LISBON SCHOOL OF ECONOMICS & MANAGEMENT
UNIVERSIDADE DE LISBOA

MASTER

ACTUARIAL SCIENCES

MASTER´S FINAL WORK

DISSERTATION

OPTIMAL REINSURANCE OF DEPENDENT RISKS

ALEXANDRA BUGALHO DE MOURA

OCTOBER - 2017

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SUPERVISION:

PROFESSOR MARIA DE LOURDES CENTENO

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Abstract

This Thesis focuses on the optimal reinsurance problem for two dependent risks, from the point of view of the ceding insurance company. We assume that the two risks are dependent by means of a copula structure. By risk we mean a line of business, a portfolio of policies or a policy.

The problem consists in finding the optimal combination of quota-share and stop loss treaties, for each risk, that maximizes the expected utility or the adjustment coefficient of the total wealth of the insurer. It is known that these two criteria are connected and moreover the adjustment coefficient is related to the ultimate probability of ruin of the insurer through the Lundberg inequality. Results are obtained numerically, using the software *Mathematica*. Sensitivity of the optimal reinsurance strategy to several values of the dependence parameter, to different distributions of the underlying risks and to a variety of reinsurance premium calculation principles are performed in three families of copulas describing different tail behaviours of the joint distribution function.

Results show that dependencies alter the optimal treaty. Different dependence structures, *i.e.* different copulas, provide different values for the optimal retention levels. In the case of the expected value principle computed on the total ceded risk, the pure stop loss contract is always optimal, but that is not the case for the remaining premium computation principles. In general, the QS retention level decreases when dependence between the risks increases. For all cases considered, the maximum adjustment coefficient decreases when dependence increases.

Keywords: Reinsurance, Dependent Risks, Copulas, Premium Calculation Principles, Expected Utility, Adjustment Coefficient

Resumo

Esta Tese foca-se no problema do resseguro ótimo para dois riscos dependentes, do ponto de vista da seguradora que cede o risco. A dependência entre os dois riscos é modelada através de cópulas. Por risco entende-se uma apólice ou conjunto de apólices, que podem ser a carteira de um ramo de negócios. O problema de otimização a resolver consiste em encontrar a combinação de tratados de *quota-share* e *stop-loss*, para cada risco, que maximiza a utilidade esperada ou o coeficiente de ajustamento do lucro total da seguradora. Sabe-se que estes dois critérios estão ligados e que o coeficiente de ajustamento está relacionado com a probabilidade da seguradora ficar insolvente em tempo finito, através da desigualdade de Lundberg. Os resultados foram obtidos numericamente, usando o *software Mathematica*. A sensibilidade da estratégia de resseguro ótimo a vários valores do parâmetro de dependência, a diferentes distribuições dos riscos subjacentes e a diversos princípios de cálculo de prémios de resseguro foi analisada para três famílias diferentes de cópulas, descrevendo diferentes comportamentos da cauda da distribuição conjunta.

Os resultados mostram que as dependências alteram o tratado de resseguro ótimo. Diferentes estruturas de dependência, *i.e.* diferentes cópulas, produzem diferentes valores para os níveis ótimos de retenção. No caso do princípio do valor esperado calculado sobre o risco total cedido, o tratado *stop-loss* puro é sempre óptimo, mas isso não acontece para os restantes princípios de cálculo de prémios. Em geral, o nível ótimo de retenção do tratado de *quota-share* decresce quando a dependência entre os riscos aumenta. Para todos os casos considerados, o coeficiente de ajustamento máximo diminui quando a dependência aumenta.

Palavras Chave: Resseguro, Riscos Dependentes, Cópulas, Princípios de Cálculo de Prémios, Utilidade Esperada, Coeficiente de Ajustamento

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1

Introduction

The management of risk has always been the business of insurers. However, due to the frequent financial crises in the past three decades, risk management has become a major focus also in finance and regulatory processes aimed at financial and insurance companies, as Basel II and Solvency II, have been enforced. Solvency II, which is a EU Directive with the purpose of ensuring and monitoring insurer solvency to protect policyholders, imposes capital standard requirements based on risk measures. Therefore, the application of quantitative risk models has become of vital importance for financial institutions, as well as regulation authorities and rating agencies, where the quantification of risk is used for deriving capital standards. From the vast range of literature intended for the financial and insurance community, it is widely accepted that dependencies play a determinant role in risk assessment and management. We refer to [52] as a reference text in quantitative risk models, with practical applications in finance and insurance, and to [30, 57] as reference texts in the mathematical modelling of dependencies with applications in insurance. In Solvency II, the standard formulae given for the calculation of the minimum amount of capital an EU insurance company must hold to reduce the risk of insolvency, known as Solvency Capital Requirement (SCR), includes dependences through a given correlation structure.

Reinsurance consists in transferring part of the liabilities of the insurance company to another insurer, the reinsurer. By doing so, the ceding insurer company protects itself from the risk of large variations in its undertaken liabilities, thus reducing the amount of capital needed to cover them. Hence, reinsurance is a risk mitigating tool, usually reducing minimum regulatory capital requirements, and an important instrument in the management of risk of an insurance company. There are other reasons for a company to reinsure part of its risk. One of them is the increase of underwriting capacity, precisely because reinsurance reduces minimum capital requirements. Also, commonly the reinsurance company provides consultation services to the insurer. The incorrect assessment of risk or the lack of liquidity can also motivate the need for reinsurance. We refer to [22] for a review on reinsurance. Of course, reinsurance has a price, the reinsurance premium, and reinsuring all the risk is not an option as in such case the insurer would be just an intermediary. In addition, there are several forms of reinsurance. So, when an insurance company decides to reinsure part of its risk, at least two questions arise “What form of reinsurance is optimal?” and “How much reinsurance is optimal?” (see [22, 19]). Answers to these questions constitute solutions to the optimal reinsurance problem, and clearly depend on the underlying risk and on the reinsurance premium.

This problem has been largely studied in the literature, however only recently dependencies among risks have been considered. As mentioned above, dependencies are of extreme importance in many cases of risk assessment. Some authors have addressed the problem of optimal reinsurance with dependencies analytically, under certain constraints, for instance, considering a fixed reinsurance premium calculated through the expected value principle [14]. Yet, in many practical cases, the complexity introduced by dependencies leads to the need of numerical approaches and many of the works on the design of optimal reinsurance strategies including dependencies use numerical techniques.

In this Thesis, we study the optimal reinsurance problem for two dependent risks, from the point of view of the ceding insurance company. We assume that each risk is dependent by means of a copula structure. By risk we mean a line of business, a portfolio of policies or a policy. We construct the optimal problem as finding the optimal combination of quota share (QS) and stop loss treaties, for each risk, that maximizes the expected utility or the adjustment coefficient of the total wealth of the insurer. It is known that these two criteria are connected [39] and moreover the adjustment coefficient is related to the ultimate probability of ruin of the insurer through the Lundberg inequality. As no analytical results could be obtained in this case, namely when considering variance related premium principles, results were obtained numerically, using the software *Mathematica*. Sensitivity of the optimal reinsurance strategy to several values of the dependence parameter, to different distributions of the underlying risks, and to a variety of reinsurance premium calculation principles were performed in three families of copulas describing different tail behaviours.

The layout of this Thesis is as follows. Chapter 2 is devoted to describing reinsurance in more detail and includes the state of the art on optimal reinsurance, with particular emphasis to reinsurance including dependencies and optimal reinsurance solutions obtained by means of numerical approaches. In Chapter 3 we introduce the copula framework to model dependence. We provide the definition of copula and main properties and discuss the copulas that will be used in this Thesis. The description of the optimal problem to be solved in this work is contained in Chapter 4. Chapter 5 is dedicated to the presentation of the numerical results and their discussion. Finally, conclusions and future perspectives are drawn in Chapter 6.

2

Optimal reinsurance and dependencies

Reinsurance is a form of insurance in which the insurance ceding company transfers part of the risk of a policy or portfolio of policies to the reinsurer. The insurer is indemnified for the specific share of the insurance claims covered by the reinsurance contract. Through reinsurance, the cedent can protect itself from extreme losses that can induce financial embarrassment or insolvency, as for instance from a large number of claims or an extremely large rare claim. Reinsurance also allows the cedent company to underwrite contracts beyond its capacity, enabling smaller companies to compete with larger ones. Indeed, by reinsuring potential big losses, large fluctuations in the insurer profits are softened. However, that comes at the cost of reinsurance premium loading. Thus, through reinsurance, the variability of the profits is expected to decrease, but the expected profit will also decrease. For this reason, when transferring risk, the cedent seeks a trade-off between profit and safety, which is dependent on the nature of the insured underlying risk and on the reinsurance premium calculation principle. Hence, the optimal reinsurance strategy is strongly dependent on these two factors. The more common forms of reinsurance are quota-share (QS), surplus, excess-of-loss (XL, on a *per risk* basis) and stop-loss (*aggregate XL*) treaties. For the sake of completeness, we summarize these four types of reinsurance.

Quota-share (QS) reinsurance The QS reinsurance is a form of proportional reinsurance such that a fixed proportion, $0 \leqslant (1-a) \leqslant 1$, of the risk of each subject policy, X, is transferred to the reinsurer. Here, aX is retained by the insurer and $(1-a)X$ is ceded; a is denoted the QS retention level. In this form of contract, usually the reinsurer receives the same share of premiums, that is, if the insurer charges P for the risk X, then the QS premium is $(1-a)P$. The reinsurer pays the cedent a commission c on the reinsurance premium, supposedly proportional to primary production and handling costs, with which the cedent manages the financial results. Hence the total QS reinsurance premium is $(1-c)(1-a)P$.

Surplus reinsurance Surplus is a form of proportional reinsurance where the ceding company determines the maximum loss to be retained for each risk in the portfolio. The maximum loss amount defined for each risk is denoted *line*. If the considered risk is higher than the defined *line*, it is ceded in a proportional basis, where the proportion varies with the size of the risk. Hence, unlike QS, here the insurer retains a fixed maximum amount for each risk and this amount defines the retained proportion, depending on the total size of the underlying policy. In this case, the proportion of each risk covered by the reinsurer (ceded risk), when they are above the retained *line*, is $(1 - a_r) =$ (policy limit - retained *line*)/policy limit. As in QS, the premium is shared and there are also commissions payed by the reinsurer.

Excess of loss (XL) reinsurance The XL treaty is a non-proportional type of reinsurance covering, up to a limit L, the part of each claim (or share) that is in excess of some specified amount M, denoted the XL cedent's retention level. So, the XL treaty covers a layer, in excess of the retained level M. In this case, the part of the risk ceded through XL is the layer $M < X < M+L$, that is, the ceded loss is $(X-M)_+ - ((X-(M+L))_+ = \min(L,(X-M)_+)$[1], whereas the retained risk is $X-(X-M)_+ - (X-(M+L))_+$. XL treaties can be defined on a *per-risk* or *per event* basis and may include several layers. The so called *Working Cover* excess layer is defined according to the claims activity expected each year.

Stop loss reinsurance The stop loss coverage is the aggregated XL treaty. In this case X represents the aggregated claim amount for a given period.

2.1 State of the art

A large amount of works can be found in literature concerning optimal reinsurance strategies, as this problem has been for long considered in the actuarial community. The first works date as far back as the 60s (e.g. with the works of Borch [9, 10], amongst others). The goal is always to find the reinsurance strategy, which is usually defined by the forms of reinsurance to be considered and the specific retention levels, that minimizes a given measure of the underlying risk, while keeping an expected profit, that is, excluding reinsuring all the risk, in which case the insurer would act as a mere intermediary.

Some authors, as [56] in the 80s but also [31, 36, 49, 43] more recently, have analysed the problem of optimal reinsurance from both the cedent and the reinsurer point of view, addressing the conflicting interests of the cedent and the reinsurer. In [31] the expected profits of the two parties are maximized for a given level of the probability of joint survival (dependent on the "joint level of risk aversion"). Here, the authors consider stop-loss reinsurance and analyse, for a linear premium income, both independent and dependent (using a rotated

[1]We are using the notation $(X-M)_+ = \max(0, X-M)$.

Clayton copula) claim severities. In [36] the authors consider a combination of QS and stop-loss reinsurance, seeking to maximize the joint survival probability under expected value reinsurance premium. In [49], the authors simultaneously analyse three optimality criteria (released capital, expected profit and expected utility of resulting wealth) applying methods from multiple attribute decision making to find the optimal retention levels, where reinsurance applies to individual claims. In [43] the author analyses the optimal retention level of "partial stop loss" contract from both the insurer and reinsurer viewpoints, assuming premium loadings are set according to the expected value principle with varying loading factors, and using the value at risk (VaR) as optimality criteria. In [69] and [51] the authors also consider the reinsurer's view point, by imposing lower (reflecting the minimum profit the reinsurer is willing to accept for undertaking the reinsurance contract) and upper (reflecting the maximum value the insurer is willing to pay for the reinsurance contract) constraints for the reinsurance premium.

Nevertheless, most of the articles found in literature deal with the problem of optimal reinsurance from the cedent's perspective, the interest of the reinsurer being enclosed in the calculation principles considered for the reinsurance premium. We refer to [22] for a review about optimal reinsurance. A very common type of reinsurance strategy found in the literature of optimal reinsurance is to envisage a combination of QS and stop (or excess of) loss contracts (see [22] and references therein, and [73] for a more recent reference including dependencies). Such a combination may include a pure QS or pure stop (or excess of) loss contract. It has been demonstrated by several authors that, if the expected value premium calculation is applied, the pure stop-loss contract is optimal for minimizing the variance or maximizing the expected utility (for the exponential utility function) or the adjustment coefficient of the retained risk (see [22] and references therein). In [16] a similar result in favour of the stop-loss is encountered when pursuing to minimize the skewness coefficient or the coefficient of variation of the retained risk in the case of expected value or standard deviation reinsurance premium loadings. Another result in favour of the stop-loss is found in [65], together with more general remarks on the optimal solution, but now for the CTE (conditional tail expectation) risk measure criteria.

In [39] (for the aggregate claim model) and [41] (for the individual claim model, assuming the number of claims belongs to the Panjer, or Katz, family) the authors obtain the optimal reinsurance strategy maximizing the adjustment coefficient or the expected utility. The premium calculation principle used is a convex functional, including the expected value, standard deviation and variance premium principles as special cases. In the case of "variance type" premium calculation principles, the optimal reinsurance contract is a specific, implicitly defined, non-linear function of the retained risk. This nonlinear function is not a formerly known type of reinsurance and it has the very interesting property that the tail of the underlying risk is shared by both the insurer and the reinsurer. If the expected value calculation principle is considered, the pure stop loss treaty is optimal. In fact, the pure stop loss, which appears as the optimal form of reinsurance in an innumerable amount of cases, namely when the expected value premium principle is used, is not realistic in practice. It means all the risk in the tail is ceded to the reinsurer which will not accept it but at a very high premium loading, in which case the stop loss is probably not optimal anymore (as shown in [39, 41]). In other words, with the expected value premium principle, the interests of the reinsurer for a stop loss contract might not be properly preserved. In [40] the authors provide numerical techniques to find the implicit function describing the optimal reinsurance, when the premium loading is proportional to an increasing function of the variance of the retained risk. Other works considering convex premium principles include [46, 47, 48, 38], where convex risk measures (e.g. the variance or semi-variance of the retained risk) are used as optimality criteria.

