Instituto Superior de Economia e Gestão
UNIVERSIDADE TÉCNICA DE LISBOA
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MESTRADO

MATEMÁTICA FINANCEIRA

TRABALHO FINAL DE MESTRADO

DISSERTAÇÃO

ILLIQUID MARKETS AND HAMILTON-JACOBI-BELLMAN EQUATIONS

CARLOS MIGUEL DOS SANTOS OLIVEIRA

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ORIENTAÇÃO:
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Abstract

In this thesis, the assumption of risky asset liquidity is relaxed. We assume that the market contains one trader sufficiently large to influence the price of the risky asset. Unlike the classical Black-Scholes equation, the Black-Scholes equations from models of illiquid markets are non-linear. In this case, it is difficult to guarantee the existence and uniqueness of classical solutions. We discuss the concept of viscosity solutions and its application in the setting by Frey and Polte (2011).

Wilmott and Schönbucher (2000) presented an equilibrium model for illiquid markets. We discuss the concept of self-financing strategy in their framework and use the Wilmott-Schönbucher model to study the consequences of collective behaviours in financial markets. We derive the corresponding Black-Scholes equation which is non-linear and has unusual boundary conditions.

Resumo

Nesta tese, a hipótese da liquidez do activo com risco é relaxada. Assumimos que o mercado contém um investidor suficientemente grande para influenciar o preço do activo com risco.

Contrariamente à equação de Black-Scholes clássica, as equações de Black-Scholes para modelos de mercados ilíquidos são não-lineares. Neste caso, é difícil garantir a existência e unicidade de solução clássica. Discutimos o conceito de soluções de viscosidade e a sua aplicação no problema proposto por Frey e Polte (2011).

Wilmott e Schönbucher (2000) apresentaram um modelo de equilíbrio para mercados ilíquidos. Nós discutimos o conceito de estratégia auto-financiada nessa abordagem e utilizamos o modelo de Wilmott-Schönbucher para estudar as consequências do comportamento colectivo nos mercados financeiros. Derivamos a correspondente equação de Black-Scholes que é não-linear e tem condições de fronteira não usuais.

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Contents

1 Introduction

The Black-Scholes model is used to price and to design hedging strategies in different securities. Black and Scholes in [2] and Merton in [14] simultaneously derived the Black-Scholes formula for European options. This theory is largely used in the financial industry, but requires strong assumptions, namely:

1. The market allows unbounded short positions, as well as fractional holdings;
2. All traders act as price takers;
3. There are no transaction costs;
4. The market is competitive;
5. The logarithm of the asset price is normally distributed.

These assumptions have been widely criticized in many works such as [9], [10] and [3] . Black [1] himself stated "I sometimes wonder why people still use the Black-Scholes formula, since it is based on such simple assumptions - unrealistically simple assumptions". The fourth assumption means that the market's selling and buying prices are equal for all agents trading in a particular asset. The second assumption signifies that all agents can buy or sell unlimited quantities of the asset without moving the price of this asset. This is sometimes refereed to as perfectly liquid market.

Many of the critiques to the Black-Scholes model are related to the log-normality of the assets' price distribution. There are many works such as [13] and [5] where the authors propose different Lévy processes to model the dynamics of the assets' price. Also, the Black-Scholes model supposes that the volatility of the returns of the assets is constant. There are many empirical studies showing that this is an unrealistic assumption and others that propose possible solutions for this problem.

The present work deals with assumptions (2) and (4). The liquidity of the market isn't a realistic assumption because in financial markets there are some companies, funds and other institutions that are so large that the effects of their actions on the asset' prices are not negligible. Oddly, there are comparatively few models that relax this assumption. These models introduce in the standard Black-Scholes model the concept of Liquidity Risk. The Liquidity Risk is the additional risk associated to the size and the timing of the transaction. There are two types of models, the reaction function models and the models with temporary price impact. These models are hurt by theoretical difficulties that are discussed, among others, by Wilmott and Schönbucher in [16] and Jarrow in [11] and [12].

While in the Black-Scholes model the partial differential equation (PDE) deduced to price contingent claims is linear, the PDE deduced from models for illiquid markets is non-linear. In many important cases, this non-linear PDE does not admit a solution in the classical sense. In many cases, this problem can be solved using viscosity solutions.

In section 2 we present the characteristics of the models of illiquid markets as well as some examples of non-linear Black-Scholes equations and a sketch of the arguments used to derive them. The section 3 is based in [7]. We discuss the properties of the Black-Scholes equation presented there, in particular, its viscosity solutions. Section 4 contains a discussion of the model proposed by Wilmott and Schönbucher [16]. Here we discuss some interesting issues as the portfolio value and the consistency of the models in illiquid markets. Finally, Section 5 deals with a collective behaviour case we discuss the concept of self-financing trading strategy as well as the Black-Scholes equation.

2 Black-Scholes equation in illiquid markets

The classical Black-Scholes theory is widely used in the financial industry although the strong assumptions of the model. In this work, we relax the assumption of the market liquidity. In illiquid markets, the Black-Scholes equations are non-linear PDEs. So in this section, in order to motivate this work, we present some examples of Black-Scholes equations deduced from models of illiquid markets.

2.1 The models

The Black-Scholes model can be used to price any contingent claim when we suppose that the assumptions (1)-(5) are verified. To model the assets' price in illiquid markets we need a model that allows for the existence of traders with different sizes and different influences in the dynamics of the assets' price. Typically these models consider two types of agents: a group of many small traders and one large trader. We identify two types of models: the reaction function models and the models with temporary price impact, as we explain below.

