insurance policy is that premiums are payable in advance, so the first payment is always due at the time the policy is effected.

The benchmark principle for calculating premiums is the equivalence principle (Gerber, 1997). This method consists in finding the premiums that set the expected present value of the income equal to the expected present value of the outgo:

$$EPV\, of\, benefit\, outgo = EPV\, of\, premium\, income$$

or equivalently $E\left[L\left(t\right)\right] = 0$.

However, there are other classical methods of calculating premiums, such as the portfolio percentile premium principle or the utility principle (Bowers et al., 1997), and more contemporary approaches, used commonly for non-traditional policies, that consists in consider the cash flows from the contract and to set the premium to satisfy a specified profit criterion (Dickson et al., 2012; Booth et al., 2005)

2.4 Provisions and Policy Values

A reserve is the amount set aside by the insurer to meet its future obligations, i.e. to pay policyholder's benefits and, where appropriate, future expenses. In reserve calculations it is possible to look to the future cash-flows forward leading to the calculation of present values, or backward leading to the calculation of accumulations (The Actuarial Profession, 2013, CT5 Contingencies). This concepts lead to two types of reserves:

- Retrospective reserve is the accumulated value of premiums received less benefits paid up to time t, on a specified basis.

- Prospective reserve is the expected present value of the loss random variable, on a specified basis.

The prospective reserve is an important element in the financial control because if the insurer holds funds equal to the reserve and the future experience follows the reserve basis then, averaging over many policies, the combination of reserve and future premiums will be sufficient to pay the future benefits and expenses. In general terms, at a certain point in time t,

$$V\left(t\right) + EPV\, at\, t\, of\, future\, premiums = EPV\, at\, t\, of\, future\, benefits + expenses$$

where $V\left(t\right)$ is the the prospective reserve at time t (Bowers et al., 1997).

In practice, the retrospective reserve will be equal to the prospective reserve when all calculations are performed on the same basis.

2.5 Thiele's equation

2.5.1 Reserving for a policy with discrete annual cash flows

Following Dickson et al. (2012), consider a policy issued to a life aged x under which premiums, expenses and claims can occur only at the start or end of the year. Suppose this policy has been in force for t years, where $t \geq 0$. Consider the $(t+1)th$ year and the following notation:

- P_t: premium payable at time t;
- E_t: premium-related expense payable at time t;
- b_{t+1}: sum insured payable at time $t+1$ if the policyholder dies in the year;
- e_{t+1}: expense of paying the sum insured at time $t+1$;
- ${}_tV$: prospective reserve for a policy in force at time t (${}_{t+1}V$ denotes the prospective reserve for a policy in force at time $t+1$);
- q_{x+t}: probability that the policyholder, alive at time t, dies in the year;
- p_{x+t}: probability that the policyholder, alive at time t, survives to age $x+t+1$;
- K_{x+t}: curtate future life time for a life aged x;
- i_t: rate of interest assumed earned in the year.

The loss random variable at time t is

$$L_t = \begin{cases} (1+i_t)^{-1}(b_{t+1}+e_{t+1}) - P_t + E_t & if\ K_{x+t} = 0\ \ (with\, probability\ q_{x+t}) \\ (1+i_t)^{-1} L_{t+1} - P_t + E_t & if\ K_{x+t} \geq 1\ \ (with\ probability\ p_{x+t}) \end{cases}$$

and therefore, the prospective reserve can be defined as

$${}_tV = E[L_t] = q_{x+t}(1+i_t)^{-1}(b_{t+1}+e_{t+1}) + p_{x+t}(1+i_t)^{-1}\ {}_{t+1}V - (P_t - E_t) \qquad (2.5.1)$$

In words, equation (2.5.1) states that the reserve at the start of the year should be equal to the present value of expected cost of the death benefits at the year end (the benefit is b_{t+1} plus expenses e_{t+1} payable with probability q_{x+t}) plus the present value of the expected cost of setting up the reserve at the year end (the reserve of amount ${}_{t+1}V$ is required with probability p_{x+t}) minus the premium cash-flows $(P_t - E_t)$.

2.5.2 Thiele's Differential Equations

The concepts presented above extend to policies where regular payments are payable continuously and sums insured are payable immediately on death (Appendix A.1). In practice, it is common to represent the prospective reserve as a system of linear differential equations describing the dynamics of reserves in life insurance in continuous time. These equations are called Thiele's differential equations and are of the form (Wolthuis, 2003; Dickson et al., 2012):

$$\frac{d}{dt}V(t) = \delta_t V(t) + P_t - E_t - (b_t + e_t - V(t))\,\mu_{x+t} \qquad (2.5.2)$$

While the left-hand side of the formula is the rate of increase in the reserve at time t, the right-hand side explains the individual factors affecting the value of $V(t)$. These factors are the following: interest is being earned on the current amount of the reserve and the rate of increase at time t is $\delta_t V(t)$; premium income, minus premium-related expenses, is increasing the reserve at rate $P_t - E_t$; claims, plus claim-related expenses, decrease this amount at rate $(b_t + e_t - V(t))\, \mu_{x+t}$.

One advantage of Thiele's equation arises from its versatility and flexibility, because it can easily accommodate variable premiums, benefits and interest rates. In this paper we will return to Thiele's equation in sub-section 3.1.2.

Chapter 3

Biometric worst-case scenarios

The main reference for this work is Christiansen (2010). The model therein presented is of the most interest, since it allows one to construct biometric worst-case scenarios that let premiums and reserves be always on the safe side with respect to given confidence bands for the biometric second-order basis. In order to do so, one should choose the first-order valuation basis that represents some worst scenario from the insurer's point of view and as such maximizes the reserve. In this chapter, we start with the theoretics of the problem, we look at the three previous approaches offered in the literature and then describe the model.

3.1 Theoretics of the Problem

Christiansen (2010) uses Thiele's equation in the Markov model of life contingencies to derive formulas concerning the expected actual development of reserves. The author follows the general approach of Milbrodt and Stracke (1997) using the argument that it is valid to the discrete and continuous methods, as well as the mixed cases. Thus, this section is focused on these two works, essential for the development of the thesis.

3.1.1 Time-Continuous Markov Model for life and other contingencies

Multi-state models are one of the most important developments in actuarial science since they simplify and provide a sound foundation for some traditional actuarial techniques. The Markov model (Ross, 1996; Wolthuis, 2003), a special type of a multi-state model, is a very useful instrument to model life insurance and annuities as it provides sufficient generality to cover most situations in the insurance of persons and satisfy the so-called Markov property under which the future development of the process depends only on the present state and not on its full history so far.

One application of the Markov model in life insurance is to determine expected present values of payments that are contingent upon the sojourn in certain states or upon transitions between the states (Wolthuis, 2003). Basically, life insurance risks are given by random maps of states. At every time after policy issue the state of the policy is recorded and this corresponds to modelling risks by jump processes, according to the following definition (The Actuarial Profession, 2013, CT4 Models):

Definition 3.1.1. A continuous-time Markov process X_t, $t \geq 0$ with a discrete, i.e. finite or countable, state space S is called a Markov jump process.

3.1.1.1 Time-inhomogeneous Markov jump processes

We start this sub-section by discussing the important features of time-inhomogeneous Markov jump processes and then introduce the integrated form of Kolmogorov backward Equations (used later in sub-section 3.1.2). In order to do so, consider a general insurance policy issued at time 0, with term T and modelled by a Markovian jump process $(X_t)_{t\in[0,T]}$ with finite state space S. Transitions between states are governed by the transition probabilities, with the transition space denoted by $J = \left\{(j,k) \in S^2 | \, j \neq k\right\}$. Assume that it is an inhomogeneous time-continuous Markov process where the transition probabilities for each fixed period of time vary in time. Thus, it is necessary to specify the beginning and the end of the interval $[s,t]$, instead of just its length $t-s$.

Definition 3.1.2. The transition probability $p_{jk}(s,t)$ is the conditional probability that the process is in state k at time t, given that the process is in state j at time s, irrespective of the way in which state k is reached (Markov property), that is:

$$p_{jk}(s,t) = P(X_t = k | X_s = j), \quad 0 \leq s \leq t \leq T, \quad (j,k) \in S^2, \quad P(X_s = j) > 0$$

$$p_{jk}(s,t) = 0, \text{ otherwise.}$$

Furthermore,

$$p_{jk}(s,s) = \delta_{jk}, \quad s \geq 0, \, (j,k) \in S^2, \tag{3.1.1}$$

where δ_{jk} is the Kronecker delta, which is equal to 0 for $j \neq k$ and equal to 1 for $j = k$.

The transition probabilities satisfy properties (1)-(2) below (Milbrodt and Stracke, 1997):

(1) $0 \leq p_{jk}(s,t) \leq 1, \quad 0 \leq s \leq t, \, (j,k) \in S^2.$

(2) $\sum_{k\in S} p_{jk}(s,t) = 1, \quad 0 \leq s \leq t, \, j \in S.$

In words, since the state space S is finite, there exists a finite-dimensional transition probability matrix denoted by $p(s,t) = (p_{jk})_{(j,k)\in S^2}$ where all elements are non negative and all rows sum to unity.

Chapman-Kolmogorov equations: If X_t is a Markov process, the transition probabilities obey the Chapman-Kolmogorov equations:

$$p_{jk}(s,t) = \sum_{i\in S} p_{ji}(s,r)\, p_{ik}(r,t) \quad 0 \leq s \leq r \leq t, \, (j,k) \in S^2 \tag{3.1.2}$$

that expresses the fact that if a process is in state j at time s and is in state k at time t the transition occurs via some state $i \in S$ at an arbitrary intermediate time r. Equivalently, written in matrix form we have

$$p(s,t) = p\,(s,r)\,p(r,t) \quad 0 \leq s \leq r \leq t,\ (j,k) \in S^2.$$

To avoid difficulties with null-sets, the concept of regular transition matrices is now introduced (Milbrodt and Stracke, 1997).

Definition 3.1.3. The transition matrix $p\,(s,t)$ is regular if it satisfies the Chapman-Kolmogorov Equations (3.1.2) and equation (3.1.1), without exceptional sets - null sets. If in addition $p_{jk}\,(s,\bullet)$ is right continuous for every $s \in [0,T]$, $(j,k) \in S^2$, then $p\,(s,t)$ is called a right continuous regular transition matrix.

Intensities of transition: Intensities of transition are the fundamental concept in continuous time. In order to differentiate the transition probabilities we will assume that the functions $p_{jk}\,(s,t)$ are continuously differentiable. This assumption implies the existence of the following quantities.

Definition 3.1.4. For $0 \leq s \leq t$, $(j,k) \in J$, the transition intensity from state j to state k is

$$\mu_{jk}\,(s) = \left[\frac{\partial}{\partial t} p_{jk}\,(s,t)\right]_{t=s} = \lim_{h \to 0} \frac{p_{jk}\,(s,s+h) - \delta_{jk}}{h}$$

and the intensity of decrement for state j is

$$\mu_{jj}\,(t) = -\sum_{k \neq j} \mu_{jk}\,(t)\,.$$

In addition, the matrix of transition intensities $\mu_{jk}\,(t)$ is $\mu\,(t) = (\mu_{jk})_{(j,k)\in S^2}$.

Assumption 3.1.5. *The intensity function for the transition from state j to state k, $\mu_{jk}\,(t)$, $j \neq k$, exists.*

Based on Alioum (2013), the concepts of cumulative transition intensity $q_{jk}\,(s,t)$ follow.

Definition 3.1.6. For $0 \leq s \leq t$, $(j,k) \in J$:

(1) The cumulative transition intensity from state j to state k is $q_{jk}\,(s,t) = \int_s^t \mu_{jk}\,(t)\,dt$.

(2) The cumulative intensity of decrement for state j is $q_{jj}\,(s,t) = -\int_s^t \mu_{jj}\,(t)\,dt$.

(3) The matrix of cumulative transition intensities $q_{jk}\,(s,t)$ is $q_J = q\,(s,t) = (q_{jk})_{(j,k)\in S^2}$.

The cumulative transition intensity matrices satisfies the following properties (Milbrodt and Stracke, 1997).

Lemma 3.1.7. *If $s \geq 0$ and $(j, k) \in S^2$ satisfy $P(X_s = j) > 0$, then*

(1) $q_{jk}(s,r) + q_{jk}(r,t) = q_{jk}(s,t), \quad s \leq r \leq t.$

(2) $\lim_{t \to s} q_{jk}(s,t) = q_{jk}(s,s) = 0.$

(3) $q_{jk}(s,t) \geq 0$, $q_{jj}(s,t) \leq 0, \quad s \leq t, \quad j \neq k.$

(4) $q_{jj}(s,t) = -\sum_{k \neq j} q_{jk}(s,t), \quad s \leq t.$

(5) $\triangle q_{jj}(t) = q_{jj}(t) - q_{jj}(t-) \geq -1, \quad t > 0.$

(6) $\triangle q_{jj}(t_0) = q_{jj}(t_0) - q_{jj}(t_0-) = -1 \Rightarrow q_{jj}(t)$ *is constant on* $[t_0, T]$.

Again, to avoid difficulties with null-sets, the concept of regular transition intensity matrices follows.

Definition 3.1.8. The cumulative transition intensity matrix $q(s,t)$ is regular if it satisfies properties (1)-(6) in Lemma 3.1.7 without exceptional sets - null sets.