Although risk measures as VaR (value at risk), CVaR (conditional value at risk) and CTE have been shown in [42] to be inadequate to measure risk, at least in the analysis of reinsurance contracts, leading to “treaties that are not enforceable and/or are clearly bad for the cedent”, they have been widely used as criteria in solving optimal reinsurance problems, as for instance [65] and [3]. The later consider the problem of optimal risk transfer when multiple reinsurance counter-parties, or an insurer group consisting of two separated entities, are taken into consideration. In [12], combinations of QS and stop loss are considered and VaR and CTE are taken as criteria to find the optimal reinsurance strategy. In [26, 27] the authors analyse several different premium principles, using as optimality criteria the VaR and CVaR. In [1] the solution of the optimal reinsurance is also analysed using VaR and CVaR as risk measures, but the idea is to find the reinsurance contract for which sensitiveness to such measures is minimized, assuming the underlying risk distribution is not known. In this case, the problem is addressed numerically using a Linear Programming approach.

While a large quantity of studies can be found regarding optimal reinsurance, only a few number consider dependence, since the independence hypothesis simplifies the demonstrations and computations. Notwithstanding, the interest in studying optimal reinsurance strategies under dependencies is increasing, driven by the need for real, robust and reliable quantitative risk models. Indeed, in [33, 34] the importance of dependencies, beyond correlations, in insurance and finance applications is illustrated, and the study of dependencies in this context is becoming an active field of research. Article [23] is one of the first works including the effects of dependence when investigating optimal forms or risk transfer. The optimal retained level for the XL reinsurance is studied considering two classes of insurance businesses, dependent through the number of claims by means of a bivariate Poisson, when the cedent intends to maximize the expected utility or the adjustment coefficient, using the expected value premium principle. In [11, 59] the authors construct models with dependence between severity and number of claims in different insurance contexts, but without considering reinsurance. In [73] the impact of dependence between claim numbers on the adjustment coefficient under the optimal combinations of QS and XL is analysed. In [7] the optimal investment-reinsurance problem for two classes of insurance businesses, dependent through the number of claims by means of common shock component correlation, is investigated using a mean-variance analysis of the insurer's wealth as criteria. In [6] reserves of two lines of business, modelled by a bivariate compound Poisson or by a common shock model, are reformulated using a controlled diffusion process. The optimal reinsurance problem of each risk (denoted in the paper as "two dimensional reinsurance policy"), is then solved through a dynamic control problem for the controlled diffusion process. Under this setting they demonstrate that the two-dimensional XL reinsurance policy is optimal in minimizing the ruin probability of the controlled diffusion process. Despite not analysing the optimal reinsurance strategy directly, the authors in [32] apply DAF (dynamic financial analysis) to study the impact of non-linear dependencies between liabilities, between assets and between liabilities and assets on reinsurance. Dependencies are modelled by means of copulas and three types of reinsurance contracts are considered: stop loss, XL and *double trigger* (in which the potential reinsurance coverage depends on both underwriting and financial risks). From their simulations, they demonstrate that reinsurance is very sensitive to non-linear dependencies, and to different types of dependencies (to different copulas). In [29] increasing stochastic dependence is used to devise strategies of coinsurance (risk sharing). In [25], the authors do not assume any particular dependence structure, as they argue it is often difficult to determine it. They propose instead to use the minimax optimal reinsurance decision formulation, in which the worst-case scenario is first identified. Here the stop loss reinsurance appears as optimal when minimizing a "general law-invariant" convex risk measure of the total retained risk. In [68] the impact of dependencies from year to year reinsurance payoffs are investigated using copulas and simulation, however optimal reinsurance is not directly addressed. In [14] positive dependencies in the individual risk are considered by means of the stochastic ordering. The reinsurance premium is fixed and computed through the expected value premium principle. By proving the convolution preservation of the convex order for positive dependence through stochastic ordering random vectors, the authors demonstrate that in this case the optimal form of reinsurance is the XL treaty, when the optimality criterion is the expectation of a convex function of the retained risk. In that paper, the authors refer to the non-proportional reinsurance as excess of loss (XL), assuming the risks are individual claims and then considering their sum. In this Thesis by risk we mean the aggregate claims of a line of business, a portfolio of policies or a policy, thus assuming that the form of non-proportional reinsurance is the stop loss.

Some of the articles accounting for dependence have protruded the use of numeral techniques, such as Dynamical Financial Analysis (DFA) [32], Linear Programming [1], dynamic control problems [6] or simulation [68], which is often based on Monte Carlo simulation. In the review [22], DFA is mentioned as a tool that, based on the well understanding of uncertainties provided by more general theoretical results, is capable of modelling sensitiveness of risk measures to reinsurance, including dependencies, among other possible variables. Indeed, recently the analysis of increasingly complex problems, of which the inclusion of dependencies is one example, to find the optimal reinsurance strategy have led several authors to apply numerical methods to find the optimal solution. Very recently, in [5], it has been advocated that when constraints on dependencies and economic and solvency factors are included in the optimal reinsurance problem, "the optimal contract can only be found numerically". Hence, they propose a numerical framework, based on the Second-Order Canonical Problem for numerical optimization, which is known to be very well behaved for convex problems. For instance in [35] the authors have felt the need to propose a numerical algorithm to estimate reliable bounds to the VaR of high dimensional portfolios, so to investigate the sensitivity of the VaR to different dependence scenarios on the factors of the portfolio. In real applications, as for instance the computation of optimal levels of Technical Provision or the Minimum Capital Requirement according to regulations, namely Solvency II, the use of numerical methods is inescapable. For instance, in [2] several scenarios of intra-group risk transfer are studied, and in [4] optimal levels of reinsurance are analysed, all using numerical approaches. Many of the literature regarding the application of numerical techniques to solve optimal reinsurance problems consider numerical methods for stochastic control theory. In [44, 15, 74] the authors apply a dynamic programming technique to implicitly find the optimal reinsurance treaty as the solution of a Hamilton-Jacobian-Bellman equation. In [15, 74] the explicit solution is found numerically through a procedure based on the dynamic programming methodology, whereas in [44] it is found by means of Markov chain approximation techniques. The latter methodology is also applied in [45] to develop numerical methods for singular controls of regime-switching diffusions, designed to find the optimal dividend payout and reinsurance policies. It is also worth mentioning the empirical approach proposed in [66] and [64], where reinsurance models are formulated based on data observation, without explicitly assuming the distribution of the underlying risks, which allows for the use of programming procedures to obtain the optimal solution.

In this work, we aim at studying the optimal reinsurance strategy in presence of dependencies with reinsurance premium loadings including the expected value principle, but also the standard deviation and the variance principle. An analytical approach for this problem was first attempted, considering dependences through copulas. However, no analytical results could be obtained. The results in [14] for the expected value principle could not be extended to the variance related premiums. Thus, we use numerical methods to analyse the optimal form of reinsurance for two dependent risks, namely numerical quadratures and global optimization algorithms (existing in *Mathematica*). We also assume reinsurance is pursued in the form of a combination of QS and stop loss (see Chapter 4) and consider the expected value, the standard deviation and the variance premium principles.

3

Modelling dependence through copulas

When two risks are assumed not to be independent, an infinite range of possible dependencies between them can be at stake. The first question is, if they are dependent, what is the best model to explain the existing dependencies.

Copulas constitute a convenient and elegant way of describing dependencies between two or more random variables. In this case, the joint distribution function is expressed as a parametric function of the marginal distribution functions. Through the copula, the joint probability function can be decomposed into the marginal probability functions and a dependence structure component. Thus, not only the joint distribution is known through the margins, as the dependence structure is decoupled from them. Also, being parametric functions of the margins, the copula offers a natural procedure for the estimation of the multivariate distribution simply by plugging in the evaluation of each marginal. See for instance [24, 28] for reviews on parametric and non-parametric copula estimation procedures and their impact in tail probabilities and robust statistical analysis, respectively. In [34] the authors state that, while in elliptical distribution context linear correlation is a natural summary of dependence, in the non-elliptical distribution context intuition about correlation breaks and deeper understanding of dependence is needed to model risks. They advocate that the best way to simulate dependent data is “when multivariate dependence structure, in the form of a copula, is fully specified by the modeller”. Using copulas, measures of non-linear dependence can be explored, as the Spearman’s or Kendall’s rank correlations. These dependence measures are copula-based measures and have the advantage, over the linear correlation coefficient, of being invariant under monotonic transformations, e.g. properly handling perfect dependence. However, they are not moment based, so not so elegantly manipulated [34, 33].

Linear-correlation is not a copula-based measure of dependence and can often be misleading when analysing dependencies [34, 33].

The definition of copula was introduced in 1959 by Sklar [61], however as mentioned by Nelsen in the 2013 reedition of his book [55], only from the end of the twentieth century the use of copulas became widespread. The definition of copula can be extended to any n-dimensional multivariate distribution (for the definition and properties of copulas in general, see for instance [34, 33, 54, 55, 30]). Nevertheless, we will stick to the two-dimensional case, for simplicity, as in this work we will consider two dependent risks.

3.1 The definition of copula and main properties

Given two random variables[1], X_1 and X_2, with distribution functions $P(X_1 \leqslant x_1) = F_{X_1}(x_1)$ and $P(X_2 \leqslant x_2) = F_{X_2}(x_2)$, respectively, a copula, $C(u_1, u_2)$, is a function that maps $(u_1, u_2) = (F_{X_1}(x_1), F_{X_2}(x_2))$, that is the unit square $[0,1] \times [0,1]$, into $[0,1]$ which will be the value of the joint distribution function:

$$\begin{array}{llcl} C: & [0,1]\times[0,1] & \longrightarrow & [0,1] \\ & (F_{X_1}(x_1), F_{X_2}(x_2)) & \mapsto & F_{X_1,X_2}(x_1,x_2) = P(X_1 \leqslant x_1, X_2 \leqslant x_2) = C(F_{X_1}(x_1), F_{X_2}(x_2)) \end{array}$$

From the Sklar's representation Theorem (see Theorem 3.2, or [30, 55, 34, 33]), the copula not only provides a way to describe the joint distribution as function of the margins, as it shows that any joint distribution can be represent by a unique copula (if the margins are continuous random variables). For the sake of completeness, we introduce the formal definition of copula, the Sklar's theorem and the main properties of copulas. An extensive introduction to copulas and main properties can be found in [55]. In [30, 34, 33], copulas and their basic properties can also be encountered, with applications in insurance.

Definition 3.1 (Bivariate copula) *A bivariate copula, C, is a non-decreasing and right-continuous function, mapping $[0,1]\times[0,1]$ into $[0,1]$ such that, for all (u_1, u_2)*

i) $\lim_{u_i \to 0} C(u_1, u_2) = 0$, $i = 1, 2$

ii) $\lim_{u_1 \to 1} C(u_1, u_2) = u_2$ *and* $\lim_{u_2 \to 1} C(u_1, u_2) = u_1$

ii) C *is supermodular:*

$$C(v_1, v_2) - C(u_1, v_2) - C(v_1, u_2) + C(u_1, u_2) \geqslant 0, \quad \textit{for} \quad u_1 \leqslant v_1 \quad \textit{and} \quad u_2 \leqslant v_2$$

[1] In this work the random variables of interest are the two risks considered. Hence, often the underlying random variables are designated by risks.

The Sklar's theorem clarifies the role of copulas in associating multivariate and marginal distribution functions (see [30, Theorem 4.2.2] or [55, 34, 33, 55]).

Theorem 3.2 (Sklar's Representation Theorem) *Let X_1 and X_2 be random variables with continuous distribution functions $F_{X_1}(x_1)$ and $F_{X_2}(x_2)$, respectively. Then, there exists a unique copula C such that, for all $(x_1, x_2) \in \mathbb{R}^2$*

$$F_{X_1,X_2}(x_1, x_2) = C(F_{X_1}(x_1), F_{X_2}(x_2)) \tag{3.1}$$

Conversely, if C is a copula and F_{X_1} and F_{X_2} are distributions of X_1 and X_2, respectively, then the function $F_{X_1,X_2}(x_1, x_2)$ defined by (3.1) *is a bivariate distribution function with margins F_{X_1} and F_{X_2}.*

Again, from (3.1) it can be seen that C couples the marginal distributions, entirely describing the dependence structure between them, separately from the margins themselves.

It is worth mentioning three important copulas:

Independence copula, where the dependence structure is non-existent:

$$C_I(u_1, u_2) = u_1 u_2, \quad (u_1, u_2) \in [0, 1] \times [0, 1]$$

Fréchet upper bound copula, which bounds all copulas, from above:

$$M(u_1, u_2) = \min(u_1, u_2), \quad (u_1, u_2) \in [0, 1] \times [0, 1]$$

Fréchet lower bound copula, which bounds all copulas, from below:

$$W(u_1, u_2) = \max(0, u_1 + u_2 - 1), \quad (u_1, u_2) \in [0, 1] \times [0, 1]$$

Whence, given a copula C, the following inequalities always hold (see for instance [55, Theorem 2.2.3])

$$W(u_1, u_2) \leqslant C(u_1, u_2) \leqslant M(u_1, u_2), \quad (u_1, u_2) \in [0, 1] \times [0, 1]$$

Remark 3.3 *In the bivariate case, W describes perfect negative dependence and M describes perfect positive dependence [33, Section 3.2].*

From [30, Proposition 4.2.12], the partial derivatives of C exist almost everywhere, the functions $u_1 \mapsto \frac{\partial}{\partial u_2} C(u_1, u_2)$ and $u_2 \mapsto \frac{\partial}{\partial u_1} C(u_1, u_2)$ being defined and non-decreasing almost everywhere and verifying, for $i = 1, 2$

$$0 \leqslant \frac{\partial}{\partial u_i} C(u_1, u_2) \leqslant 1, \qquad u_j \in [0, 1], \;\; j \neq i$$

The copula partial derivatives are strictly related with the conditional probabilities of X_1 and X_2, as demonstrated in the following Property [30, Property 4.2.13].

Property 3.4 (Conditional probabilities) *Given a random vector (X_1, X_2) with joint distribution given by copula C,* (3.1)*, then we have*

$$\begin{aligned} P(X_2 \leqslant x_2 | X_1 = x_1) &= \frac{\partial}{\partial u_1} C(u_1, u_2) \bigg|_{(u1,u2)=\left(F_{X_1}(x_1), F_{X_2}(x_2)\right)} \\ P(X_1 \leqslant x_1 | X_2 = x_2) &= \frac{\partial}{\partial u_2} C(u_1, u_2) \bigg|_{(u1,u2)=\left(F_{X_1}(x_1), F_{X_2}(x_2)\right)} \end{aligned}$$

To obtain the joint density function, we resort to the second order crossed partial derivate of the copula, which also exists almost everywhere, and is denoted the copula density c:

$$c(u_1, u_2) = \frac{\partial^2}{\partial u_1 \partial u_2} C(u_1, u_2), \quad (u_1, u_2) \in [0, 1] \times [0, 1]$$

Next property clarifies the role of the copula in explaining the dependence structure [30, Property 4.2.14].