Reaction function models: In this approach, the asset price depends on the quantity held by the large trader. For example $S_t = \Phi(t, \phi(t), W)$, where t is the current time, $\phi(t)$ is the quantity of stock held by the large trader at time t and W is a random state. Here the impact of trading strategies on the asset price is permanent in the sense that it lasts as long as the large trader keeps his position. That is, the effect does not vanish after he stops trading. In some works such as [15], [8] and [17] the reaction functions, also called feedback functions, are obtained using an equilibrium approach. In Equilibrium models the reaction function is obtained through explicit microeconomic equilibrium. Wilmott and Schönbucher [16] define the aggregate excess of demand by small traders, $\chi(S, W, t)$, as a function of the asset price S, Brownian motion W, and the current time t. The net demand by the large trader is a function of the asset price and current time, $f(S,t)$. Therefore the equilibrium is given by $\chi(S,W,t)+f(S,t)=0$, and the reaction function is derived by inverting this expression. We discuss all these points in section 3.

Models with temporary price impact: In this class of models the price reacts to the quantity of assets traded in the market at the time t. The impact of the large trader's strategies on the stock price is temporary in the sense that it ceases when he stops trading. An important model for temporary impacts is presented in [18], which developed the concept of stochastic supply curve, $S(t,x,W)$ for an asset price that is a function of the current time, t, the size of the large trader's purchase, x, and a random state, W. When the size of the transaction is 0, $S(t,0,W)$ represents the market price of the asset. Otherwise, $S(t,x,W)$ represents the market price modified by the purchase order x. If the application $x \mapsto S(t,x,W)$ is increasing, then we have $S(t,x,W)=\varphi(x,S(t,0,W))$.

The main difference between these two types of models is the form as the large trader strategy influences the asset price. In the first case the price is influenced by the quantity held by the large trader while in the second case it is influenced by the quantity that he wants buy or sell in every instant of time.

2.2 Deducing Black-Scholes PDE

In this section we explain one methodology to obtain the Black-Scholes PDE. Suppose we have a contingent claim with a price process $F(S,t)$. The price process of the underlying asset follows:

$$dS_t = \mu(S,t)dt + \sigma(S,t)dW. \tag{1}$$

We assume a self-financing strategy (f, c), where $f(S,t)$ and $c(S,t)$ are the number of titles of the risky asset and number of bonds held by the trader. The value of this strategy is $Y_t = f(S,t)S_t + c(S,t)B_t$ and its dynamics is

$$dY_t = f(S,t)dS_t + c(S,t)dB_t. \tag{2}$$

If this self-financing strategy replicates the payoff of the contingent claim, we will have $F(S_T, T) = Y_T$ almost surely. To avoid arbitrage possibilities the contingent claim price and the portfolio value have to be the same for all $0 \leq t \leq T$ almost surely. Therefore the stochastic component of the portfolio value and price process have to be the same as well as the deterministic component. The Black-Scholes PDE is directly obtained from this condition. Some authors, such as Frey and Polte in [7] deduce the Black-Scholes PDE assuming $r = 0$, which facilitates the derivation.

The Black-Scholes PDE's can be viewed as a particular case of an Hamilton-Jacobi-Bellman (HJB) equation which appears in optimal control problems. In the case of the Black-Scholes PDE, we have a HJB equation where the control can assume a unique value. In general the HJB equation is non-linear unlike the classical Black-Scholes PDE which is linear. All the models considered in this work generate non-linear Black-Scholes (HJB) equations.

2.3 Examples of Black-Scholes PDE

Now we give two examples of Black-Scholes PDE. The first example arises in a reaction function model. In [15], it is assumed that the reaction function, $\Phi(t, \phi, W)$, is $\Phi(W, \phi) = W \exp(\rho\phi)$ where ρ is a liquidity parameter. If the market is liquid then we will have $\rho = 0$. The larger the parameter ρ the more illiquid is the market. Here W is a Brownian motion and represents some fundamental value and that is considered relevant to explain the dynamic of the asset. Finally, $\phi_t = \varphi(t, S_t)$ represents the quantity of asset that the large trader wants to hold. Naturally we have the equality $S_t = \Phi(W_t, \phi_t)$, so the dynamic of S can be obtained applying Itô lemma to the last equality,

$$dS_t = \rho S_t d\phi_t + \mu S_t + \sigma S_t dW_t \tag{3}$$

$$d\phi_t = \frac{\partial}{\partial t}\varphi(t, S_t)dt + \frac{\partial}{\partial S}\varphi(t, S_t)dS_t + \frac{\partial^2}{\partial S^2}\varphi(t, S_t)(dS_t)^2. \tag{4}$$

We consider that the interest rate is zero such as Frey and Polte in [7], so the self-financing strategy (2) can be rewritten as $dY_t = \varphi(t, S_t)dS_t$. Notice that if we substitute (4) into (3) we will obtain the stochastic term as $\frac{\sigma S}{1-\rho S \frac{\partial}{\partial S}\varphi(t,S)}$. When we use the technique described in the last subsection we will obtain the Black-Scholes PDE

$$\frac{\partial}{\partial t}u + \frac{1}{2}S^2 \frac{\sigma^2}{(1-\rho S\frac{\partial^2}{\partial S^2}u)^2}\frac{\partial^2}{\partial S^2}u = 0. \tag{5}$$

The second example that we present arises in a model with temporary impact. This example is presented in [18], an important work in this category of models. We think that it is interesting to present the self-financing strategy derived in theorem A3 of [18] that incorporates the concept of liquidity cost. Considering Y_t the value of the portfolio, that verifies $Y_t = c_t + \phi_t S(t, 0)$ where ϕ_t is the stock position and c_t is the bond position. The self-financing strategy, Y_t, verifies:

$$Y_t = Y_0 + \int_0^t \phi_{u-} dS(u, 0) - \sum_{0\le u\le t} \delta\phi_u \big(S(u, \delta\phi_u) - S(u, 0)\big) - \int_0^t \frac{\partial S}{\partial x}(u, 0)d[\phi, \phi]_u \tag{6}$$

where $d[\phi, \phi]_u$ represents the quadratic variation of ϕ_u.

