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Some simple and classical approximations to ruin probabilities applied to the perturbed model

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Abstract

We recover the classical actuarial risk model with a diffusion component like in the model introduced by Dufresne and Gerber (1991). We just target the computation of the ruin probability in infinite time by approximation, picking up some simple and classical methods, built for the basic or classical Cramér-Lundberg model, that can be adapted for the perturbed model in an easy way.

In particular, we consider the well-known methods by De Vylder, Dufresne & Gerber's bounds approach, Beekman-Bowers' and the Tijms' approximations. We further consider the adaptation of the Fourier/Laplace transforms method worked for instance by Lima *et al.* (2002).

With the help of several examples we attempt the accuracy of the approximations, some light and fat tail distributions for the individual claim severity were used. We evaluate the ultimate ruin probability and separate the probability of ruin due to the introduction of the oscillating component.

Keywords: Perturbed risk model; diffusion; ruin probability; ruin probability approximations; ruin probability by oscillation.

Resumo

Trabalhamos o modelo clássico de risco perturbado por difusão, em particular, o modelo introduzido por Dufresne and Gerber (1991). O objectivo é aproximar a probabilidade de ruína em tempo infinito para este modelo usando algumas aproximações simples e clássicas, originalmente apresentadas para o modelo clássico de Cramér-Lundberg e que são facilmente adaptáveis para este modelo.

Em particular é trabalhada a aproximação de De Vylder, o método de Dufresne & Gerber que permite a construção de limites inferiores e superiores para a probabilidade de ruína, a aproximação de Beekman-Bowers e a aproximação de Tijms. É ainda considerada uma adaptação do método das transformadas de Fourier/Laplace, usado por exemplo em Lima *et al.* (2002).

Recorrendo a vários exemplos é testada a precisão das aproximações, usando distribuições de cauda leve e de cauda pesada. É calculada a probabilidade de ruína em tempo infinito e separada a probabilidade de ruína devida à introdução da componente oscilatória.

Palavras-chave: Modelo de risco perturbado; difusão; probabilidade de ruína; aproximações à probabilidade de ruína; ruína causada por oscilação.

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Introduction

The so called risk models in actuarial science are much focused in modelling the surplus process of the insurance business. In these models, like in any other mathematical models we have to make simplifications of the real world. One simplification that is made in the classical risk model is that no other factors than the received premiums and the paid claims affect the surplus. The object of study in this work is the perturbed model introduced in actuarial science by Dufresne and Gerber (1991). By adding a diffusion component to the classical surplus process, it allows to consider other factors to affect the surplus such as the uncertainty of the premium income, changes in the number of clients, fluctuations of the interest rate or deviations in the claims payouts, without discarding all the other assumptions.

The aim of this work is to obtain approximations for the probability of the surplus process becoming negative in infinite time horizon (ultimate ruin). Closed ruin probability formulae are known in some special cases and so we aim to approximate this probability using a variety of different methods. In this work we consider some simple and well known methods that are applied in the classical risk model and we *translate* them to the perturbed model.

The work that follows in the next pages can be seen as a sequel of previous works like Silva (2006) and Jacinto (2008). The first author adapted to the perturbed model the De Vylder's method and the Dufresne & Gerber's upper and lower bounds. The latter presented some numerical illustrations of these methods and a first try to adapt the Beekman-Bowers' approximation.

In this dissertation we first present the perturbed model, we review and correct the methods adapted by Silva (2006) and we try to improve the Beekman-Bowers' approximation presented by Jacinto (2008). We add to these methods the Tijms' approximation and also one possible application of the Fourier transform to obtain ruin probabilities in this model.

In the last chapter are presented some typical and illustrative examples in order to evaluate the accuracy of the different methods by comparing the results with exact ones available, or with the upper and lower bounds. In addition to the approximate figures of the ultimate ruin probability, we present also figures for the ultimate ruin probability due to the diffusion component and the weight that this form of ruin has to the total ruin probability in each of the examples.

It is assumed during this work that the reader is familiar with ruin theory and specially with the classical continuous time model, also known as the Cramér-Lundberg model. Good first references are Dickson (2005), Gerber (1979), Kaas *et al*. (2008) or even Klugman *et al.* (2008).

1 The Perturbed Model

In this chapter we introduce the model as presented by Dufresne and Gerber (1991). We define the perturbed surplus process, the probability of ruin and its decomposition, the adjustment coefficient and some particular cases where it is possible to calculate the exact ruin probability. We also define the process of aggregate loss, the maximal aggregate loss and also one possible decomposition for it. Last but not least, asymptotic results for the ruin probabilities are presented. Proofs of the theorems, when not presented can be seen on that paper.

1.1 Model description

The perturbed surplus process at time t is defined as:

$$\begin{aligned} V(t) &= U(t) + \sigma W(t), \\ U(t) &= u + ct - S(t), \quad t \geq 0, \end{aligned}$$

where $U(t)$ defines the well known classical surplus process, c is the rate at which premiums are received, $u = V(0) = U(0)$ is the initial surplus, $S(t) = \sum_{i=0}^{N(t)} X_i$, $X_0 \equiv 0$ are the aggregate claims up to time t, $N(t)$ is the number of claims received up to time t, X_i is the i-th individual claim, $W(t)$ is the diffusion component and σ^2 the variance parameter. $\{W(t), t \geq 0\}$ is a standard Wiener process (or Brownian Motion), $\{N(t), t \geq 0\}$ is a Poisson process with parameter λ and $\{X_i\}_{i=1}^{\infty}$ is a sequence of $i.i.d.$ random variables, independent from $\{N(t)\}$ with common distribution function $P(.)$ with $P(0) = 0$ and $p_k = E[X^k]$. The corresponding density function is denoted as $p(.)$. We assume that $\{S(t)\}$ and $\{W(t)\}$ are independent. We also assume that $c = (1+\theta)\lambda p_1$, where $\theta > 0$ is the premium loading coefficient.

The diffusion component introduces an extra source of uncertainty to the classical model due to its oscillating nature and so the process $V(t)$ becomes more uncertain than $U(t)$. For more details about the Weiner process and the moments of $V(t)$, please see Appendix A.

1.2 Probability of ruin

Definition 1 *Let* $T = \inf\{t : t \geq 0 \text{ and } V(t) \leq 0\}$, *be the moment of ruin, with the assumption that* $T = \infty$ *if* $V(t) > 0$, $\forall t$.

Definition 2 *The probability of ultimate ruin from initial surplus* u *is*

$$\psi(u) = \Pr(T < \infty | V(0) = u) = \psi_d(u) + \psi_s(u)$$

where $\psi_d(u) = \Pr(T < \infty \text{ and } V(T) = 0 \,|V(0) = u)$ *is the probability of ruin caused by oscillation and* $\psi_s(u) = \Pr(T < \infty \text{ and } V(T) < 0 \,|V(0) = u)$ *is the probability that ruin occurs due to a claim.*

The survival probability is denoted as $\delta(u) = 1 - \psi(u)$. Due to the oscillating nature of the process we have that $\psi_d(0) = \psi(0) = 1$ (see Appendix E). In Dufresne and Gerber (1991) important defective renewal equations for $\delta(u)$, $\psi(u)$, $\psi_d(u)$ and $\psi_s(u)$ are derived.

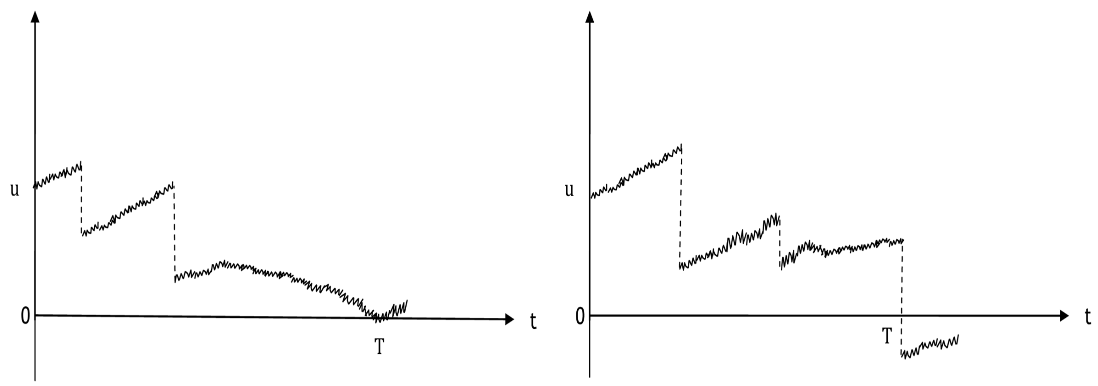

Figure 1.1: The two types of ruin, due to oscillation and because of a claim.

Theorem 1 *The survival probability for the process* $V(t)$ *obeys to the following defective renewal equation:*

$$\delta(u) = qH_1(u) + (1-q)\int_0^u \delta(u-x)h_1 * h_2(x)dx, \quad u \geq 0, \tag{1.1}$$

were $q = 1 - \left(\frac{\lambda}{c}\right) p_1 \Leftrightarrow q = \frac{\theta}{1+\theta}$ *and* $h_1 * h_2(.)$ *is the convolution of the following density functions, with the respective distribution functions denoted* $H_1(.)$ *and* $H_2(.)$*,*

$$\begin{aligned} h_1(x) &= \zeta e^{-\zeta x},\, x > 0 \\ h_2(x) &= p_1^{-1} \left[1 - P(x)\right],\, x > 0 \\ \zeta &= 2c/\sigma^2. \end{aligned}$$

■

Equation (1.1) has some components that are well known from the classical model given by $U(t)$. The expression for q is the survival probability with no initial reserve ($u = 0$) and $h_2(.)$ is the density function of the individual records from the aggregate loss.

Theorem 2 *The probability of ruin caused by oscillation and by claim, for the process* $V(t)$*, obey to the following defective renewal equations, respectively:*

$$\psi_d(u) = 1 - H_1(u) + (1-q) \int_0^u \psi_d(u-x) h_1 * h_2(x) dx,\, u \geq 0, \tag{1.2}$$

$$\begin{aligned} \psi_s(u) &= (1-q)\left[H_1(u) - H_1 * H_2(u)\right] + \\ &+ (1-q) \int_0^u \psi_s(u-x) h_1 * h_2(x) dx,\, u \geq 0, \end{aligned} \tag{1.3}$$

where q*,* $H_1(.)$ *and* $H_2(.)$ *are the same as defined above.* ■

Theorem 3 *The ruin probability for the process* $V(t)$ *obeys to the following defective renewal equation:*

$$\begin{aligned} \psi(u) &= q\left[1 - H_1(u)\right] + (1-q)\left[1 - H_1 * H_2(u)\right] + \\ &+ (1-q) \int_0^u \psi(u-x) h_1 * h_2(x) dx,\, u \geq 0, \end{aligned} \tag{1.4}$$

where q*,* $H_1(.)$ *and* $H_2(.)$ *are the same as defined above.*

Proof. Using the relation, $\psi(u) = 1 - \delta(u)$ in (1.1) or summing (1.2) with (1.3). ■

These equations will have an important role in the approximations presented in the next chapters. For a brief idea about renewal equations and renewal theory, please see Appendix D.