Property 3.5 *If the marginal distributions F_{X_1} and F_{X_2} are continuous functions with respective marginal densities f_{X_1} and f_{X_2}, then the joint density function of (X_1, X_2) is given by*

$$f_{X_1,X_2}(x_1, x_2) = \underbrace{f_{X_1}(x_1) f_{X_2}(x_2)}_{\substack{\text{independence} \\ \text{joint pdf}}} \times \underbrace{c\left(F_{X_1}(x_1), F_{X_2}(x_2)\right)}_{\text{dependence structure}} \tag{3.2}$$

From Property 3.5 it is evident that the joint density function can be decoupled in two parts, the part corresponding to independence and the part enclosing the dependence structure. The dependence structure is fully described by the copula density, which is, for that reason, also known as *dependence function*.

Another important and nice property of copulas is their invariance regarding increasing functions (see [30, Proposition 4.4.4] or [55, 34, 33, 55]).

Property 3.6 *Let C be a copula associating the random variables (X_1, X_2) and h_1 and h_2 non-decreasing functions. Then the random vector $(h_1(X_1), h_2(X_2))$ also possesses copula C.*

It is due to Property 3.6 that copula-based dependence measures, such as the Kendall's and Spearman's rank correlations, are invariant to strictly increasing functions.

Notice that the definition of copula, as well as its properties provided above, presupposes the marginal random variables are continuous. Yet, it can be extended to general arbitraty margins [30, Section 4.2.5]. In this case, the copula representation still holds, but the uniqueness of representation is lost [30, Theorem 4.2.19]. In our case, we assume the two underlying risks are continuous random variables X_1 and X_2. However, the retained risk after the combination of QS and stop loss, $Y_i = min(a_i X_i, M_i)$, $i = 1, 2$, $0 \leqslant a_i \leqslant 1$, $M_i \geqslant 0$, is a mixed random variable. Function $Y_i = min(a_i X_i, M_i)$, $i = 1, 2$, is a non-decreasing function of X_i, hence, from Property 3.6, the copula associating risks X_1 and X_2 is invariant to Y_1 and Y_2 and the dependence structure is maintained for the retained risks. That is, if the joint distribution of (X_1, X_2) is described by copula C, $F_{X_1,X_2}(x_1, x_2) = C(F_{X_1}(x_1), F_{X_2}(x_2))$, then the joint distribution of (Y_1, Y_2) is also described by copula C, $F_{Y_1,Y_2}(y_1, y_2) = C(F_{Y_1}(y_1), F_{Y_2}(y_2))$. Notwithstanding, in this work we compute all the necessary expectations relying on the underlying risks, instead of the retained risks, thus always referring to the continuous joint distribution of X_1 and X_2 (see Chapter 4, (4.16)).

3.2 The Archimedean family of copulas

As mentioned in [67], "copulas differ not so much in the degree of association they provide", usually through the copula parameter, "but rather in which part of the distributions the association is strongest". In the context of insurance, the choice of the copula, apart from the level of dependence given by the parameter, will depend on the line of business or the type of risks that are correlated. Usually, when analysing dependencies of risks in insurance, correlations on the right tail tend to be more important. In actuarial applications, clearly positive dependence is of vital importance. In [13], the authors relate, by means of the application of invariant properties, the positive dependence (through the stochastic ordering or through the upper orthant ordering) with the positive dependence of its copula.

In this work, we consider copulas of the Archimedean family, which constitute one of the most important family of copulas [55, Introduction]. Due to their simple form, ease of construction and generalization to higher dimensions, as well as other appealing properties, Archimedean copulas have been widely studied (see for instance [34, 33, 63, 53, 54, 55, 62, 8]) and applied in several fields. Namely, in many finance and insurance applications, there seems to be a stronger relation between big losses, than between big gains [33]. This makes elliptical distributions, and thus elliptical copulas, inappropriate in these contexts. The family of Archimedean copulas include a vast number of copulas and allow for a wide range of different dependence structures. In [8] the authors perform several simulations showing that "between the Archimedean and Elliptical copulas, the Archimedean copulas were the most likely to fit the simulated pairs of random variables". In [37, 72, 33] applications of Archimedean copulas in insurance context can be found. In [72] the authors state that "Archimedean copulas are one of the most popular classes of copulas that are used in actuarial science and finance for modelling risk dependencies and for using them to quantify the magnitude of tail dependence". [62] refers to them as being "rich in various distributional attributes that are desired when modelling". In [60], the authors use hierarchical Archimedean

copula	generator	domain of α	copula
Clayton	$\phi(t) = (t^{-\alpha} - 1)/\alpha$	$\alpha > 0$	$C_\alpha(u_1, u_2) = (u_1^{-\alpha} + u_2^{-\alpha} - 1)^{-1/\alpha}$
Frank	$\phi(t) = -\log\left(\frac{e^{-\alpha t}-1}{e^{-\alpha}-1}\right)$	$\alpha \neq 0$	$C_\alpha(u_1, u_2) = -\frac{1}{\alpha}\log\left(1 + \frac{(e^{-\alpha u_1}-1)(e^{-\alpha u_2}-1)}{e^{-\alpha}-1}\right)$
Gumbel	$\phi(t) = (-\ln t)^\alpha$	$\alpha \in [1, \infty[$	$C_\alpha(u_1, u_2) = \exp\left(-\left[(-\log u_1)^\alpha + (-\log u_2)^\alpha\right]^{1/\alpha}\right)$

Table 3.1: Examples of Archimedean copulas.

copulas to better model joint distributions of asset's returns.

An Archimedean copula (see [30, Section 4.5], and also [55]) is defined by means of the so-called generator function, $\phi : [0, 1] \to \mathbb{R}^+$, possibly infinite, with continuous first and second derivatives on $(0, 1)$ and such that $\phi(1) = 0$, $\phi'(t) < 0$ (strictly decreasing) and $\phi''(t) > 0$ (convex), for all $t \in [0, 1]$. Under these assumptions, the function

$$C(u_1, u_2) = \phi^{-1}\left(\phi(u_1) + \phi(u_2)\right), \tag{3.3}$$

can be proved to be a copula [34, 33, 55] and is referred to as Archimedean copula. If $\phi(0) = \infty$, we say that C is *strict*, otherwise C is *non-strict* [54, Section 4]. When C is strict, $C(u_1, u_2) > 0$, for all $(u_1, u_2) \in [0, 1] \times [0, 1]$. Otherwise, the copula is defined by (3.3) if $\phi(u_1) + \phi(u_2) \leqslant \phi(0)$, and is zero otherwise. Notice that the generator function of any Archimedean copula is defined up to a multiplicative constant.

Within the Archimedean class of copulas, the most applied and found in literature are the Clayton's, the Frank's and the Gumbel's families of copulas. Table 3.1 shows the generator functions and respective copulas for these three families. Observe that the generator function depends on a parameter, α, that measures the level of dependence. We use the under script α in the copula notation, C_α, indicating the dependence parameter.

The Gumbel's copula has upper tail dependence [33, Example 6.10], the Clayton's copula has lower tail dependence [33, Example 6.11], while Frank's copula does not have upper nor lower tail dependence [33, Example 6.12]. In this work, we consider Clayton's and Frank's copulas. However, instead of Gumbel's copula, we consider the Pareto's copula (see [30, Example 4.2.7] and [58]). In [30, Example 4.2.7] the Pareto's copula is derived as the "natural" bivariate distribution of two Pareto distributions with the same shape parameter α. In [67] it is referred to as "heavy right tail copula". Indeed, the Pareto's copula is given by

$$C_\alpha(u_1, u_2) = u_1 + u_2 - 1 + \left((1 - u_1)^{-1/\alpha} + (1 - u_2)^{-1/\alpha} - 1\right)^{-\alpha} \tag{3.4}$$

which is the survival Clayton's copula with dependence parameter $1/\alpha$. The survival copula associated to a copula C is (see [30, Definition 4.4.1])

$$\overline{C}(u_1, u_2) = u_1 + u_2 - 1 + C(1 - u_1, 1 - u_2) \tag{3.5}$$

and it is a copula itself, fulfilling all conditions in Definition 3.1 when evaluated at $(1 - u_1, 1 - u_2) \in [0, 1] \times [0, 1]$. We have that $\overline{C}(1 - u_1, 1 - u_2) = C(u_1, u_2) + 1 - u_1 - u_2$ which, recalling that $u_1 = F_{X_1}(x_1)$, $u_2 = F_{X_2}(x_2)$ and $C(u_1, u_2) = F_{X_1,X_2}(x_1, x_2)$, results in $\overline{C}(1-u_1, 1-u_2) = P(X_1 > x_1, X_2 > x_2)$. Thus, survival copulas can be used to express the joint survival probability function. That is the case here with the Pareto's copula with respect to the Clayton's copula: $C_\alpha^{Pareto}(u_1, u_2) = \overline{C}_{1/\alpha}^{Clayton}(u_1, u_2)$. Conversely, Clayton's copula with dependence parameter $1/\alpha$ is the survival copula of a Pareto's copula with dependence parameter α, since $\overline{C}_\alpha^{Pareto}(u_1, u_2) = \left(u_1^{-1/\alpha} + u_2^{-1/\alpha} - 1\right)^{-\alpha}$

Article [50] investigates the heaviness of multivariate Pareto distribution tails. They demonstrate that, although multivariate Pareto distributions can be seen as a scale mixture of multivariate exponential distributions (which yield copulas of the Marshall-Olkin type [30, Example 4.2.8]), since the mixture is performed with heavy tail mixing random variable, "such heavy tail property thickens the exponential tails" yielding the "heavy tail behaviour of the multivariate Pareto", which is "among the distributions which possess the multivariate heavy tail". They also relate the multivariate Pareto distribution with the survival copula of Archimedean copulas. In [31] the authors consider a rotated Clayton copula, which for a rotation of 180^0 yields the Clayton survival copula.

For Clayton's copula, the closest the dependence parameter α is to zero, the more independent the two random variables are, and the largest the dependence parameter α is, the higher the (positive) dependence between the variables (recall Remark 3.3):

$$\lim_{\alpha\to 0} C_\alpha^{Clayton}(u_1, u_2) = u_1 u_2, \qquad \text{and} \qquad \lim_{\alpha\to +\infty} C_\alpha^{Clayton}(u_1, u_2) = \min(u_1, u_2) = M(u_1, u_2). \tag{3.6}$$

In the case of Frank's copula, independence is also attained in the limit when the dependence parameter α goes to zero, and the furthest from zero the parameter α is, the higher the dependence:

$$\begin{gathered} \lim_{\alpha\to 0} C_\alpha^{Frank}(u_1, u_2) = u_1 u_2, \\ \lim_{\alpha\to +\infty} C_\alpha^{Frank}(u_1, u_2) = \min(u_1, u_2) = M(u_1, u_2) \\ \lim_{\alpha\to -\infty} C_\alpha^{Frank}(u_1, u_2) = \max(0, u_1 + u_2 - 1) = W(u_1, u_2) \end{gathered} \tag{3.7}$$

In this case, recalling Remark 3.3, dependence is negative if $\alpha < 0$ and positive if $\alpha > 0$. Concerning the Pareto's copula, independence is achieved in the limit when α goes to infinity, and the closer the parameter α is from zero, the higher the (positive) dependence.

$$\lim_{\alpha\to +\infty} C_\alpha^{Pareto}(u_1, u_2) = u_1 u_2, \qquad \text{and} \qquad \lim_{\alpha\to 0} C_\alpha^{Pareto}(u_1, u_2) = \min(u_1, u_2) = M(u_1, u_2). \tag{3.8}$$

4

Setting the optimization problem

We seek to find the optimal combination between QS and stop loss reinsurance, in the sense that the stop loss contract is set on top of the QS contract, so that the expected utility or the adjustment coefficient of the wealth of the cedent company is maximized.

We consider two risks, X_1 and X_2, with distribution functions $F_{X_1}(x_1)$ and $F_{X_2}(x_2)$, respectively. By risk we mean a line of business, a portfolio of policies or a policy. We assume that the two risks are dependent through a copula, denoted C, such that the joint distribution is given by $C_\alpha(x_1, x_2) = C_\alpha(F_{X_1}(x_1), F_{X_2}(x_2))$, and the joint density function is given by (3.2) and denoted by $f_{X_1,X_2}(x_1, x_2)$. We use the notation $dC_\alpha(x_1, x_2) = f_{X_1,X_2}(x_1, x_2)dx_1dx_2$.

We assume that the insurer reinsures each risk by means of a QS contract topped by a stop loss treaty. Thus, the retained risks are $Y_i = Y_i(a_i, M_i) = \min(a_iX_i, M_i)$, $i = 1, 2$, where $0 \leqslant a_i \leqslant 1$, represents the QS retained level of risk i, $i = 1, 2$, and $M_i \geqslant 0$, denotes the stop loss retained level, above which all the risk is ceded to the reinsurer, for risk i, $i = 1, 2$. Therefore, the total wealth of the insurer after reinsurance is given by

$$W(a_1, M_1, a_2, M_2) = W(a_1, M_1) + W(a_2, M_2) = (1 - e_1)P_1 - P_{R1} - Y_1 + (1 - e_2)P_2 - P_{R2} - Y_2 \tag{4.1}$$

where $P_i > 0$, represents the premium received by the insurer for each risk i, $i = 1, 2$, and $e_i > 0$, $i = 1, 2$ are the corresponding insurer expenses; $P_{Ri} = P_{Ri}(a_i, M_i) > 0$ denotes the premium charged by the reinsurer for each risk i, $i = 1, 2$.

4.1 The reinsurance premium

The premium charged by an insurer is composed by the pure premium, that is the expected value of the covered risk, plus a premium loading. In this work, we analyse optimal reinsurance strategies for the more common expected value calculation principle, where the loading is proportional to the expected value of the risk, but also for the variance and standard deviation calculation principles. The later belong to the so-called (see [39]) variance related premium principles, as the premium loading is an increasing function of the variance of the covered risk.

Noticing that the amount of each risk ceded to the reinsurer, per risk i, $i = 1, 2$, is $X_i - Y_i$, with $Y_i = \min(a_i X_i, M_i)$, we can whence compute the reinsurance premium on each total ceded risk.