For the extended Black-Scholes economy we have $S(t, x) = \exp(\alpha x)S(t, 0)$ with $\alpha > 0$. Here $S(t, 0)$ is a geometric Brownian motion with volatility σ. In this case, for

$\phi_t = \varphi(t, S_t)$, the dynamic of the self-financing strategy is

$$dY_t = \varphi(t, S(t,0))dS(u,0) - \alpha S(t,0)\varphi_S^2(t, S(t,0))\sigma^2 S^2(t,0)dt. \tag{7}$$

The Black-Scholes equation, obtained by the usual arguments, as it can be seen in [7], is

$$\frac{\partial}{\partial t}u + \frac{1}{2}S^2\sigma^2\left(1 + 2\alpha S\frac{\partial^2}{\partial S^2}u\right)\frac{\partial^2}{\partial S^2}u = 0. \tag{8}$$

3 Viscosity solutions to Black-Scholes equation in illiquid markets

In many cases, it is difficult to guarantee the existence of classical solutions for non-linear PDEs. When classical solutions fail to exist, solutions in some weaker sense may still exist. The concepts of weak solutions require weaker assumptions to guarantee the existence of solution than classical solutions. In stochastic control problems, the concept of viscosity solution is frequently used.

In this section we present problems proposed by Frey and Polte [7] and discuss the existence and uniqueness of viscosity solutions to these problems.

3.1 Problem Setting

In the last section we presented two examples of Black-Scholes PDE. We can notice that these two examples yield PDE's of similar structure, namely

$$\frac{\partial}{\partial t}u + \frac{1}{2}S^2 v\left(S, \frac{\partial^2}{\partial S^2}u\right) = 0. \tag{9}$$

Frey and Polte [7] studied equations of this type on the domain $Q = [0,T[\times(\underline{S},\overline{S})$, with boundary condition

$$u = h \quad (t,S) \in \partial Q, \tag{10}$$

and $0 < \underline{S} < \overline{S} < +\infty$, $T \in (0,+\infty)$, $\partial Q = \{T\} \times (\underline{S},\overline{S}) \cup (0,T) \times \{\underline{S},\overline{S}\}$.

Following [7], we take the following regularity assumptions:

(A1) The payoff $h : [\underline{S},\overline{S}] \to \mathbb{R}$ is continuous . The function $v : [\underline{S},\overline{S}] \times \mathbb{R} \to \mathbb{R}$ is continuous on the set $dom(v) = \{(S,q) \in [\underline{S},\overline{S}] \times \mathbb{R} : v(S,q) < \infty\}$.

(A2) For fixed $S \in [\underline{S},\overline{S}]$ the mapping $v(S,.) : q \to v(S,q)$ is convex and lower semicontinuous. Moreover, $v(S,0) = 0$, and there is a constant $\lambda^0 > 0$ with $\frac{\partial}{\partial q}v^-(S,0) \leq \lambda^0 \leq \frac{\partial}{\partial q}v^+(S,0)$ for all $S \in [\underline{S},\overline{S}]$, where $\frac{\partial}{\partial q}v^-$ and $\frac{\partial}{\partial q}v^+$ denote de left and right derivatives of the convex function $v(S,.)$.

In the first example of section 2.3, we have $v(S,q) = \frac{\sigma^2}{(1-\rho Sq)^2}q$. The assumption (A1) is verified trivially. The convexity of the function $v(S,q)$ in the second argument is not verified. The expression of the second derivative of $v(S,q)$ with respect to the second argument, $\frac{\partial^2}{\partial q^2}v(S,q)$ can be negative,

$$\frac{\partial^2}{\partial q^2}v(S,q) = \sigma^2\rho S\frac{4+2\rho Sq}{(1-\rho Sq)^4} < 0 \text{ when } q < -\frac{2}{\rho S}. \tag{11}$$

On the other hand, the first derivative verifies $\frac{\partial}{\partial q}v(S,q) = \sigma^2\frac{1+\rho Sq}{(1-\rho Sq)^3}$ whereby is verified that $v_q^-(S,0) \leq \lambda^0 \leq v_q^+(S,0)$ with $\lambda^0 > 0$.

In the second example, $v(S,q) = \sigma^2 q(1+2\alpha Sq)$ and the first assumption is trivially verified. The second assumption is also verified. Notice that

$$\frac{\partial}{\partial q} v(S,q) = \sigma^2(1 + 4\alpha S q). \tag{12}$$

So, there is $\lambda^0 > 0$ such that $\lambda^0 = \frac{\partial}{\partial q} v(S,0) = \sigma^2$. Furthermore, the application $q \mapsto v(S,q)$ is convex and continuous. It is easily seen that $\frac{\partial^2}{\partial q^2} v(S,q) = 4\sigma^2 \alpha S > 0$.

For a Dirichlet problem of type (9)-(10), the existence of a classical solution can be proved only under additional regularity assumptions, namely $h \in C^3$ (see theorem 3.1 in [7]) . This condition is too strong for practical applications because the more common payoff functions, the European call and put options, are not differentiable.

In many cases where the Black-Scholes equation does not admit a classical solution, the fair price of a contingent claim is still a solution of the Black-Scholes equation, but in the viscosity sense. This is a weaker concept of solution that we present below.

We consider $F : \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}^d \mapsto \mathbb{R}$, where $\mathcal{O} \subset \mathbb{R}^d$ is open and $\mathcal{S}^d$ is the space of symmetric matrices of size d. The concept of elliptic function is essential to the theory of viscosity solutions.

Definition 3.1. Let F be a function defined as above. We say that $F(x,r,p,A)$ is elliptic if

$$F(x,r,p,A) - F(x,r,p,B) \leq 0, \tag{13}$$

whenever $A - B$ is positive semi-definite.