1.3 The adjustment coefficient and a simple upper bound

Using a martingale argument, Dufresne and Gerber (1991) defined the adjustment coefficient and an upper bound similar to Lundberg's inequality. The adjustment coefficient plays an important role in the asymptotic formulas for the ruin probabilities, which also happens in the classical model. In what follows we assume that the moment generating function of the claim amount distribution, $M_X(s) = E[e^{sX}]$, exists for $-\infty < s < \eta$ and that $\lim_{s \to \eta} M_X(s) = +\infty$.

Definition 3 *The adjustment coefficient R is the only positive root of the equation:*

$$\lambda M_X(R) + \frac{\sigma^2}{2} R^2 = \lambda + cR, \;\; r < \eta \tag{1.5}$$

We can see that the equation has only one positive root, letting $h(r) = rc - \lambda M_X(r) + \lambda - \frac{\sigma^2}{2} r^2 \Rightarrow h(0) = 0$. $h^{'}(r) = c - \lambda M^{'}_X(r) - \sigma^2 r \Rightarrow h^{'}(0) = c - \lambda p_1 > 0$, by the hypotheses of the model. $h^{''}(r) = -\lambda M^{''}_X(r) - \sigma^2 = -\lambda E[X^2 e^{rX}] - \sigma^2 < 0$. So $h(r)$ is concave and as $\lim_{r \to \eta} h(r) = -\infty \Rightarrow$ the equation $h(r) = 0$ has two roots, one is $r = 0$ and the other is the positive root R.

With this result it is also possible to derive a simple upper bound for the ruin probability.

Theorem 4

$$\psi(u) < e^{-Ru}, \;\; u > 0$$

where R is the adjustment coefficient and u is the initial reserve. ■

1.4 Exact ruin probabilities for mixture of exponentials

As in the classical model, the exact ruin probability can be calculated when the claim amount distribution is exponential or mixture of exponentials.

Theorem 5 *If the claim amount distribution has probability density function of the form*

$$p(x) = \sum_{i=1}^{n} A_i \beta_i e^{-\beta_i x}, \;\; with \; \sum_{i=1}^{n} A_i = 1 \; for \; x > 0 \tag{1.6}$$

then the exact ultimate ruin probability must be of the form

$$\psi(u) = \sum_{k=1}^{n+1} C_k e^{-r_k u}, u \geq 0 \tag{1.7}$$

where

$$C_h = \prod_{i=1}^{n} \left(\frac{r_h - \beta_i}{\beta_i} \right) \prod_{\substack{k=1 \\ k \neq h}}^{n+1} \left(\frac{r_k}{r_h - r_k} \right), \ for\ h = 1, \ldots, n+1 \tag{1.8}$$

with $\sum_{h=1}^{n+1} C_h = 1$ *and* $r_{1,} \ldots, r_{n+1}$ *being the solutions of the equation*

$$\lambda \sum_{i=1}^{n} \frac{A_i}{\beta_i - r} + \frac{\sigma^2}{2} r = c \tag{1.9}$$

■

Of course that if $n = 1$, we have the case where the claim size distribution is exponential with parameter β. Under the same conditions it is also possible to obtain the exact ruin probability caused by oscillation.

Theorem 6 *If the claim amount distribution has probability density function of the form* (1.6)*, then the exact ruin probability by oscillation must be of the form*

$$\psi_d(u) = \sum_{k=1}^{n+1} C_k^d e^{-r_k u}, u \geq 0 \tag{1.10}$$

with

$$C_h^d = \frac{r_h}{q\zeta} C_h = \frac{\prod_{i=1}^{n} (r_h - \beta_i)}{\prod_{k=1, k \neq h}^{n} (r_h - r_k)}, \ for\ h = 1, \ldots, n+1 \tag{1.11}$$

where $r_{1,} \ldots, r_{n+1}$ *are the solutions of equation* (1.9)*.* ■

The probability of ruin caused by a claim can be simply calculated using the relation $\psi_s(u) = \psi(u) - \psi_d(u)$. This decomposition can allow us to see the contribution of claims and oscillations to the occurrence of ruin. It is expectable that under the same perturbation by diffusion (same σ), the weight of the perturbation to the occurrence of ruin will be lower in a heavy tail distribution than in a light one.

1.5 Maximal aggregate loss

Considering the process $V(t)$, we can construct the process of aggregate loss at time t, $\{L(t), t \geq 0\}$:

$$L(t) = S(t) - ct - \sigma W(t) \Leftrightarrow V(t) = u - L(t) \tag{1.12}$$

and we can define the maximal aggregate loss,

Definition 4 *Let $\{L(t), t \geq 0\}$ be the process of aggregate loss,* (1.12)*, then the maximal aggregate loss is the random variable L such that $L = \max\{L(t), t \geq 0\}$.*

We can express the survival probability in terms of L. As $L(0) = 0 \Rightarrow L \geq 0$,

$$\begin{aligned}\Pr[L &\leq u] = \Pr[L(t) \leq u, \forall t \geq 0] = \Pr[S(t) - ct - \sigma W(t) \leq u, \forall t \geq 0] \\ &= \Pr[V(t) \geq 0, \forall t \geq 0] = \delta(u)\end{aligned}$$

This way we can see that the survival probability is the distribution function of the random variable L. Contrary to the classical model, $\delta(u)$ does not have a mass point at zero, because $\Pr[L \leq 0] = \delta(0) = 0$.

Dufresne and Gerber (1991) decomposed the maximal aggregate loss random variable into two kinds of "record highs". Letting M be the number of records of the process $\{L(t)\}$ that are caused by claims and letting $t_1, \ldots, t_M$ denote the times when these claims occur with $t_0 = 0$ and $t_{M+1} = +\infty$, we can define:

$$L_i^{(1)} = \max\{L(t), t < t_{i+1}\} - L(t_i) \ for\ i = 0, 1, \ldots, M$$

and

$$L_i^{(2)} = L(t_i) - L(t_{i-1}) - L_{i-1}^{(1)} \ for\ i = 1, \ldots, M$$

where $\max\{L(t), t < t_{i+1}\}$ is the record due to the Wiener process that occurs before the time t_{i+1}. The random variables $L_i^{(1)}$ and $L_i^{(2)}$ represent the amounts that result into the $(i+1)$-th and the i-th record highs of the aggregate loss process $\{L(t)\}$ due

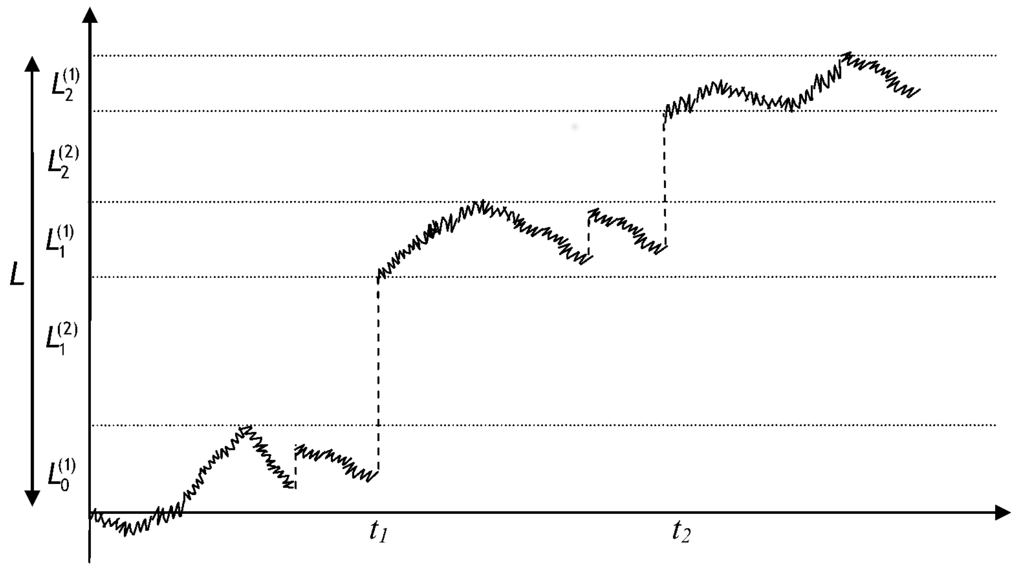

Figure 1.2: Decomposition of the maximal aggregate loss.

to oscillation and a claim, respectively, as it can be seen in figure 1.2. These two types of random variables allow us to write the maximal aggregate loss as:

$$L = L_0^{(1)} + L_1^{(1)} + L_1^{(2)} + \ldots + L_M^{(1)} + L_M^{(2)} = L_0^{(1)} + \sum_{i=1}^{M}\left(L_i^{(1)} + L_i^{(2)}\right) \tag{1.13}$$

Note that $L = L_0^{(1)}$ if $M = 0$. The main idea behind this decomposition is that a certain record is simply the sum of the differences between successive records until we obtain the "maximal record".

Theorem 7 *$\{L_i^{(1)}\}_{i=0,\ldots,M}$ and $\{L_i^{(2)}\}_{i=1,\ldots,M}$ are two sequences of independent and identically distributed random variables with common probability density functions $h_1(x)$ and $h_2(x)$, respectively and also independent from M. So the distribution functions of each $L_i^{(1)}$ and $L_i^{(2)}$ are,*

$$\begin{aligned}\Pr[L_i^{(1)} &\leq x] = H_1(x) = 1 - e^{-\zeta x}, x \geq 0 \\ \Pr[L_i^{(2)} &\leq x] = H_2(x) = p_1^{-1}\int_0^x [1 - P(y)]\, dy, x \geq 0\end{aligned}$$

■

Theorem 8 *The random variable M is geometrically distributed,*

$$P[M = m] = q(1-q)^m \; for \; m = 0, 1, \ldots$$

where q represents the probability that there are no record highs that are caused by a claim. This implies that $\delta(u)$ is a compound geometric distribution,

$$\delta(u) = \sum_{m=0}^{\infty} q(1-q)^m H_1^{*(m+1)} * H_2^{*m}(u) \tag{1.14}$$

where q, $H_1(.)$ and $H_2(.)$ are the same as defined above. ■

The convolution formula (1.14) generalizes a similar result in the classical model. Theoretically (1.14) can allow us to calculate the ruin probability but in practice this is often impossible due to the calculation of the convolutions of $H_1(.)$ and $H_2(.)$ with themselves and the convolution between them. However, this convolution formula is the base for one of the approximation methods presented below.

We can calculate the moments of L, if they exist. The expected value and variance were easily deducted by Jacinto (2008) considering that $L = L_0^{(1)} + \sum_{i=1}^{M}\left(L_i^{(1)} + L_i^{(2)}\right)$ is a compound distribution,

$$E[L] = E[L_0^{(1)}] + E[M]\left(E[L_i^{(1)}] + E[L_i^{(2)}]\right)$$

$$Var[L] = Var[L_0^{(1)}] + E[M]\left(Var[L_i^{(1)}] + Var[L_i^{(2)}]\right) + Var[M]\left(E[L_i^{(1)}] + E[L_i^{(2)}]\right)^2$$

Noting that

$$E[L_i^{(1)}] = \tfrac{1}{\zeta}, \quad Var[L_i^{(1)}] = \tfrac{1}{\zeta^2}, \quad E[L_i^{(2)k}] = \tfrac{p_{k+1}}{(k+1)p_1}, \quad Var[L_i^{(2)}] = \tfrac{p_3}{3p_1} - \left(\tfrac{p_2}{2p_1}\right)^2$$

And that

$$E[M] = \tfrac{1-q}{q} \quad \text{and} \quad Var[M] = \tfrac{1-q}{q^2}$$

Using the fact that $q = 1 - \left(\frac{\lambda}{c}\right) p_1$ and $\zeta = \frac{2c}{\sigma^2}$ we arrive to the following expressions,

$$E[L] = \frac{\sigma^2}{2c} + \frac{\lambda p_1\left(\frac{\sigma^2}{2c} + \frac{p_2}{2p_1}\right)}{c\left(1 - \frac{\lambda}{c}p_1\right)} \tag{1.15}$$

$$Var[L] = \frac{\sigma^4}{4c^2} + \frac{\lambda p_1\left(\frac{\sigma^2}{2c} + \frac{p_2}{2p_1}\right)^2}{c\left(1 - \frac{\lambda}{c}p_1\right)^2} + \frac{\lambda p_1\left(\frac{\sigma^4}{4c^2} - \frac{p_2^2}{4p_1^2} + \frac{p_3}{3p_1}\right)}{c\left(1 - \frac{\lambda}{c}p_1\right)} \tag{1.16}$$

Formulas (1.15) and (1.16) allow us to obtain this two moments in a easy way, using only values that are known from the beginning, if they exist. To obtain high-order moments we can use the moment generating function.