Expected value principle:

$$P_{Ri} = E(X_i - Y_i) + \delta_i E(X_i - Y_i) = (1 + \delta_i)E(X_i - Y_i) \tag{4.2}$$

Variance principle:

$$P_{Ri} = E(X_i - Y_i) + \delta_i Var(X_i - Y_i) \tag{4.3}$$

Standard deviation principle:

$$P_{Ri} = E(X_i - Y_i) + \delta_i \sqrt{Var(X_i - Y_i)} \tag{4.4}$$

where $\delta_i > 0$, $i = 1, 2$, is the loading coefficient. This is how the authors in [14], using the expected value principle, as well as in [39, 41], for variance related principles, compute the reinsurance premium. In this case, the moments to be computed are developed as follows.

$$E(X_i - Y_i) = E(X_i) - a_i \int_0^{M_i/a_i} (1 - F_{X_i}(x))\, dx, \qquad i = 1, 2 \tag{4.5}$$

and

$$\begin{aligned} Var(X_i - Y_i) \quad = \quad & (1 - a_i)^2 \int_0^{M_i/a_i} x^2 f_{X_i}(x)\, dx + \int_{M_i/a_i}^{+\infty} (x - M_i)^2 f_{X_i}(x)\, dx \\ & - \left[E(X_i) - a_i \int_0^{M_i/a_i} (1 - F_{X_i}(x))\, dx \right]^2, \qquad i = 1, 2 \end{aligned} \tag{4.6}$$

However, when a combination of QS and stop loss is taken into account, the QS and stop loss premiums can be considered separately. This is the procedure followed for instance in [18, 19], and it corresponds to the common practice. In fact, as described in Chapter 2, the QS premium is usually proportional to the ceded risk minus a commission. In this case, the QS premium is the proportion of the premium received by the insurer P_i correspondent to the ceded risk, $(1 - a_i)P_i$, subtracting the commission, $c_i > 0$: $P_{QSi} = (1 - a_i)(1 - c_i)P_i$. The stop loss premium will be computed on the ceded risk after QS: $Z_i = \max(a_i X_i - M_i, 0)$, $i = 1, 2$. Thereby, the total reinsurance premium turns out as follows.

Expected value principle:

$$P_{Ri} = P_{QSi} + (1 + \delta_i)E(Z_i) \tag{4.7}$$

Variance principle:

$$P_{Ri} = P_{QSi} + E(Z_i) + \delta_i Var(Z_i) \tag{4.8}$$

Standard deviation principle:

$$P_{Ri} = P_{QSi} + E(Z_i) + \delta_i \sqrt{Var(Z_i)} \tag{4.9}$$

In this case, the moments to be computed are given as follows.

$$E(Z_i) = a_i \int_{M_i/a_i}^{+\infty} (1 - F_{X_i}(x))\, dx, \qquad i = 1, 2 \tag{4.10}$$

and

$$\begin{aligned} Var(Z_i) \quad = \quad & 2\, a_i^2 \int_{M_i/a_i}^{+\infty} x\,(1 - F_{X_i}(x))\, dx - 2\, a_i\, M_i \int_{M_i/a_i}^{+\infty} (1 - F_{X_i}(x))\, dx \\ & - \left[a_i \int_{M_i/a_i}^{+\infty} (1 - F_{X_i}(x))\, dx \right]^2, \qquad i = 1, 2 \end{aligned} \tag{4.11}$$

Here, we will study and compare optimal reinsurance strategies in both cases where the premium is computed on the total ceded risk or separately for QS and stop loss.

4.2 The expected utility and the adjustment coefficient

As mentioned in Chapter 2, several authors have considered to use the expected utility [18, 19, 23, 39, 41, 49] of wealth as optimality criteria when ascertaining the optimal reinsurance strategy. In [20, 39] it has been demonstrated that the coefficient of the expected utility (for the exponential utility function) is strictly related to the adjustment coefficient, which in turn is connected to the ultimate probability of ruin in discrete time.

The probability of ruin, denoted $\Phi(u)$ for a given initial amount of reserves $u > 0$, relates to the adjustment coefficient, R, through the well known Lundberg Inequality:

$$\Phi(u) \leqslant e^{-uR} \tag{4.12}$$

For this reason, the adjustment coefficient is also known as the Lundberg exponent. Clearly, from (4.12), the larger the adjustment coefficient is, the smaller the upper bound of the probability of ultimate ruin is. Furthermore, for some reinsurance forms, the behaviour of the probability of ruin, $\Phi(u)$, is very similar to the behaviour of its upper bound in the Lundberg inequality [20]. Thus, maximizing the adjustment coefficient R instead of minimizing the probability of ruin $\Phi(u)$ is reasonable. Because of this, many authors have considered maximizing the adjustment coefficient as optimality criteria for reinsurance [17, 21, 23, 39, 41, 20, 73].

4.2.1 Maximizing the expected utility

The goal is to determine the optimal reinsurance contract for a risk-averse insurer which purpose is to maximize the expected utility of its wealth. We consider the exponential utility function, for risk averse investors, defined through

$$U(x) = \frac{1-\ e^{-\beta\, x}}{\beta} \tag{4.13}$$

where $\beta = -U''(x)/U'(x) > 0$ is the coefficient of risk aversion. In this case, the expected utility of the wealth for a given (fixed) coefficient of aversion β is:

$$E[U(W(a_1, M_1, a_2, M_2))] = \frac{1}{\beta}\left(1 - E\left[e^{-\beta W(a_1,M_1,a_2,M_2)}\right]\right) \tag{4.14}$$

Maximizing the expected utility (4.14) corresponds to find the reinsurance strategy, (a_1, M_1, a_2, M_2), that maximizes $E[U(W)]$ for a given (fixed) coefficient of risk aversion β. Recalling (4.1), this is equivalent to minimize the following functional.

$$\begin{aligned}
E\left[e^{-\beta W(a_1,M_1,a_2,M_2)}\right] &:= G(\beta, a_1, M_1, a_2, M_2) = \qquad (4.15)\\
&= \int_0^{+\infty}\int_0^{+\infty} e^{-\beta((1-e_1)P_1 - P_{R1}(a_1,M_1) - Y_1(a_1,M_1) + (1-e_2)P_2 - P_{R2}(a_2,M_2) - Y_2(a_2,M_2))} dC_\alpha(x_1, x_2)\\
&= e^{-\beta((1-e_1)P_1 + (1-e_2)P_2)} e^{\beta(P_{R1}(a_1,M_1) + P_{R2}(a_2,M_2))} \int_0^{+\infty}\int_0^{+\infty} e^{\beta(Y_1(a_1,M_1)+Y_2(a_2,M_2))} dC_\alpha(x_1, x_2)
\end{aligned}$$

for a given (fixed) β. Looking closer into the double integral in (4.15), and recalling that $Y_i = \min(a_i X_i, M_i)$, $i = 1, 2$, we have

$$\begin{aligned}
\int_0^{+\infty}\int_0^{+\infty} & e^{\beta(Y_1(a_1,M_1)+Y_2(a_2,M_2))} dC_\alpha(x_1, x_2) = \qquad (4.16)\\
&= \int_0^{M_1/a_1}\int_0^{M_2/a_2} e^{\beta(a_1x_1 + a_2x_2)} dC_\alpha(x_1, x_2) + \int_0^{M_1/a_1}\int_{M_2/a_2}^{+\infty} e^{\beta(a_1x_1 + M_2)} dC_\alpha(x_1, x_2)\\
&\quad + \int_{M_1/a_1}^{+\infty}\int_0^{M_2/a_2} e^{\beta(M_1 + a_2x_2)} dC_\alpha(x_1, x_2) + \int_{M_1/a_1}^{+\infty}\int_{M_2/a_2}^{+\infty} e^{\beta(M_1+M_2)} dC_\alpha(x_1, x_2)
\end{aligned}$$

Notice that, besides the double integrals (4.16), also single integrals need to be computed to evaluate the value of $G(\beta, a_1, M_1, a_2, M_2)$, (4.15). These are related with the reinsurance premium calculations, which are based on moments of the retained risk (4.5), (4.6), (4.10) and (4.11).

4.2.2 Maximizing the adjustment coefficient

The adjustment coefficient, R, of the retained risk after reinsurance is defined as the unique positive root, if it exists, of $G(R, a_1, M_1, a_2, M_2) = 1$, where G is given by (4.15). The coefficient of adjustment is related to the coefficient of risk aversion of the exponential utility, as it corresponds to the value of the risk aversion coefficient for which the expected utility (4.14) is zero [39]. It follows immediately from (4.14) and (4.15) that a reinsurance policy maximizes the expected utility, for a given (fixed) risk aversion coefficient β, if and only if it minimizes the functional $G(\beta, a_1, M_1, a_2, M_2)$, for the same fixed value of β. Furthermore, in [39] it is demonstrated that, under general regularity assumptions on the functional G verified in our case, a reinsurance policy maximizes the adjustment coefficient, $\widehat{R}$, if and only if:

i) The expected utility, with coefficient of risk aversion $\widehat{R}$, is maximum for that policy and

ii) $G(\widehat{R}, a_1, M_1, a_2, M_2) = 1$

Thus, as suggested in [39], the problem of maximizing the adjustment coefficient can be split in two sub problems:

1. For each $\beta > 0$, find the reinsurance strategy, (a_1, M_1, a_2, M_2) that minimizes $G(\beta, a_1, M_1, a_2, M_2)$
2. Solve $G(\beta, a_1, M_1, a_2, M_2) = 1$ with respect to the single variable β.

Whence, given the algorithm to find the optimal reinsurance maximizing the expected utility it is straightforward to obtain the reinsurance strategy maximizing the adjustment coefficient. However, maximizing the adjustment coefficient requires the solution of several expected utility maximization problems, until the desired root is found. For this reason, maximizing the adjustment coefficient comprises higher computational costs.

5

Numerical results and discussion

As described in Chapter 4, we aim at finding the combination of QS and stop loss reinsurance, (a_1, M_1, a_2, M_2), that minimizes functional $G(\beta, a_1, M_1, a_2, M_2)$ (4.15). If coefficient β is fixed, then we are maximizing the expected utility, whereas if we impose the constraint $G(\beta, a_1, M_1, a_2, M_2) = 1$ for varying values of the coefficient β, we are maximizing the adjustment coefficient, that in turn is equal to the corresponding value of β.

All the double and single integrals involved in the evaluation of $G(\beta, a_1, M_1, a_2, M_2)$ are solved using *Mathematica* numerical integration, which applies global adaptive Gauss-Kronrod quadrature rules (see [70] for numerical integration with *Mathematica*).

The resolution of the minimization problems were carried out using numerical algorithms for non-linear constrained global optimization already implemented in *Mathematica*, namely the Nelder Mead and Differential Evolution algorithms. Strictly speaking, the Nelder Mead algorithm is not a global optimization method, but it tends to work quite well if the objective function does not have many local minima, which is the case here. Whenever Nelder Mead did not produce good results, the Differential Evolution algorithm, which is a genetic algorithm, was applied, but at a significant higher computational cost (see [71] for non-linear global optimization with *Mathematica*). Here, global optimization is required as the functional to minimize (4.15) does not necessarily has a single minimum in the constraint domain.

The numerical procedure, namely the numerical optimization problem, is amenable for improvement. No particular features of the functional to minimize were taken into consideration and general global optimization was applied. Some problems were encountered, specially when dependence between the two risks increases. In these situations, a case by case analysis was sometimes required due to the existence of *plateaux* regions in the functional to minimize, specially regarding the stop loss retention values. Nevertheless, results were achieved and analysis of optimal reinsurance for two dependent risks were performed.

In the following, the premium received by the insurer is computed by means of the expected value principle with a loading coefficient of $\gamma_i = 0.2$, $i = 1, 2$. For the underlying risks, X_1 and X_2, we will consider different distributions, but in such way that the expected value is always 1. Hence, the premium loading charged by the insurer is $\gamma_i \, E(X_i) = \gamma_i$, $i = 1, 2$. We assume expenses are 5% of the premium, $e_i = 0.05$, $i = 1, 2$. Whenever the QS premium is computed on a proportional basis, separately from the stop-loss premium, the commission is $c_i = 0.03$, $i = 1, 2$. Indeed, the QS reinsurance commission should be lower than the insurer expenses $c_i < e_i$, meaning it is impossible to reinsure the whole risk through QS with a certain profit. This implies that the QS premium loading is $E(X_i)\ [(1 - c_i)(1 + \gamma_i) - 1] = 0.164 \, E(X_i)$. When maximizing the expected utility, we consider a coefficient of risk aversion $\beta = 0.1$.

Regarding the stop loss premium, the loading coefficients are chosen so that the premium loadings are comparable for all three premium principles (expected value, standard deviation and variance principles). For instance, for the Pareto distribution with expected value 1 and shape parameter 3, the variance is 3 and the standard deviation is $\sqrt{3}$. These are very discrepant values, so if the loading coefficients are the same, the premium loadings will be very different. Namely, the variance loading will be much higher than the standard deviation or expected value loadings. Also, the loading coefficient should not be the same for the premium of the pure stop loss, where only the tail is ceded, and for the premium computed on the whole ceded risk, that is, on the combination of QS and stop loss, where part of the risk, other than the tail, is ceded. The loading coefficient in the latter case should be lower than in the former situation.

Because of all these reasons, the loading coefficients were chosen differently for all three premium calculation principles. In Table 5.1 are presented the premium loadings. With these values, the premium loading when all the risk is transfered by means of a pure stop loss contract, *i.e* when $a_i = 1$ and $M_i = 0$, is the same for all three premium principles. From (4.5) and (4.6), for QS and stop loss premiums computed together, and from (4.10) and (4.11), for QS and stop loss premiums computed separately, if the risk is all ceded through a pure stop loss contract, *i.e.* if $a_i = 1$ and $M_i = 0$, the premium loading for QS and stop loss separately or together is the same. Indeed, in this case the moments envolved correspond to the moments of the underlying risk. However, if QS and stop loss are considered separately, (4.10) and (4.11), that is true only when $a_i = 1$ (and $M_i = 0$), whereas if the premium is computed for the QS and stop loss together, (4.5) and (4.6), that is true no matter the value of a_i (as long as $M_i = 0$).

premium principle	loading coefficient
expected value	δ
variance	$\delta\, E(X)/Var(X)$
standard deviation	$\delta\, E(X)/\sqrt{Var(X)}$

Table 5.1: Loading coefficients for the three premium principles considered, where δ is the loading coefficient for the expected value principle and X is the underlying risk.

We first consider two Pareto distributions with expected value 1 and shape parameter 3, meaning that the variance is 3 and the probability density function is given by $f(x) = \frac{24}{(2+x)^4}$. The loading coefficients in this case are shown in Table 5.2. Here, independently of the premium calculation principle, the optimal retention levels of both QS and stop loss contracts are the same for both risks, as they are equal. However, they obviously vary with the premium calculation principle and the dependence parameter α.

premium principle	QS and stop loss separately	QS and stop loss together
expected value	0.3	0.2
variance	0.1	0.0666667
standard deviation	0.173205	0.11547

Table 5.2: Loading coefficients, for the three premium principles, considering two Pareto risks with expected value 1 and shape parameter 3.

Results for the optimal reinsurance maximizing the expected utility with coefficient of risk aversion $\beta = 0.1$ and the loading coefficients in Table 5.2 are presented in Figures 5.1, 5.2 and 5.3.