Let F be elliptic and consider the Dirichlet problem

$$F(x, u(x), Du(x), D^2u(x)) = 0 \qquad x \in \mathcal{O}, \tag{14}$$

$$u(x) = g(x) \quad x \in \partial\mathcal{O}. \tag{15}$$

For any function $u : \mathcal{O} \mapsto \mathbb{R}$ let $\overline{u}$ and $\underline{u}$ be defined as

$$\overline{u}(x) = \limsup_{z \to x} u(z), \quad \underline{u}(x) = \liminf_{z \to x} u(z) \quad \forall x \in \mathcal{O} \tag{16}$$

Now, we introduce the definition of viscosity solution:

Definition 3.2. Let $F : \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}^d \mapsto \mathbb{R}$ be an elliptic function and let $u : \mathcal{O} \mapsto \mathbb{R}$ be a locally bounded function. u is a viscosity subsolution of the Dirichlet problem (14)-(15) if:

i) $$F(x_0, \overline{u}(x_0), D\phi(x_0), D^2\phi(x_0)) \leq 0$$

for all pairs $(x_0, \phi) \in \mathcal{O} \times C^2(\mathcal{O})$ where x_0 is a local maximizer of the difference $(\overline{u} - \phi)$ on $\mathcal{O}$, and

ii) $$\limsup_{z \to x, z \in \mathcal{O}} u(z) \leq g(x), \qquad \forall x \in \partial\mathcal{O}.$$

u is a viscosity supersolution of the Dirichlet problem (14)-(15) if:

iii) $$F(x_0, \underline{u}(x_0), D\phi(x_0), D^2\phi(x_0)) \geq 0$$

for all pairs $(x_0, \phi) \in \mathcal{O} \times C^2(\mathcal{O})$ where x_0 is a local minimizer of the difference $(\underline{u} - \phi)$ on $\mathcal{O}$, and

iv) $$\liminf_{z \to x, z \in \mathcal{O}} u(z) \geq g(x), \qquad \forall x \in \partial\mathcal{O}.$$

Finally u is a viscosity solution if it is simultaneously a viscosity subsolution and supersolution.

3.2 A modified problem

The ellipticity of the left-hand side of the equation (14) is fundamental to develop the theory of viscosity solutions. We will study the ellipticity of the equation (9) to verify the adaptability of our problem to the viscosity solutions theory.

Considering $F((t,x),r,p,A) = -p_1 - \frac{1}{2}x^2 v(x,a)$[1] we notice that

$$F((t,x),r,p,A) - F((t,x),r,p,B) = -p_1 - \frac{1}{2}x^2 v(x,a) - (-p_1 - \frac{1}{2}x^2 v(x,b)) \tag{17}$$

$$= -\frac{1}{2}x^2(v(x,a) - v(x,b)). \tag{18}$$

So, this means that the equation (9) is elliptic if and only if $v(x,q)$ is an increasing function in the second argument, but this is not guaranteed by assumptions (A1)-(A2) alone.

This motivate Frey and Polte [7] to propose a Modified Problem, with better properties than the Original Problem (14)-(15).

The construction of Frey and Polte's Modified Problem relies on the theory of conjugate functions. The conjugate function of $q \mapsto v(S,q)$ is

$$v^*(S,\lambda) = \sup\{\lambda q - v(S,q) : q \in \mathbb{R}\} \quad \lambda \in \mathbb{R}$$

and the second conjugate function is

$$v^{**}(S,q) = \sup\{\lambda q - v^*(S,q) : \lambda \in \mathbb{R}\}$$

Since $q \mapsto v(S,q)$ is assumed to be convex, it follows that $v = v^{**}$.

[1] In this case, A= $\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$.

The Modified Problem is

$$-\frac{\partial}{\partial t}u - \frac{1}{2}S^2\tilde{v}\left(S, \frac{\partial^2}{\partial S^2}u\right) = 0 \quad (t,S) \in Q \tag{19}$$

$$u(t,S) = h(S) \quad (t,S) \in \partial Q \tag{20}$$

where

$$\tilde{v}(S,q) = \sup\{\lambda q - v^*(S,\lambda) : \lambda \in [\underline{v}, \overline{v}]\}. \tag{21}$$

The approach in [7] consists in viewing the modified Dirichlet problem (19)-(20) as the Hamilton Jacobi Bellman (HJB) equation associated with the following optimal stochastic control problem.

Consider the state process S with dynamic

$$dS_t = \sqrt{\lambda_t}S_t dw_t \tag{22}$$

where $\{\lambda_t\}_{0\le t\le T}$ is a progressively mensurable process with values in the set $[\underline{v}, \overline{v}]$ and the functional to be maximized is

$$J(t,S,\lambda) = E_{t,S}\left(\int_t^\tau -\frac{1}{2}S_\theta^2 v^*(S_\theta, \lambda_\theta)d\theta + h(S_\tau)\right) \tag{23}$$

where $\tau = \inf\{t \ge 0 : (t, S_t) \notin \overline{Q}\}$.

It can be checked that the HJB equation for the problem (22)-(23) is

$$\frac{\partial}{\partial t}u + \sup\left\{\frac{1}{2}S^2\lambda\frac{\partial^2}{\partial u^2}u - \frac{1}{2}S^2 v^*(S,\lambda) : \lambda \in [\underline{v}, \overline{v}]\right\} = 0 \tag{24}$$

which is equivalent to (19).

The problem (22)-(23) consists of choosing the path of stochastic volatility, λ_t, in order to maximize $E_{t,S}\left(h(\tau, S_\tau)\right)$ subtracting the instantaneous control cost $\frac{1}{2}S_u^2 v^*(S_u, \lambda_u)$.

The Modified Problem has better properties than the Original Problem. So, we will explain some properties of $\tilde{v}$ and of equation (19).

From duality theory, the function $\tilde{v}(S,q)$ coincides with $v(S,q)$ at every point (S,q) such that $\underline{v} \leq \frac{\partial}{\partial q}v(S,q) \leq \overline{v}$. Further, we have $\frac{\partial}{\partial q}\tilde{v}(S,q) = \underline{v}$ whenever $\frac{\partial}{\partial q}v(S,q) \leq \underline{v}$ and $\frac{\partial}{\partial q}\tilde{v}(S,q) = \overline{v}$ whenever $\frac{\partial}{\partial q}\tilde{v}(S,q) \geq \overline{v}$. It follows that $\tilde{v}(S,q)$ is convex and $\underline{v} \leq \frac{\partial}{\partial q}\tilde{v}(S,q) \leq \overline{v}$ for all (S,q). Therefore $\tilde{v}(S,.)$ is Lipschitz with $|\tilde{v}(S,q_1) - \tilde{v}(S,q_2)| \leq \overline{v}|q_1 - q_2|$.