Assumption 3.1.9. *The Markovian jump process*$(X_t)_{t\in[0,T]}$ *has a regular cumulative transition intensity matrix q.*

Kolmogorov backward integral equations: Define the probability of staying uninterruptedly in the current state j in the interval $[s,t]$ as: $p_{\overline{jj}}(s,t) = e^{\int_s^t \mu_{jj}(r)dr}$. Let the residual holding time R_s be the amount of time between s and the next jump and let $X_s^+ = X_{s+R_s}$. Conditional on R_s and X_s^+ and using the law of total probability

$$\begin{aligned} p_{jk}(s,t) &= P[X_t = j | X_s = k] \\ &= \delta_{jk} p_{\overline{jj}}(s,t) + \textstyle\sum_{i \neq j} \int_s^t p_{\overline{jj}}(s,r)\, \mu_{ji}(r)\, P[X_t = k | X_s = j, R_s = r - s, X_s^+ = i]\, dr \end{aligned}$$

and therefore

$$p_{jk}(s,t) = \delta_{jk} p_{\overline{jj}}(s,t) + \sum_{i \neq j} \int_s^t p_{\overline{jj}}(s,r)\, \mu_{ji}(r)\, p_{ik}(r,t)\, dr. \tag{3.1.3}$$

$p_{\overline{jj}}(s,r)\, \mu_{ji}(r)\, p_{ik}(r,t)$ is the probability of remaining in state j from time s to time r, then making a transition to state i at time r, and finally going from state i to state k between times r and t. To take into account the possible values of R_s we integrate from $r = s$ to $r = t$, and to take into account all possible intermediate states we sum over all possible values of $i \neq j$.

Milbrodt and Stracke (1997) present the backward integral equations using the cumulative transition intensities, as follows:

$$p_{jk}(s,t) = \delta_{jk} + \sum_{i \in S} \int_s^t p_{ik}(w,t)\, dq_{ji}(w). \tag{3.1.4}$$

3.1.1.2 Insurance benefits and premiums

Let us consider the general insurance policy introduced in 3.1.1.1. Contractual payments between the insurer and the policyholder are taken to be on a continuous time basis. At any time $t \in [0, T]$ the policy provides (Wolthuis, 2003):

- Lump sum benefits $b_{jk}(t)$ upon a transition from state j to k, payable at time $DT(t) \geq t$, for $(j,k) \in J$. $b_{jk}(t)$ are deterministic non negative functions with bounded variation and $DT(t)$, $DT : (0,\infty) \rightarrow (0,\infty)$, is an increasing function introduced by Milbrodt and Stracke (1997) in order to model the difference that may occur between the payment date and the time of transition. To simplify notation, assume that $DT(T) = T$.

- Annuity payments $B_j(t)$ during sojourn in a state j defined in a cumulative manner, i.e. $B_j(t)$ is the total amount paid in time $[0,t]$. $B_j(t)$ are right continuous deterministic functions with bounded variation. While benefits paid to the insured have a positive sign, premiums paid by the insured have a negative sign.

For reasons of simplicity, expenses and single premiums are disregarded. However, the theory may be easily developed to incorporate these topics. For instance, expenses may be considered as additional benefits.

3.1.1.3 Interest model

In the literature a large number of interest models is available (Brigo and Mercurio, 2006). Christiansen (2010), assumes that the investment portfolio of the insurance company earns interest according to the compound interest model with interest intensity function φ and cumulative intensity Φ. In the general case, interest cumulative intensity Φ can be defined based on the interest function $r(t)$ as bellow (Milbrodt and Stracke, 1997).

Definition 3.1.10. Let $r(t)$ be an interest function, non decreasing and right-continuous, equal to 1 at time zero. Then $\Phi(t) = \int_0^t \frac{1}{r(s-)} dr(s)\, ,\ t \geq 0$.

It follows that the value at time s of a unit payable at time $t > s$ is

$$\upsilon(s,t) = \left(\prod_{(s,t]} (1 + d\Phi) \right)^{-1} \tag{3.1.5}$$

for partitions $s < t_0 < t_{1<} ... < t_n = t$.

Furthermore, when the cumulative intensity function Φ is a step-function, one can separate the jumps of $\Phi(t)$ from its continuous part and get the generalized exponential formula

$$\upsilon(s,t) = e^{(-\Phi^c(t) - \Phi^c(s))} \prod_{\tau \in (s,t]} (1 + \triangle\Phi(\tau))^{-1} \tag{3.1.6}$$

where $\Phi^c(t) = \Phi(t) - \sum_{\tau \le t} \triangle\Phi(\tau)$ for all t is the continuous part of Φ and $\triangle\Phi(t) = \Phi(t) - \Phi(t-)$.

Further details about (3.1.5) and (3.1.6) can be found in Jacod (1975) and Gill (1980, Lemma 3.2.1 and Appendix 4).

3.1.2 Reserves and Thiele's Equations revisited

Naturally, the definition of prospective reserve for a policy modelled using a multi-state model continues to be the expected value of the future loss random variable, with one obvious additional requirement. Due to the Markov property, the reserve depends on the current state of the policy and the time elapsed since entering this state. Formally, considering the three basic elements of the time-continuous Markov model presented in the previous section, the Definition 3.1.11 follows.

Definition 3.1.11. The prospective reserve for the policy that is in state i at time s, given that q is a regular matrix, is defined by

$$V_i(s) = \sum_{j \in S} \int_{(s,T]} v(s,t)\, p_{ij}(s,t)\, dB_j(t) + \sum_{(j,k) \in J} \int_{(s,T]} v(s,\ DT(t))\, b_{jk}(t)\, p_{ij}(s,t-)\, dq_{jk}(t). \tag{3.1.7}$$

Both terms in the right-hand side of the equation concern expected present values over the interval $(s,T]$. The first term is the expected present value of annuity payments made during the sojourn in states of the Markov chain, and the second term is the expected present value of the lump-sum insurance benefits. Random variables are both the times of transition and the states to where the transitions occur.

Under appropriate smoothness conditions, Milbrodt and Stracke (1997) show that Thiele's differential equations are obtained differentiating (3.1.7):

$$\frac{d}{dt} V_i(t) = -b_i(t) + (\varphi(t) - \mu_{ii}(t))\, V_i(t) - \sum_{k \neq i} (v(t, DT(t))\, b_{ik}(t) + V_k(t))\, \mu_{ik}(t) \tag{3.1.8}$$

Formula (3.1.8) can be interpreted in the same way as formula (2.5.2). During sojourns in state i, the reserve changes as a result of interest being earned at rate $\varphi(t)\, V_i(t)$, and benefits being paid at rate $b_i(t)$. Transitions from state i to any other state k at time t, also lead to changes in the prospective reserve: a decrease of $b_{ik}(t)$ as the insurer has to pay at time $DT(t)$ any lump sum benefit contingent on jumping from state i to state k; a decrease of $V_k(t)$ as the insurer has to set up the appropriate reserve in the new state; and an increase of $V_i(t)$ as this amount is no longer needed (for all possible transitions we have $\mu_{ii}(t)\, V_i(t)$).

The authors named equations (3.1.7) "Thiele's integral equations of type 1" and established another system of integral equations for the prospective reserve referred to as "Thiele's integral equations of type 2". For the second ones, the existence of a regular cumulative transition intensity and transition intensity matrices that satisfy the backward integral equations identically, i.e.. without exceptional sets, is required.

As a matter of fact, if the cumulative transition intensity matrix $q\,(s,t)$ is regular, then the transition probability matrix $p\,(s,t)$ has a representation of the form

$$p\,(s,t) = \prod_{(s,t]} (\mathbb{I} + dq), \tag{3.1.9}$$

where $\mathbb{I}$ denotes the appropriate unit matrix. Furthermore, according to Andersen et al. (1991), the product-integral of p over intervals of the form $[0,t]$ can be defined as

$$p(s,t) = \lim_{\max|t_i - t_{i-1}| \to 0} \prod \left(\mathbb{I} + q\,(t_i) - q\,(t_{i-1})\right),$$

where $s < t_0 < t_1 < ... < t_n = t$ is a partition of $[s,t]$.

From this derivations, Milbrodt and Stracke (1997, Lemma 4.7) prove the regularity conditions required in the Lemma below.

Lemma 3.1.12. *Let X_t be Markov, q a regular cumulative transition matrix for X_t and p defined by the product in* (3.1.9)*. Then p is a right continuous regular transition matrix for X_t that satisfies the backward integral equations* (3.1.3) *and* (3.1.4) *identically.*

For Thiele's integral equations of type 2, stronger integrability conditions, which do only make sense if q is regular, are also required. Following Milbrodt and Stracke (1997), these are listed in Assumption (3.1.13).

Assumption 3.1.13.

(1) $\sum_{j \in S} |B_j|\,(T) < \infty$.

(2) $\sum_{(j,k) \in J} \int_{(0,T]} \upsilon\,(t,\, DT\,(t))\, b_{jk}\,(t)\, dq_{jk}\,(t) < \infty$.

(3) $\sum_{j \in S} \sum_{(j,k) \in J} \int_{(0,T]} \int_{(t,T]} \upsilon\,(0,\, s)\, d|B_j|\,(s)\, r\,(t)\, dq_{jk}\,(t) < \infty$.

(4) $\sum_{(i,l) \in J} \sum_{(j,k) \in J} \int_{(0,T]} \int_{(t,T]} \upsilon\,(0,\, DT\,(s))\, b_{jk}\,(s)\, dq_{jk}\,(s)\, r\,(t)\, dq_{il}\,(t) < \infty$.

At this stage, it is now possible to present Thiele's integral equation of type 2 (Milbrodt and Stracke, 1997, Theorem 4.8).

Theorem 3.1.14. *(Thiele's integral equation of type 2) Let X_t be Markov with a regular cumulative transition intensity matrix q and p be a right continuous regular transition matrix for X_t, which satisfies the backward integral equations* (3.1.3) *identically. Assume that the integrability conditions (1)-(4) hold and fix a version of the prospective reserve by* (3.1.7)*. Then for every $s \in [0,\, T]$, $i \in S$:*

$$V_i\,(s) = B_i\,(T) - B_i(s) - \int_{(s,T]} V_i\,(t-)\, d\Phi\,(t) + \sum_{j:j \neq i} \int_{(s,T]} R_{ij}\,(t)\, dq_{ij}\,(t) \tag{3.1.10}$$

where $R_{ij}(t)$*is the so-called sum-at-risk associated with a possible transition from state* i *to state* j *at time* t,

$$R_{ij}(t) = \upsilon\left(t, DT\left(t\right)\right) b_{ij}\left(t\right) + V_j\left(t\right) + \triangle B_j(t) - V_i\left(t\right) - \triangle B_i\left(t\right),$$

$\triangle B_j\left(t\right) = B_j\left(t\right) - B_j\left(t-\right)$.

The interpretation of the system of integral equations in terms of infinitesimal sojourn payments $B_i\left(T\right) - B_i\left(s\right)$, infinitesimal interest premiums $V_i\left(t-\right) d\Phi\left(t\right)$ and infinitesimal sums-at-risk $R_{ij}\left(t\right) dq_{ij}\left(t\right)$ is the same as for the classical Thiele's equation. We should point out that Thiele's integral equations of type 1 imply Thiele's integral equations of type 2 and vice versa. Proofs are presented in Milbrodt and Stracke (1997).

If further Φ and the q_{jk} have the intensities φ and μ_{jk}, then (3.1.10) becomes:

$$V_i\left(s\right) = B_i\left(T\right) - B_i(s) - \int_{(s,T]} V_i\left(t-\right) \varphi\left(t\right) dt + \sum_{j:j\neq i} \int_{(s,T]} R_{ij}\left(t\right) \mu_{ij}\left(t\right) dt. \qquad (3.1.11)$$

Additionally, from (3.1.7) we get an initial condition for Thiele's integral equation system: $V_i\left(T\right) = 0$ for all $i \in S$.

From now on, following Christiansen's approach (2010), the prospective reserve is defined as in (3.1.10) and (3.1.11).

3.2 The existing approaches

As already referred, before Christiansen (2010), literature offered three main frameworks for the construction of a biometric first-order valuation basis. In this section, we give a more detailed survey of these approaches.

3.2.1 Sum-at-risk

Lidstone (1905) studied the effect on reserves of variations in valuation basis and contract terms in discrete time. Lidstone's ideas are extended to a continuous time version, using Thiele's differential equations, by Norberg (1985). Later, Hoem (1988), Ramlau-Hansen (1988) and Linnemann (1993) studied the expected profit resulting from changes in valuation basis.

The basic safe-side requirement introduced for the Markov model by Hoem (1988) is

$$V_i^*\left(t\right) \leq V_i\left(t\right),$$

where $V_i^*\left(t\right)$ is the prospective reserve for a policy that is in state i at time t, calculated using the second-order basis. It means that the reserve on the first-order basis is always sufficient to cover the reserve needed, according to the second-order basis. Note that the difference $V_i\left(t\right) - V_i^*\left(t\right)$ can be interpreted as the expected profit of the insurer.

Assuming premiums, benefits and the prospective reserve at time T to be equal on first and second order basis, Linnemann (1993) shows that changes in the prospective reserves for a closed insurance portfolio caused by alterations in the technical basis are given by

$$V_i(t) - V_i^*(t) = \int_t^T \upsilon^*(t,u) \sum_k p_{ik}^*(t,u)\, g_k^*(u)\, du, \tag{3.2.1}$$

where $g_k^* = (\varphi^* - \varphi) V_k(u) - \sum_l (\mu_{kl}^*(u) - \mu_{kl}(u)) R_{kl}(u)$. Observe that if $g_k^* \geq 0$ for all k and u, then $V_i(t) \geq V_i^*(t)$ for all i and t and the basic safe-side requirement is fulfilled. A sufficient condition is, of course, that $\varphi^* \geq \varphi$ and the second-order biometric basis is smaller than the first-order basis at ages for which the sum-at-risk is positive, and vice versa (Ramlau-Hansen, 1988).