Theorem 9

$$M_L(s) = \frac{s\zeta\,(c-\lambda p_1)}{\lambda\zeta + cs\,(\zeta - s) - \lambda\zeta M_X(s)} \tag{1.17}$$

Proof. See Appendix E. ■

1.6 Asymptotic results

The three defective renewal equations (1.2), (1.3) and (1.4) have a possible asymptotic solution if we apply renewal theory techniques such as the ones presented in Appendix D. The solutions for the ruin probabilities demand the existence of the adjustment coefficient and so we can only consider its use when the moment generating function of X exists. These asymptotic solutions will be the base for one of the approximations presented below.

Theorem 10 *If the adjustment coefficient, R, exists, then as $u \to \infty$ we have,*

$$\psi_d(u) \sim C^d e^{-Ru} \tag{1.18}$$

$$\psi_s(u) \sim C^s e^{-Ru} \tag{1.19}$$

$$\psi(u) \sim C e^{-Ru} \tag{1.20}$$

where,

$$C^d = \frac{\int_0^\infty e^{Rx}\,[1-H_1(x)]\,dx}{(1-q)\int_0^\infty xe^{Rx} h_1 * h_2(x)dx}$$

$$C^s = \frac{\int_0^\infty e^{Rx}(1-q)\,[H_1(x) - H_1 * H_2(x)]\,dx}{(1-q)\int_0^\infty xe^{Rx} h_1 * h_2(x)dx}$$

and

$$C = C^d + C^s = \frac{\int_0^\infty e^{Rx}\,[q\,[1-H_1(x)] + (1-q)\,[1-H_1 * H_2(x)]]}{(1-q)\int_0^\infty xe^{Rx} h_1 * h_2(x)dx}$$

Proof. See Appendix E. ■

2 Approximation methods

2.1 De Vylder's approximation

The main idea behind this approximation is to use the existence of a closed formula for the ruin probability when the individual claim amount distribution is exponential. This simple and practical approximation was originally proposed for the classical model by De Vylder (1978) where he replaces the original surplus process, $U(t)$, by another one, let's say, $U^*(t) = u + c^*t - S^*(t)$, matching the first three moments. This new process is characterized by the fact that $S^*(t)$ is a compound Poisson distribution with parameter λ^* and the claim amount distribution is exponential with parameter β. The third parameter that characterize the process is c^*, the new rate at which premiums are received. Equating the moments of the two processes we can find the expressions for the new parameters.

In the context of the perturbed model, a first adaptation of this approximation can be seen in Silva (2006) and some numerical illustrations can be seen at Jacinto (2008). The idea is similar to the original approximation but in the perturbed model we have an extra parameter, σ^2. Considering the process $V^*(t) = U^*(t) + \sigma^* W(t)$, we can find the parameters β, λ^*, c^* and $\sigma^{*\,2}$ by equating the corresponding four central moments of $V(t)$ and $V^*(t)$. The moments of the process $V(t)$ can be seen in Appendix A and as the first four raw moments of the exponential distribution are $\frac{1}{\beta}$, $\frac{2}{\beta^2}$,$\frac{6}{\beta^3}$ and $\frac{24}{\beta^4}$, the system to be solved is:

$$\begin{cases} u + ct - \lambda t p_1 = u + c^* t - \lambda^* t \frac{1}{\beta} \\ \sigma^2 t + \lambda t p_2 = \sigma^{*\,2}\, t + \lambda t \frac{2}{\beta^2} \\ -\lambda t p_3 = -\lambda^* t \frac{6}{\beta^3} \\ \lambda t p_4 + 3\lambda^2 t^2 p_2^2 + 6\lambda t^2 p_2 \sigma^2 + 3\sigma^4 t^2 = \lambda^* t \frac{24}{\beta^4} + 3\lambda^{*2} t^2 \left(\frac{2}{\beta^2}\right)^2 + 6\lambda^* t^2 \frac{2}{\beta^2}\, \sigma^{*\,2} + 3\, \sigma^{*\,4}\, t^2 \end{cases}$$

Which leads to the following solution,

$$\beta = 4\frac{p_3}{p_4}; \qquad \lambda^* = 32\lambda \frac{p_3^4}{3p_4^3}; \qquad c^* = 8\lambda \frac{p_3^3}{3p_4^2} + c - \lambda p_1; \qquad \sigma^{*\,2} = \sigma^2 + \lambda p_2 - 4\lambda \frac{p_3^2}{3p_4}.$$

As we can see by the final expressions of the parameters this method only requires the existence of the first four raw moments of the original claim amount distribution, which is not very restrictive when comparing it with more complex methods.

Having this parameters we can now apply formulas (1.7) and (1.8) in order to obtain the approximation, $\psi_{DV}(u)$, given by,

$$\psi_{DV}(u) = C_1 e^{-r_1} + C_2 e^{-r_2}, \qquad C_1 = \tfrac{r_1-\beta}{\beta}\tfrac{r_2}{r_1-r_2}, \qquad C_2 = \tfrac{r_2-\beta}{\beta}\tfrac{r_1}{r_2-r_1}\ ,$$

where r_1 and r_2 are the solutions of the following equation

$$\frac{\sigma^{*\,2}}{2}r + \frac{\lambda^*}{\beta - r} = c^*.$$

With this method we can also obtain a simple approximation to the ruin probability due to oscillation, $\psi_d(u)$. Denoting the approximation as $\psi_{d,DV}(u)$ and applying formulas (1.10) and (1.11), we obtain,

$$\psi_{d,DV}(u) = C_1^d e^{-r_1} + C_2^d e^{-r_2}, \qquad C_1^d = \tfrac{r_1}{q^*\zeta^*}C_1, \qquad C_2^d = \tfrac{r_2}{q^*\zeta^*}C_2\ ,$$

where r_1 and r_2 are the same as for the case of $\psi_{DV}(u)$ but now q^* and ζ^* are given by,

$$q^* = 1 - \tfrac{\lambda^*}{c^*}\tfrac{1}{\beta}, \qquad \zeta^* = 2\tfrac{c^*}{\sigma^{*\,2}}$$

The approximation to the ruin probability due to a claim can be obtained simply using $\psi_{s,DV}(u) = \psi_{DV}(u) - \psi_{d,DV}(u)$.

2.2 Dufrene & Gerber's upper and lower bounds

Since that the direct calculation of the ruin probability through formula (1.14) is most of the times impossible to obtain due to the calculation of the convolutions, Silva (2006) proposed a method to obtain bounds for the ruin probability that generalizes another one proposed for the classical model by Dufresne and Gerber (1989). The original method is based on defining appropriate discrete distributions to replace on the classical convolution formula for the survival probability and then, using the fact that $\delta(u)$ is a compound geometric distribution, apply the well know recursive Panjer's formula proposed by Panjer (1981). For the perturbed model we can use a similar method now based on formula (1.14). With this method we can obtain an upper and lower bound to the ruin probability because we use two types of discretization.

Following the methods by Dufresne and Gerber (1989), Silva (2006) starts by defining discrete random variables based on the decomposition of the maximal aggregate loss,

$$\begin{aligned} L^j &= L_0^{j,(1)} + \sum_{i=1}^{M} \left(L_i^{j,(1)} + L_i^{j,(2)} \right), \\ \text{with } L^j &= L_0^{j,(1)} \text{ if } M = 0 \text{ and } j = l, u \end{aligned}$$

with

$$\begin{aligned} L_i^{l,(k)} &= \vartheta \left[\frac{L_i^{(k)}}{\vartheta} \right] \quad \vartheta \epsilon (0;1) \\ L_i^{u,(k)} &= \vartheta \left[\frac{L_i^{(k)} + \vartheta}{\vartheta} \right] \quad \vartheta \epsilon (0;1) \end{aligned}$$

for $\{k = 1, i = 0, ..., M\}$ and for $\{k = 2, i = 1, ..., M\}$. $[x]$ represents the integer part of x.

Thus the idea is to round the summands in (1.13) to the next lower multiple of ϑ which gives L^l and to the next higher multiple of ϑ, which gives L^u. Clearly,

$$L^l \leq L \leq L^u$$

which implies that

$$\Pr(L^l \geq u) \leq \psi(u) \leq \Pr(L^u \geq u)$$

This discretization of L implies that the probability functions of the discrete random variables $L_i^{l,(1)}, L_i^{l,(2)}, L_i^{u,(1)}$ and $L_i^{u,(2)}$ are, respectively:

$$\begin{aligned} h_{1,k}^l &= \Pr\left(L_i^{l,(1)} = k\vartheta \right) = H_1(k\vartheta + \vartheta) - H_1(k\vartheta), \;\; k = 0, 1, ... \\ h_{2,k}^l &= \Pr\left(L_i^{l,(2)} = k\vartheta \right) = H_2(k\vartheta + \vartheta) - H_2(k\vartheta), \;\; k = 0, 1, ... \;, \end{aligned}$$

$$\begin{aligned} h_{1,k}^u &= \Pr\left(L_i^{u,(1)} = k\vartheta \right) = H_1(k\vartheta) - H_1(k\vartheta - \vartheta), \;\; k = 1, 2, ... \\ h_{2,k}^u &= \Pr\left(L_i^{u,(2)} = k\vartheta \right) = H_2(k\vartheta) - H_2(k\vartheta - \vartheta), \;\; k = 1, 2, ... \end{aligned}$$

Where $H_1(.)$ and $H_2(.)$ are the same as defined above. Note that $h^l_{1,k}$ is the probability of $L^{l,(1)}_i$ being equal to $k\vartheta$, i.e., that $L^{(1)}_i$ is between $k\vartheta$ and $k\vartheta+\vartheta$. Similar interpretations can be done to $h^l_{2,k}, h^u_{1,k}$ and $h^u_{2,k}$.