Since $\underline{v} > 0$, it follows that $\tilde{v}(S,q)$ is strictly increasing with respect to q. Therefore, equation (19) is elliptic.

The parabolicity of the PDEs is a good property when we study the solution of this equation. We can verify that the equation (19) is parabolic while the equation (9) is not necessarily parabolic. We can rewrite the left hand side of(19) in Taylor formula around $\frac{\partial^2}{\partial S^2}u = 0$,

$$-\frac{\partial}{\partial t}u - \frac{1}{2}S^2\tilde{v}(S,0) - \frac{1}{2}S^2\frac{\partial}{\partial q}\tilde{v}(S,0)\frac{\partial^2}{\partial S^2}u + R\left(S, \frac{\partial}{\partial t}u, \frac{\partial^2}{\partial S^2}u\right) = 0 \tag{25}$$

$$-\frac{\partial}{\partial t}u - \frac{1}{2}S^2\frac{\partial}{\partial q}\tilde{v}(S,0)\frac{\partial^2}{\partial S^2}u + R\left(S, \frac{\partial}{\partial t}u, \frac{\partial^2}{\partial S^2}u\right) = 0 \tag{26}$$

If we write the coefficient values' matrix of the second derivatives of u we will obtain

$$\Gamma_{2,2} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{2}S^2\frac{\partial}{\partial q}\tilde{v}(S,0) \end{pmatrix}$$

The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = -\frac{1}{2}S^2\frac{\partial}{\partial q}\tilde{v}(S,0)$, as $\frac{\partial}{\partial q}\tilde{v}(S,0) \in [\underline{v}, \overline{v}]$ the second eigenvalue is always negative. We conclude that the equation (19) is parabolic. Otherwise, the equation (9) can be non-parabolic because we do not have the guarantee that $\frac{\partial}{\partial q}v(S,0)$ has a unique sign.

For each $0 \leq \underline{v} \leq \overline{v} + \infty$, let $\tilde{v}_{[\underline{v},\overline{v}]}$ denote the correspondent function $\tilde{v}$. $\tilde{v}$ has the following monotonicity property.

Proposition 3.1. *If* $[\underline{v}_1, \overline{v}_1] \subset [\underline{v}_2, \overline{v}_2]$ *then* $\tilde{v}_{[\underline{v}_1,\overline{v}_1]} \leq \tilde{v}_{[\underline{v}_2,\overline{v}_2]}$, $\lim_{\substack{\underline{v}\to 0^+ \\ \overline{v}\to+\infty}} \tilde{v}_{[\underline{v},\overline{v}]}$ *is the greatest monotonic convex function no greater than* v.

Proof. The proof follows directly from the definition of $\tilde{v}$. □

3.3 Existence and uniqueness of viscosity solutions

Below, we explain some properties of viscosity solutions, presented by Frey and Polte.

To prove the existence and uniqueness of viscosity solution, we proceed by several intermediate propositions.

Proposition 3.2. *The Modified Problem has viscosity subsolution,* ϕ*, and viscosity supersolutions,* ψ *such that*

$$\underline{\phi}(t,S) = h(S), \quad \overline{\psi} = h(S) \quad \forall (t,S) \in \partial Q \tag{27}$$

Proof. Consider a sequence $\{\eta_n \in C^\infty\}_{n\in\mathbb{N}}$ such that $\eta_n \leq h$ and η_n converges uniformly to h in $[\underline{S}, \overline{S}]$. If we define $A_n = \max_{S\in[\underline{S},\overline{S}]} \left(-\frac{S^2}{2}\tilde{v}\left(S, \frac{\partial^2}{\partial S^2}\eta_n\right)\right)^+$ then, $u_n = A_n(t-T)+\eta_n(S)$ is a viscosity subsolution. To verify this,

$$-\frac{\partial}{\partial t}u_n - \frac{1}{2}S^2\tilde{v}\left(S, \frac{\partial^2}{\partial S^2}u_n\right) = -A_n - \frac{1}{2}S^2\tilde{v}\left(S, \frac{\partial^2}{\partial S^2}\eta_n\right) \leq 0$$

Then $u(t,S) = \sup_{n\in\mathbb{N}} u_n(t,S)$ is a viscosity subsolution and

$$\lim_{(t,S)\to(T,\hat{S})} u(t,S) = h(\hat{S}). \tag{28}$$

Now, consider a new function $\tilde{u}_n = \frac{C_n}{2}(S-\underline{S})(S-\overline{S}) + \eta_n(S)$ where $C_n \geq \max_{S\in[\underline{S},\overline{S}]}\left(-\frac{\tilde{v}(S,0)}{\overline{v}} + \frac{\overline{v}}{\underline{v}}\left|\frac{\partial^2}{\partial S^2}\eta_n(S)\right|\right)$. This function is also a viscosity subsolution,

$$-\frac{\partial}{\partial t}\tilde{u}_n - \frac{1}{2}S^2\tilde{v}\left(S, \frac{\partial^2}{\partial S^2}\tilde{u}_n\right) \leq -\frac{1}{2}S^2\left(\tilde{v}(S,0) + \underline{v}C_n - \overline{v}\left|\frac{\partial^2}{\partial S^2}\eta_n\right|\right) \leq 0$$

Then, $\tilde{u}(t,S) = \sup_{n\in\mathbb{N}} \tilde{u}_n(t,S)$ is a viscosity subsolution and

$$\lim_{(t,S)\to(\hat{t},\underline{S})} u(t,S) = h(\underline{S}) \quad \lim_{(t,S)\to(\hat{t},\overline{S})} u(t,S) = h(\overline{S}). \tag{29}$$

We conclude that $\phi = max\,(u,\tilde{u})$ is a viscosity subsolution and verifies (28) and (29).