When the purpose is to study changes in the prospective reserves caused only by alterations in the biometric technical basis, equation (3.2.1) reduces to

$$V_i(t) - V_i^*(t) = \int_t^T \upsilon^*(t,u) \sum_k p_{ik}^*(t,u) \left(- \sum_l (\mu_{kl}^*(u) - \mu_{kl}(u)) R_{kl}(u) \right) du. \tag{3.2.2}$$

For a given first-order basis with corresponding sum-at-risk, all the referred authors showed that reserves are on the safe-side if: $\mu_{kl}(u) \geq \mu_{kl}^*(u)$ whenever $R_{ij}(u) \geq 0$ and $\mu_{kl}(u) \leq \mu_{kl}^*(u)$ whenever $R_{ij}(u) \leq 0$. Unfortunately, we can not say anything about the alternative scenarios. Thus, there are situations where the sum-at-risk method guarantees to be on the safe-side only for a small number of possible second-order basis within the confidence bounds. Following Christiansen (2010), the solution to this problem is to set the first-order basis equal to the upper bounds of the confidence bands where the first-order sums-at-risk are positive and to the lower bounds where it is negative. However, the sum-at-risk method does not explain how to find such first-order basis.

3.2.2 Derivatives: local linearisations

Using derivatives, Dienst (1995) studied the impact in the premium caused by changes in disability probabilities and Bowers et al. (1997) analyse the sensitivity of the expected loss with respect to the interest rate. Christiansen (2008a,b) gives a general formula for the sensitivity of the prospective reserve and the premium of an individual insurance contract with respect to changes in a large number of parameters of the technical basis.

Following Christiansen (2008a), the gradient vector of the prospective reserve $V_i(t)$ associated with a possible transition from state j to state k at time s is

$$\nabla q_{jk} V_i(t,s) = 1_{(t,T]}(s)\, p_{ij}(t,s-)\, \upsilon(t,s)\, R_{jk}(s)$$

and the prospective reserve $V_i(t, q_J)$ (written in this way to show its dependence on the biometric valuation basis) can be locally approximated by the first-order Taylor expansion

$$V_i(t,\, q_J + H) \backsimeq V_i(t, q_J) + \int \nabla q_{jk} V_i(t,u) dH,$$

where H is a local shift and $\nabla q_{jk} V_i(t)$ measures the local sensitivity of the prospective reserve $V_i(t)$ to changes of the transition rates at any time $s \in [0, T]$.

The problem is that as differentiation is in general a local concept, we can only study infinitesimal changes of the transition rates. Hence, the method based on derivatives works only for narrow confidence bands and yields not exact but only approximate results.

3.2.3 Derivatives: one-step approach

The method proposed by Kalashnikov and Norberg (2003) consists in differentiating the reserve with respect to one arbitrary real parameter θ, which may be an element of the valuation basis or an element of the design of the contract. They consider that the estimator $\hat{\theta}$ of the p-dimensional parameter θ is consistent and asymptotically normally distributed with mean θ and variance matrix $\Sigma(\theta)$. Then, by combination of the so-called Scheffé method and the Delta method (Sverdrup, 1986), confidence intervals with asymptotic confidence level $1-\varepsilon$ of the form

$$V_i(t,\theta) \in V_i\left(t,\hat{\theta}\right) \pm \sqrt{\chi^2_{(p,1-\varepsilon)} DV_i\left(t,\hat{\theta}\right) \hat{\Sigma} DV_i^T\left(t,\hat{\theta}\right)}$$

can be obtained, where $DV_i(t,\theta) = \left(\frac{\partial}{\partial\theta_1} V_i(t,\theta), \ldots, \frac{\partial}{\partial\theta_p} V_i(t,\theta)\right)$, $\hat{\Sigma}$ is a consistent estimator of $\Sigma(\theta)$ and $\chi^2_{(p,1-\varepsilon)}$ is the $1-\varepsilon$ fractile of the chi-squared distribution with p degrees of freedom.

The confidence intervals for reserves are obtained directly in one step in contrast with other methods that first find the confidence intervals for the valuation basis and then use these to construct the confidence bands for reserves. However, Christiansen (2010) argues that the one-step approach may be very appealing, but it runs counter to the traditional rules of insurance regulation in many countries. For instance, under the Solvency II regime, after computing the reserves the effect of changes in valuation basis is studied, by applying stress scenarios. We will expand on this in the next chapter.

3.3 The model

As explained above, the three previous methods have weaknesses and are not exact. To fill this gap, Christiansen (2010) gave a new approach for the construction of worst-case scenarios: based on Thiele's integral equation, another integral equation is developed whose solution yields the maximal prospective reserve with respect to all cumulative transition intensities within some confidence band. In this section, we introduce the method proposed by Christiansen (2010). The proofs of the results in this section will be omitted but can be found in the original work.

3.3.1 The confidence bands

The first step is to impose some bounds for the actuarial assumptions, for instance by applying statistical methods on the past data, and reducing the future uncertainties to certain intervals. In the case where intensities do not exist, the author defines the confidence bands for the intensities of the form

$$L_{jk}(t) - L_{jk}(s) \leq q_{jk}(t) - q_{jk}(s) \leq U_{jk}(t) - U_{jk}(s), \tag{3.3.1}$$

$t \geq s$, and considers that L and U are regular cumulative transition intensity matrices.
Assuming that the transition matrix q_J is differentiable, it means that the intensities matrix μ_J exists and the confidence bands can be written as follows

$$l_{jk}(t) \leq \mu_{jk}(t) \leq u_{jk}(t), \tag{3.3.2}$$

where $l_{jk}(t)$ and $u_{jk}(t)$ are integrable functions.
For the states a="active" and d="dead", the prospective reserve of a whole life insurance is maximized by the bonds u_{ad} and U_{ad}, while for a whole life annuity the maximal bonds are l_{ad} and L_{ad}. However, for policies with mixed character or more than two states the situation is much more complex. The method in study is specially informative for theses cases.

From now on, in order to emphasize the dependence of prospective reserve $V_i(s)$ on the biometric valuation basis, the prospective reserve that corresponds to μ_J or q_J is denoted as $V_i(s, \mu_J)$ or $V_i(s, q_J)$, respectively.

3.3.2 The worst-case integral equation system

Now that confidence bands are defined, it is necessary to find for which μ_J within these bounds the prospective reserve is maximal. This set of transition intensities represent the biometric worst-case scenario for the insurer and is denoted by $\overline{\mu}_{jk}$. Based on the sum-at-risk theory, assume that it satisfies the following

$$\overline{\mu}_{jk}(t) = \begin{cases} u_{jk}(t) & if\ R(t, \overline{\mu}_J) > 0 \\ l_{jk}(t) & if\ R(t, \overline{\mu}_J) < 0 \end{cases} \tag{3.3.3}$$

for all $t \geq s$ and $(j,k) \in J$.

As our purpose is to find the biometric first-order basis that sets premiums and reserves on the safe-side, from equation (3.2.2) it is possible to conclude that if the prospective reserve is computed using the biometric scenario (3.3.3), then the basic safe-side requirement $V_i(s, \overline{\mu}_J) \geq V_i(s, \mu_J)$ is fulfilled for all μ_J with

$$\begin{aligned} \mu_{jk}(t) \leq \overline{\mu}_{jk}(t) = u_{jk}(t) \qquad & if\ R_{jk}(t, \overline{\mu}_J) > 0 \\ \mu_{jk}(t) \geq \overline{\mu}_{jk}(t) = l_{jk}(t) \qquad & if\ R_{jk}(t, \overline{\mu}_J) < 0 \end{aligned}$$

and it is the worst-case scenario for the insurer that maximizes the prospective reserve. In the following we use the intuitive notation $\overline{R}_{ij} = R_{ij}(\cdot, \overline{\mu}_J)$ and $\overline{V}_i = V_i(\cdot, \overline{\mu}_J)$. Taking these results into account, the author presents the called "worst-case integral equation system" (see Appendix A.2)

$$\begin{aligned}\overline{V}_i(s) &= B_i(T) - B_i(s) - \int_{(s,T]} \overline{V}_i(t-)\, d\Phi(t) \\ &+ \sum_{j:j\neq i} \left(\int_{(s,T]} \tfrac{1}{2} \mid \overline{R}_{ij} \mid (t)\, (u_{ij} - l_{ij})(t)\, dt + \int_{(s,T]} \tfrac{1}{2} \overline{R}_{ij}(t)\, (u_{ij} + l_{ij})(t)\, dt \right)\end{aligned} \tag{3.3.4}$$

or equivalently, in the cumulative intensity notation,

$$\begin{aligned}\overline{V}_i(s) &= B_i(T) - B_i(s) - \int_{(s,T]} \overline{V}_i(t-)\, d\Phi(t) \\ &+ \sum_{j:j\neq i} \left(\int_{(s,T]} \tfrac{1}{2} \mid \overline{R}_{ij} \mid (t)\, d(U_{ij} - L_{ij})(t) + \int_{(s,T]} \tfrac{1}{2} \overline{R}_{ij}(t)\, d(U_{ij} + L_{ij})(t) \right),\end{aligned} \tag{3.3.5}$$

for all $i \in S$ and $s \in [0,T]$, with initial condition $\overline{V}_i(T) = 0$. Note that the new Thiele's integral equation does not directly depends on μ_{jk} anymore and as such a maximizing scenario $\overline{\mu}_J$, with respect to (3.3.2) or (3.3.1), can be constructed as a solution of the previous integral equation systems (3.3.4) or (3.3.5).

3.3.3 Existence and uniqueness of solutions

Christiansen (2010) shows that if the integrals in (3.1.10) exist, then $s \mapsto V_i(s)$ has finite total variation on $[0,T]$, that is

$$\|V_i\|_{V_{[0,T]}} \leq \|B_i\|_{V_{[0,T]}} + \int_{(0,T]} \mid V_i(t-) \mid d\Phi(t) + \sum_{j:j\neq i} \int_{(0,T]} \mid R_{ij}(t) \mid dq_{ij}(t) < \infty,$$

and as such $\left(V_i \mid_{[0,T]}\right)_{i\in S}$ can be seen as an element of the Banach space

$$BV_{[0,T]}^{|S|} = \left\{ f : [0,T] \to \mathbb{R}^{|S|} \mid \sum_{i=1}^{|S|} \|f_i\|_{V_{[0,T]}} < \infty,\ f(T) = 0,\ f(t) = f(t+0) \right\}$$

with norm $\|f\|_{V_{[0,T]}^{|S|}} = \sum_{i=1}^{|S|} \|f_i\|_{V_{[0,T]}}$, where $\| \cdot \|_{V_{[0,T]}}$ is the total variation on $[0,T]$.

Taking these two results into account, the author proves the existence and the uniqueness of the solution of the worst-case integral equation systems (3.3.4) and (3.3.5) and that this solution is always maximal.

Theorem 3.3.1. *There always exists a unique solution* $\left(\overline{V}_j\right)_{j\in S} \in BV_{[0,T]}^{|S|}$ *for integral equation system* (3.3.5).

Theorem 3.3.2. *If* $\left(\overline{V}_j\right)_{j\in S} \in BV_{[0,T]}^{|S|}$ *is a solution of integral equation system* (3.3.5), *then* $\overline{V}_j(s) \geq V_j(s, q_J)$ *for all* $j \in S$, $s \in [0,T]$ *and any regular cumulative transition intensity matrix* q_J *that satisfies* (3.3.1).

Under the assumption that intensities exist, the worst-case scenario is constructed by choosing the upper bound where the sum-at-risk of the solution of (3.3.4) is positive and the lower bound where it is negative. In the general case, the worst-case scenario is constructed as follows (Christiansen, 2010).

Corollary 3.3.3. *Let* $\left(\overline{V}_j\right)_{j\in S} \in BV^{|S|}_{[0,T]}$ *be a solution of integral equation system* (3.3.5). *Then* $\widetilde{q}$ *defined by*

$$\begin{cases} \tilde{q}_{jj}(t) = -\sum_{k\neq j} \tilde{q}_{jk}(t) \\ \tilde{q}_{jk}(t) = \int_{(0,T]\cap\left\{\overline{R}_{jk}<0\right\}} dL_{jk} + \int_{(0,T]\cap\left\{\overline{R}_{jk}\geq 0\right\}} dU_{jk} \end{cases}$$

with $(j,k) \in J$, $t \in [0,T]$, *is a regular cumulative transition intensity matrix that maximizes the prospective reserves* $V_j(s)$ *for all* $j \in S$ *and all* $s \in [0,T]$.

Furthermore, an interesting result is that the worst-case scenario $\tilde{q}$ is invariant with respect to time and space. It means that if we calculate $\tilde{q}$ in state i at the beginning of the contract period ($t = 0$), it remains the worst-case scenario at any other time $t \in [0,T]$, even though the increase of information about the policyholder's pattern of states and the progression of time.

However, the uniqueness of the solution depends on the assumption that $\left\{\overline{R}_{ij} \neq 0\right\}$. In the case where $\left\{\overline{R}_{ij} = 0\right\}$, $\tilde{q}_{jk}(t)$ or $\tilde{\mu}_{jk}(t)$ can be arbitrarily defined within the bounds (3.3.1) and (3.3.2), respectively.

3.3.4 Solving the Worst-Case Integral Equation

Finally, the author proposes two approaches in order to obtain the solution of the worst-case integral equation system: an approach based on intensities and a discrete time approach.

3.3.4.1 Approach based on intensities

The assumptions above require the cumulative intensities q_{jk} and Φ to be differentiable with intensities μ_{jk} and φ, respectively. The annuity payments that fall due during sojourns in a state j are modelled by a right-continuous function, that can be written as:

$$B_j(t) = \int_{(0,t]} b_j(u)\, du + \sum_{0\leq u\leq t} \triangle B_j(u),$$

with $b_j(u) = B_j^{'}(u)$ continuous at u and $\triangle B_j(u) = B_j(u) - B_j(u-)$, where the discrete time payments $\triangle B_j(u) > 0$ may occur in any state j at any time $0 = t_0 < t_1 < ... < t_n = T$. Therefore, in order to solve the worst-case integral equation system we apply the following algorithm.