If we write the probability functions of L^l and L^u as

$$f^j_k = \Pr\left(L^j = k\vartheta\right), \ k = 0, 1, ... \text{ for } j = l, u.$$

we have then the following bounds for $\psi(.)$

$$1 - \sum_{k=0}^{m-1} f^l_k \leq \psi(m\vartheta) \leq 1 - \sum_{k=0}^{m} f^u_k, \quad m = 0, 1, ...u/\vartheta, \quad u = 0, 1, ..$$

The calculation of these bounds is possible using the Panjer's recursion formula for the compound geometric distribution (see Panjer (1981)). We arrive then to the following distributions of f^l_k and f^u_k (please see Silva (2006) for further details),

$$f^l_k = \begin{cases} \frac{qh^l_{1,0}}{1-(1-q)h^l_{1,0}h^l_{2,0}}, \ k = 0 \\ \frac{1}{1-(1-q)h^l_{1,0}h^l_{2,0}} \begin{bmatrix} qh^l_{1,k} + (1-q)\, h^l_{1,0} \sum_{j=1}^{k} f^l_{k-j} h^l_{2,j} \\ + (1-q) \sum_{i=1}^{k} \sum_{j=0}^{k-i} f^l_{k-i-j} h^l_{1,i} h^l_{2,j} \end{bmatrix}, \ k = 1, 2, ... \end{cases}$$

and

$$f^u_k = \begin{cases} 0, \ k = 0 \\ qh^u_{1,1}, \ k = 1 \\ qh^u_{1,k} + (1-q) \sum_{i=1}^{k-1} \sum_{j=1}^{k-i} f^u_{k-i-j} h^u_{1,i} h^u_{2,j}, \ k = 2, 3, ... \end{cases}$$

This method is very useful to test the accuracy of other approximations for the cases where we do not have exact results for the ruin probability.

2.3 Beekman-Bowers' approximation

This approximation is a modification of an approximation presented for the classical model by Beekman (1969) and the main idea is to use a gamma distribution in the renewal equation for the survival probability. In the classical model this equation is given by $\delta(u) = \delta(0) + \psi(0) \int_0^u \delta(u-x) \frac{[1-P(x)]}{p_1} dx$, please see for instance Dickson (2005) for further details. What the Beekman-Bowers' approximation does in this case is to approximate the survival probability by $\delta_{BB}(u) = \delta(0)+$

$\psi(0)\int_0^u \frac{\beta}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}dx$. The integral in this expression represents now a cumulative distribution function of a gamma(α,β) at point u. The parameters of this gamma are chosen by matching the moments of the r.v. L with the moments of $\delta_{BB}(.)$.

To adapt this approximation to the perturbed model, instead of replacing the whole integral by a gamma distribution in (1.1), as presented by Jacinto (2008), we start with the idea of replacing the convolution $\delta * h_2(.)$ by a cumulative distribution function of a gamma(α,β). Using the properties of convolutions and defining $H_3(u)=\int_0^u \frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}dx$, we can write the approximation as,

$$\begin{aligned}\delta(u) &= qH_1(u)+(1-q)h_1*\delta*h_2(u)\\ &\Rightarrow \delta_{BB}(u)=qH_1(u)+(1-q)h_1*H_3(u)\end{aligned}\tag{2.1}$$

In order to obtain α and β, we can still use the relation between L and $\delta(.)$. We can find the moments of this new approximation applying the Laplace transform on (2.1) and equating them with the moments of L. The definition and some useful properties of the Laplace transform can be seen in Appendix C.

Taking Laplace transforms on (2.1) we obtain,

$$\overline{\delta}_{BB}(s)=q\overline{h}_1(s)+(1-q)\overline{h}_1(s)\overline{h}_3(s)$$

where

$$\overline{\delta}_{BB}(s)=\int_0^\infty e^{-sy}d\delta_{BB}(y),\qquad \overline{h}_1(s)=\int_0^\infty e^{-sy}h_1(y)dy,\qquad \overline{h}_3(s)=\int_0^\infty e^{-sy}dH_3(y)$$

Note that the Laplace transform here is calculated using the density functions, in order to be possible to calculate the moments. As $h_1(y)$ is the density function of an exponential distribution with parameter ζ, the corresponding Laplace transform is given by $\overline{h}_1(s)=\frac{\zeta}{\zeta+s}$. For the gamma density, the Laplace transform is given by $\overline{h}_3(s)=\left(\frac{\beta}{\beta+s}\right)^\alpha$. So $\overline{\delta}_{BB}(s)$ can be written as,

$$\overline{\delta}_{BB}(s)=q\frac{\zeta}{\zeta+s}+(1-q)\frac{\zeta}{\zeta+s}\left(\frac{\beta}{\beta+s}\right)^\alpha\tag{2.2}$$

Taking the first two derivatives on (2.2) in order to s and evaluating them at point

0 we arrive to,

$$\begin{aligned}
\frac{d}{ds}_{|s=0}\overline{\delta}_{BB}(s) &= -\frac{q}{\zeta} - (1-q)\frac{\beta+\alpha\zeta}{\beta\zeta} \\
\frac{d^2}{ds^2}_{|s=0}\overline{\delta}_{BB}(s) &= \frac{2q}{\zeta^2} + (1-q)\frac{2\beta^2+2\alpha\beta\zeta+\alpha(1+\alpha)\zeta^2}{\beta^2\zeta^2}
\end{aligned}$$

Using the relation $Var[L] = E[L^2] - E^2[L]$ and equating these two new expressions with the expressions for $E[-L]$ and $E[L^2]$ (see (1.15) and (1.16)) we can find α and β and approximate the ruin probability using (2.1). This leads us to the idea that this approximation might not be as good as the De Vylders's approximation since there we are matching four moments and here we are only matching two.

2.4 Tijms' approximation

Tijms (1994) proposed an approximation that takes advantage of the asymptotic behaviour for large u and smooth it for small u. This approximation was originally presented in the context of queuing theory but it has been adapted by many authors for the ruin probability in the classical model. See for instance Dickson (2005) or Klugman *et al.* (2008).

The idea is to add an exponential term to (1.20) in order to improve the accuracy for small u. We define the approximation as

$$\psi_T(u) = Ce^{-Ru} + Ae^{-Su}, u \geq 0$$

where A is chosen in such a way that the probability with no initial reserve matches the exact probability, i.e., $\psi(0) = \psi_T(0)$. As $\psi(0) = 1$, we have $A = (1-C)$. Also, as $\psi(.)$ is the survival function of L, S is chosen in order that $\int_0^\infty \psi_T(u)du = E[L]$. So we have,

$$E[L] = \frac{C}{R} + \frac{(1-C)}{S} \Leftrightarrow S = \frac{R\,(1-C)}{RE[L]-C}$$

This approximation can only be used for the cases where the moment generating function of X exists, in order to exist the adjustment coefficient. It is an approximation easy to set and understand, the major problem is the computation of C.

The approximation is likely to produce good results if the choice of S through the expected value of L is enough to smooth the asymptotic result for small values of u.

2.5 Fourier transform and ruin

This section is inspired on a previous work by Lima *et al.* (2002) where the authors used the Fourier transform to obtain ruin probabilities in the classical model. Here we present the transform, some properties, the key results used by them and adapt the methodology to the perturbed model.

We must note that this method is not an approximation like the others presented above. Ideally, with the expressions developed here we should be able to compute exact ruin probabilities but we obtain differences to the exact values due to the accumulation of numerical errors in the computations and not due to the lack of suitability of the method to a particular claim amount distribution.

2.5.1 Fourier transform

The Fourier transform of a continuous function, $f(x)$, defined for $x \geq 0$ whose integral exists for all $x > 0$ is defined as

$$\phi_{f(x)}(s) = \int_0^{+\infty} e^{isx} f(x)dx$$

where $i^2 = -1$. We note that if $f(.)$ is a density function then, $\phi_{f(.)}(s)$ is the corresponding characteristic function. The transform has some useful properties that are used in this work. They are presented below and the proofs can be seen for instance at Poularikas (1996).

Property 1 Let $f(.)$ and $g(.)$ be defined on $\Re_0^+$ as above and $h(x) = af(x) + bg(x)$, where a and b are two constants. Then

$$\phi_{h(x)}(s) = a\phi_{f(x)}(s) + b\phi_{g(x)}(s)$$

Property 2 Let $F(.)$ be defined on $\Re_0^+$ as above, $\lim_{x\to\infty} F(x) = 1$ and $f(x) = F'(x)$. Then

$$\phi_{1-F(x)}(s) = \frac{i(1-F(0))}{s} - \frac{i}{s}\phi_{f(x)}(s)$$

Property 3 Let $\{f_j(.)\}_{j=1}^n$ be functions defined on $\Re_0^+$ as above, and let $h(.)$ be the n-th convoluted function, $h(x) = f_1 * f_2 * \cdots * f_n(x)$. Then

$$\phi_{h(x)}(s) = \prod_{i=1}^{n} \phi_{f_i(x)}(s)$$

Another interesting fact about the Fourier transform is that it can always be splitted into two parts. The real part and the complex part, which can be verified through Euler's formula, $e^{ix} = \cos(x) + i\sin(x)$. We can then write,

$$\phi_{f(x)}(s) = \phi^r_{f(x)}(s) + i\phi^c_{f(x)}(s) = \int_0^{+\infty} \cos(sx)f(x)dx + i\int_0^{+\infty} \sin(sx)f(x)dx$$

where $\phi^r_{f(x)}(s)$ represents the real part of the transform and $\phi^c_{f(x)}(s)$ the complex part. Also, considering the Fourier cosine transform,

$$\varphi(s) = \int_0^{\infty} \cos(sx)f(x)dx$$

we can recover the integral of the original function with the inverse transform:

$$\int_0^x f(y)dy = \frac{2}{\pi}\int_0^{\infty} \frac{\sin(xs)}{s}\varphi(s)ds$$

where $f(x)$ is a continuous non-negative function defined on $\Re^+$, whose integral exists for all $x > 0$. From this well known result we can write the following theorem.

Theorem 11 *Let $f(x)$ be a continuous non-negative function defined on $\Re_0^+$ whose integral exists for all $x > 0$, and $f(x) = F'(x)$. Then,*

$$F(x) = F(0) + \frac{2}{\pi}\int_0^{\infty} \frac{\sin(xs)}{s}\phi^r_{f(x)}(s)ds \tag{2.3}$$

■

The proof of this result can be seen at Poularikas (1996), for instance. The fact that we can obtain the cosine transform from the Fourier transform, which corresponds to the real part of the transform, makes (2.3) a very useful method to obtain ruin probabilities. The major problem when applying this method is that the integrand function in (2.3) can be a rapidly oscillating function, which leads to a series of numerical problems when computing the integral. Lima *et al.* (2002) solved this problem using what they called the *dicotomic approach algorithm*, which revealed extremely good results at the time. In our case we used the *Mathematica* package to invert the Fourier transform, which was capable of reproduce most of the original figures.