Pareto's copula with two identical risks and the loading coefficients of Table 5.2

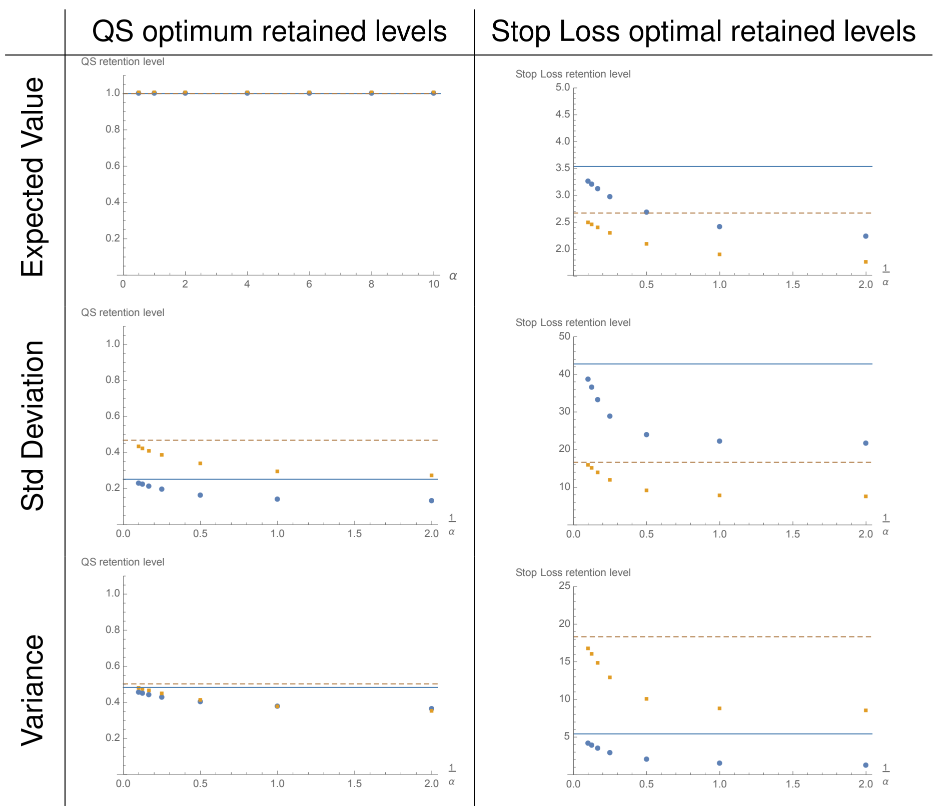

Figure 5.1: QS (left) and stop loss (right) optimal retained levels, computing stop loss and QS premiums separately (blue) and together (yellow), as function of the dependence parameter $1/\alpha$. **Top:** Expected value principle. **Middle:** Standard deviation principle. **Bottom:** Variance principle. The horizontal lines correspond to the optimal retention levels in case of independence. Maximizing the expected utility with coefficient of risk aversion $\beta = 0.1$.

Clayton's copula with two identical risks and the loading coefficients of Table 5.2

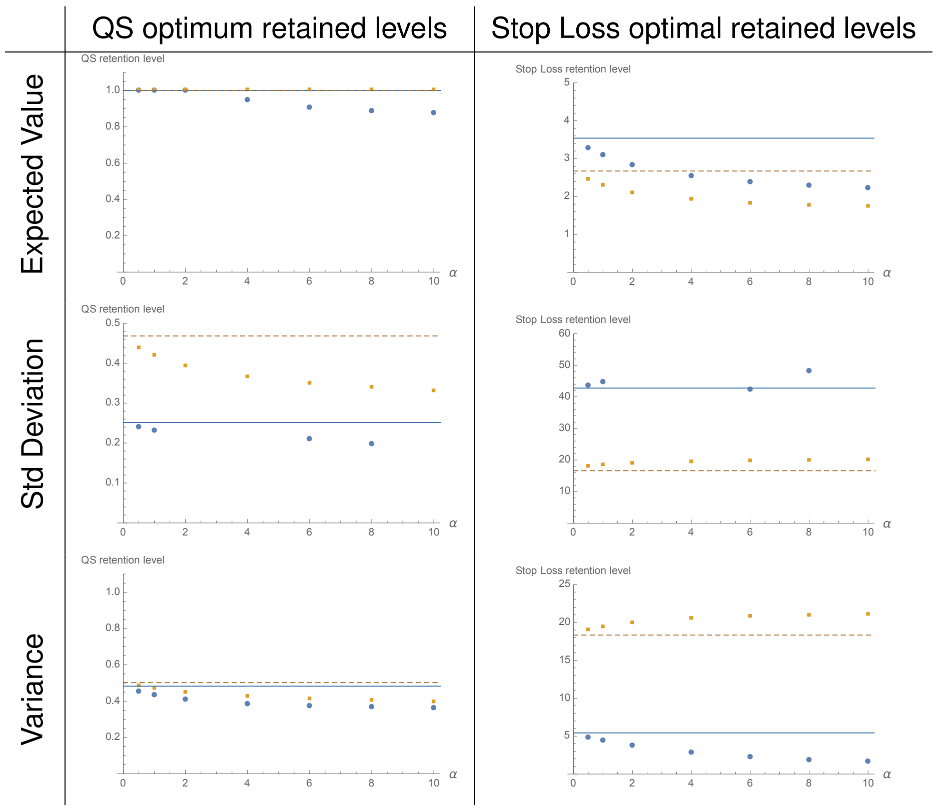

Figure 5.2: QS (left) and stop loss (right) optimal retained levels, computing stop loss and QS premiums separately (blue) and together (yellow), as function of the dependence parameter α. **Top:** Expected value principle. **Middle:** Standard deviation principle. **Bottom:** Variance principle. The horizontal lines correspond to the optimal retention levels in case of independence. Maximizing the expected utility with coefficient of risk aversion $\beta = 0.1$.

Frank's copula with two identical risks and the loading coefficients of Table 5.2

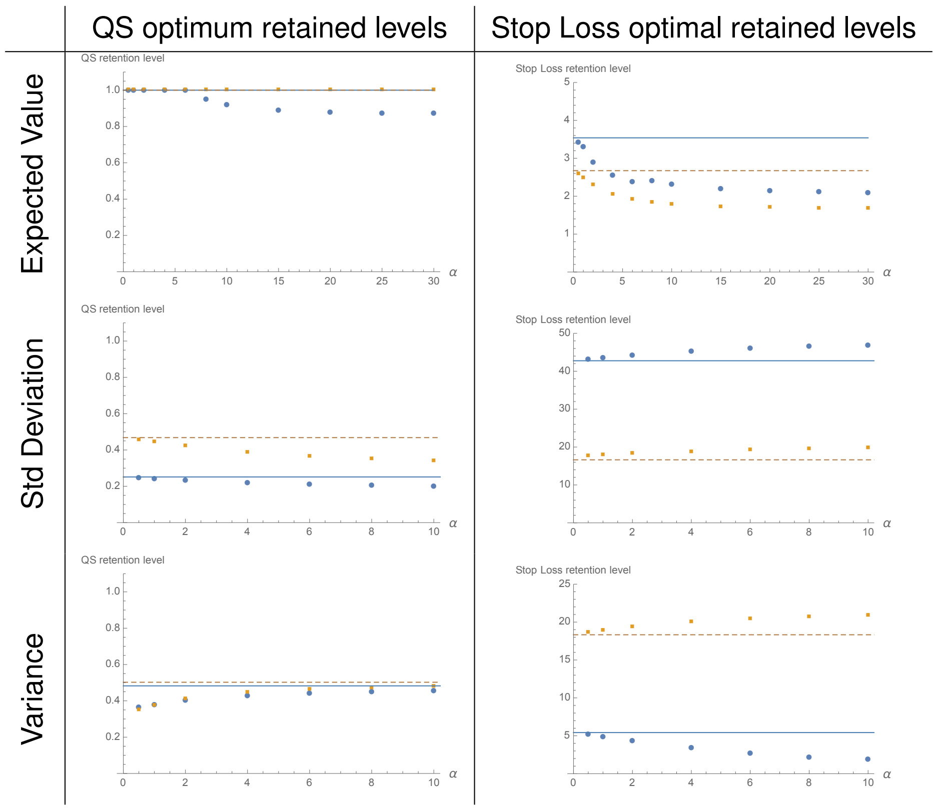

Figure 5.3: QS (left) and stop loss (right) optimal retained levels, computing stop loss and QS premiums separately (blue) and together (yellow), as function of the dependence parameter α. **Top:** Expected value principle. **Middle:** Standard deviation principle. **Bottom:** Variance principle. The horizontal lines correspond to the optimal retention levels in case of independence. Maximizing the expected utility with coefficient of risk aversion $\beta = 0.1$.

The results show that dependence impacts the optimal levels of retention. Both QS and stop loss optimal retention levels vary as dependence increases. It should be noted that, although optimal stop loss retention levels change with dependence, the absolute values do not vary much. Nevertheless, the impact of dependence is evident.

For all three copulas, the optimal reinsurance using the expected value principle together for the QS and stop loss is the pure stop loss contract, regardless the dependence parameter value. This was expected, from the results in [39]. In the case of Pareto's copula, for the expected value principle, if QS and stop loss premiums are computed separately, then the stop loss contract is still optimal, regardless the dependence parameter value. However, for Clayton's and Frank's copulas, as the dependence parameter increases, the QS retention level decreases.

Still for the expected value principle, computed on the whole ceded risk or just on the stop loss contract, the stop loss optimal retention level decreases as dependence increases. For the Pareto's copula, this is also true for the standard deviation and variance principles.

For all copulas and premium principles, the QS optimal retention level is below the optimal value in case of independence and decreases as dependence increases. For the Pareto's copula, the decrease, with dependence, in the optimal QS retained level is accompanied by a decrease in the optimal stop loss retention level. This is expectable, as the Pareto's copula has right tail behaviour. This is not the case for Calyton's and Frank's copulas, namely for the standard deviation and variance principles. These phenomena are related with the fact that both Clayton's and Frank's copulas have no right tail dependence, which is the part of the distribution covered by the stop loss contract. For these two copulas, when variance (computed on the whole ceded risk) and standard deviation (computing QS and stop loss premiums separately or together) principles are considered, the optimal solution in presence of dependence is to transfer more risk through QS than in the independent case and less risk through stop loss than in the independent case. This behaviour is not observed in the Pareto's copula.

In Figures 5.4, 5.5 and 5.6, these results are presented comparing the three premium principles. When computing QS and stop loss premiums together, the standard deviation and variance premium principles produce similar results, with the standard deviation optimal retained levels below the variance ones. In this case, the optimal retained levels when using the expected value principle are significantly different from those using the standard deviation and variance principles. Indeed, the pure stop loss is the optimal treaty for the expected value principle when computing QS and stop loss premiums together. Thus, the optimal stop loss retained level in this case becomes quite lower than that of the standard deviation and variance principles, where a combination with QS is optimal. This is observable for all three copulas considered.

When premiums are computed separately, the behaviour of the optimal retained levels is less predictable, since the QS loading is lower than the stop loss loading, and transferring more risk through QS and less risk through stop loss may be optimal.

Pareto's copula with two identical risks and the loading coefficients of Table 5.2

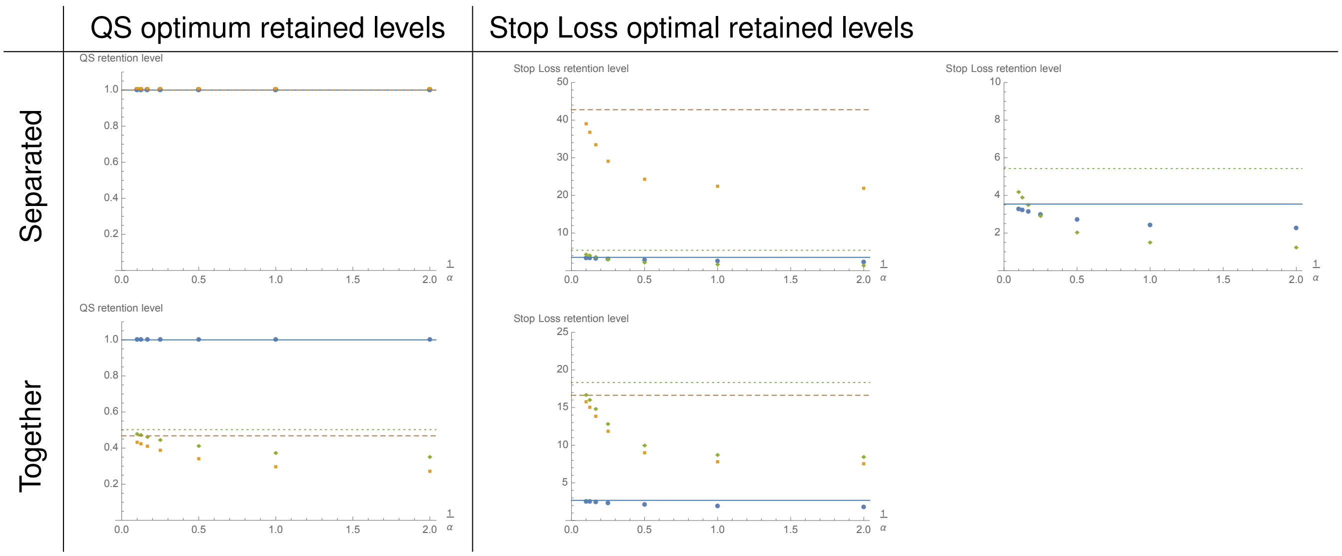

Figure 5.4: Optimal retention levels for the expected value (blue), standard deviation (yellow) and variance (green) principles, computing QS and stop loss premiums separately (top) and together (bottom), as function of the dependence parameter $1/\alpha$. Maximizing the expected utility with coefficient of risk aversion $\beta = 0.1$.

Clayton's copula with two identical risks and the loading coefficients of Table 5.2

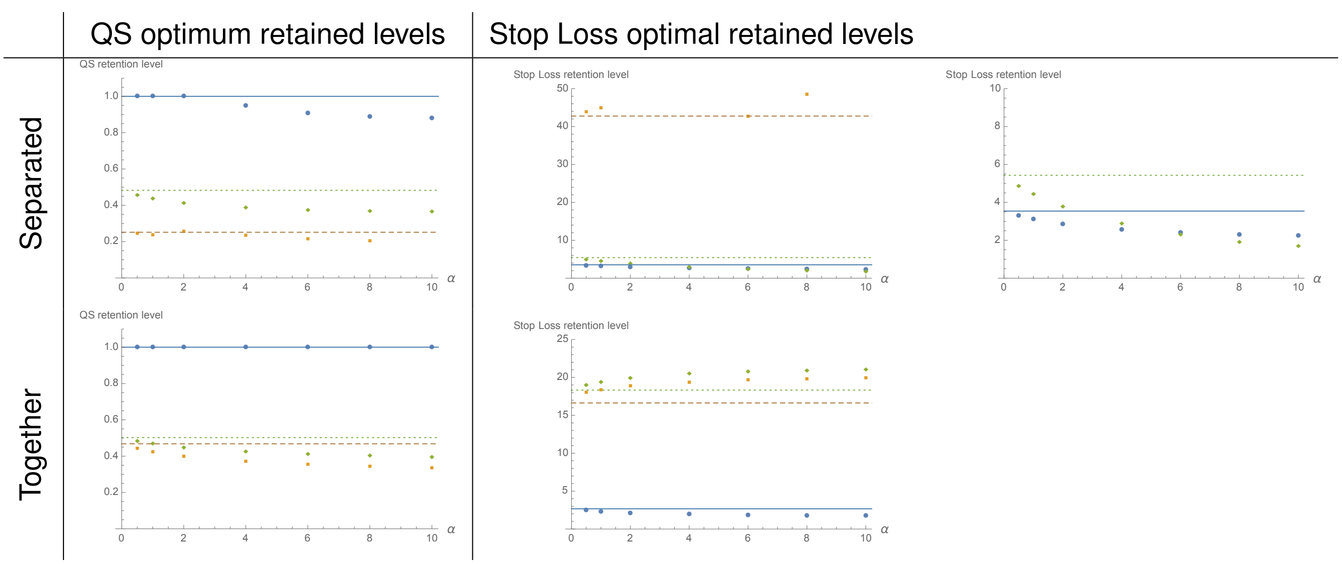

Figure 5.5: Optimal retention levels for the expected value (blue), standard deviation (yellow) and variance (green) principles, computing QS and stop loss premiums separately (top) and together (bottom), as function of the dependence parameter α. Maximizing the expected utility with coefficient of risk aversion $\beta = 0.1$.