Algorithm 3.3.4.

1) Start from the initial condition $V_j(t_n) = 0$

2) Calculate $V_j(t_n-)$, $V_j(t_{n-1}-)$, ..., $V_j(0-)$ *by applying*

$$\overline{V}_j(t-) = \begin{cases} \overline{V}_j(t) + \triangle B_j(t) & t \in [0,T], \, j \in S, \, t \in \{t_1, ..., t_n\} \\ \overline{V}_j(t) & otherwise \end{cases} \tag{3.3.6}$$

3) For the time intervals $(t_{i-1}, t_i]$ *between discrete time payments, calculate* $V_j(t_{n-1})$, $V_j(t_{n-2})$, ..., $V_j(0)$ *by applying*

$$\frac{d}{ds}\overline{V}_j(s) = -b_j(s) + \overline{V}_j(s)\varphi(s) - \sum_{k:k\neq j}\left(\frac{1}{2} \mid \overline{R}_{jk} \mid (s)(u_{jk} - l_{jk})(s) + \frac{1}{2}\overline{R}_{jk}(s)(u_{jk} + l_{jk})(s)\right) \tag{3.3.7}$$

This differential equation system can be solved numerically using standard methods, as Euler Method (Griffiths and Higham, 2010). The idea is to start from $V_j(t_n) = 0$ and calculate $V_j(t_n-)$,$V_j(t_{n-1})$,$V_j(t_{n-1}-)$, $V_j(t_{n-2})$, ...,$V_j(0)$, $V_j(0-)$ by applying (3.3.6) and (3.3.7) in an alternating manner.

3.3.4.2 Discrete time approach

When assumed that transitions between states occur only on a discrete time set $\{0, 1, 2, ..., T\}$, the cumulative transition intensity matrix q_J is constant between jumps and they are weighted by $\triangle q_{jk}(t) = p_{jk}(t-, t) = p_{jk}(t-1, t)$. Thus, the confidence bounds are defined for transition probabilities as follows

$$\triangle L_{jk}(t) \leq p_{jk}(t-1, t) \leq \triangle U_{jk}(t), \tag{3.3.8}$$

for $(j,k) \in J$ and $t \in \{1, 2, ..., T\}$.
The annuity payments that fall due during sojourns in state j are of the form

$$B_j(t) = \sum_{u=0}^{[t]} \triangle B_j(u).$$

In order to solve (3.3.5), we use the algorithm below.

Algorithm 3.3.5.

1) Start from the initial condition $V_j(T) = 0$*;*

2) Calculate $V_j(t-1)$, $V_j(t-2)$, ..., $V_j(0)$ by applying:

$$\begin{aligned} \overline{V}_j(t-1) = \quad & \nu(t-1,t)\Bigg(\overline{V}_j(t) + \triangle B_j(t) + \sum_{k:k\neq j}\left(\tfrac{1}{2} \mid \overline{R}_{jk} \mid (t)\triangle(U_{jk} - L_{jk})(t)\right. \\ & \left. + \tfrac{1}{2}\overline{R}_{jk}(t)\triangle(U_{jk} + L_{jk})(t)\right)\Bigg) \end{aligned} \tag{3.3.9}$$

3) For the intervals in between the integer times $\{0, 1, 2, ..., T\}$: $V_i(s) = \upsilon([s], s)^{-1} V_i([s])$.

Chapter 4

Case Studies

We will now apply the model and results in 3.3 to two case studies. This chapter is the core of the work in the sense that the cases discussed in Christiansen (2010) are extended in order to accommodate more complex products, and at the same time are updated according to the most recent developments of the Solvency II regime. While case study 1 deals with a combination of an annuity and a life insurance product evaluated in discrete time, case study 2 is a combination of a disability income insurance with a critical illness insurance policy, evaluated in continuous time. These two policies are not common products in the Portuguese insurance market but are elaborate cases that allow us to illustrate the topic in study in a very comprehensive way. In the following, we will present some aspects of Solvency II that are particularly important nowadays and were not known when Christiansen (2010) was published, performing this way an update of the study.

4.1 Solvency II Regime

Solvency II project (Directive 2009/138/EC) is the new regulation framework of the European Union for insurance and reinsurance companies that will replace the Solvency I regime (Directive 2002/13/EC and Directive 2002/83/EC). Its main target is to ensure the financial soundness of insurance undertakings and guarantee their survival during difficult periods, protecting policyholders and keeping stability of the financial system as a whole.

A first step, the Solvency II Framework Directive 2009/138/EC with the general principles of the regime was adopted on November 2009. However, it had to be adapted in response to the new supervisory structure introduced in the EU's Treaty of Lisbon (2007). On 11 March 2014 the European Parliament adopted the Omnibus II Directive (2014/51/EC) that completes the Solvency II Directive and finalizes the new framework for insurance regulation and supervision in the EU. The application of the Solvency II Directive is scheduled for 1 January 2016.

One important feature of the new regime is the establishment of quantitative requirements regarding own funds, in particular the Solvency Capital Requirement (SCR). Article 101.º of the Solvency II Directive (2009/138/EC) requires that the SCR "shall correspond to the Value-at-Risk of the basic own funds of an insurance or reinsurance undertaking subject to a confidence level of 99.5% over a one-year period.".

Furthermore, the European standard formula given by the directive uses a modular approach where the risk modules are in turn built on sub-modules (Fig. A.1, Appendix A.3). The SCR for each risk is calculated by re-evaluating best estimate (BE) liabilities (Directive 2009/138/EC, Article 77º) under a specific stress scenario and corresponds to the variation of the basic own funds. These are then aggregated to arrive at the overall SCR, using the so-called square-root formula:

$$SCR = \sqrt{\sum_{ij} Corr_{ij} \times SCR_i \times SCR_j} \tag{4.1.1}$$

where $Corr_{ij}$ represents the correlation factor between risks i and j, given on EIOPA (2014b) (Table A.1, Appendix A.3). EIOPA is the European Insurance and Occupational Pensions Authority.

In order to refine the methodologies, parameters and assumptions, and to help determining the quantitative requirements for new solvency rules, several quantitative impact studies (QIS) have been carried out by EIOPA since 2005. The QIS are the primary means for testing the design of the future European Standard Formula, as well as the main route for finding the correct calibration reflecting the VaR 99.5% over a 1-year time horizon. In preparing the insurance market for the entry into force of the new solvency regime, the ISP (Circular N.º 1/2014) launched a mandatory quantitative impact study nationwide, aimed at all companies under to its prudential supervision.

In this work, the focus is on the biometric risks such as mortality risk (mort), longevity risk (long) and disability or morbidity risk (dis). The Solvency II stress scenarios used to quantify these risks, presented in (EIOPA, 2014b), are the following:

- 15% increase in mortality rates for each age for SCR_{mort};
- 20% decrease in mortality rates for each age for SCR_{long};
- 35% increase in disability rates for the next year, together with a permanent 25% increase in disability rates at each age in following years, plus (when applicable) a permanent 20% decrease in morbidity/disability recovery rates.

Under Solvency II the worst-case for the insurer, i.e. the largest value of future obligations, corresponds to the BE plus the SCR obtained from these set of stressed scenarios.

However, the standard formula is calibrated at European level and given the diversity of markets and products, such calibration may reveal inappropriate for some individual insurers. In this case, and subject to prior authorization, each insurance company can choose between setting up its own internal model to calculate the SCR or using the European standard formula.

As a matter of fact, for an insurance company that wishes to exchange the standard SCR by an internal model-based calculation, or simply intends to study the adequacy of the standard formula to its risk profile, the worst-case scenario method presented can show how such "internal stress scenarios" can be derived.

In the next section all the previous elements will be illustrated and applied in order to show how theory and practice may be so harmoniously combined. The situations in analysis, although particular, are at the same time profound and allow us to derive significant conclusions.

4.2 Case Study 1

We start with the policy used by Christiansen (2010) in example 5.1 in order to discuss the impact of using:

- the Portuguese mortality basis: Case Study (CS) 1.1;
- the term structure of interest rates (TSIR) (Brigo and Mercurio, 2006) to discount cash-flow and calculate premiums: CS 1.2.

After that, in CS 1.3, we change the term and the amount of the benefits to amounts more realistic. Lastly, we study two new products with extra covers:

- the disability cover in CS 1.4;
- the survival cover in CS 1.5.

For the different situations described, we present three methods: I - the BE for the reserve (c.f. section 4.1); II - the worst-case method using the discrete time approach Algorithm 3.3.5; III - the sum-at-risk method defining the transition intensity equal to the lower bound where the sum-at-risk with respect to the BE is negative and equal to the upper bound where it is positive (sub-section 3.2.1).

4.2.1 CS 1.1 and CS 1.2

Consider a 30 year old man who buys a policy on 31 December 2013 with the following benefits:

- $B_a(t) = 1$, $t \geq 67$: a pension of 1 is paid yearly in advance from age 67 on till death;
- $b_{ad}(t) = 17$, $t \in \{31, 32, ..., 87\}$: a sum insured of 17 is paid in case of death before age 87.

Mortality Basis: According to the last information released by ISP (2013), the most used life tables in Portugal for the new life insurance policies with death and survival covers are the Swiss GKM80 and GKF95, respectively. Thus, we take the GKF95 table as the lower bound $\triangle L_{ad}$ and the GKM80 table as the upper bound $\triangle U_{ad}$ in (3.3.8), instead of the German life tables DAV2004R and DAV2008T used by the author. Given the mixed character of this policy, it seems reasonable to continue using the arithmetic average of that tables as BE: $(50\% \, GKF95 + 50\% \, GKM80)$ (Fig. A.2, Appendix A.4).

Financial Basis: Under Solvency II, cash-flows should be discounted using the relevant TSIR (Directive 2009/138/EC, Article 77.º). During EIOPA 2014 Stress Test Exercise, EIOPA (2014a) provided a curve for Portugal (Fig. A.6, Appendix A.4).

In order to measure the impact of financial basis we considered the following situations:

CS 1.1 Flat term structure: An interest rate of 2.25% is used in premiums and reserve calculations (Christiansen, 2010);

CS 1.2 TSIR: The risk-free interest rate term structure provided by EIOPA (2014a) is used to discount the reserve from one period to the previous one: $\nu(t-1,t)$ in (3.3.9). As in real practice, premiums are computed using a constant technical interest rate, we assume that it is equal to the internal rate implicit in all outflows discounted using the zero coupon curve plus a bonus factor of 0.5%. The obtained rate is 3.796%.

As usual, the annual premium is paid in advance till retirement or death, whichever occurs first, and is computed using the equivalence principle method and the BE mortality basis. The results are in Table 4.1.

	CS 1.1 - Flat Structure	**CS 1.2 - TSIR**
Premium	0.39639	0.26208
Technical Interest Rate	2.25%	3.796%

Table 4.1: CS 1.1 and CS 1.2 - Premiums and technical interest rates

As might be expected, the resulting premiums are different from the one obtained by Christiansen (2010): 0.390147. The new mortality basis has a small positive impact on premiums (CS 1.1) given the highest mortality rates of Swiss BE (Fig. A.4, Appendix A.4). It becomes more significant when combined with the negative interest rate impact (CS 1.2). Table 4.2 shows the reserve at contract time 0−, i.e., at age 30−, in state active $V_a(0-)$ for the three different methods.

Method	**Mortality Basis**	**Reserve CS 1.1 (Flat Structure)**	**Reserve CS 1.2 (STIR)**
I - Best Estimate	50% GKF95 + 50% GKM80	0	0.25155
II - Worst-case Scenario Method	Lower bound: GKF95 Upper bound: GKM80	0.65464	0.66323
III - Sum-at-risk Method	Lower bound: GKF95 Upper bound: GKM80	0.63576	0.65591

Table 4.2: CS 1.1 and CS 1.2 - Prospective reserve $V_a(0^-)$

(The results obtained by the author under each of the previous methods are 0, 1.1647 and 1.0034, respectively.)

In CS 1.1, the BE is null because the premiums basis is equal to the valuation basis. In order to analyse the impact of using the Portuguese mortality basis, we observe that the life table GKF95 is less conservative than DAV2004R for annuities, while GKM80 is more conservative than DAV2008T until age 83 for policies with occurrence character (Fig. A.3, Appendix A.4). Thus, given the mixed character of the policies in study, the application of Swiss life tables leads always to lower reserves, even when we use GKF95 and GKM80 separately. Further, even using the most conservative of these tables instead of the worst-case or the sum-at-risk methods may underestimate the risk of mortality changes.

If we use the TSIR, the BE is greater than zero, meaning the insurer expects to make a loss on this policy from the outset. It sounds uncomfortable but is not uncommon in practice, explained by the fact that valuation basis may be more conservative than the premium basis. The introduction of the term structure of interest rate leads to an increase in the reserves, under the three methods.

In both situations, the worst-case scenario method required the highest reserve pointing out that although the stress scenarios seem quite demanding that is not necessarily the case. From now one, we will consider the Swiss life tables and the TSIR as the applicable basis.

4.2.2 CS 1.3, CS 1.4 and CS 1.5

Now, consider that the policy was bought by a 35 year old man and the products purchased are as follows.

CS 1.3: Whole life Annuity and Temporary Insurance

- $B_a(t) = 14\,000$, $t \geq 66$: a pension of 14 000 is paid yearly in advance from age 66 (the retirement age in Portugal for the year 2014) till death;
- $b_{ad}(t) = 200\,000$, $t \in \{39, 40, ..., 75\}$: a sum insured of 200 000 is paid in case of death before age 75, after a deferred period of 3 years during which no benefits are paid.

CS 1.4: Whole life Annuity and Temporary Insurance with Disability Benefits

Death benefits are the same as in CS 1.3 provided that the disability benefit has not already been paid; There is also a disability benefit: $b_{ai}(t) = 150\,000$, $t \in \{37, 38, ..., 66\}$: a sum insured of 150 000 is paid in case of permanent disability before age 66, after a deferred period of 1 year during which no benefits are paid.