2.5.2 Applying the inversion formula

In order to use formula (2.3) to obtain ruin probabilities we have to obtain first expressions for $\psi'_d(u)$, $\psi'_s(u)$ and $\psi'(u)$. Starting from (1.2), (1.3) and (1.4) and using the Leibniz's differentiation rule we obtain respectively,

$$\begin{aligned}
\psi'_d(u) &= -h_1(u) + (1-q)h_1 * h_2(u) + (1-q)\int_0^u \psi'_d(u-x)h_1 * h_2(x)dx \\
\psi'_s(u) &= (1-q)h_1(u) - (1-q)h_1 * h_2(u) + (1-q)\int_0^u \psi'_s(u-x)h_1 * h_2(x)dx \\
\psi'(u) &= -qh_1(u) + (1-q)\int_0^u \psi'(u-x)h_1 * h_2(x)dx
\end{aligned}$$

Applying the Fourier transform to these expressions we obtain, using Properties 1 and 3,

$$\begin{aligned}
\phi_{\psi'_d(u)}(s) &= \frac{-\phi_{h_1(u)}(s) + (1-q)\phi_{h_1(u)}(s)\phi_{h_2(u)}(s)}{1-(1-q)\phi_{h_1(u)}(s)\phi_{h_2(u)}(s)} \\
\phi_{\psi'_s(u)}(s) &= \frac{(1-q)\phi_{h_1(u)}(s) - (1-q)\phi_{h_1(u)}(s)\phi_{h_2(u)}(s)}{1-(1-q)\phi_{h_1(u)}(s)\phi_{h_2(u)}(s)} \\
\phi_{\psi'(u)}(s) &= \frac{-q\phi_{h_1(u)}(s)}{1-(1-q)\phi_{h_1(u)}(s)\phi_{h_2(u)}(s)}
\end{aligned}$$

These expressions can be rearranged according to $\phi_{h_1(u)}(s)=\phi^r_{h_1(u)}(s)+i\phi^c_{h_1(u)}$, and using Property 2, $\phi_{h_2(u)}(s)=\frac{i}{sp_1}-\frac{\phi_{p(u)}(s)}{sp_1}i=\frac{\phi^c_{p(u)}(s)}{sp_1}+\frac{(1-\phi^r_{p(u)}(s))}{sp_1}i$, we obtain:

$$\begin{aligned}(1-q)\phi_{h_1(u)}(s)\phi_{h_2(u)}(s) &= \frac{1-q}{sp_1}\left(\phi^r_{h_1(u)}(s)\phi^c_{p(u)}(s)-\phi^c_{h_1(u)}(s)+\phi^c_{h_1(u)}(s)\phi^r_{p(u)}(s)\right)\\ &\quad +i\frac{1-q}{sp_1}\left(\phi^r_{h_1(u)}(s)-\phi^r_{h_1(u)}(s)\phi^r_{p(u)}(s)+\phi^c_{h_1(u)}(s)\phi^c_{p(u)}(s)\right)\\ &= \frac{1-q}{sp_1}J+i\frac{1-q}{sp_1}I\end{aligned}$$

Where J and I correspond to the expressions in brackets. This leads to,

$$\phi_{\psi'_d(u)}(s) = \frac{-\phi^r_{h_1(u)}(s)-i\phi^c_{h_1(u)}+\frac{1-q}{sp_1}J+i\frac{1-q}{sp_1}I}{1-\frac{1-q}{sp_1}J-i\frac{1-q}{sp_1}I} \tag{2.4}$$

$$\phi_{\psi'_s(u)}(s) = \frac{(1-q)\,\phi^r_{h_1(u)}(s)+i\,(1-q)\,\phi^c_{h_1(u)}-\frac{1-q}{sp_1}J-i\frac{1-q}{sp_1}I}{1-\frac{1-q}{sp_1}J-i\frac{1-q}{sp_1}I} \tag{2.5}$$

$$\phi_{\psi'(u)}(s) = \frac{-q\phi^r_{h_1(u)}(s)-iq\phi^c_{h_1(u)}}{1-\frac{1-q}{sp_1}J-i\frac{1-q}{sp_1}I} \tag{2.6}$$

We must now split the transforms into the real and the complex parts in order to apply (2.3) and obtain expressions of the following type,

$$\psi(u)=\psi(0)+\frac{2}{\pi}\int_0^{+\infty}\frac{\sin(us)}{s}\phi^r_{\psi'(u)}(s)ds$$

If we do

$$A=-q\phi^r_{h_1(u)}(s) \quad B=-q\phi^c_{h_1(u)} \quad C=1-\tfrac{1-q}{sp_1}J \quad D=\tfrac{1-q}{sp_1}I$$

we can write (2.6) as,

$$\phi_{\psi'(u)}(s)=\frac{A+iB}{C-iD}=\frac{AC-BD+i(BC+AD)}{C^2+D^2}$$

and (2.3) becomes, in this case,

$$\psi_F(u)=\psi(0)+\frac{2}{\pi}\int_0^{+\infty}\frac{\sin(us)}{s}\frac{AC-BD}{C^2+D^2}ds \tag{2.7}$$

We present only the case for $\psi_F(u)$, but similar reasonings can be done to obtain expressions to $\psi_{d,F}(u)$ and $\psi_{s,F}(u)$, starting from (2.4) and (2.5), respectively. Nevertheless we present the numerical results for the three types of ruin.

3 Numerical illustrations

For purposes of testing the accuracy of the different methods we present here some numerical examples. In all of them we are going to assume that the premium rate is equal to $c = 2$, every individual claim amount distribution has mean $p_1 = 1$, the Poisson parameter is $\lambda = 1$ and the volatility is also $\sigma = 1$. This leads to $\zeta = 4$, which implies that $H_1(.)$, $\phi^r_{h_{1(.)}}(.)$ and $\phi^c_{h_{1(.)}}(.)$ are always equal to $H_1(x) = 1 - e^{-4x}$, $\phi^r_{h_{1(x)}}(s) = \frac{4^2}{4^2+s^2}$ and $\phi^c_{h_{1(x)}}(s) = \frac{4s}{4^2+s^2}$. $H_2(.)$, $\phi^r_{p(.)}(.)$ and $\phi^c_{p(.)}(.)$ assume different forms according to the individual claim amount distribution. All the figures were computed using the *Mathematica* software.

3.1 Exponential distribution

We consider here that the claim amount distribution follows an exponential distribution with parameter $\beta = 1$. So, $p(x) = e^{-x}$ and $P(x) = 1 - e^{-x}$. This way we get $H_2(x) = 1 - e^{-x}$. This distribution is considered to have a light tail, specially with such small parameter β. This fact makes us believe that the contribution of claims to the occurrence of ruin might be less important in this case than in the case of a heavier-tail distribution.

Table 3.1 shows the exact ruin probability and the exact ruin probability caused by oscillation. The third column shows the weight of this form of ruin in the total ruin probability. We can see that it has quite some impact when compared to the mixture of exponentials presented in the next section, that has a heavier tail.

Although we know the exact ruin probabilities we present the numerical results of the methods proposed during this work, to test the behaviour of the approximations in the exponential case. Table 3.2 shows the results for Beekman-Bowers', Tijms' and Fourier methods. The De Vylder's approximation is not presented because in this case it is reduced to the exact ruin probability and the bounds are not needed because we can use the exact probability to test the results. In order to be possible to approximate the ruin probability using the Fourier transform we must split $\phi_{p(x)}(s)$

u	$\psi(u)$ (I)	$\psi_d(u)$ (II)	$(II)/(I)$
0	1,00000	1,00000	100%
1	0,40470	0,09688	24%
2	0,25853	0,05676	22%
3	0,16674	0,03655	22%
4	0,10755	0,02358	22%
5	0,06938	0,01521	22%
6	0,04475	0,00981	22%
7	0,02887	0,00633	22%
8	0,01862	0,00408	22%
9	0,01201	0,00263	22%
10	0,00775	0,00170	22%
11	0,00500	0,00110	22%
12	0,00322	0,00071	22%
13	0,00208	0,00046	22%
14	0,00134	0,00029	22%
15	0,00087	0,00019	22%

Table 3.1: Exact figures, Exponential.

into the real and the complex part. In the exponential(β) case we have,

$$\phi^r_{p(x)}(s) = \frac{\beta^2}{\beta^2+s^2} \qquad \phi^c_{p(x)}(s) = \frac{\beta s}{\beta^2+s^2}$$

We can see by the ratio between (I) and (II) that the Beekman-Bowers approximation is, of the three, the worst. Tijms' approximation and the Fourier method revealed to be excellent approximations. This is mainly due to the fact that in this case $\phi^r_{p(x)}(s)$ and $\phi^c_{p(x)}(s)$ are non problematic functions to apply into (2.7) and so the software calculates the integral nice and easy, without accumulate to many numerical errors. Table 3.3 shows the application of the Fourier method for $\psi_d(u)$ and $\psi_s(u)$ which also revealed to be an excellent approximation.

3.2 Mixture of exponentials

The claim amount distribution of this example is taken from Wikstad (1971), case IIA. The distribution function of the mixture of exponentials presented there has the following form:

$$P(x) = 1 - 0.8881815e^{-5.514588x} - 0.1078392e^{-0.190206x} - 0.0039793e^{-0.014631x}$$

u	$\psi(u)$ (I)	$\psi_{BB}(u)$ (II)	$(I)/(II)$	$\psi_T(u)$ (III)	$(I)/(III)$	$\psi_F(u)$ (IV)	$(I)/(IV)$
0	1,00000	1,00000	1,00000	1,00000	1,00000	1,00000	1,00000
1	0,40470	0,39819	1,01633	0,40470	1,00000	0,40470	1,00000
2	0,25853	0,26155	0,98847	0,25853	1,00000	0,25853	1,00000
3	0,16674	0,17096	0,97529	0,16674	1,00000	0,16674	1,00000
4	0,10755	0,11049	0,97345	0,10755	1,00000	0,10755	1,00000
5	0,06938	0,07089	0,97866	0,06938	1,00000	0,06937	1,00000
6	0,04475	0,04526	0,98874	0,04475	1,00000	0,04475	1,00000
7	0,02887	0,02879	1,00253	0,02887	1,00000	0,02887	1,00000
8	0,01862	0,01827	1,01929	0,01862	1,00000	0,01862	1,00000
9	0,01201	0,01156	1,03859	0,01201	1,00000	0,01201	1,00000
10	0,00775	0,00731	1,06010	0,00775	1,00000	0,00775	1,00000
11	0,00500	0,00461	1,08364	0,00500	1,00000	0,00500	1,00000
12	0,00322	0,00291	1,10906	0,00322	1,00000	0,00322	1,00000
13	0,00208	0,00183	1,13627	0,00208	1,00000	0,00208	1,00000
14	0,00134	0,00115	1,16520	0,00134	1,00000	0,00134	1,00000
15	0,00087	0,00072	1,19580	0,00087	1,00000	0,00087	1,00000

Table 3.2: Beekman-Bowers', Tijms' and Fourier methods, Exponential.

u	$\psi_d(u)$ (I)	$\psi_{d,F}(u)$ (II)	$(I)/(II)$	$\psi_s(u)$ (III)	$\psi_{s,F}(u)$ (IV)	$(III)/(IV)$
0	1,00000	1,00000	1,00000	0,00000	0,00000	-
1	0,09688	0,09688	0,99999	0,30782	0,30782	1,00000
2	0,05676	0,05676	1,00001	0,20177	0,20177	1,00000
3	0,03655	0,03655	1,00000	0,13018	0,13018	1,00000
4	0,02358	0,02358	1,00000	0,08397	0,08398	0,99997
5	0,01521	0,01521	1,00000	0,05417	0,05417	1,00000
6	0,00981	0,00981	1,00000	0,03494	0,03494	1,00000
7	0,00633	0,00633	1,00000	0,02254	0,02254	1,00000
8	0,00408	0,00408	1,00000	0,01454	0,01454	1,00000
9	0,00263	0,00263	1,00000	0,00938	0,00938	1,00000
10	0,00170	0,00170	1,00000	0,00605	0,00605	1,00000
11	0,00110	0,00110	1,00000	0,00390	0,00390	1,00000
12	0,00071	0,00071	1,00000	0,00252	0,00252	1,00000
13	0,00046	0,00046	1,00001	0,00162	0,00162	1,00000
14	0,00029	0,00029	1,00001	0,00105	0,00105	1,00001
15	0,00019	0,00019	1,00002	0,00068	0,00068	0,99999

Table 3.3: Fourier method for $\psi_d(u)$ and $\psi_s(u)$, Exponential.