To prove the existence of viscosity supersolution that verifies (28) end (29) we consider a sequence $\{\gamma_n \in C^\infty\}_{n\in\mathbb{N}}$ such that $\gamma_n \geq h$ and γ_n converges uniformly to h in $[\underline{S},\overline{S}]$. If we define $B_n = \inf_{S\in[\underline{S},\overline{S}]}\left(-\frac{S^2}{2}\tilde{v}\left(S, \frac{\partial^2}{\partial S^2}\gamma_n\right)\right)^-$ then, $w_n = B_n(t-T) + \gamma_n(S)$ is a viscosity supersolution as well as the function $w(t,S) = \inf_{n\in\mathbb{N}} w_n(t,S)$ and this verifies (28). On the other hand, $\tilde{w}_n = \frac{D_n}{2}(S-\underline{S})(S-\overline{S}) + \gamma_n(S)$, with $D_n \leq \inf_{S\in[\underline{S},\overline{S}]}\left(-\tilde{v}(S,0) - \frac{\overline{v}}{\underline{v}}\left|\frac{\partial^2}{\partial S^2}\gamma_n(S)\right|\right)$, is a viscosity supersolution as well as the function $\tilde{w}(t,S) = \inf_{n\in\mathbb{N}} \tilde{w}_n(t,S)$ and this verifies (29). We conclude that $\psi = max\,(v,\tilde{v})$ is a viscosity supersolution and verifies (28) and (29). □

We can establish some relations between the viscosity subsolutions of two modified equations.

Proposition 3.3. *If* $[\underline{v}_1,\overline{v}_1] \subset [\underline{v}_2,\overline{v}_2]$ *then, every viscosity subsolution of* $-\frac{\partial}{\partial t}u - \frac{1}{2}S^2\tilde{v}_{[\underline{v}_1,\overline{v}_1]}\left(S, \frac{\partial}{\partial S^2}u\right) = 0$ *is also viscosity subsolution of* $-\frac{\partial}{\partial t}u - \frac{1}{2}S^2\tilde{v}_{[\underline{v}_2,\overline{v}_2]}\left(S, \frac{\partial}{\partial S^2}u\right) = 0.$

Proof. Pick $\phi \in C^2$ and using J^1 as a viscosity subsolution of the first equation and $(t_1, x_1) = arg\max(J^1 - \phi)$, so by definition

$$-\frac{\partial}{\partial t}\phi(t_1,x_1) - \frac{1}{2}x_1^2\tilde{v}_{[\underline{v}_1,\overline{v}_1]}\left(x_1, \frac{\partial^2}{\partial S^2}\phi(t_1,x_1)\right) \leq 0. \tag{30}$$

As $\tilde{v}_{[\underline{v}_2,\overline{v}_2]}(S,q) \geq \tilde{v}_{[\underline{v}_1,\overline{v}_1]}(S,q)$ the inequality is verified

$$-\frac{\partial}{\partial t}\phi(t_1,x_1) - \frac{1}{2}x_1^2\tilde{v}_{[\underline{v}_2,\overline{v}_2]}\left(x_1, \frac{\partial^2}{\partial S^2}\phi(t_1,x_1)\right) \leq -\frac{\partial}{\partial t}\phi(t_1,x_1) - \frac{1}{2}x_1^2\tilde{v}_{[\underline{v}_1,\overline{v}_1]}\left(x_1, \frac{\partial^2}{\partial S^2}\phi(t_1,x_1)\right) \leq 0 \tag{31}$$

So we conclude that J^1 is viscosity subsolution of the second equation. □

A consequence of this proposition is that when the PDE of the Original Problem is elliptic, a viscosity subsolution of the Modified Problem is a viscosity subsolution of the Original Problem.

Now we want to prove the existence and uniqueness of viscosity solution of the Modified Problem. We start with the proof of existence of viscosity solution of this problem.

Lemma 3.1. *Consider the function*

$$F\left((t,x), \frac{\partial}{\partial S^2}u\right) = -\frac{1}{2}x^2\tilde{v}\left(S, \frac{\partial}{\partial S^2}u\right). \tag{32}$$

If the following assumption is verified,

(A3) The functions $v^*(S,\lambda)$ and $\frac{\partial}{\partial S}v^*(S,\lambda)$ are continuous on $[\underline{S},\overline{S}] \times [\underline{v},\overline{v}]$,

then there is a continuous function $w : [0,\infty) \mapsto [0,\infty)$ that satisfies $w(0) = 0$ such that

$$F((s,y),Y) - F((t,x),X) \leq w\left(|x-y| + |t-s| + \frac{(x-y)^2 + (t-s)^2}{\epsilon}\right) \tag{33}$$

for every $(t,x),(s,y) \in \mathcal{O}$*,* $\epsilon > 0$*, and symmetric matrices* X*,* Y *satisfying*

$$-\frac{3}{\epsilon}\begin{pmatrix} Id & \mathbf{0} \\ \mathbf{0} & Id \end{pmatrix} \leq \begin{pmatrix} X & \mathbf{0} \\ \mathbf{0} & -Y \end{pmatrix} \leq \frac{3}{\epsilon}\begin{pmatrix} Id & -Id \\ -Id & Id \end{pmatrix}. \tag{34}$$

Proof. The proof follows of lemma V.7.1 in [6] when we consider $F((t,x), \frac{\partial}{\partial S^2}u) = -\frac{1}{2}x^2 v(x, \frac{\partial}{\partial S^2}u)$. □

Notice that assumptions (A1) and (A2) do not guarantee the continuity of v^*. The next example illustrates this situation. Consider $v : \mathbb{R}^2 \mapsto \mathbb{R}$ in C^∞ such that

$$v(S,q) = aq + b(S)q^2, \quad a > 0, \quad b(s) \geq 0.$$

In this case, $v^*(S,\lambda) = \frac{(\lambda - a)^2}{2b(S)}$. If $b(S)$ has roots, v^* is discontinuous.

Now, we introduce the comparison principle for a Dirichlet problem.

Definition 3.3. A Dirichlet problem (14),(15) satisfies the comparison principle if the inequality

$$\phi(x) \leq \varphi(x) \tag{35}$$

holds for any viscosity subsolution, ϕ, and any viscosity supersolution, φ, of the Dirichlet problem.