CS 1.5: Whole life Annuity and Endowment Insurance with disability Benefits

Benefits as in CS 1.4. Now the survival cover at age 75 is $B_a(75) = 14\,000 + 25\,000$, provided that the life is able at that time.

Note that product in CS 1.3 is similar to the one considered in CS 1.1 and 1.2 but with terms and benefits that seem to be most appropriate. We use it as the starting point to analyse the individual impact of introducing the new covers.

Disability basis: According to ISP (2013), disability tables are often given by reinsurers. Thus, we take the Swiss Re 2001 as the lower bound $\triangle L_{ai}$ and the Individual TPD Reference Table of Swiss Re (TPD) as the upper bound $\triangle U_{ai}$ in (3.3.8), and again seems to be reasonable to consider as BE ($50\%\, SwissRe2001 + 50\%\, TPD$) (Fig. A.5, Appendix A.4). As the maximum age in both tables is 64, we assume that disability rate at age 65 is equal to that at age 64 since the disability rates are already stabilised.

The annual premium is paid in advance till retirement, disability or death, whichever occurs first. The results are in Table 4.3.

	CS 1.3	**CS 1.4**	**CS 1.5**
Premium	3 833.71	3 942.14	4 127.15
Technical Interest Rate	3.657%	3.632%	3.638%

Table 4.3: CS 1.3, 1.4 and 1.5 - Premiums and technical interest rates

As expected, the introduction of a new cover leads to an increase in premiums.
Table 4.4 present the reserves in state active at time 0^- under the three usual methods.

Method	**Mortality Basis**	**Disability Basis**
I - Best Estimate	50% GKF95 + 50% GKM80	50% SwissRE2001 + 50% TPD
II - Worst-case Scenario Method	Lower bound: GKF95 Upper bound: GKM80	Lower bound: SwissRe2001 Upper bound: TPD
III - Sum-at-risk Method	Lower bound: GKF95 Upper bound: GKM80	Lower bound: SwissRe2001 Upper bound: TPD

Method	**CS 1.3**	**CS 1.4**	**CS 1.5**
I - Best Estimate	2 740.59	2 436.05	2 620.86
II - Worst-case Scenario Method	11 276.00	10 445.34	10 240.43
III - Sum-at-risk Method	11 099.97	10 289.12	9 978.43

Table 4.4: CS 1.3, 1.4 and 1.5 - Prospective reserve $V_a\,(0^-)$

Naturally, the introduction of the new covers produces changes in the BE reserve. This is explained by the change in technical interest rate that leads to an increase in premiums. If we consider the same interest rate of 3.657% in premiums calculations, the reserve will always increase to 2 854.23 and 2 964.18 in CS 1.4 and 1.5.

Under methods II and III, if the insurer increases the premiums as in Table 4.3, it can offer extra covers to the policyholders and the diversification of risks leads to an improvement on its worst-case scenario. Furthermore, note that the reserves obtained by method II are always higher than those required by method III. Thus, it is possible to conclude that within the confidence bands, there are scenarios that are worse, i.e. that lead to a higher reserve, than the worst scenarios used by the sum-at-risk method (section A.4.2,

Appendix A.4). The use of it instead of the worst-case method does not guarantee that we are on the safe side and may underestimate the risk of biometric rate changes.

From now on focus is on CS 1.5. Consider the "internal stress scenarios" based on our confidence bands (sub-section 4.2.1) and the Solvency II stress scenarios presented in EIOPA (2014b). EIOPA (2014b, SCR.7.12.) suggests that for policies providing benefits both in case of death and survival, contingent on the life of the same insured person, the mortality and longevity scenarios should be applied to the policy as a whole without decomposing it into the annuity and the life insurance. Therefore, after obtaining the stress scenarios and calculating the SCR's, they are aggregate with the "square-root formula" using the correlation factors in matrix Table. A.1, Appendix A.3. Table 4.5 summarizes the results.

Scenario A - Internal stress scenarios		
	Mortality Basis	Disability Basis
Mortality Stress Scenario	GKM80	BE
Longevity Stress Scenario	GKF95	BE
Disability Stress Scenario	BE	TPD
Scenario B - Solvency II stress scenarios		
	Mortality Basis	Disability Basis
Mortality Stress Scenario	Increase of 15% in BE	BE
Longevity Stress Scenario	Decrease of 20% in BE	BE
Disability Stress Scenario	BE	Increase of 35%/ 25% in BE

Scenario	**Method**	**SCR**	**Reserve**
A	Standard formula of Solvency II	1 698.97	4 319.83
	Mortality Stress Scenario	1 467.06	4 087.92
	Longevity Stress Scenario	1 276.71	3 897.57
	Disability Stress Scenario	51.87	2 672.73
B	Standard formula of Solvency II	653.85	3 274.70
	Mortality Stress Scenario	68.49	2 689.35
	Longevity Stress Scenario	514.53	3 135.39
	Disability Stress Scenario	402.40	3 023.26

Table 4.5: CS 1.5 - Standard formula of Solvency II

Observe that while the prevailing risk in scenario A is the mortality risk, in scenario B is the longevity risk. Remembering that the worst-case scenario (Table 4.4) requires a reserve of 10 240.43, the value obtained in scenario A is lower and a possible explanation is that although the square-root formula considers the correlation between the risks, it disregards mixed mortality scenarios. Furthermore, according to Theorem 3.3.2 scenarios $\bar{\mu}_{ad}$ and $\bar{\mu}_{ai}$ obtained by method II are the worst scenarios that can occur in terms of increasing liabilities with respect to the used confidence bands, so the standard formula of

Solvency II applied to the "internal stress scenarios" may underestimate the risk of changes in the mortality and disability rates.

The reserve in scenario B is also lower than the one required by the worst-case scenario method. However, the same conclusion is only valid to Solvency II stress scenarios if the insurer considers that the confidence bands GKF95/GKM80 and SwissRe2001/TPD are the most appropriate and realistic. If it is the case, may be the calibration of standard formula of Solvency II reveals inappropriate for the insurer and it should consider its own internal model to calculate the SCR.

After evaluating the obtained overall results, in case of policies with both survival and occurrence character, the worst-case method seems to be the most adequate in order to obtain reserves on the safe side under the assumptions made. Further details about this Case Study can be seen in Appendix A.4.

4.3 Case Study 2

In Christiansen (2010), the continuous time approach (sub-section 3.3.4.1) is illustrated in example 5.2. In this Case Study, we return to that illustration but considering a more complex product that provides a death benefit, a disability income benefit and a critical illness benefit as follows:

- $b_{ai}(t) = 20\,000$: a benefit of 20 000 is payable immediately on the active life becoming critically ill;
- $b_{si}(t) = 18\,000$: a benefit of 18 000 is payable immediately on the sick life becoming critically ill;
- $b_{ad}(t) = b_{sd} = 10\,000$: a benefit of 10 000 is payable immediately on the death, provided that the life has not already been paid a critical illness benefit;
- $b_s(t) = 1\,000$: a disability income annuity of 1 000 is payable every month while the life is disabled. Note that the first payment will be made 1 month after the policy starts, if the policyholder is in state sick at that time.

The policy may be described by the four-state Markov model depicted in Fig. 4.1.

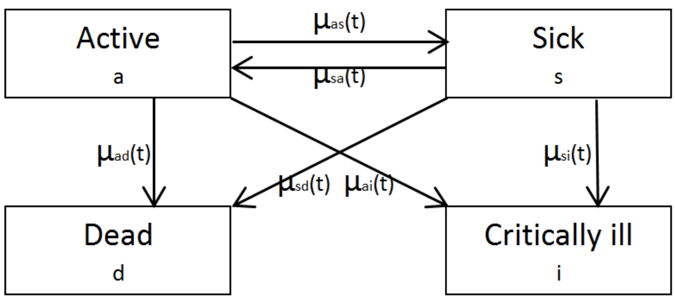

Figure 4.1: Disability Income Protection & Critical Illness Model

Assume that the policy was bought by a 30 year-old female in state active a and ends at age 66 or the first time the policyholder reaches states i or d, whichever occurs first. Thus, in each time t the reserve in states a and s must be computed. Furthermore, the single premium is paid at the time the contract is effected using (4.3.1) as bellow

$$P = 20\,000\,\bar{A}^{ai}_{30:\overline{36}|} + 18\,000\,\bar{A}^{si}_{30:\overline{36}|} + 10\,000\left(\bar{A}^{ad}_{30:\overline{36}|} + \bar{A}^{sd}_{30:\overline{36}|}\right) + 1\,000\,a^{(12)s}_{30:\overline{36}|} \qquad (4.3.1)$$

We will assume an interest rate of $i = 3.5\%$, i.e. $\varphi(t) = log(1.035)$. Note that an increase/decrease in the interest rate leads to a decrease/increase in reserves.

As it was not possible to find transition intensities fitted to real data to describe the model, the following theoretical functions based on Dickson et al. (2012) are used as our BE (Fig. A.8, Appendix A.5).

$$\begin{array}{lll} \tilde{\mu}_{as}(t) = 0.0004 + 0.0000035 \times \exp(0.14t) & \tilde{\mu}_{ai}(t) = 0.05 \times \tilde{\mu}_{as}(t) & \tilde{\mu}_{sd(t)} = \tilde{\mu}_{ad}(t) \\ \tilde{\mu}_{ad}(t) = 0.0005 + 0.000076 \times \exp(0.09t) & \tilde{\mu}_{sa}(t) = 0.1 \times \tilde{\mu}_{as}(t) & \tilde{\mu}_{si}(t) = \tilde{\mu}_{as}(t) \end{array} \qquad (4.3.2)$$

The upper and lower bounds of confidence bands were chosen randomly between [0.65, 1.35] (the maximum amplitude of the biometric Solvency II shocks) with step 0.5

$$\begin{array}{rcccl} l_{as}(t) = 0.7 \times \tilde{\mu}_{as}(t) & \leq & \mu_{as}(t) & \leq & u_{as}(t) = 1.3 \times \tilde{\mu}_{as}(t) \\ l_{ai}(t) = 0.7 \times \tilde{\mu}_{ai}(t) & \leq & \mu_{ai}(t) & \leq & u_{ai}(t) = 1.3 \times \tilde{\mu}_{ai}(t) \\ l_{ad}(t) = 0.8 \times \tilde{\mu}_{ad}(t) & \leq & \mu_{ad}(t) & \leq & u_{ad}(t) = 1.15 \times \tilde{\mu}_{ad}(t) \\ l_{sa}(t) = 0.8 \times \tilde{\mu}_{sa}(t) & \leq & \mu_{sa}(t) & \leq & u_{sa}(t) = 1.15 \times \tilde{\mu}_{sa}(t) \\ l_{si}(t) = 0.7 \times \tilde{\mu}_{si}(t) & \leq & \mu_{si}(t) & \leq & u_{si}(t) = 1.15 \times \tilde{\mu}_{si}(t) \\ l_{sd}(t) = 0.9 \times \tilde{\mu}_{sd}(t) & \leq & \mu_{sd}(t) & \leq & u_{sd}(t) = 1.05 \times \tilde{\mu}_{sd}(t)\,. \end{array} \qquad (4.3.3)$$

As the policy is bought by a woman in state active, the premium should be equal to the reserve in that state $V_a(0)$. Table 4.6 summarizes the results for the three methods, using Euler method (Griffiths and Higham, 2010) with a step of 0.0001.

Method	**Reserve**
I - Best Estimate	8 466.33
II - Worst-case Scenario Method	10 708.15
III - Sum-at-risk Method	10 679.31

Table 4.6: CS 2 - Prospective reserve $V_a(0^-)$

At this stage, it is important to study what happens if the second-order valuation basis is different from the first order valuation basis.

The worst-case method (Algorithm 3.3.4) makes the premium increase by 2 241.82, while according to the sum-at-risk method the increase should be 2 212.97 (section A.5.2, Appendix A.5). As there is no reason to exclude method II if we allow for method III, the use of the sum-at-risk method instead of the worst-case method may underestimate the risk of rate changes in biometric transition intensities.

Under Solvency II regime, a critical illness insurance that also provides a benefit in case of death should be classified as life insurance obligations because the main risk driver is usually death (rather than contracting the illness), while a pure income protection should be classified as health insurance obligations. We consider that it is not appropriate to unbundle contracts and then our policy is classified as life obligations. According to EIOPA (2014b, SCR 7.38), the disability shock should be applied to all transition rates from one status to a more severe health status: $\tilde{\mu}_{as}$, $\tilde{\mu}_{ai}$ and $\tilde{\mu}_{si}$. In the same way, the mortality and longevity stresses are applied to transition intensities $\tilde{\mu}_{ad}$ and $\tilde{\mu}_{sd}$. As usual, we deal with internal and Solvency II stress scenarios, aggregating SCR's with the "square-root formula". Table 4.7 summarizes the results.