Frank's copula with two identical risks and the loading coefficients of Table 5.2

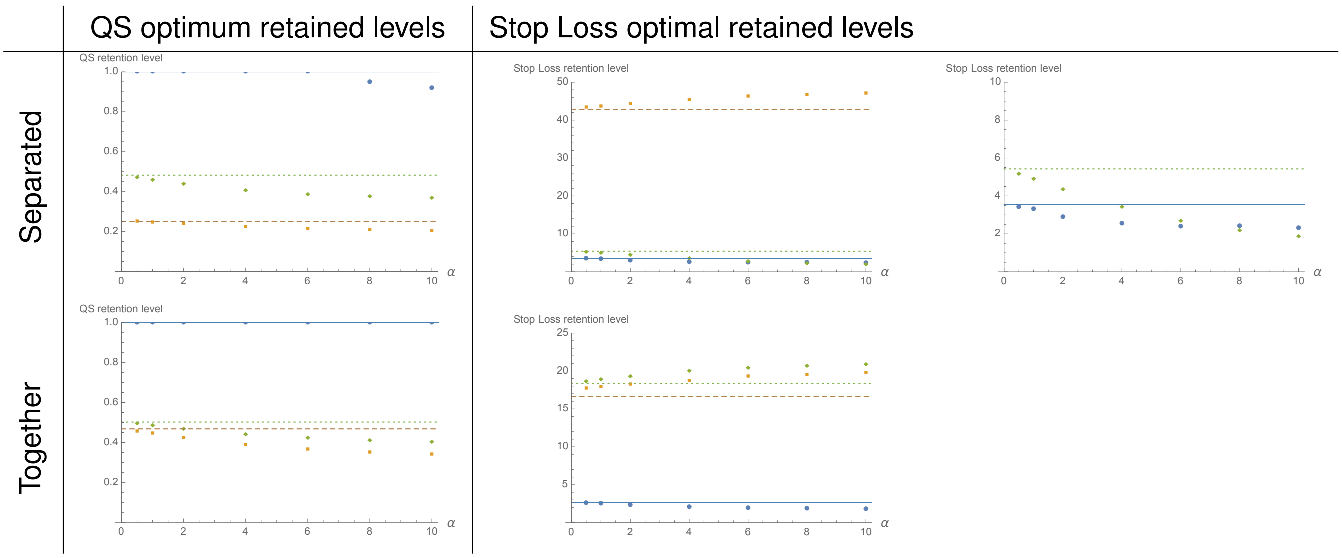

Figure 5.6: QS (left) and stop loss (right) optimal retention levels for the expected value (blue), standard deviation (yellow) and variance (green) principles, computing QS and stop loss premiums separately (top) and together (bottom), as function of the dependence parameter α. Maximizing the expected utility with coefficient of risk aversion $\beta = 0.1$.

The results obtained so far regard optimal retention levels maximizing the expected utility. Thus, they are dependent on the choice of the coefficient of the risk aversion β, which was chosen to be $\beta = 0.1$. If maximizing the adjustment coefficient is considered as optimality criteria instead, the optimal reinsurance treaty is not dependent on this coefficient anymore. As described in Chapter 4, to obtain the optimal reinsurance treaty maximizing the adjustment coefficient, it is enough to find the optimal solution for the expected utility problem with the coefficient of risk aversion $\beta > 0$ such that the expected utility value in (4.14) is equal to 0. In order to solve equation $G(R, a_1, M_1, a_2, M_2) = 1$, for (a_1, M_1, a_2, M_2) minimizing $G(R, a_1, M_1, a_2, M_2)$, a bisection method was applied. Amongst the root finding numerical methods, bisection is the simplest. Although its convergence is not very fast when compared with Newton-type methods, it has the advantage of not requiring the computation of derivatives of the functional. Also, convergence to a tolerance of 10^{-6} was reached within an average of 10 iterations, as the initial points were easily chosen close enough to the solution. Situations where convergence was more difficult regard instances where converge of the constraint global optimization algorithm to the minimum of functional G was slow. This was the case of Clayton's copula, when using the standard deviation principle computing QS and stop loss premiums separately.

Figures 5.7, 5.8 and 5.9 show the optimal reinsurance results maximizing the adjustment coefficient for the loading coefficients in Table 5.2.

Pareto's copula with two identical risks and the loading coefficients of Table 5.2

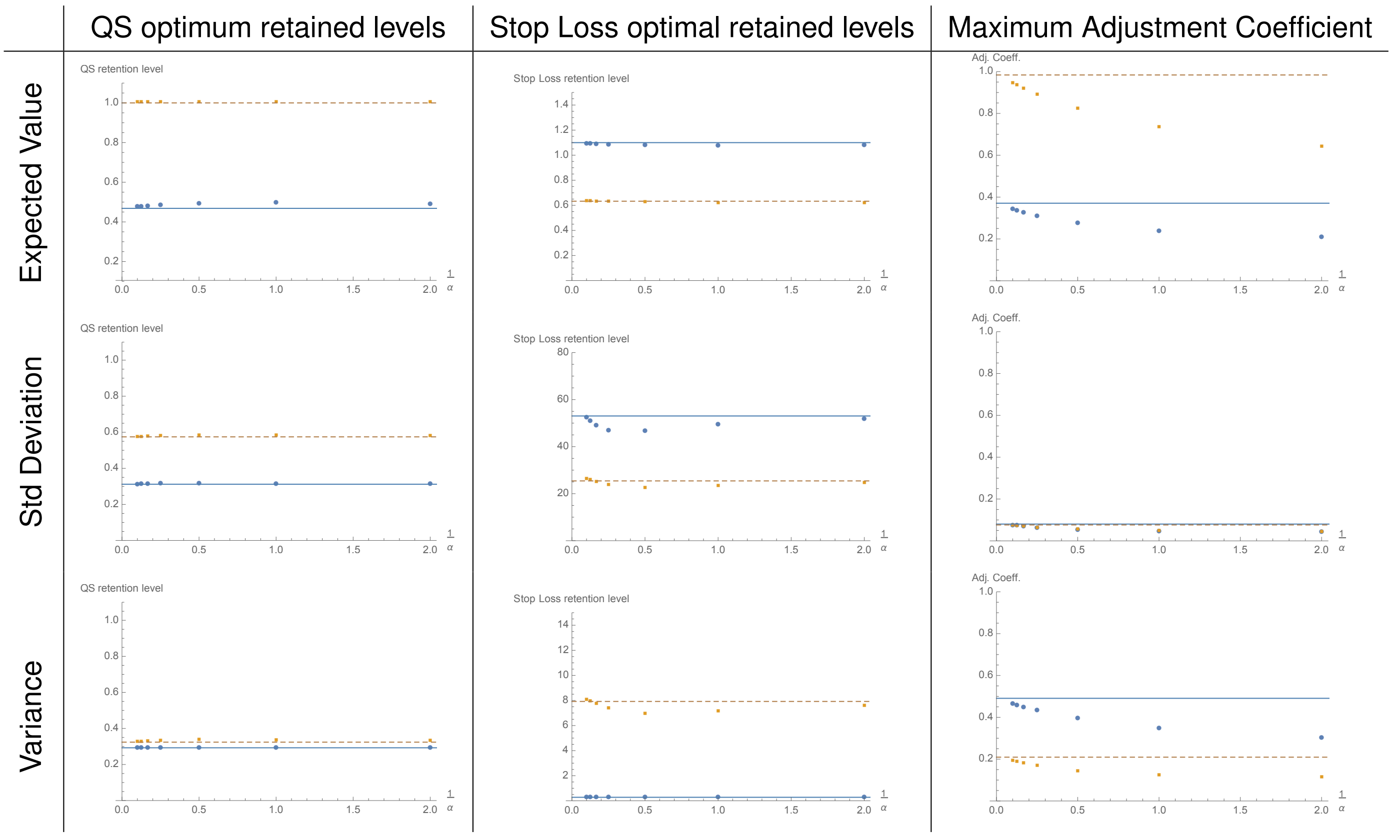

Figure 5.7: QS (left) and stop loss (middle) optimal retained levels, and maximum adjustment coefficient (right), computing stop loss and QS premiums separately (blue) and together (yellow), as function of the dependence parameter $1/\alpha$. **Top:** Expected value principle. **Middle:** Standard deviation principle. **Bottom:** Variance principle. The horizontal lines correspond to the optimal retention levels in case of independence. Maximizing the adjustment coefficient.

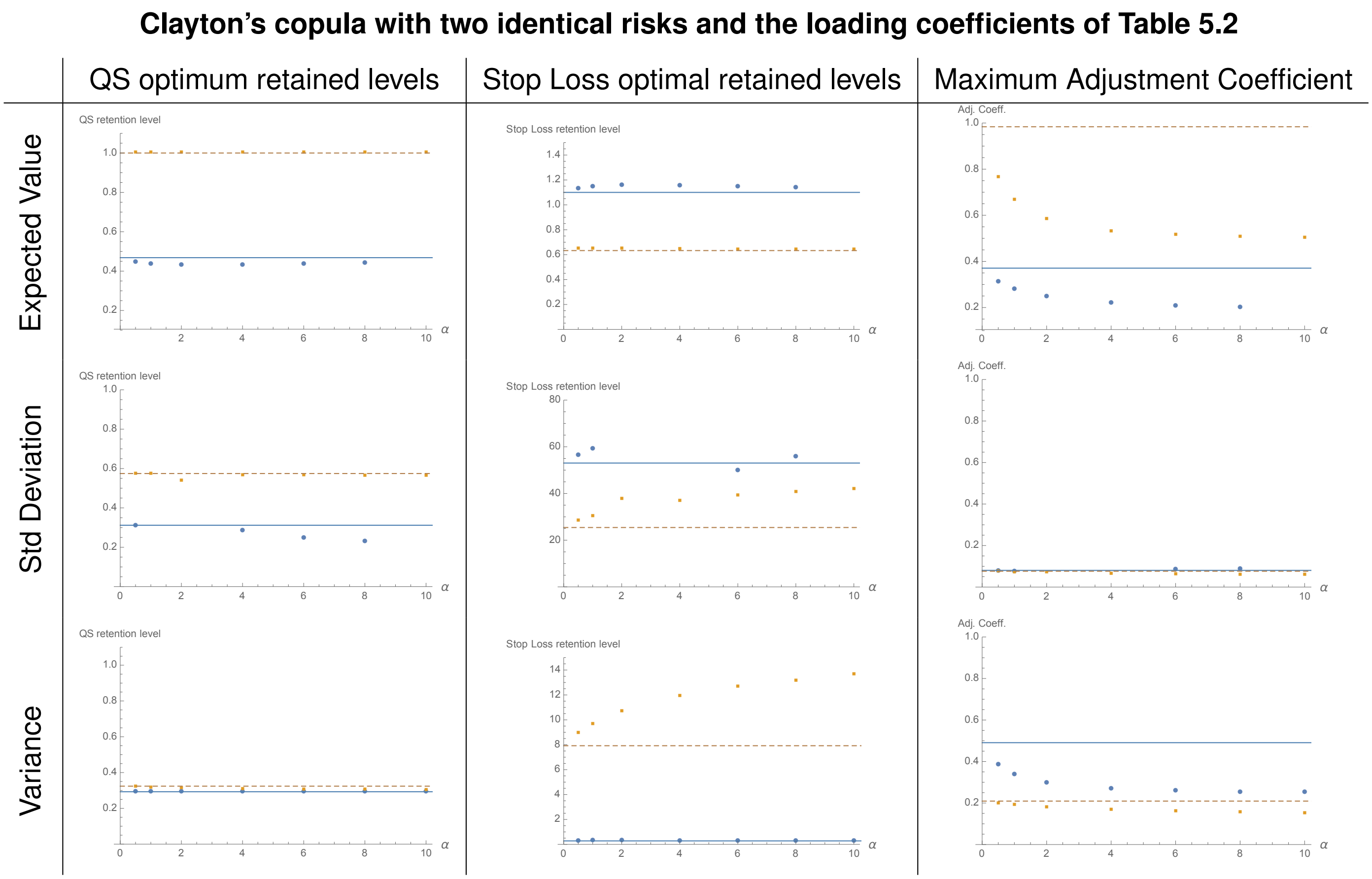

Figure 5.8: QS (left) and stop loss (middle) optimal retained levels, and maximum adjustment coefficient (right), computing stop loss and QS premiums separately (blue) and together (yellow), as function of the dependence parameter α. **Top:** Expected value principle. **Middle:** Standard deviation principle. **Bottom:** Variance principle. The horizontal lines correspond to the optimal retention levels in case of independence. Maximizing the adjustment coefficient.