To prove the comparison principle for the Modified Problem (19), (20) we use a strong monotonicity condition that is not verified by the equation (19). This problem is solved using a change variable.

Proposition 3.4. *Let $\tilde{u} = e^{t-T}u$. u is a viscosity solution of the Modified Problem (19), (20), if and only if $\tilde{u}$ is a viscosity solution of the problem*

$$-\frac{\partial}{\partial t}\tilde{u} + \tilde{u} - \frac{1}{2}e^{t-T}S^2\tilde{v}\left(S, e^{T-t}\frac{\partial^2}{\partial S^2}\tilde{u}\right) = 0 \quad (t,S) \in Q \tag{36}$$

$$\tilde{u}(t,S) = e^{t-T}h(S) \quad (t,S) \in \partial Q \tag{37}$$

Proof. Consider the point $(t_0, x_0) = arg\max(\overline{u} - \phi)$, then this point verified $(t_0, x_0) = arg\max(e^{t-T}\overline{u} - e^{t-T}\phi)$. Then, since u is a viscosity subsolution

$$\begin{aligned}
&-\frac{\partial}{\partial t}(e^{t-T}\phi)(t_0,x_0) + e^{t_0-T}u(t_0,x_0) - \frac{1}{2}e^{t_0-T}x_0^2\tilde{v}\left(x_0, e^{T-t_0}\frac{\partial^2}{\partial S^2}\left(e^{t-T}\phi\right)(t_0,x_0)\right)\\
&= e^{t_0-T}\left(-\frac{\partial}{\partial t}\phi(t_0,x_0) - \frac{1}{2}x_0^2\tilde{v}\left(x_0, \frac{\partial^2}{\partial S^2}\phi(t_0,x_0)\right)\right) \leq 0.
\end{aligned}$$

So, $\tilde{u} = e^{t-T}u$ is a viscosity subsolution of the Modified Problem (36),(37). With the same argument we prove that $\tilde{u} = e^{t-T}u$ is a viscosity supersolution of the same problem. □

Notice that the result can be easily generalized for all $\tilde{u} = \gamma(t)u$ with $\gamma(t) \in C^1(\mathbb{R})$, $\gamma(t) > 0$ and $\frac{\partial}{\partial t}\gamma(t) > 0$.

The next theorem shows that our Modified Problem (19),(15) verifies the comparison principle.

Theorem 3.4. *Consider $u : Q \to \mathbb{R}$ and $w : Q \to \mathbb{R}$ viscosity subsolutions and supersolutions, respectively, of the Modified Problem (36), (37) with u an upper semicontinuous function and w a lower semicontinuous function. Then*

$$\sup_Q (u - w) \leq 0. \tag{38}$$

Proof. We let us begin by assuming the opposite, $\max_{(t,S)\in\overline{Q}}(u(t,S) - w(t,S)) > 0$. The

$arg\max(u-w) \in \{0\} \times (\underline{S}, \overline{S}) \cup (0,T) \times (\underline{S}, \overline{S})$. The function $\varphi = u - \frac{\gamma}{t}$ is also a viscosity subsolution and for sufficiently small $\gamma > 0$, $\max_{(t,S)\in\overline{Q}}(\varphi(t,S) - w(t,S)) > 0$. In this case $arg\max(\varphi - w)$ is attained in $(0,T) \times (\underline{S}, \overline{S})$. Consider the function

$$\phi_\epsilon(t,x,s,y) = \varphi(t,x) - w(s,y) - \frac{(x-y)^2 + (t-s)^2}{2\epsilon}.$$

$$(\hat{t}_\epsilon, \hat{x}_\epsilon, \hat{s}_\epsilon, \hat{y}_\epsilon) = arg\max \phi_\epsilon$$

Notice that $\phi_\epsilon(\hat{t}_\epsilon, \hat{x}_\epsilon, \hat{s}_\epsilon, \hat{y}_\epsilon) \geq \max_{(t,S)\in\overline{Q}}(\varphi(t,S) - w(t,S))$. Moreover,

$$\lim_{\epsilon\to 0} \phi_\epsilon(\hat{t}_\epsilon, \hat{x}_\epsilon, \hat{s}_\epsilon, \hat{y}_\epsilon) = \max_{(t,S)\in\overline{Q}}(\varphi(t,S) - w(t,S)) = \phi(\hat{t}_0, \hat{x}_0)$$
$$\lim_{\epsilon\to 0} \frac{(x_\epsilon - y_\epsilon)^2 + (t_\epsilon - s_\epsilon)^2}{\epsilon} = 0$$

and for all $\epsilon > 0$ there is $X_\epsilon, Y_\epsilon \in \mathcal{S}^2$ such that

$$\left(\frac{(\hat{x}_\epsilon - \hat{y}_\epsilon) + (\hat{t}_\epsilon - \hat{s}_\epsilon)}{\epsilon}, X_\epsilon\right) \in J^+_{\mathcal{Q}}\varphi(\hat{t}_\epsilon, \hat{x}_\epsilon), \quad \left(\frac{(\hat{x}_\epsilon - \hat{y}_\epsilon) + (\hat{t}_\epsilon - \hat{s}_\epsilon)}{\epsilon}, Y_\epsilon\right) \in J^-_{\mathcal{Q}}w(\hat{s}_\epsilon, \hat{y}_\epsilon)$$

$$-\frac{3}{\epsilon}\begin{pmatrix} Id & \mathbf{0} \\ \mathbf{0} & Id \end{pmatrix} \leq \begin{pmatrix} X_\epsilon & \mathbf{0} \\ \mathbf{0} & -Y_\epsilon \end{pmatrix} \leq \frac{3}{\epsilon}\begin{pmatrix} Id & -Id \\ -Id & Id \end{pmatrix}.$$

that is guaranteed by lemma A.3.