Scenario A - Internal stress scenarios		
	Mortality Basis	Disability Basis
Mortality Stress Scenario	$u_{ad}(t)$, $u_{sd}(t)$	BE
Longevity Stress Scenario	$l_{ad}(t)$, $l_{sd}(t)$	BE
Disability Stress Scenario	BE	$u_{as}(t)$, $u_{ai}(t)$, $u_{si}(t)$, $l_{sa}(t)$
Scenario B - Solvency II stress scenarios		
	Mortality Basis	Disability Basis
Mortality Stress Scenario	Increase of 15% in $\tilde{\mu}_{ad}$, $\tilde{\mu}_{sd}$	BE
Longevity Stress Scenario	Decrease of 20% in $\tilde{\mu}_{ad}$, $\tilde{\mu}_{sd}$	BE
Disability Stress Scenario	BE	Increase of 35%/25% in $\tilde{\mu}_{as}$, $\tilde{\mu}_{ai}$, $\tilde{\mu}_{si}$ Decrease of 20% in $\tilde{\mu}_{sa}$

Scenario	**Method**	**SCR**	**Reserve**
A	Standard formula of Solvency II	1 983.34	10 449.68
	Mortality Stress Scenario	26.89	8 493.23
	Longevity Stress Scenario	0	8 440.36
	Disability Stress Scenario	1 976.45	10 442.78
B	Standard formula of Solvency II	1 631.10	10 097.43
	Mortality Stress Scenario	0	8 453.47
	Longevity Stress Scenario	16.05	8 482.38
	Disability Stress Scenario	1 631.02	10 097.35

Table 4.7: CS 2 - Standard formula of Solvency II

Note that the prevailing risk in both scenarios is the disability risk. From the results, it is possible to see that the premiums obtained in scenarios A and B are less than those required by the worst-case method. Within the confidence bands (4.3.3), the worst-case method shows that additional liabilities of 2 241.82 can occur and again the square-root formula (4.1.1) applied to the "internal stress scenarios" underestimates the risk of mortality and disability rate changes to this product. Additionally, if the insurer believes that the confidence bands (4.3.3) are the most adequate may be the standard formula of Solvency II applied to the external stress scenarios reveals inappropriate underestimating the risks in study.

Therefore, the final premium should be 10 708.15, under the assumption that the confidence bands (4.3.3) are the ones that best reflect the reality of the insurer.

4.3.1 Sensitivity analysis

As the transition intensities (4.3.2) are a mere example, in order to development some kind of sensitivity analysis we performed the calculations again using the transition intensities $\tilde{\mu}_{as}(t)$ and $\tilde{\mu}_{ad}(t)$ based on Ramlau-Hansen (1991), constructing the remaining ones as in (4.3.2):

$$\begin{array}{lll} \tilde{\mu}_{as}(t) = 0.0004 + 10^{0.06t-5.46} & \tilde{\mu}_{ai}(t) = 0.05 \times \tilde{\mu}_{as}(t) & \tilde{\mu}_{sd(t)} = \tilde{\mu}_{ad}(t) \\ \tilde{\mu}_{ad}(t) = 0.0005 + 10^{0.038t-4.12} & \tilde{\mu}_{sa}(t) = 0.1 \times \tilde{\mu}_{as}(t) & \tilde{\mu}_{si}(t) = \tilde{\mu}_{as}(t) \end{array} \tag{4.3.4}$$

and the confidence bands (4.3.3). Table 4.8 summarizes the results.

Method	**Reserve**
I - Best Estimate	7 928.34
II - Worst-case Scenario Method	10 031.84
III - Sum-at-risk Method	10 010.04

Table 4.8: CS 2 Sensitivity analysis - Prospective reserve $V_a(0^-)$

The new transitions intensities are less conservative than (4.3.2) (Fig. A.8, Appendix A.5), leading to lower reserves under all methods. Even so, while method I leads to a premium of 7 928.34, methods II and III show that additional liabilities of 2 103.51 and 2 081.70 can occur.

After evaluating the obtained overall results, in case of policies that are a combination of disability income insurance and critical illness insurance, the worst-case method seems to be the most adequate in order to obtain reserves and premiums on the safe side. The worst-case scenarios obtained by the worst-case and the sum-at-risk methods as well as the the evolution of the reserves in states active and sick, $V_a(t)$ and $V_s(t)$ can be seen in Appendix A.5.

Chapter 5

Conclusion

Over the course of this thesis, we mostly explore and apply the worst-case scenario method proposed by Christiansen (2010). It allows to find the first order valuation basis that represents the worst-scenario from the insurer's point of view within all scenarios contained in a given confidence region. By worst-scenario we mean the one that maximizes the prospective reserve for given benefits and premiums.

Following the method, after imposing some bounds for our actuarial assumptions and setting the confidence bands, we define the first order basis to be equal to the upper bounds of these confidence bands where the sum-at-risk are positive and equal to the lower bound where it is negative. By that, it is possible to formulate a maximization problem for the prospective reserve finding a worst-case valuation basis, denoted "worst-case integral equation system", and give a solution for it (the problem has an unique solution which is maximal in all relevant situations). Contrarily to other methods that can be found in the literature, the worst-case integral equation system is fully solved using a discrete approach and an approach based on intensities, and always yields exact results even if the confidence bands are arbitrarily wide. Thus, the worst-case method proposed by Christiansen (2010) allows to find the first order valuation basis that set premiums and reserves on the safe side and so to quantify the biometric risk that the insurer suffers associated to adverse experience.

The method was illustrated with two case studies demonstrating the usefulness for the calculation of premiums and technical reserves. For all the situations presented in both cases, the reserves required by the worst-case method scenario method are greater than the values obtained by the sum-at-risk method. In means that, within the confidence bands there are often biometric scenarios that are worse (leading to higher reserves) than the worst scenarios used by the second method. Thus, as if we allow for one method, there is no reason to exclude the other, we can conclude that the use of the sum-at-risk method instead of the worst-case method may underestimate the risk of rate changes in biometric transition intensities. In addition, we see that for policies with mixed character such as in case study 1, the choice of a table from the upper and lower bounds, even if it falls over the table that seems to be more conservative, it does not imply the highest liability increases that can occur. Therefore, after evaluating the obtained overall results, the worst-case method seems to be the most adequate in order to obtain prospective reserves and premiums on the safe side with respect to the used confidence bands.

Solvency II, the new solvency regime of the EU also uses worst-case scenarios (Solvency II stress scenarios) for the calculation of solvency capital requirements. Applying the Standard formula of Solvency II to "internal stress scenarios" and to Solvency II stress scenarios, we discuss how the method in study could improve the calculation of biometric solvency reserves. In both case studies, when the standard formula is applied to the internal stress scenarios, the obtained reserve is lower than the one required by the worst-case method. It was showed that the worst-case method leads to the worst scenarios that can occur in terms of increasing liabilities with respect to the used confidence bands and cannot be exceeded any more. Thus, we can conclude that the use of the standard formula of Solvency II may underestimate the risk of mortality/disability rate changes depending on the type of policy in study. When the standard formula of Solvency II is applied to Solvency II stress scenarios, the confidence bands used are not the same. Therefore, if the insurer assumes that these new confidence bands are the most appropriate and realistic, once again, the standard formula of Solvency II may underestimate the biometric risk.

We should point out that the standard formula is calibrated at European level and given the diversity of markets and products, such calibration may be inappropriate for some individual insurers. It is the case in both situations: if the insurer believes that the internal confidence bands are the most appropriate and realist, then the aggregation matrix suggested by EIOPA (2014b) and/or the stress scenarios are not the most adequate to this type of products. Therefore, the insurer should consider its own internal model to calculate the reserves and use the worst-case method to define the internal stress scenarios.

The worst-case calculations performed over the work are based on single life insurance policies. However, the invariance property of the worst-case scenario leads to the conclusion that results remain still valid on a portfolio level, for homogeneous portfolios. It means that the only difference that we allow for is that policies may start at different calendar times. Since we do not consider inhomogeneous portfolios, extending the results to heterogeneous portfolios is an interesting field for future research.

In closing, we believe this study is a contribution to better understanding the worst-case scenario method proposed by Christiansen (2010) and to enhance the usefulness of this for the calculations of premiums and reserves on the safe side and, as such, the possible application in calculation of biometric solvency reserves for Solvency II under the Portuguese framework.

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Appendix A

A.1 Reserving for a policy with continuous cash flows

Following Dickson et al. (2012), consider a policy where regular payments (premiums and/or annuities) are payable continuously and sums insured are payable immediately on death. If the policy is issued to a life aged x and let

- P_t the annual rate of premium payable at time t;
- E_t the annual rate premium-related expense payable at time t;
- b_t the sum insured payable at time t if the policyholder dies at exact time t;
- e_t the expense of paying the sum insured at time t;
- $V(t)$ the prospective reserve for a policy in force at time t;
- μ_{x+t} the force of mortality at age $x+t$;
- δ_t the force of interest per year assumed earned at time t.

Then the prospective reserve can be written as

$$V(t) = \int_t^{\infty} \frac{\upsilon(r)}{\upsilon(t)} (b_r + e_r)\, {}_{r-t}p_{x+t}\, \mu_{x+r} dr - \int_t^{\infty} \frac{\upsilon(r)}{\upsilon(t)} (P_r - E_r)\, {}_{r-t}p_{x+t} dr,$$

where $\upsilon(t) = \exp\left\{-\int_0^{\infty} \delta_s ds\right\}$.
This equality can be used to calculate $V(t)$ by numerical integration.

A.2 The worst-case integral equation system

We start by replacing the transition intensities $\mu_{ij}(t)$ in Thiele's integral equation system (3.1.11) with scenario $\bar{\mu}_{ij}(t)$ (3.3.3). The last integral in (3.1.11) has the form:

$$\int_{(s,T]} \overline{R}_{ij}(t)\,\overline{\mu}_{ij}(t)\,dt \quad = \int_{(s,T]\cap\left\{\overline{R}_{ij}<0\right\}} \overline{R}_{ij}(t)\,l_{ij}(t)\,dt + \int_{(s,T]\cap\left\{\overline{R}_{ij}>0\right\}} \overline{R}_{ij}(t)\,u_{ij}(t)\,dt$$

Noting that

$$\begin{aligned}\int_{(s,T]\cap\left\{\overline{R}_{ij}<0\right\}} \overline{R}_{ij}(t)\,l_{ij}(t)\,dt \quad &= \tfrac{1}{2}\int_{(s,T]} \overline{R}_{ij}(t)\,l_{ij}(t)\,dt + \tfrac{1}{2}\int_{(s,T]} \overline{R}(t)\,l_{ij}(t)\,dt \\ &= \tfrac{1}{2}\int_{(s,T]} \overline{R}_{ij}(t)\,l_{ij}(t)\,dt - \tfrac{1}{2}\int_{(s,T]} \mid \overline{R}_{ij} \mid (t)\,l_{ij}(t)\,dt\end{aligned}$$

and

$$\begin{aligned}\int_{(s,T]\cap\left\{\overline{R}_{ij}>0\right\}} \overline{R}_{ij}(t)\,u_{ij}(t)\,dt \quad &= \tfrac{1}{2}\int_{(s,T]} \overline{R}_{ij}(t)\,u_{ij}(t)\,dt + \tfrac{1}{2}\int_{(s,T]} \overline{R}(t)\,u_{ij}(t)\,dt \\ &= \tfrac{1}{2}\int_{(s,T]} \overline{R}_{ij}(t)\,u_{ij}(t)\,dt + \tfrac{1}{2}\int_{(s,T]} \mid \overline{R}_{ij} \mid (t)\,u_{ij}(t)\,dt\end{aligned}.$$

Thus, the right-hand side of the integral can be written as

$$\begin{aligned}\int_{(s,T]} \overline{R}_{ij}(t)\,\overline{\mu}_{ij}(t)\,dt \quad &= \tfrac{1}{2}\int_{(s,T]} \overline{R}_{ij}(t)\,l_{ij}(t)\,dt - \tfrac{1}{2}\int_{(s,T]} \mid \overline{R}_{ij} \mid (t)\,l_{ij}(t)\,dt \\ &+\tfrac{1}{2}\int_{(s,T]} \overline{R}_{ij}(t)\,u_{ij}(t)\,dt + \tfrac{1}{2}\int_{(s,T]} \mid \overline{R}_{ij} \mid (t)\,u_{ij}(t)\,dt\end{aligned}$$

and Thiele's integral equation system (3.1.11) have the form

$$\begin{aligned}\overline{V}_i(s) \quad &= B_i(T) - B_i(s) - \int_{(s,T]} \overline{V}_i(t-)\,d\Phi(t) \\ &+ \sum_{j:j\neq i} \Big(\int_{(s,T]} \tfrac{1}{2} \mid \overline{R}_{ij} \mid (t)\,(u_{ij} - l_{ij})(t)\,dt + \int_{(s,T]} \tfrac{1}{2}\overline{R}_{ij}(t)\,(u_{ij} + l_{ij})(t)\,dt\Big)\end{aligned}$$

called the worst-case integral equation system.

A.3 Overall structure of the SCR and Correlation Matrix

The calculation of the Solvency Capital Requirement according to the standard formula is divided into modules as follows:

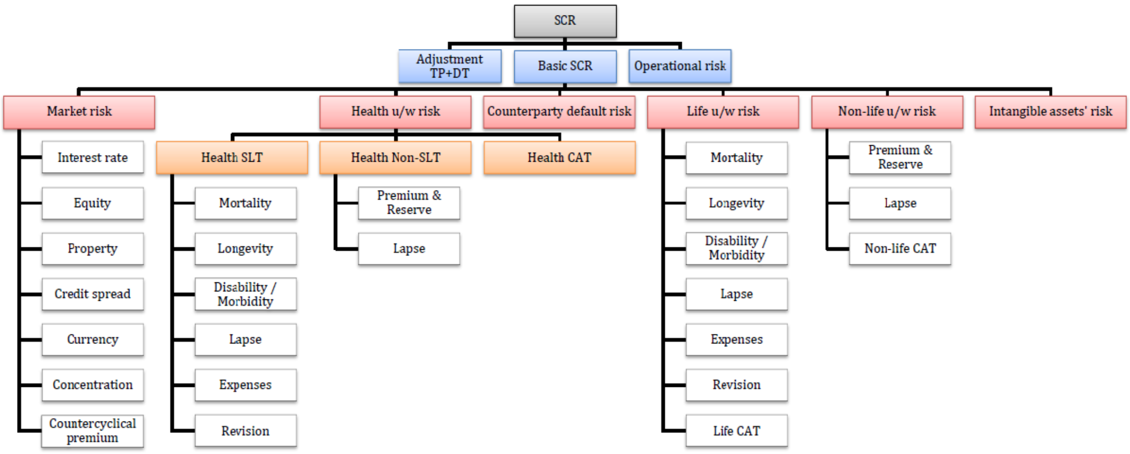

Figure A.1: Overall structure of the Solvency Capital Requirement. Source: EIOPA (2014b)

The mortality risk, longevity risk and disability risk in the life module are aggregated using the following correlation factors provided by EIOPA (2014b).