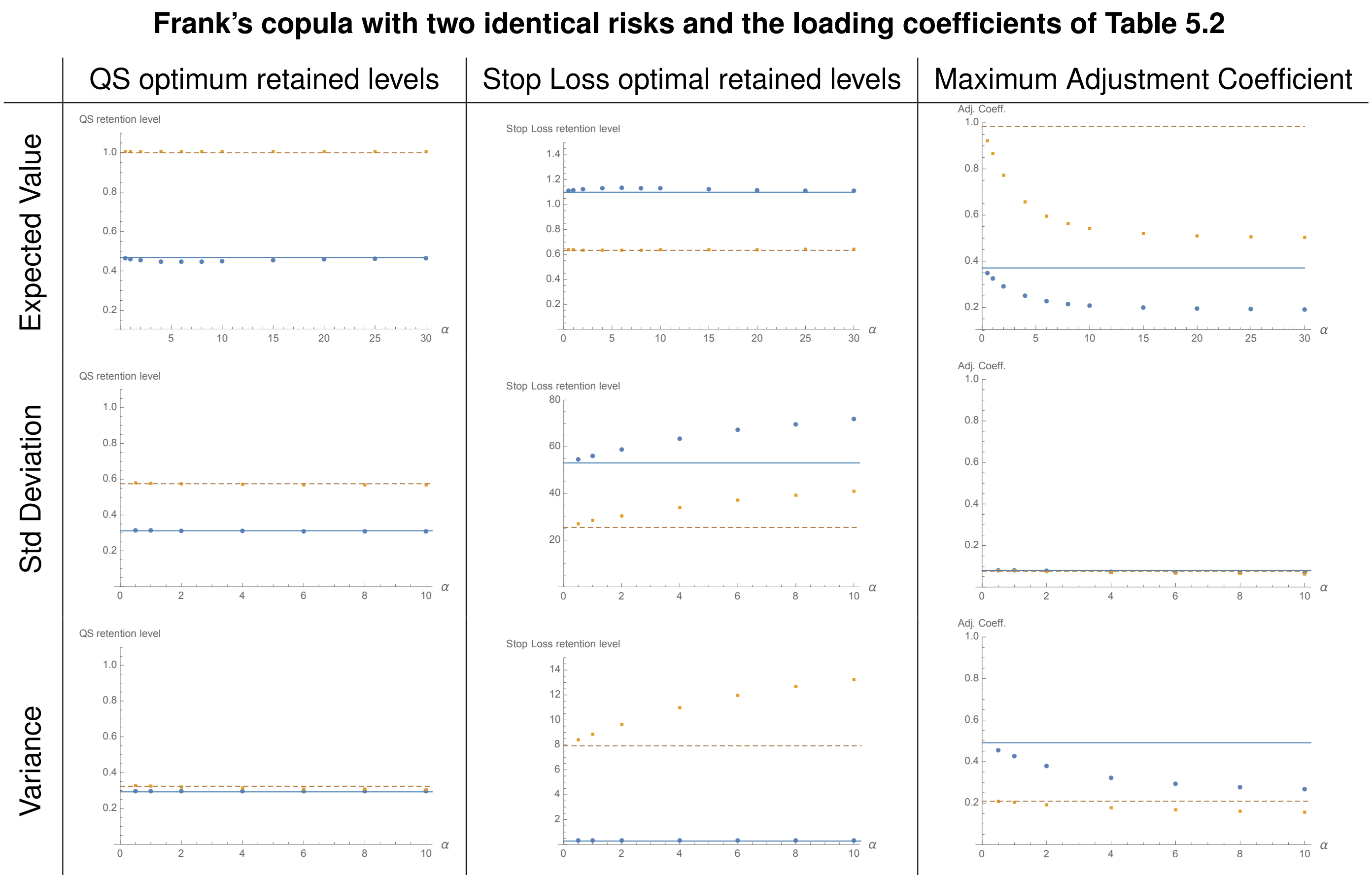

Figure 5.9: QS (left) and stop loss (middle) optimal retained levels, and maximum adjustment coefficient (right), computing stop loss and QS premiums separately (blue) and together (yellow), as function of the dependence parameter α. **Top:** Expected value principle. **Middle:** Standard deviation principle. **Bottom:** Variance principle. The horizontal lines correspond to the optimal retention levels in case of independence. Maximizing the adjustment coefficient.

Again, as expected from the results in [39], the optimal treaty when the expected value principle is applied to QS and stop loss together is the pure stop loss, for all three copulas and all values of the dependence parameter. That is not the case for the variance related premium principles, and it is not the case for the expected value principle when QS and stop loss premiums are computed separately, even in the independent case. This is because the QS loading is lower than the stop loss loading. The optimal retention levels vary with the dependence parameter, although not so significantly as for the case where the risk aversion coefficient was fixed. Instead, the impact of dependence is very relevant in the value of the maximum adjustment coefficient. It can be observed, for all copulas and premium principles considered, that the adjustment coefficient decreases as dependence increases. This means, from (4.12), that the higher the dependence, the higher the upper bound of the ultimate probability of ruin in discrete time.

Due to the choice of the premium loading coefficients, the optimum reinsurance levels are comparable and Figures 5.10, 5.11 and 5.12 represent these results comparing the three premium principles considered.

Pareto's copula with two identical risks and the loading coefficients of Table 5.2

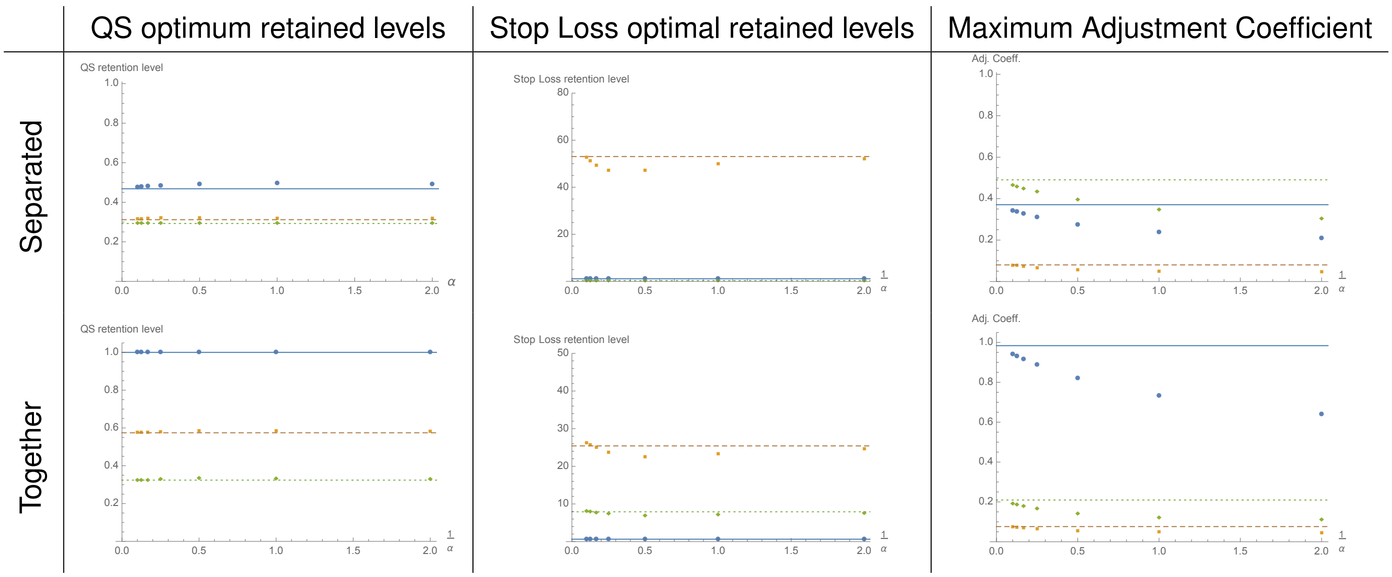

Figure 5.10: QS (left) and stop loss (middle) optimal retention levels, and maximum adjustment coefficient (right) for the expected value (blue), standard deviation (yellow) and variance (green) principles, computing QS and stop loss premiums separately (top) and together (bottom), as function of the dependence parameter $1/\alpha$. Maximizing the adjustment coefficient.

Clayton's copula with two identical risks and the loading coefficients of Table 5.2

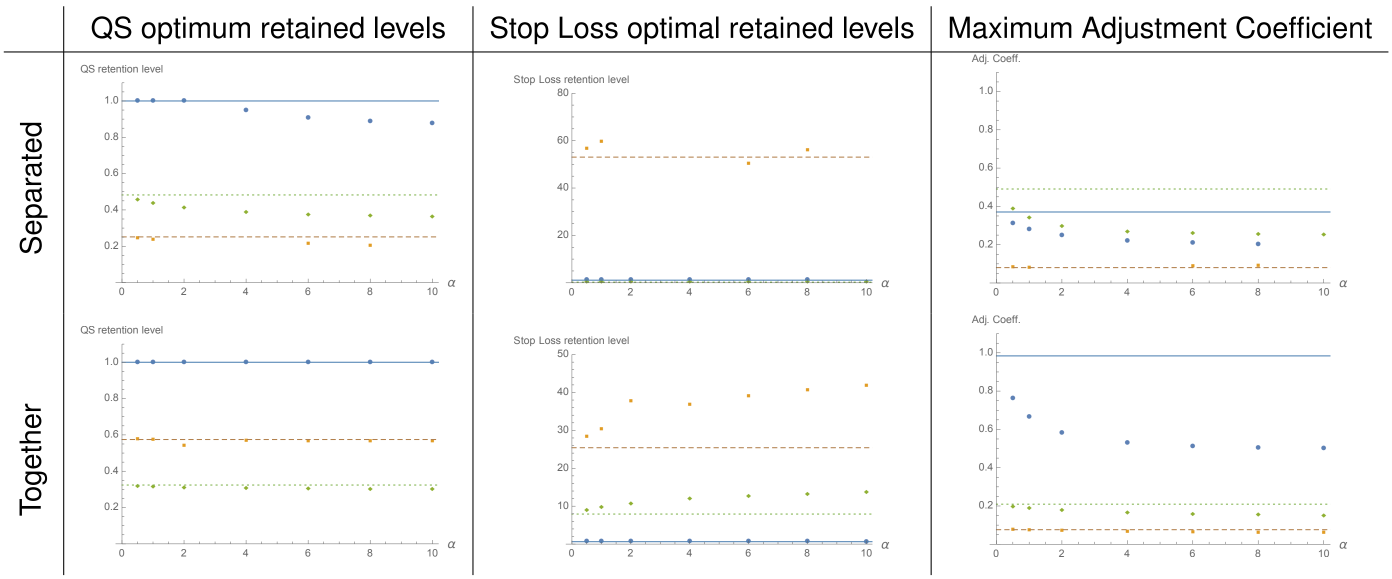

Figure 5.11: QS (left) and stop loss (middle) optimal retention levels, and maximum adjustment coefficient (right) for the expected value (blue), standard deviation (yellow) and variance (green) principles, computing QS and stop loss premiums separately (top) and together (bottom), as function of the dependence parameter α. Maximizing the adjustment coefficient.

Frank's copula with two identical risks and the loading coefficients of Table 5.2

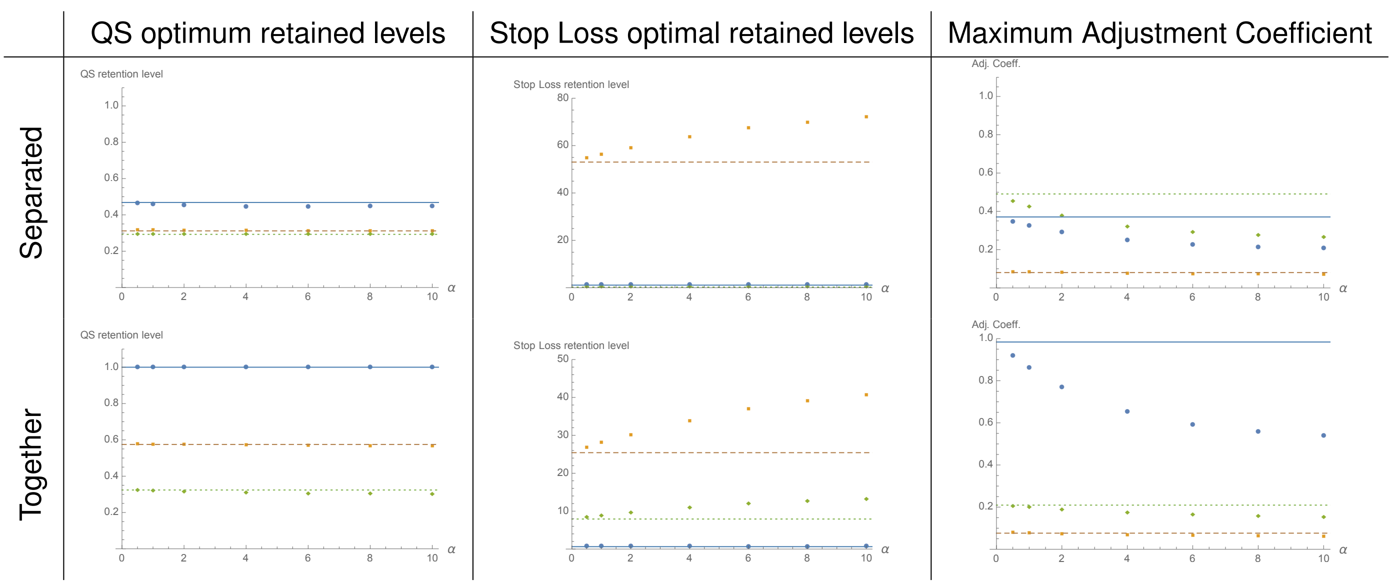

Figure 5.12: QS (left) and stop loss (middle) optimal retention levels, and maximum adjustment coefficient (right) for the expected value (blue), standard deviation (yellow) and variance (green) principles, computing QS and stop loss premiums separately (top) and together (bottom), as function of the dependence parameter α. Maximizing the adjustment coefficient.

The remarks made for the case of a fixed coefficient of aversion apply here, although now the differences in the standard deviation and variance principles, computing QS and stop loss premiums together, are more accentuated. The adjustment coefficient always decreases when dependence increases. It can be observed that the maximum adjustment coefficient using the standard deviation principle is always below those using the expected value and variance principles. If QS and stop loss premiums are computed together, then the maximum adjustment coefficient using the expected value principle is higher. If premiums are computed separately, then the maximum adjustment coefficient using the variance principle is higher. This is verified for all three copulas.

It is worth noticing the differences in the optimal reinsurance for the different copulas. Differences are particularly significant between the Pareto's copula and Clayton's and Frank's copulas. This is because Pareto's copula has right tail dependence, while Claytons and Frank's copulas do not. Differences in comparing Pareto's copula results with Clayton's and Frank's copulas are also related with the fact that, for the Pareto's copula, dependence increases when the dependence parameter α decreases to zero (3.8), while for Clayton's and Frank's copulas the situation is reversed, (3.6) and (3.7). One question that arises is "What are reasonable values of the dependence parameter?". Of course, it will depend on the problem at hand. An interesting future work would be to apply this approach to compute the optimal QS and stop loss retention levels to a real case study, estimating the dependence parameter from data.

Afterwards we have considered two risks with different tail heaviness: two Pareto distributions with expected value 1 and shape parameters 3 and 12, respectively. In this case, the variances are 3 and 1.2, respectively, and the cumulative distribution functions are $F(x) = 1 - \left(1 + \frac{x}{2}\right)^{-3}$ and $F(x) = 1 - \left(1 + \frac{x}{11}\right)^{-12}$, respectively. For this case, we have only considered dependence by means of the Pareto's copula, where dependence is stronger on the right tail. Regarding the loading coefficients, we apply the same reasoning as before, which is described in Table 5.1 and leads to the loading coefficients presented in Table 5.3. Figures 5.13, 5.14, 5.15 and 5.16 present the results for the two different risks with these loading coefficients.

premium principle	QS and stop loss separately		QS and stop loss together	
	X_1	X_2	X_1	X_2
expected value	0.3	0.3	0.2	0.2
variance	0.1	0.25	0.0666667	0.166667
standard deviation	0.173205	0.273861	0.11547	0.182574

Table 5.3: Loading coefficients, for the three premium principles, considering two Pareto risks with expected value 1 and shape parameter 3 (X_1) and 12 (X_2).