This distribution is described by Wikstad as an attempt to model the Swedish non-industrial fire insurance data from 1948-1951. It is a highly skewed distribution, with variance of 42,1982 and skewness of 27,6873. Here we get $H_2(x) = 1 - 0.16106e^{-5.514588x} - 0.56696e^{-0.190206x} - 0.271977e^{-0.014631x}$. The exact figures for $\psi(u)$ and $\psi_d(u)$ can be seen at table 3.4. We can note that in this case the weight of the diffusion to ruin is much smaller than in the case of the exponential distribution.

u	$\psi(u)$ (I)	$\psi_d(u)$ (II)	$(II)/(I)$
0	1,00000	1,00000	100%
1	0,45978	0,04326	9%
2	0,41447	0,01473	4%
3	0,38805	0,01220	3%
4	0,36508	0,01081	3%
5	0,34466	0,00963	3%
6	0,32647	0,00859	3%
7	0,31023	0,00767	2%
8	0,29571	0,00687	2%
9	0,28269	0,00616	2%
10	0,27101	0,00554	2%
11	0,26050	0,00499	2%
12	0,25101	0,00451	2%
13	0,24243	0,00408	2%
14	0,23465	0,00371	2%
15	0,22758	0,00338	1%

Table 3.4: Exact figures, Mixture of Exponentials.

The results of the approximation methods can be seen in table 3.5. The figures do not seem to be impressive in any case. We remember that these three methods are based on the idea of equating a few moments in order to obtain closed formula approximations. As the distribution is highly skewed, the few moments here do not seem to be enough to obtain good results. If we computed the Tijms' approximation for a large enough u we would get good results but it would be essentially due to the asymptotic result. However the inversion of the Fourier transform produced good results as we can see in the tables 3.6 and 3.7. The real and the complex parts of

u	$\psi(u)$ (I)	$\psi_{DV}(u)$ (II)	$(I)/(II)$	$\psi_{BB}(u)$ (III)	$(I)/(III)$	$\psi_T(u)$ (IV)	$(I)/(IV)$
0	1,00000	1,00000	1,00000	1,00000	1,00000	1,00000	1,00000
1	0,45978	0,73340	0,62691	0,42460	1,08284	0,75482	0,60912
2	0,41447	0,55758	0,74334	0,38253	1,08351	0,58596	0,70734
3	0,38805	0,44134	0,87925	0,36101	1,07491	0,46942	0,82666
4	0,36508	0,36420	1,00241	0,34439	1,06008	0,38875	0,93911
5	0,34466	0,31274	1,10209	0,33053	1,04277	0,33270	1,03597
6	0,32647	0,27812	1,17383	0,31852	1,02497	0,29352	1,11226
7	0,31023	0,25458	1,21859	0,30785	1,00771	0,26592	1,16663
8	0,29571	0,23831	1,24085	0,29823	0,99153	0,24626	1,20077
9	0,28269	0,22682	1,24632	0,28944	0,97668	0,23207	1,21816
10	0,27101	0,21848	1,24042	0,28133	0,96331	0,22162	1,22289
11	0,26050	0,21222	1,22750	0,27380	0,95142	0,21374	1,21874
12	0,25101	0,20732	1,21075	0,26675	0,94099	0,20765	1,20885
13	0,24243	0,20333	1,19230	0,26013	0,93195	0,20277	1,19559
14	0,23465	0,19995	1,17357	0,25389	0,92425	0,19875	1,18067
15	0,22758	0,19697	1,15538	0,24797	0,91776	0,19531	1,16520

Table 3.5: De Vylder's, Beekman-Bowers' and Tijms' methods, Mixture of Exponentials.

$\phi_{p(x)}(s)$ in a mixture of n-exponentials are given by

$$\phi^r_{p(x)}(s) = \sum_{i=1}^{n} \frac{A_i\beta_i^2}{\beta_i^2+s^2} \qquad \phi^c_{p(x)}(s) = \sum_{i=1}^{n} \frac{sA_i\beta_i}{\beta_i^2+s^2}$$

The results are not so good as for the case of the exponential distribution due, we believe, essentially two things: 1) $\phi^r_{p(x)}(s)$ and $\phi^c_{p(x)}(s)$ are harder to compute in (2.7) than before; 2) the values of the $\beta_i' s$ and $A_i' s$ in this distribution are much more likely to produce and accumulate numerical errors than just a $\beta = 1$ and a $A = 1$. Despite the small differences to the exact values, this method is still an excellent approximation.

3.3 Gamma distribution

We consider now that the claim amount distribution is gamma(2,2), with density function $p(x) = 4xe^{-2x}$ and distribution function $P(x) = 1 - (1 + 2x)e^{-2x}$ which implies $H_2(x) = 1 - (1 + x)e^{-2x}$. We do not have exact figures for this distribution so in order to test the accuracy of the methods we computed Dufresne & Gerber's bounds with $\vartheta = 0.01$ to use as a comparison basis. To apply the Fourier method

u	$\psi(u)$ (I)	$\psi_F(u)$ (II)	$(I)/(II)$
0	1,00000	1,00000	1,00000
1	0,45978	0,45978	1,00000
2	0,41447	0,41447	1,00000
3	0,38805	0,38805	1,00000
4	0,36508	0,36508	1,00000
5	0,34466	0,34466	1,00002
6	0,32647	0,32646	1,00002
7	0,31023	0,31023	1,00000
8	0,29571	0,29572	0,99995
9	0,28269	0,28266	1,00010
10	0,27101	0,27101	0,99999
11	0,26050	0,26051	0,99996
12	0,25101	0,25103	0,99993
13	0,24243	0,24243	1,00002
14	0,23465	0,23465	1,00001
15	0,22758	0,22756	1,00005

Table 3.6: Fourier method, Mixture of Exponentials.

u	$\psi_d(u)$ (I)	$\psi_{d,F}(u)$ (II)	$(I)/(II)$	$\psi_s(u)$ (III)	$\psi_{s,F}(u)$ (IV)	$(III)/(IV)$
0	1,00000	1,00000	1,00000	0,00000	0,00000	-
1	0,04326	0,04326	0,99998	0,41651	0,41652	1,00000
2	0,01473	0,01473	1,00004	0,39975	0,39975	0,99999
3	0,01220	0,01220	1,00025	0,37585	0,37585	0,99999
4	0,01081	0,01081	1,00042	0,35427	0,35427	0,99999
5	0,00963	0,00961	1,00181	0,33503	0,33505	0,99997
6	0,00859	0,00857	1,00194	0,31788	0,31789	0,99997
7	0,00767	0,00767	0,99999	0,30256	0,30256	1,00000
8	0,00687	0,00690	0,99542	0,28884	0,28882	1,00006
9	0,00616	0,00609	1,01118	0,27653	0,27657	0,99986
10	0,00554	0,00554	0,99991	0,26547	0,26547	0,99999
11	0,00499	0,00501	0,99571	0,25551	0,25550	1,00004
12	0,00451	0,00454	0,99179	0,24651	0,24649	1,00008
13	0,00408	0,00406	1,00445	0,23835	0,23836	0,99995
14	0,00371	0,00370	1,00213	0,23095	0,23095	0,99998
15	0,00338	0,00334	1,00955	0,22420	0,22422	0,99991

Table 3.7: Fourier method for $\psi_d(u)$ and $\psi_s(u)$, Mixture of Exponentials.

in this case, we used the following decomposition of the gamma(α, β) transform,

$$\phi^r_{p(x)}(s) = \eta^\alpha \cos(\alpha\rho) \qquad \phi^c_{p(x)}(s) = \eta^\alpha \sin(\alpha\rho)$$

$$\text{with } \eta = \frac{\beta}{\sqrt{\beta^2+s^2}} \text{ and } \rho = \arccos(\eta)$$

Table 3.8 shows the results of the approximations. We can see that the only really poor approximation is the Beekman-Bowers', with almost every values outside the bounds. Tijms' and De Vylder's approximations only miss the bound at $u = 1$ and the Fourier method reproduce figures inside the bounds for all values of u.

u	Lower Bound	$\psi_{DV}(u)$	$\psi_{BB}(u)$	$\psi_T(u)$	$\psi_F(u)$	Upper Bound
0	1,00000	1,00000	1,00000	1,00000	1,00000	1,00000
1	0,38643	0,39199	0,38231	0,39394	0,38867	0,39092
2	0,21650	0,21809	0,22337	0,21912	0,21869	0,22089
3	0,12024	0,12155	0,12660	0,12198	0,12196	0,12369
4	0,06667	0,06774	0,07009	0,06790	0,06790	0,06915
5	0,03696	0,03775	0,03825	0,03780	0,03780	0,03865
6	0,02049	0,02104	0,02067	0,02104	0,02104	0,02161
7	0,01136	0,01173	0,01109	0,01171	0,01171	0,01208
8	0,00630	0,00654	0,00592	0,00652	0,00652	0,00675
9	0,00349	0,00364	0,00315	0,00363	0,00363	0,00377
10	0,00194	0,00203	0,00167	0,00202	0,00202	0,00211
11	0,00107	0,00113	0,00088	0,00112	0,00112	0,00118
12	0,00059	0,00063	0,00046	0,00063	0,00063	0,00066
13	0,00033	0,00035	0,00024	0,00035	0,00035	0,00037
14	0,00018	0,00020	0,00013	0,00019	0,00019	0,00021
15	0,00010	0,00011	0,00007	0,00011	0,00011	0,00012

Table 3.8: Dufresne & Gerber's Bounds, De Vylder's, Beekman-Bowers', Tijms' and Fourier Methods, Gamma.

Since that the De Vylder's approximation revealed to be adjusted to this distribution we decided to present also the approximation to $\psi_d(u)$ by this method, along with the Fourier method. Table 3.9 shows the results. Here we do not have a comparison basis but judging from the previous results we believe that $\psi_{d,F}(u)$ and $\psi_{s,F}(u)$ are the closest to the exact values. Given this, we present also the ratio between $\psi_{d,F}$ and ψ_F where we can see that the contribution of oscillations to the occurence of ruin is again considerable when compared with the mixture of exponentials presented in the previous section and it has also more weight than in the case of the exponential distribution, which has a heavier tail than the gamma.

u	$\psi_{d,DV}(u)$	$\psi_{d,F}(u)$	$\psi_{s,F}(u)$	$\psi_{d,F}(u)/\psi_F(u)$
0	1,00000	1,00000	0,00000	100%
1	0,10480	0,11221	0,27647	29%
2	0,05738	0,06354	0,15515	29%
3	0,03198	0,03570	0,08626	29%
4	0,01782	0,01989	0,04801	29%
5	0,00993	0,01107	0,02673	29%
6	0,00554	0,00616	0,01488	29%
7	0,00309	0,00343	0,00828	29%
8	0,00172	0,00191	0,00461	29%
9	0,00096	0,00106	0,00257	29%
10	0,00053	0,00059	0,00143	29%
11	0,00030	0,00033	0,00080	29%
12	0,00017	0,00018	0,00044	29%
13	0,00009	0,00010	0,00025	29%
14	0,00005	0,00006	0,00014	29%
15	0,00003	0,00003	0,00008	29%

Table 3.9: De Vylder's and Fourier methods for $\psi_d(u)$ and $\psi_s(u)$, Gamma.