	Mortality	**Longevity**	**Disability**
Mortality	1		
Longevity	-0.25	1	
Disability	0.25	0	1

Table A.1: Correlation matrix for biometric life insurance risks. Source: EIOPA (2014b)

A.4 Case Study 1

A.4.1 Technical Basis

Mortality Basis

Fig. A.2 shows mortality tables GKF95 and GKM80 and the arithmetic average of two tables ($50\% \, GKF95 + 50\% \, GKM80$), in a logarithmic basis.

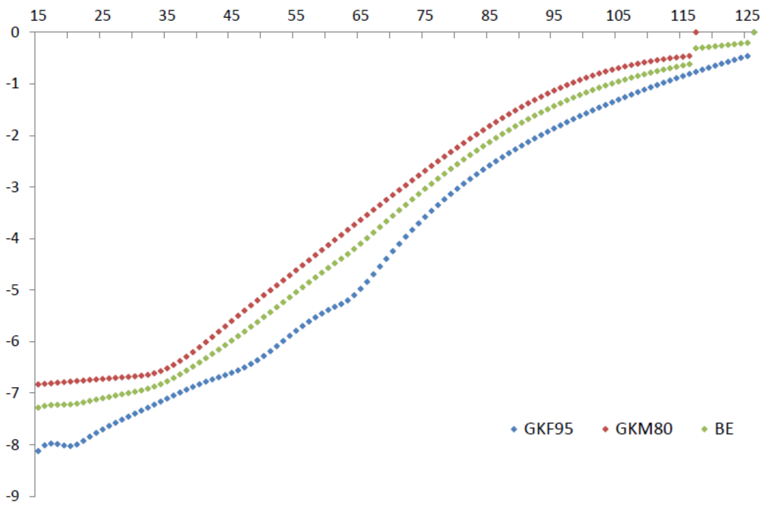

Figure A.2: Log mortality rates: BE, GFK95 and GKM80

In Fig. A.3 it is possible to compare the German mortality tables DAV2004R and DAV2008T with Swiss mortality tables GKF95 and GKM80.

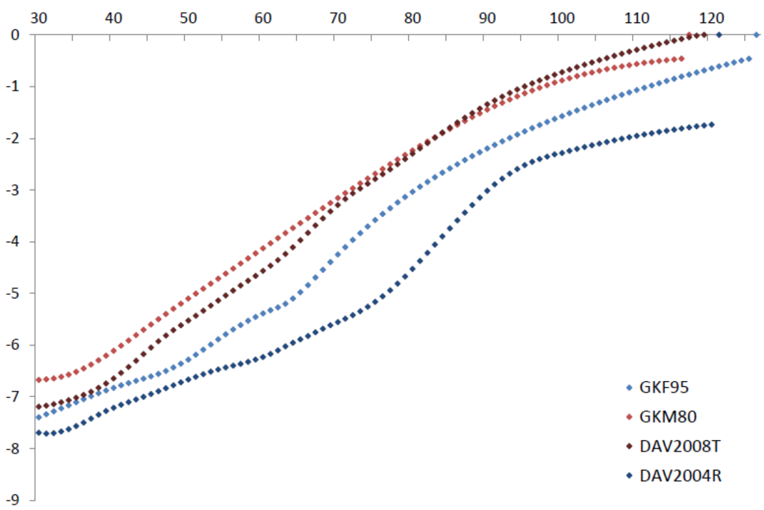

Figure A.3: Log mortality rates: GFK95 & GKM80; DAV2004R & DAV2008T

It is also possible to compare the respective BE's in Fig. A.4.

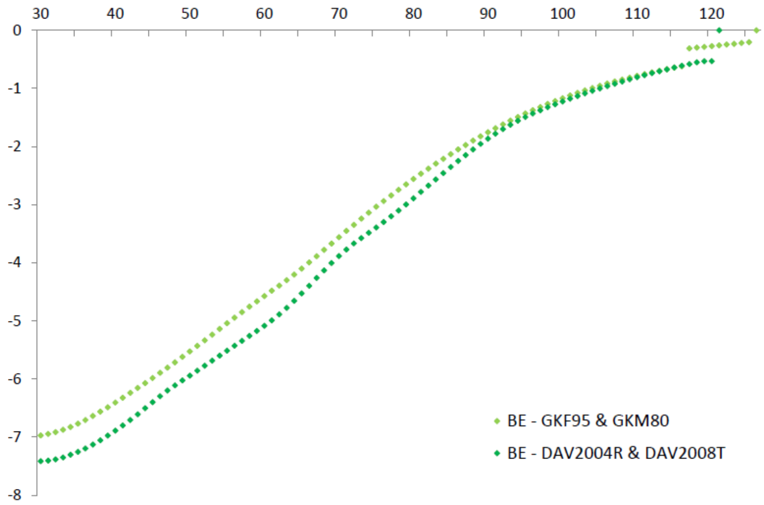

Figure A.4: Log mortality rates: BE of GFK95 & GKM80; BE of DAV2004R & DAV2008T

From the figures it is possible to see that the Swiss table GKF95 is less conservative than the German table DAV2004R for annuities while GKM80 is more conservative than DAV2008T until age 83 for life insurance with occurrence character. The BE resulting from Swiss life tables has higher mortality rates than the German BE.

Disability Basis

Fig. A.5 shows the disability tables TPD and SwissRe2001 and the arithmetic average of two tables ($50\% \, SwissRE2001 + 50\% \, TPD$), in a logarithmic basis.

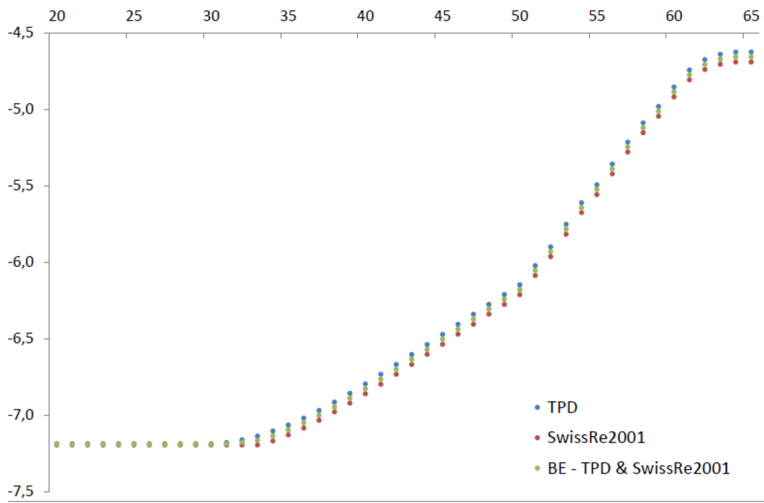

Figure A.5: Log disability rates: BE, TPD and SwissRe2001

Term Structure of Interest Rate

Fig. A.6 presents the risk-free interest rate term structure for Portugal as of 31 December 2013, without volatility adjustment provided by EIOPA during the 2014 Stress Test Exercise.

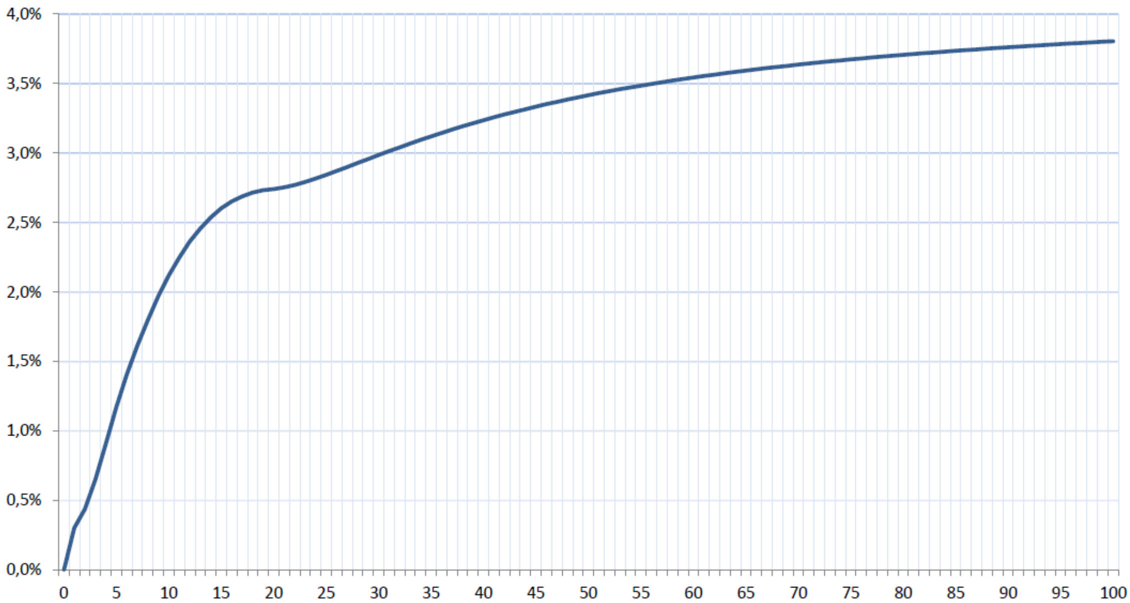

Figure A.6: Term Structure of Interest Rates - 31 December 2013

A.4.2 The worst-case scenarios

The worst mortality scenarios obtained by the worst-case scenario and the sum-at-risk methods in CS 1.1, CS 1.2 and CS 1.3 are the following:

Worst-case Scenario

$$\bar{\mu}_{ad}(t) = \begin{cases} u_{ad}(t) & : t \in [31,59] \cup [80,87] \\ l_{ad}(t) & : else \end{cases}$$

CS 1.1

Sum-at-risk Method

$$\mu_{ad}(t) = \begin{cases} u_{ad}(t) & : t \in [31,60] \cup [78,87] \\ l_{ad}(t) & : else \end{cases}$$

CS 1.1

$$\bar{\mu}_{ad}(t) = \begin{cases} u_{ad}(t) & : t \in [31,64] \cup [77,87] \\ l_{ad}(t) & : else \end{cases}$$

CS 1.2

$$\mu_{ad}(t) = \begin{cases} u_{ad}(t) & : t \in [31,65] \cup [75,87] \\ l_{ad}(t) & : else \end{cases}$$

CS 1.2

$$\bar{\mu}_{ad}(t) = \begin{cases} u_{ad}(t) & : t \in [39,64] \cup [69,75] \\ l_{ad}(t) & : else \end{cases}$$

CS 1.3

$$\mu_{ad}(t) = \begin{cases} u_{ad}(t) & : t \in [39,75] \\ l_{ad}(t) & : else \end{cases}$$

CS 1.3

Below, the worst mortality and disability scenarios defined by the worst-case scenario and by the sum-at-risk method are presented:

Worst-case Scenario

$$\bar{\mu}_{ad}(t) = \begin{cases} u_{ad}(t) & : t \in [39, 64] \cup [69, 75] \\ l_{ad}(t) & : else \end{cases}$$

$$\bar{\mu}_{ai}(t) = \begin{cases} u_{ai}(t) & : t \in [37, 59] \\ l_{ai}(t) & : else \end{cases}$$

CS 1.4

Sum-at-risk Method

$$\mu_{ad}(t) = \begin{cases} u_{ad}(t) & : t \in [39, 75] \\ l_{ad}(t) & : else \end{cases}$$

$$\mu_{ai}(t) = \begin{cases} u_{ai}(t) & : t \in [37, 61] \\ l_{ai}(t) & : else \end{cases}$$

CS 1.4

$$\bar{\mu}_{ad}(t) = \begin{cases} u_{ad}(t) & : t \in [39, 63] \cup [71, 75] \\ l_{ad}(t) & : else \end{cases}$$

$$\bar{\mu}_{ai}(t) = \begin{cases} u_{ai}(t) & : t \in [37, 59] \\ l_{ai}(t) & : else \end{cases}$$

CS 1.5

$$\mu_{ad}(t) = \begin{cases} u_{ad}(t) & : t \in [39, 65] \cup [68, 75] \\ l_{ad}(t) & : else \end{cases}$$

$$\mu_{ai}(t) = \begin{cases} u_{ai}(t) & : t \in [37, 60] \\ l_{ai}(t) & : else \end{cases}$$

CS 1.5

In all case studies there is a difference between the worst mortality scenario and disability scenario provided by the sum-at-risk method and the worst-case method. Moreover, observe that the worst mortality scenarios in CS 1.3 and CS 1.4 are equal.

A.4.3 Standard Formula of Solvency II

Given the the internal stress scenarios and Solvency II stress scenarios in Table A.2,

Scenario A - Internal stress scenarios		
	Mortality Basis	Disability Basis
Mortality Stress Scenario	GKM80	BE
Longevity Stress Scenario	GKF95	BE
Disability Stress Scenario	BE	TPD
Scenario B - Solvency II stress scenarios		
	Mortality Basis	Disability Basis
Mortality Stress Scenario	Increase of 15% in BE	BE
Longevity Stress Scenario	Decrease of 20% in BE	BE
Disability Stress Scenario	BE	Increase of 35%/ 25% in BE

Table A.2: CS 1 - Internal stress scenarios and Solvency II stress scenarios

Table A.3 shows the results of the application of the square-root formula to these stress scenarios in CS 1.1 and CS 1.2.