Pareto's copula with two different risks and the loading coefficients of Table 5.3

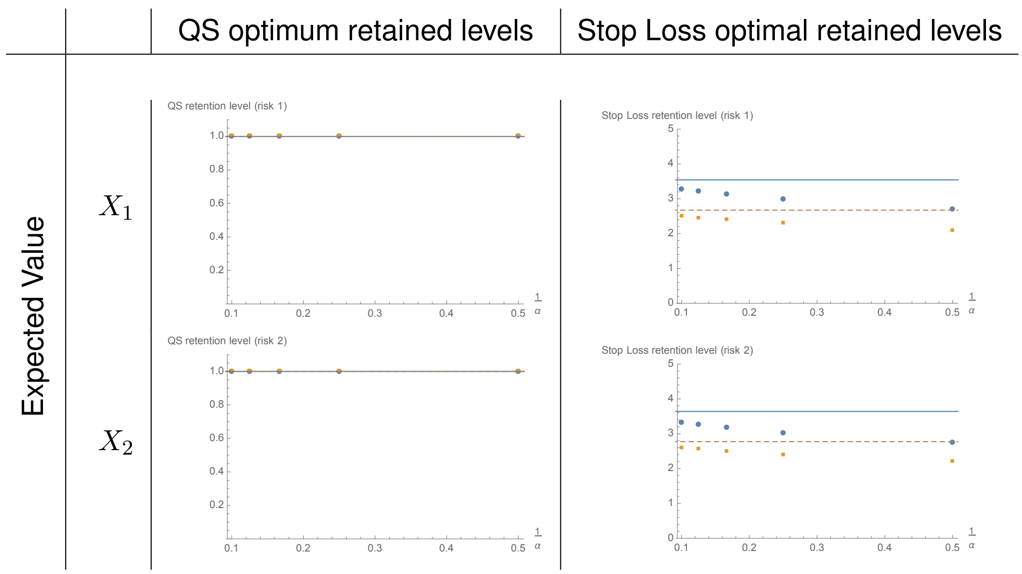

Figure 5.13: QS (left) and stop loss (right) optimal retained levels for each risk (X_1, with heavy tail, and X_2 with lighter tail), computing stop loss and QS premiums separately (blue) and together (yellow), as function of the dependence parameter $1/\alpha$. Expected value principle. The horizontal lines correspond to the optimal retention levels in case of independence. Maximizing the expected utility with coefficient of risk aversion $\beta = 0.1$.

Pareto's copula with two different risks and the loading coefficients of Table 5.3

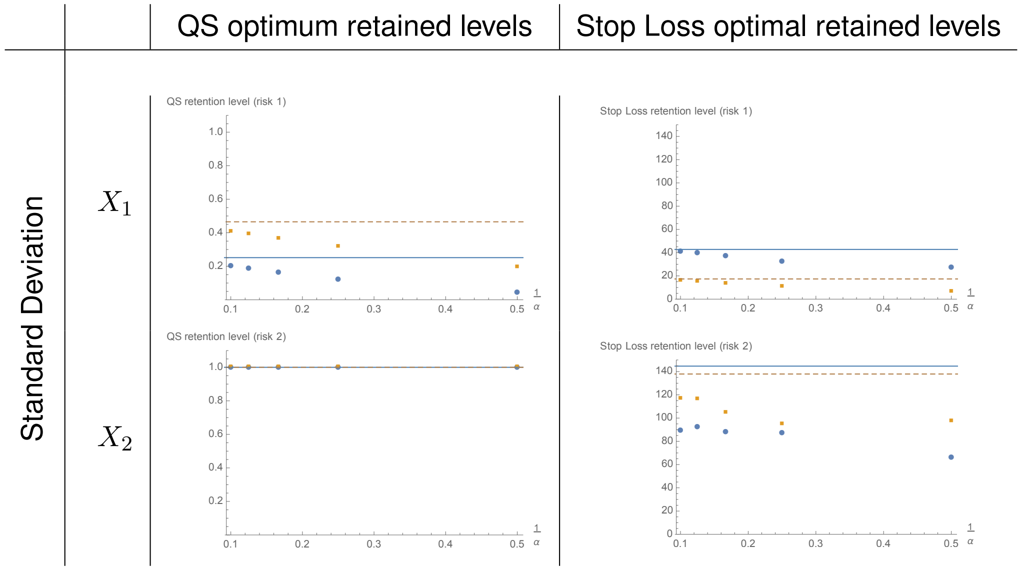

Figure 5.14: QS (left) and stop loss (right) optimal retained levels for each risk (X_1, with heavy tail, and X_2 with lighter tail), computing stop loss and QS premiums separately (blue) and together (yellow), as function of the dependence parameter $1/\alpha$. Standard deviation principle. The horizontal lines correspond to the optimal retention levels in case of independence. Maximizing the expected utility with coefficient of risk aversion $\beta = 0.1$.

Pareto's copula with two different risks and the loading coefficients of Table 5.3

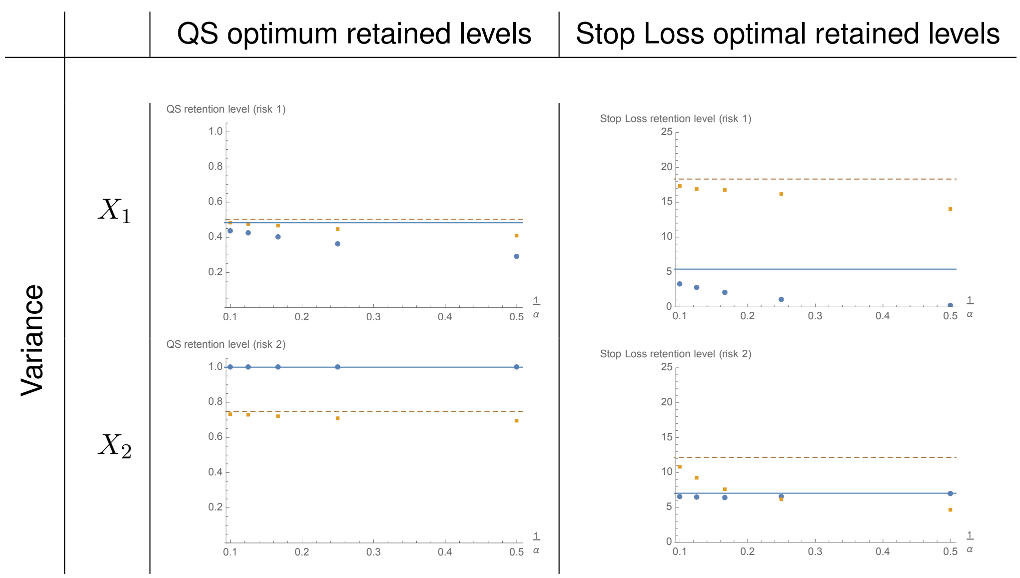

Figure 5.15: QS (left) and stop loss (right) optimal retained levels for each risk (X_1, with heavy tail, and X_2 with lighter tail), computing stop loss and QS premiums separately (blue) and together (yellow), as function of the dependence parameter $1/\alpha$. Variance principle. The horizontal lines correspond to the optimal retention levels in case of independence. Maximizing the expected utility with coefficient of risk aversion $\beta = 0.1$.

Pareto's copula with two different risks and the loading coefficients of Table 5.3

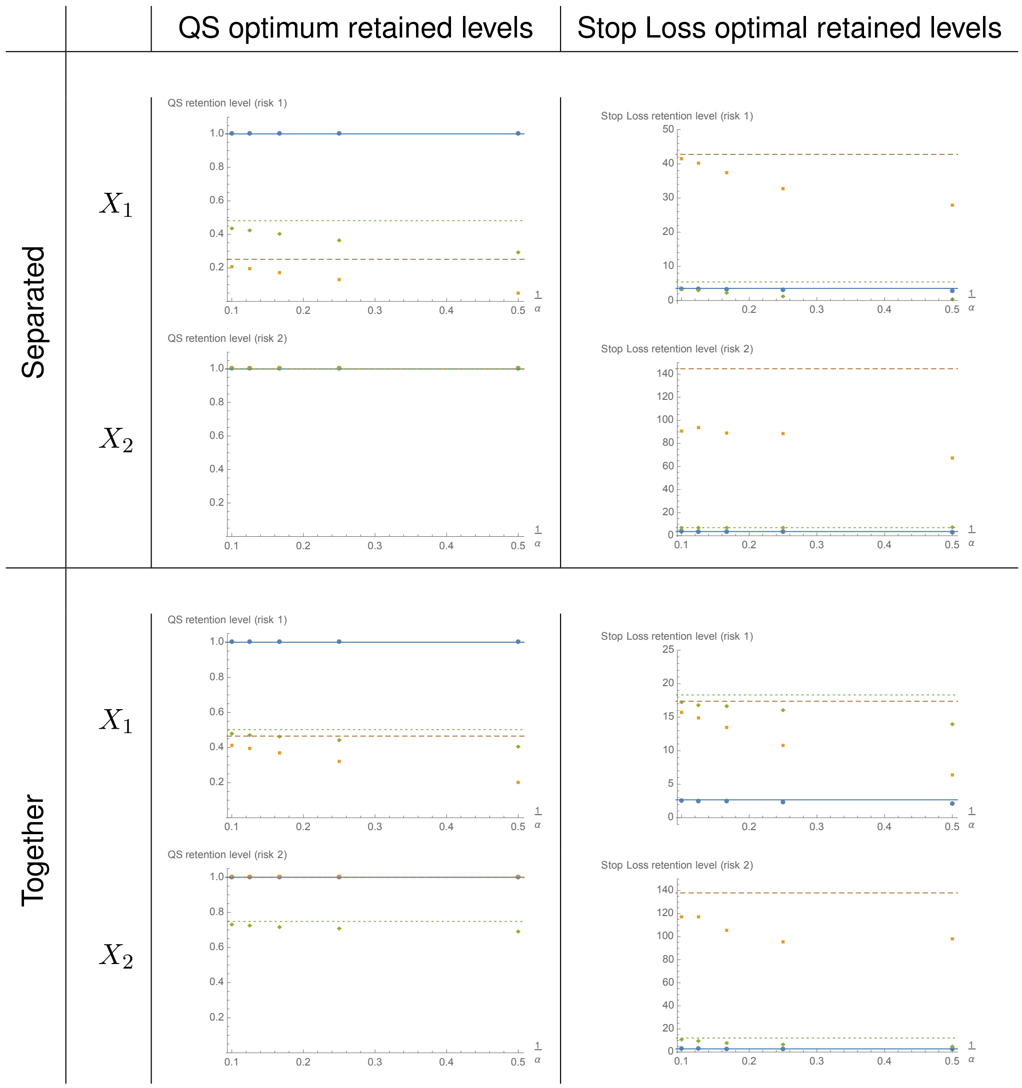

Figure 5.16: QS (left) and stop loss (right) optimal retention levels for each risk (X_1, with heavy tail, and X_2 with lighter tail), for the expected value (blue), standard deviation (yellow) and variance (green) principles, computing QS and stop loss premiums separately (top) and together (bottom), as function of the dependence parameter $1/\alpha$. The horizontal lines correspond to the optimal retention levels in case of independence. Maximizing the expected utility with coefficient of risk aversion $\beta = 0.1$.

Whenever the expected value principle is applied, either on the total ceded risk or just on the stop loss contract, the pure stop loss treaty is optimal for both risks. The optimal stop loss retention levels are similar for both risks and decrease as dependence increases.

Regarding the standard deviation principle, the pure stop loss contract is optimal for the second (light tail) risk, when computing the reinsurance premium both on the total ceded risk or separately for QS and stop loss. The optimal stop loss retention values for this pure stop loss contract on the second risk are significantly high, compared to the first risk or with other premium calculation principles, and decrease as dependence increases. For the second (heavy tail) risk, the optimal reinsurance contract is not the pure stop loss anymore and the optimal stop loss retention levels are much lower than those of the second risk, though still higher compared to other premium principles. The optimal QS levels are quite low and both QS and stop loss optimal retention levels decrease as dependence increases. With the standard deviation premium principle, much of the first risk is transferred, while much of the second risk is kept.

For what concerns the variance principle, the pure stop loss contract is optimal for the second risk only when the QS premium is computed on a proportional basis. Again, the optimal QS and stop loss retention values decrease with dependence. The optimal stop loss retention levels of the first risk are significantly different when QS and stop loss premiums are computed together or separately. This difference is less accentuated for the second risk, where the stop loss contract is optimal when computing QS and stop loss premiums separately.

In general, for all three premium principles and for both risks the optimal QS and stop loss retained levels decrease as dependence increases. In most cases the pure stop loss contract is optimal for the second (light tail) risk. Thus, in most cases only the tail of the second risk is transferred. On the contrary, for the first (heavy tail) risk, the pure stop loss contract is optimal only for the expected value principle, meaning that for the standard deviation and variance principle it is optimal to transfer more of the first (heavy tail) risk.

6

Conclusions

Optimal combinations of QS and stop loss reinsurance, in the sense that the expected utility or the adjustment coefficient of the cedent's insurer wealth are maximized, were numerically investigated. The study was performed for two Pareto risks under the parametric dependence of copulas. This controlled environment, together with the numerical approach, allowed for several comparison studies. A sensitivity analysis of the optimal reinsurance treaty to a variety of factors under dependence of the underlying risks was carried out. Several levels of dependence (by varying the copula dependence parameter) as well as different dependence structures (through different types of copulas) were considered. Different values of the shape parameter of the Pareto underlying risks were also investigated, so to consider the influence of lighter or heavier tails. The impact of different reinsurance premium loading principles on the retained levels of the quota-share and stop loss treaties was also analysed.

Clearly dependencies alter the optimal treaty, as compared with the independent case. Different dependence structures, *i.e.* different families of copulas, yield significantly different optimal solutions. As expected, the optimal treaty is also highly sensitive to the premium calculation principle and relevant differences are encountered between premiums calculated on the total ceded risk or separately for QS and stop loss. In some cases, this behaviour is accentuated in the presence of dependencies.

From the results, in the case of the expected value principle computed on the total ceded risk, the stop loss contract is always optimal. This was not the case for the remaining premium principles considered. It was not also the case for the expected value principle computed separately on the QS and stop loss treaties. This was because the QS loading was lower than the stop loss loading. From the results, it was observed that, in general, the higher the dependence between the two risks, the lower the optimal QS retention levels. For the Pareto's copula, with right tail behaviour, these decrease in the optimal QS retained level is accompanied by a decrease in the optimal stop loss retention level. For Clayton's and Frank's copulas, without right tail behaviour, the optimal stop loss level does not necessarily accompany the decrease of the optimal QS level when dependence increases. For all cases considered, the maximum adjustment coefficient decreases when dependence increases.

The results here presented can be useful in bringing insight on the impact of dependence on the optimal reinsurance strategy. Such insight can be helpful in the design of more general theoretical results on optimal reinsurance of dependent risks. It can also be beneficial when analysing *real world* case studies of applications.

The framework is built to consider sensitiveness of the optimal reinsurance strategy to other factors, allowing usage in further cases. For instance, including layers in the stop loss contract, as commonly observed in this type of reinsurance. Starting with a single layer, this would add to our problem two new variables, L_i, $i = 1, 2$, corresponding to the upper limits of the layer for each risk, increasing the complexity. Further analysis, with different underlying distributions under the copulas here considered, could be performed, as in some cases intuition regarding the optimal treaty under the existence of dependences may mislead. Finally, it would be of interest to apply these procedures to real data, after estimating the copula associating the risks.

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