3.4 Pareto distribution

In this last example we consider that the claim amount distribution follows a Pareto(5,4) with density function $p(x) = \frac{5\times 4^5}{(4+x)^6}$ and distribution function $P(x) = 1 - \left(\frac{4}{x+4}\right)^5$. This leads to $H_2(x) = 1 - \left(\frac{4}{x+4}\right)^4$. Tijms' approximation is not applicable because the moment generating function of the Pareto distribution does not exist. The Fourier method is applicable but as the Pareto distribution does not have an explicit form for the characteristic function we need to calculate numerically the integrals $\int_0^{+\infty} \cos(sx)p(x)dx$ and $\int_0^{+\infty} \sin(sx)p(x)dx$ to apply the inversion formula (2.7). Table 3.10 shows the results for De Vylder's, Beekman-Bowers' and Fourier methods. In this case we also do not have exact figures for the ruin probability so we calculated Dufresne & Gerber's bounds, again with $\vartheta = 0.01$.

Both Beekman-Bowers' and De Vylder's approximations revealed a poor fit to this distribution with the majority of the values outside the bounds. Although

u	Lower Bound	$\psi_{DV}(u)$	$\psi_{BB}(u)$	$\psi_F(u)$	Upper Bound
0	1,00000	1,00000	1,00000	1,00000	1,00000
1	0,40867	0,45521	0,38282	0,41036	0,41206
2	0,27697	0,24441	0,27165	0,27853	0,28011
3	0,19577	0,15464	0,20096	0,19707	0,19838
4	0,14124	0,11033	0,15017	0,14229	0,14336
5	0,10339	0,08437	0,11286	0,10423	0,10509
6	0,07656	0,06680	0,08516	0,07723	0,07792
7	0,05724	0,05373	0,06443	0,05778	0,05832
8	0,04317	0,04353	0,04886	0,04359	0,04402
9	0,03280	0,03537	0,03712	0,03314	0,03348
10	0,02511	0,02879	0,02824	0,02537	0,02564
11	0,01935	0,02344	0,02151	0,01956	0,01977
12	0,01501	0,01909	0,01640	0,01518	0,01534
13	0,01172	0,01555	0,01251	0,01185	0,01198
14	0,00920	0,01266	0,00956	0,00931	0,00941
15	0,00727	0,01032	0,00730	0,00736	0,00744

Table 3.10: Dufresne & Gerber's Bounds, De Vylder's and Beekman-Bowers' methods, Pareto.

this distribution does not have a very high variance, its skewness of approximately 4.65, which is greater than in the case of the exponential or the gamma presented before, can help to explain the poor fit of these two methods. The Fourier method produced values inside the bound for all u. The approximations to the decomposed probabilities using this method can be seen in table 3.11 along with the ratio between $\psi_{d,F}$ and ψ_F. We can see that the contribution of the diffusion component to the occurence of ruin is again less considerable, when compared with the exponential or gamma distributions, which is expectable due to the fact that the Pareto distribution is a well known example of a heavy tail distribution.

u	$\psi_{d,F}(u)$	$\psi_{s,F}(u)$	$\psi_{d,F}(u)/\psi_F(u)$
0	1,00000	0,00000	100%
1	0,09042	0,31994	22%
2	0,05009	0,22844	18%
3	0,03296	0,16411	17%
4	0,02261	0,11968	16%
5	0,01590	0,08833	15%
6	0,01138	0,06585	15%
7	0,00826	0,04952	14%
8	0,00606	0,03753	14%
9	0,00448	0,02866	14%
10	0,00334	0,02203	13%
11	0,00251	0,01705	13%
12	0,00190	0,01327	13%
13	0,00145	0,01040	12%
14	0,00111	0,00820	12%
15	0,00085	0,00650	12%

Table 3.11: Fourier method for $\psi_d(u)$ and $\psi_s(u)$, Pareto.

Conclusions

Looking at the figures of the approximations presented, a first conclusion we should underline is the poor fit of the Beekman-Bowers' approximation method, no matter the distribution example. Similar conclusions had been taken by Jacinto (2008), who first tried this kind of approximation. The approximations by De Vylder and Tijms look capable of producing good results if the claim amount distribution is *well behaved*. It is, at least, the case of the exponential or gamma distributions, when opposed to the mixture of exponentials and Pareto distributions. These two methods have the advantage of being very simple to compute and they do not require much software power. In other cases, good approximations can be obtained by computing Dufresne & Gerber's method of bounds, or using the Fourier transform method. Although these two approaches require a more powerful software, like *Mathematica*, in order to calculate recursive sums or more complex integrals, they provide much more precise results.

In short, we have adapted from the classical Cramér-Lundberg model to the perturbed model some simple methods to approximate the ultimate ruin probability. Some of the methods behave simultaneously well in both cases, others behave well in the classical model but not in the perturbed model and others do not work satisfactory in either case. Also, the results obtained to the ruin probability caused by diffusion seem to leave the idea that the diffusion component can have a substantial part in the ultimate ruin probability, specially if the claim amount distribution is light tailed. Obviously, if we increase the value of the perturbation, σ, this probability will go up. As the classical model and the illustrations for different σ are not dealt in this work, see for instance Jacinto (2008) for some numerical examples.

Concerning the ultimate ruin probability in the classical risk process perturbed by diffusion we think that we have obtained satisfactory results. Future works can be developed dropping the Poisson assumption an generalize to other renewal risk models with a perturbation by diffusion.

A Wiener process and moments of V(t)

Definition 5 *A stochastic process* $\{W(t), t \geq 0\}$ *is said to be a Standard Wiener Process if*

i) $W(0) = 0$;

ii) $\{W(t), t \geq 0\}$ has stationary and independent increments;

iii) for every $t > 0$, $W(t)$ is normally distributed with mean 0 and variance t

We can generalize the standard Wiener process into a Wiener process with diffusion, *i.e.* if $\{W(t)\}_{t\geq 0}$ is a standard Wiener process, $\{\sigma W(t)\}_{t\geq 0}$ is a Wiener process with diffusion coefficient $\sigma^2 > 0$.

From iii) we can conclude that the density function of $W(t)$ is given by

$$f_{W(t)}(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$$

and from ii) we can conclude that for $t_0 < t_1 < \cdots < t_n$, $W(t_1) - W(t_0), \cdots, W(t_n) - W(t_{n-1})$ are independent and due to stationarity, $W(t+s) - W(s)$, is normally distributed with mean 0 and variance t, for $s > 0$.

The moments of $S(t) = \sum_{t=1}^{N(t)} X_i$ are well known from the actuarial literature, the deduction of them can be seen for instance at Gerber (1979). The first four raw moments are given by,

$$\begin{aligned}
E[S(t)] &= \lambda t p_1 \\
E[S^2(t)] &= \lambda^2 t^2 p_1^2 + \lambda t p_2 \\
E[S^3(t)] &= \lambda^3 t^3 p_1^3 + 3\lambda^2 t^2 p_1 p_2 + \lambda t p_3 \\
E[S^4(t)] &= \lambda^4 t^4 p_1^4 + 6\lambda^3 t^3 p_1^2 p_2 + 3\lambda^2 t^2 p_2^2 + 4\lambda^2 t^2 p_1 p_3 + \lambda t p_4
\end{aligned}$$

Joining the moments of $W(t)$ with the moments of $S(t)$ we can calculate the central moments of the perturbed surplus process $\{V(t)\}$,

$$\begin{aligned}
E[V(t)] &= u + ct - \lambda t p_1 \\
V[V(t)] &= \sigma^2 t + \lambda t p_2 \\
E[(V(t) - E[V(t)])^3] &= -\lambda t p_3 \\
E[(V(t) - E[V(t)])^4] &= \lambda t p_4 + 3\lambda^2 t^2 p_2^2 + 6\lambda t^2 p_2 \sigma^2 + 3\sigma^4 t^2
\end{aligned}$$

B Convolutions

Definition 6 *The convolution operation between two general functions $f\ (.)$ and $g(.)$ is defined by*

$$f * g(z) = \int_{-\infty}^{+\infty} f(x)g(z-x)dx \quad x \in \Re$$

In probability theory, we can use the convolution operator to find the distribution of a random variable that is a sum of other two.

Theorem 12 *If X and Y are two continuous and independent random variables with probability distribution function $F_X(x)$, $F_Y(y)$ and probability density function $f_X(x)$, $f_Y(y)$, respectively. Then the probability distribution function of $Z = X + Y$ is given by*

$$F_Z(z) = F_X * F_Y(z) = \int_{-\infty}^{+\infty} F_X(z-y)f_Y(y)dy = \int_{-\infty}^{+\infty} F_Y(z-x)f_X(x)dx$$

the respective density function of Z is given by

$$f_Z(z) = f_X * f_Y(z) = \int_{-\infty}^{+\infty} f_X(z-y)f_Y(y)dy = \int_{-\infty}^{+\infty} f_Y(z-x)f_X(x)dx$$

Proof. $F_Z(z) = \Pr(X+Y \leq z) = \iint\limits_{x+y\leq z} f_X(x)f_Y(y)dxdy = \int_{-\infty}^{+\infty}\int_{-\infty}^{z-y} f_X(x)f_Y(y)dxdy = \int_{-\infty}^{+\infty} F_X(z-y)f_y(y)dy$. Differentiating $F_Z(z)$ in order do z we get $f_Z(z) = \frac{d}{dz}\int_{-\infty}^{+\infty} F_X(z-y)f_Y(y)dy = \int_{-\infty}^{+\infty} f_X(z-y)f_Y(y)dy$. ■

If X and Y are two positive random variables it is important to note that $f_X(.)$ and $f_Y(.)$ are concentrated on $(0, +\infty)$ and therefore the convolution is reduced to $f_X * f_Y(z) = \int_0^z f_X(z-y)f_Y(y)dy = \int_0^z f_Y(z-x)f_X(x)dx$. The same happens with $F_X * F_Y(z)$.

Definition 7 *The n-fold convolution of $F_X(x)$, denoted by $F_X^{*n}(x)$, represents the distribution function of the sum of n mutually independent random variables with common distribution $F_X(x)$ and it is defined iteratively. For $n = 0$ we have,*

$$F_X^{*0}(x) = \begin{cases} 0 \ \textit{if}\ x < 0 \\ 1 \ \textit{if}\ x \geq 0 \end{cases}$$

and for $1, 2, \cdots, n$ *we have,*

$$F_X^{*n}(x) = \int_{-\infty}^{+\infty} F_X^{*(n-1)}(x-y)dF_X(y) = F_X^{*(n-1)} * F_X(x)$$

If we are only in the presence of non negative random variables, $F_X^{*n}(x)$ is reduce to $F_X^{*n}(x) = \int_0^x F_X^{*(n-1)}(x-y)dF_X(y)$. Also note that in this case $F_X^{*1}(x) = F_X(x)$.

C Laplace transform

We just present here some basic properties but for a deeper insight about this transform see for instance Poularikas (1996).

Let $f(x)$ be a continuous function defined for $x \geq 0$ whose integral exists for all $x > 0$. Its Laplace transform is defined as

$$\bar{f}(s) = \int_0^{+\infty} e^{-sx} f(x) dx$$

if the integral is convergent. If $f(x)$ is a probability density function of a non negative random variable then the Laplace transform exists at least for $s \geq 0$. We can also see the Laplace transform as $L_X(s) = E[e^{-sX}]$ and with this calculate the raw moments of a random variable:

$$\frac{d^k}{ds^k} L_X(s) = E[(-X)^k e^{-sX}]$$

evaluating at $s = 0$ we get $E[(-X)^k]$. The Laplace transform is analogous to the Moment Generating function of a random variable, the advantage is that if the random variable is non negative, the Laplace transform exists at least for $s \geq 0$.