Scenario	Method	CS 1.1		CS 1.2	
		SCR	Reserve	SCR	Reserve
A	Standard formula of Solvency II	0.09768	0.09768	0.28970	0.54125
	Mortality Stress Scenario	0.09768	0.09768	0.28970	0.54125
	Longevity Stress Scenario	0	-0.06800	0	-0.08190
B	Standard formula of Solvency II	0.05300	0.05300	0.11001	0.36156
	Mortality Stress Scenario	0.05300	0.05300	0.11001	0.36156
	Longevity Stress Scenario	0	-0.06554	0	0.09682

Table A.3: CS 1.1 and CS 1.2 - Standard formula of Solvency II

Table A.4 shows the results of the application of the square-root formula to the internal stress scenarios and Solvency II stress scenarios in CS 1.3 and CS 1.4.

Scenario	Method	CS 1.3		CS 1.4	
		SCR	Reserve	SCR	Reserve
A	Standard formula of Solvency II	2 086.84	4 827.43	2 187.58	4 623.64
	Mortality Stress Scenario	2 146.98	4 887.58	2 227.98	4 664.04
	Longevity Stress Scenario	719.65	3 460.24	319.34	2 755.40
	Disability Stress Scenario	-	-	63.99	2 500.04
B	Standard formula of Solvency II	357.79	3 098.39	648.29	3 084.35
	Mortality Stress Scenario	238.05	2 978.65	311.04	2 747.10
	Longevity Stress Scenario	333.17	3 073.76	164.30	2 600.35
	Disability Stress Scenario	-	-	495.07	2 931.13

Table A.4: CS 1.3 and CS 1.4 - Standard formula of Solvency II

In CS 1.3 there is no disability benefit, so there is no value to disability stress scenario. From the results we can conclude that the Standard Formula of Solvency II underestimates the risk of biometric rate changes in all that situations presented in Case Study 1.

A.4.4 The evolution of reserves

Fig. A.7 shows the evolution of the prospective reserve in state active under the assumptions described in CS 1.5.

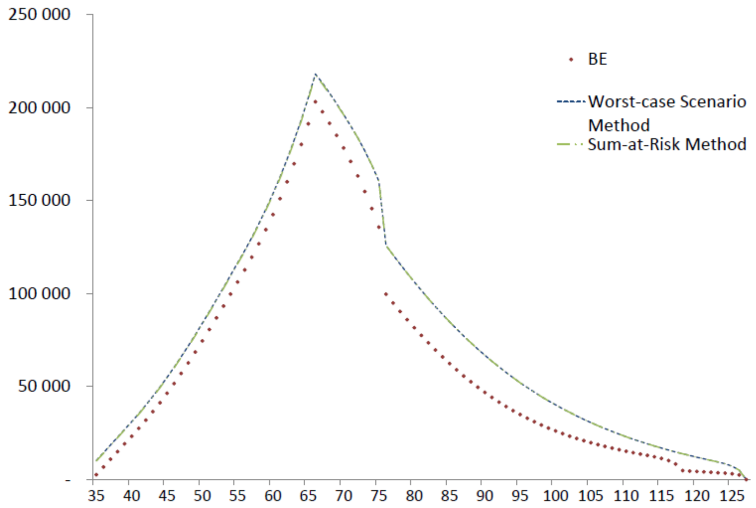

Figure A.7: CS 1.5 - Prospective reserve $V_a(t)$

Observe that the reserves calculated using the worst-case scenario method and the sum-at-risk method seem to be close due to the total amount of the reserves but they are different. The worst-case method leads to a higher reserve than the value obtained from the sum-at-risk method. Under the three methods, the reserve increases and then decreases. The largest value occurs at age 65, time $t = 30$ when the last premium is paid.

A.5 Case Study 2

A.5.1 Technical Basis

Fig. A.8 shows the transitions intensities (4.3.2) (continuous line - (j,k)) and (4.3.4) (dot line - $(j,k)'$) used in Case Study 2, in a logarithmic basis.

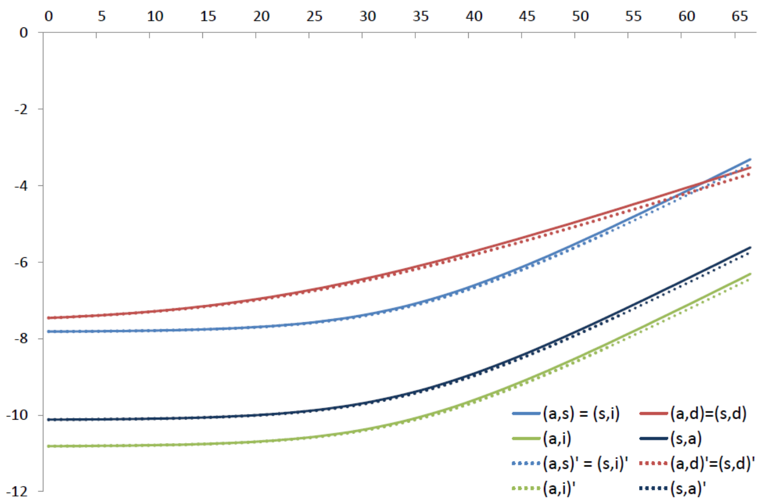

Figure A.8: Log transition intensities

The results shows that transition intensities (4.3.2) are more conservative than transition intensities (4.3.4).

A.5.2 The worst-case scenarios

The worst biometric scenarios obtained by the worst-case scenario method and the sum-at-risk method, using transition intensities (4.3.2), are the following:

Worst-case Scenario

$$\bar{\mu}_{ad}(t) = \begin{cases} l_{ad}(t) & : t \in (30.0001, 55.2393) \\ u_{ad}(t) & : t \in else \end{cases}$$

$$\bar{\mu}_{sd}(t) = \begin{cases} l_{sd}(t) & : t \in (30.0001, 65.25) \cup \\ & \quad \{65.3333\} \\ u_{sd}(t) & : else \end{cases}$$

$$\bar{\mu}_{si}(t) = \begin{cases} l_{si}(t) & : t \in (30.00001, 64.5) \cup \\ & (64.5203, 64.5833) \cup \{64.6667\} \\ u_{si}(t) & : t \in else \end{cases}$$

$$\bar{\mu}_{as}(t) = u_{as}(t)$$
$$\bar{\mu}_{ai}(t) = u_{ai}(t)$$
$$\bar{\mu}_{sa}(t) = l_{sa}(t)$$

Sum-at-risk Method

$$\mu_{ad}(t) = \begin{cases} l_{ad}(t) & : t \in (41.9647, 47.8879) \\ u_{ad}(t) & : t \in else \end{cases}$$

$$\mu_{sd}(t) = \begin{cases} l_{sd}(t) & : t \in (30.0001, 65.25) \cup \\ & \quad \{65.3333\} \\ u_{sd}(t) & : else \end{cases}$$

$$\mu_{si}(t) = \begin{cases} l_{si}(t) & : t \in (30.0001, 64.5) \cup \\ & \quad \{64.5833, 64.6667\} \\ u_{si}(t) & : t \in else \end{cases}$$

$$\mu_{as}(t) = u_{as}(t)$$
$$\mu_{ai}(t) = u_{ai}(t)$$
$$\mu_{sa}(t) = l_{sa}(t)$$

Under transition intensities (4.3.4), the worst-case scenarios obtained are:

Worst-case Scenario

$$\bar{\mu}_{ad}(t) = \begin{cases} l_{ad}(t) & : t \in (30.0001,\, 53.5074) \\ u_{ad}(t) & : t \in else \end{cases}$$

$$\bar{\mu}_{sd}(t) = \begin{cases} l_{sd}(t) & : t \in (30.0001,\, 65.25) \cup \\ & \quad \{65.3333\} \\ u_{sd}(t) & : else \end{cases}$$

$$\bar{\mu}_{si}(t) = \begin{cases} l_{si}(t) & : t \in (30.0001,\, 64.50) \cup \\ & \quad \{64.5833,\, 64.6667\} \\ u_{si}(t) & : t \in else \end{cases}$$

$$\bar{\mu}_{as}(t) = u_{as}(t)$$

$$\bar{\mu}_{ai}(t) = u_{ai}(t)$$

$$\bar{\mu}_{sa}(t) = l_{sa}(t)$$

Sum-at-risk Method

$$\mu_{ad}(t) = u_{ad}(t)$$

$$\mu_{sd}(t) = \begin{cases} l_{sd}(t) & : t \in (30.0001,\, 65.25) \cup \\ & \quad \{65.3333\} \\ u_{sd}(t) & : else \end{cases}$$

$$\mu_{si}(t) = \begin{cases} l_{si}(t) & : t \in (30.0001,\, 64.50) \cup \\ & \quad \{64.58333\} \\ u_{si}(t) & : else \end{cases}$$

$$\mu_{as}(t) = u_{as}(t)$$

$$\mu_{ai}(t) = u_{ai}(t)$$

$$\mu_{sa}(t) = l_{sa}(t)$$

A.5.3 Standard Formula of Solvency II

Table A.6 shows the results of the application of the square-root formula to the internal stress scenarios and Solvency II stress scenarios (Table A.5) using transition intensities (4.3.4).

Scenario A - Internal stress scenarios		
	Mortality Basis	Disability Basis
Mortality Stress Scenario	$u_{ad}(t)$, $u_{sd}(t)$	BE
Longevity Stress Scenario	$l_{ad}(t)$, $l_{sd}(t)$	BE
Disability Stress Scenario	BE	$u_{as}(t)$, $u_{ai}(t)$, $u_{si}(t)$, $l_{sa}(t)$
Scenario B - Solvency II stress scenarios		
	Mortality Basis	Disability Basis
Mortality Stress Scenario	Increase of 15% in $\tilde{\mu}_{ad}$, $\tilde{\mu}_{sd}$	BE
Longevity Stress Scenario	Decrease of 20% in $\tilde{\mu}_{ad}$, $\tilde{\mu}_{sd}$	BE
Disability Stress Scenario	BE	Increase of 35%/25% in $\tilde{\mu}_{as}$, $\tilde{\mu}_{ai}$, $\tilde{\mu}_{si}$ Decrease of 20% in $\tilde{\mu}_{sa}$

Table A.5: CS 2 - Internal stress scenarios and Solvency II stress scenarios

Scenario	Method	SCR	Reserve
A	Standard formula of Solvency II	1 889.98	9 818.32
	Mortality Stress Scenario	34.80	7 963.14
	Longevity Stress Scenario	0	7 889.93
	Disability Stress Scenario	1 880.98	9 809.32
B	Standard formula of Solvency II	1 555.79	9 484.13
	Mortality Stress Scenario	1.46	7 929.79
	Longevity Stress Scenario	0	7 924.95
	Disability Stress Scenario	1 555.43	9 483.76

Table A.6: CS 2 Sensitivity analysis - Standard formula of Solvency II

Results leads to the same conclusions than when transition intensities (4.3.2) are used: the the Standard Formula of Solvency II underestimates the risk of biometric rate changes.

A.5.4 The evolution of reserves

Fig. A.9 shows the evolution of the prospective reserve in state active, using transition intensities (4.3.2).

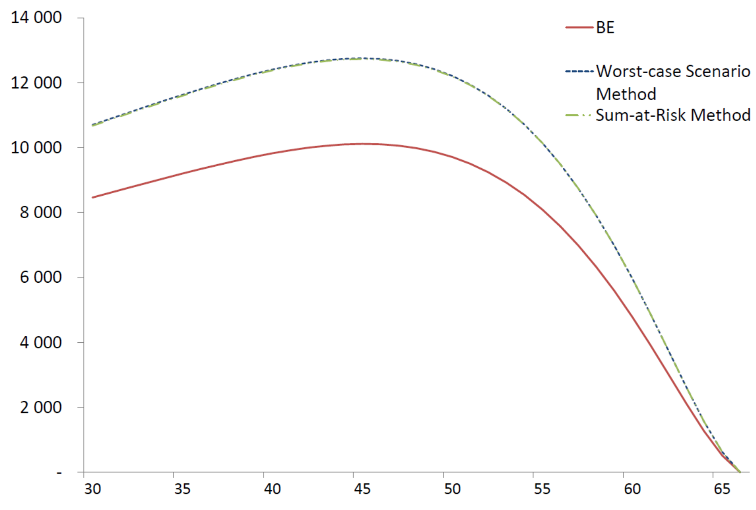

Figure A.9: CS 2 - Prospective reserve $V_a(t)$

The evolution of the prospective reserve in state stick using transition intensities in (4.3.2) is presented in Fig. A.10.

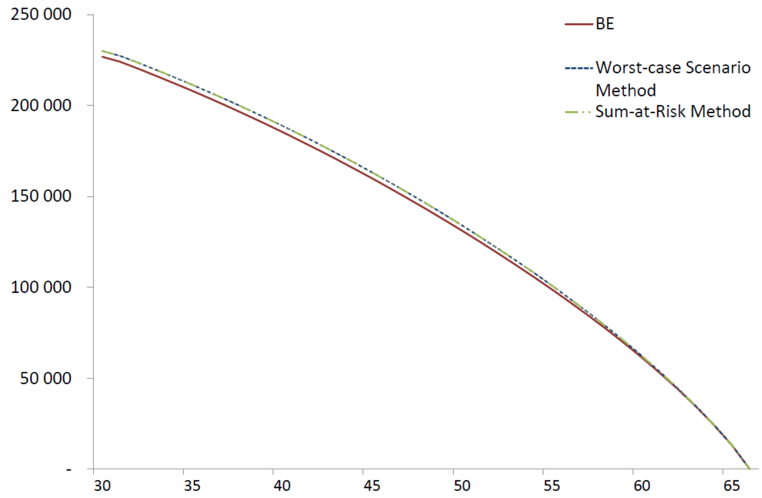

Figure A.10: CS 2 - Prospective reserve $V_s\,(t)$

Once again, the reserves calculated using the worst-case scenario method and the sum-at-risk method seem to be close but the first method leads to a higher reserve than the second one.