Property 1 Let $f(.)$ and $g(.)$ be functions with Laplace transform and let a and b be constants. Then,

$$\int_0^{+\infty} e^{-sy}[af(y) + bg(y)]dy = a\bar{f}(s) + b\bar{g}(s)$$

Property 2 Let $F(x) = \int_0^x f(y)dy$, then

$$\bar{F}(s) = \frac{1}{s}\bar{f}(s)$$

Property 3 Let $f(y)$ be a continuous and differentiable function defined for $x \geq 0$. Then,

$$\int_0^{+\infty} e^{-sy}\left(\frac{d}{dy}f(y)\right) dy = s\bar{f}(s) - f(0)$$

Property 4 Let $\{f_j(.)\}_{j=1}^n$ be functions which the Laplace transforms exist and let $h(x)$ be the n-fold convolution of them, i.e. $h(x) = f_1 * f_2 * \cdots * f_n(x)$. Then the Laplace transform of $h(x)$ is

$$\bar{h}(s) = \prod_{j=1}^{n} \bar{f}_j(s)$$

D Renewal theory

We present here some basic notions of Renewal Theory that are used to obtain the asymptotic results for the ruin probabilities. A deeper insight about this matter can be found for instance at Feller (1971).

Definition 8 *Let $H(x)$ be a proper distribution function concentrated on $(0, +\infty)$, such that $H(0) = 0$ and $H(\infty) = 1$. Because of the assumed positivity we can safely write*

$$\mu = \int_0^\infty y dH(y) = \int_0^\infty (1 - H(y))\, dy$$

where $\mu \leq \infty$. When $\mu = \infty$ we interpret the symbol μ^{-1} as 0.

Definition 9 *A proper renewal equation is defined as an equation of the form*

$$Z(x) = z(x) + \int_0^x Z(x-y) dH(y), \ \ x \geq 0 \tag{D.1}$$

For $x \geq 0$, the quantities $H(x)$ and $z(x)$ are known and $Z(x)$ is unknown. One of the major goals of the Renewal Theory is to study the asymptotic behaviour of the solution Z of the equation $(D.1)$. Feller (1971) proved the following renewal theorem that gives a solution to this problem.

Theorem 13 *If $z(x)$ is directly Riemann integrable and $H(.)$ is non arithmetic it follows that*

$$Z(x) \rightarrow \mu^{-1} \int_0^\infty z(y) dy$$

as $x \rightarrow \infty$. ■

An important generalization of the renewal process that is very useful in ruin theory is obtained when $H(.)$ is a defective distribution, i.e. $H(\infty) < 1$. Such process is called terminating or transient process. The defect $1 - H(\infty)$ is the probability of extinction.

Definition 10 *A defective renewal equation is defined as an equation of the form*

$$Z(x) = z(x) + \int_0^x Z(x-y)dH(y), \ x \geq 0$$

with $H(\infty) < 1$.

A very useful and standard argument used in renewal theory, when we have a transient processes is that there exists a number κ such that,

$$\int_0^\infty e^{\kappa y} dH(y) = 1 \tag{D.2}$$

If the integral exists, the root κ is unique and as the distribution $H(.)$ is defective, $\kappa > 0$.

Defining

$$dH^{\#}(y) = e^{\kappa y} dH(y)$$

we have by $(D.2)$ that $H^{\#}(y)$ is now a proper probability distribution because $H^{\#}(\infty) = 1$. Also, we associate with any general function $f(.)$, another function $f^{\#}(.)$ defined as,

$$f^{\#}(x) = e^{\kappa x} f(x)$$

Applying this transformation to each element of a defective renewal equation we obtain the following equation, that is now a proper renewal equation (please see Feller (1971) for further details),

$$Z^{\#}(x) = z^{\#}(x) + \int_0^x Z^{\#}(x-y)dH^{\#}(y), \, x \geq 0$$

and if $z^{\#}(x)$ is directly Riemann integrable, the Renewal Theorem implies, for a non arithmetic $H^{\#}(.)$ that,

$$e^{\kappa x} Z(x) \to \frac{1}{\mu^{\#}} \int_0^\infty e^{\kappa y} z(y) dy$$

where

$$\mu^{\#} = \int_0^\infty y e^{\kappa y} dH(y)$$

E Proofs

We present here the proof of $\psi_d(0) = 1$, the expression for $M_L(s)$ (expression (1.17)) and the deduction of the asymptotic result for $\psi_d(u)$ (expression (1.18)).

Proof of $\psi_d(0) = \psi(0) = 1$. Defining $\tau_d = \inf\{t \geq 0 : W(t) + ct < 0\}$ we have that,

$$\{\tau_d \leq t\} \supseteq \{W(t) + ct < 0\} \Rightarrow \Pr\{\tau_d \leq t\} \geq \Pr\{W(t) + ct < 0\} = \Phi(-ct)$$

As $\Phi(.)$ is the distribution function of a Normal(0,1), $\Phi(-ct)$ is always positive. Taking the limit we have:

$$\Pr\{\tau_d = 0\} = \lim_{t \to 0^+} \Pr\{\tau_d \leq t\} \geq \Phi(-ct).$$

by the Blumenthal's law (see for instance Mörters and Peres (2010)) we have that $\Pr\{\tau_d = 0\} = 1$, *i.e*, the Weiner process with drift can almost surely take negative values immediately after its beginning.

Given this and given that $N(0) = 0$, we have $\psi_d(0) = \psi(0) = 1$. ■

Proof of $M_L(s)$. If we consider $L_i^* = L_i^{(1)} + L_i^{(2)}$ for $i = 1, \ldots, M$ we can write $M_L(s)$ as:

$$M_L(s) = E[e^{sL_0^{(1)} + s\sum_{i=0}^{M} L_i^*}] = E[e^{sL_0^{(1)}}]E[e^{s\sum_{i=0}^{M} L_i^*}] = M_{L_0^{(1)}}(s)M_M[\ln(M_{L^*}(s))] \tag{E.1}$$

where

$$M_M(s) = \frac{q}{1 - (1-q)e^s} = \frac{c - \lambda p_1}{c - e^s \lambda p_1} \tag{E.2}$$

$$M_{L_0^{(1)}} = \frac{\zeta}{\zeta - s} \tag{E.3}$$

Noting that the probability density function of L_i^* is $h_1 * h_2(.)$, please see Appendix

B to see the properties of convolutions, we obtain,

$$\begin{aligned}
M_{L^*}(s) &= \int_0^\infty e^{sx} h_1 * h_2(x)dx = \int_0^\infty \int_0^x e^{sx} h_1(y) h_2(x-y) dy dx \\
&= \int_{y=0}^\infty h_1(y) \int_{x=y}^\infty e^{sx} \frac{[1-P(x-y)]}{p_1} dx dy \\
&= p_1^{-1} \int_0^\infty h_1(y) \int_y^\infty \int_{x-y}^\infty e^{sx} p(v) dv dx dy = \\
&= p_1^{-1} \int_0^\infty h_1(y) \int_{v=0}^\infty p(v) \int_y^{v+y} e^{sx} dx dv dy \\
&= (sp_1)^{-1} \int_0^\infty h_1(y) \times \left[\int_0^\infty p(v) e^{sv+sy} dv - \int_0^\infty p(v) e^{sy} dv \right] dy \\
&= (sp_1)^{-1} \int_0^\infty h_1(y) \times \left[e^{sy} M_X(s) - e^{sy} \right] dy = \\
&= (sp_1)^{-1} \int_0^\infty h_1(y) e^{sy} \times \left[M_X(s) - 1 \right] dy \\
&= (sp_1)^{-1} \left[M_X(s) - 1 \right] M_{L_0^{(1)}}(s) \qquad \text{(E.4)}
\end{aligned}$$

so if we apply $(E.2)$, $(E.3)$ and $(E.4)$ into $(E.1)$, we obtain (1.17). ■

Proof of $\psi_d(u)$. We present the deduction for (1.18) but (1.19) and (1.20) can be obtain in a similar way starting from (1.3) and (1.4), respectively. The starting point here is the defective renewal equation (1.2). According to the notation presented in Appendix D we have,

$$Z(u) = \psi_d(u), \quad z(u) = 1 - H(u), \quad H(x) = (1-q) \int_0^x h_1 * h_2(y) dy$$

Equation (1.2) is defective because $H(\infty) = (1-q) < 1$. So, supposing that exists a number κ such that $\int_0^\infty e^{\kappa x} dH(x) = 1$, we have

$$(1-q) \int_0^\infty e^{\kappa x} h_1 * h_2(x) dx =$$

$$
\begin{aligned}
&= (1-q)\int_0^\infty e^{\kappa x}\int_0^x h_1(y)h_2(x-y)dydx \\
&= (1-q)\int_0^\infty h_1(y)\int_y^\infty e^{\kappa x}\frac{1-P(x-y)}{p_1}dxdy \\
&= \frac{\lambda}{c}\int_0^\infty h_1(y)\int_y^\infty e^{\kappa x}\int_{x-y}^\infty p(v)dvdxdy \\
&= \frac{\lambda}{c}\int_0^\infty h_1(y)\int_0^\infty p(v)\int_y^{v+y} e^{\kappa x}dxdvdy \\
&= \frac{\lambda}{c}\int_0^\infty h_1(y)\int_0^\infty p(v)\frac{e^{\kappa(v+y)}-e^{\kappa y}}{\kappa}dvdy \\
&= \frac{\lambda}{c\kappa}\int_0^\infty h_1(y)e^{\kappa y}\int_0^\infty p(v)\left[e^{\kappa v}-1\right]dvdy \\
&= \frac{\lambda}{c\kappa}\int_0^\infty \zeta e^{-\zeta y}\times e^{\kappa y}\left[\int_0^\infty p(v)e^{\kappa v}dv-\int_0^\infty p(v)dv\right]dy \\
&= \frac{\lambda}{c\kappa}\times\frac{\zeta}{\zeta-\kappa}\times\left[\int_0^\infty p(v)e^{\kappa v}dv-1\right] = \frac{2\lambda}{2c\kappa-\kappa^2\sigma^2}\times\left[\int_0^\infty p(v)e^{\kappa v}dv-1\right]
\end{aligned}
$$

And by the definition of the adjustment coefficient, equation (1.5), we can note that the only possible solution for $\int_0^\infty e^{\kappa x}dH(x)=1$, is when $\kappa=R$. Applying e^{Rx} to each function of equation (1.2), we arrive to

$$e^{Ru}\psi_d(u) = e^{Ru}\left[1-H_1(u)\right] + (1-q)\int_0^u e^{R(u-x)}\psi_d(u-x)e^{Rx}h_1 * h_2(x)dx$$

Which is now a proper renewal equation for the function $e^{Ru}\psi_d(u)$ and according to the renewal theorem we have that

$$e^{Ru}\psi_d(u) \to \frac{\int_0^\infty e^{Rx}\left[1-H_1(x)\right]dx}{(1-q)\int_0^\infty xe^{Rx}h_1 * h_2(x)dx} = C^d$$

which is the same as

$$\psi_d(u) \sim C^d e^{-Ru}$$

■

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