For the function $H\left((t,x), u, D^2u\right) = u - \frac{1}{2}e^{t-T}S^2\tilde{v}\left(S, e^{T-t}\frac{\partial^2}{\partial S^2}u\right)$ the following is true:

$$\max_{(t,x)\in\overline{Q}}(\varphi(t,x) - w(t,x)) \leq \varphi(\hat{t}_\epsilon, \hat{x}_\epsilon) - w(\hat{s}_\epsilon, \hat{y}_\epsilon)$$
$$= H\left((\hat{t}_\epsilon, \hat{x}_\epsilon), \varphi(\hat{t}_\epsilon, \hat{x}_\epsilon), X_\epsilon\right) - H\left((\hat{t}_\epsilon, \hat{x}_\epsilon), w(\hat{s}_\epsilon, \hat{y}_\epsilon), X_\epsilon\right)$$
$$= H\left((\hat{t}_\epsilon, \hat{x}_\epsilon), \varphi(\hat{t}_\epsilon, \hat{x}_\epsilon), X_\epsilon\right) - H\left((\hat{s}_\epsilon, \hat{y}_\epsilon), w(\hat{s}_\epsilon, \hat{y}_\epsilon), Y_\epsilon\right)$$
$$+ H\left((\hat{s}_\epsilon, \hat{y}_\epsilon), w(\hat{s}_\epsilon, \hat{y}_\epsilon), Y_\epsilon\right) - H\left((\hat{t}_\epsilon, \hat{x}_\epsilon), w(\hat{s}_\epsilon, \hat{y}_\epsilon), X_\epsilon\right).$$

By the definition of viscosity subsolution and supersolution, it follows that the first difference in the last equality is non positive and lemma 3.1 guarantees that $H\left((\hat{s}_\epsilon, \hat{y}_\epsilon), w(\hat{s}_\epsilon, \hat{y}_\epsilon), Y_\epsilon\right) - H\left((\hat{t}_\epsilon, \hat{x}_\epsilon), w(\hat{s}_\epsilon, \hat{y}_\epsilon), X_\epsilon\right) \leq w\left(|\hat{x}_\epsilon - \hat{y}_\epsilon| + |\hat{t}_\epsilon - \hat{s}_\epsilon| + \frac{(\hat{x}_\epsilon - \hat{y}_\epsilon)^2 + (\hat{t}_\epsilon - \hat{s}_\epsilon)^2}{\epsilon}\right)$. So, we conclude that

$$\max_{(t,x)\in\overline{Q}} (\varphi(t,x) - w(t,x)) \leq m\left(|x_\epsilon - y_\epsilon| + |t_\epsilon - s_\epsilon| + \frac{(x_\epsilon - y_\epsilon)^2 + (t_\epsilon - s_\epsilon)^2}{\epsilon}\right).$$

This is a contradiction to the initial assumption. □

Proposition 3.5. *If the functions v^*, $\frac{\partial}{\partial S} v^*$ are continuous on $[\overline{S}, \underline{S}] \times [\underline{v}, \overline{v}]$, the Modified Problem (19),(20) has a unique viscosity solution.*

Proof. Proposition 3.2 guarantees that there is a viscosity subsolution u and supersolution w that verify

$$\liminf_{(t,S)\to(\tilde{t},\tilde{S}),(t,S)\in Q} w = \limsup_{(t,S)\to(\tilde{t},\tilde{S}),(t,S)\in Q} \varphi = h(S), \qquad \forall(\tilde{t},\tilde{S}) \in \partial Q.$$

Moreover by theorem 3.4, these viscosity subsolution and supersolution verify the comparison principle. Then the existence of viscosity solution follows of the theorem A.4 (Ishii theorem) in appendix A.

Suppose that there are two viscosity solutions $\varphi(t,S)$ and $\psi(t,S)$ that verify $\varphi(t,S) < \psi(t,S)$. By definition $v(t,S)$ and $u(t,S)$ are, simultaneously, viscosity subsolutions and supersolutions. Then, by theorem 3.4, there are viscosity supersolutions and subsolutions that verify $\varphi(t,S) < \psi(t,S)$. So, we have a contradiction with the comparison principle. □

4 The Wilmott-Schönbucher model

In the literature there are comparatively few models taking into account the illiquidity of the risky asset market. Here we present a model proposed by Wilmott and Schönbucher in [16]. We discuss the concept of portfolio value as well as the consistency of the model, i.e., the possibility of the model to collapse due to the influence of the large trader.

4.1 Short introduction to the model

The model considers two types of assets, a risky one, with price S and a risk-free one, with price B. The risk-free asset is taken as numeraire with B_0 normalized to 1. The market of the risk-free asset is perfectly liquid but the market of the risky asset is not. There are two types of agents in the market: A single large trader and a large set of small traders.

The aggregate demand of the risky asset by the small traders at time t is a function $D(S,W,t)$, where S denotes the price of the risky asset and W is a random parameter. Similarly, the aggregate supply by the small traders is a function $Su(S,W,t)$. All information that arrives to small traders is contained in W. Thus, the small traders don't have any knowledge about the presence of the large trader in the market. The excess demand is by definition the difference between demand and supply,

$$\chi(S,W,t) = D(S,W,t) - Su(S,W,t). \tag{39}$$

In the absence of the large trader, the equilibrium price at time t is the solution of the equilibrium equation $\chi(S,W_t,t) = 0$. Assuming $\chi(S,W,t)$ is smooth and

$$\frac{\partial \chi(S,W,t)}{\partial S} < 0, \quad \forall (S,W,t) \in]0,+\infty[\times \mathbb{R} \times [0,+\infty[, \tag{40}$$

the equilibrium price is a unique $C^{2,1}$ function $S = S(W,t)$. Economically, the condition (40) means that when the price goes up the excess of demand goes down, as occurs when supply increases and demand decreases with price.

The quantity of risky asset that the large trader wishes to hold at time t is a function $f(S,t)$ of the price. It is assumed that this function is smooth and it does not depend explicitly of W. In the presence of the large trader, the equilibrium price at time t is the solution of:

$$\chi(S,W,t) + f(S,t) = 0. \tag{41}$$

If the illiquid market counterpart of (40)

$$\frac{\partial \chi(S,W,t)}{\partial S} + \frac{\partial f}{\partial S}(S,t) < 0, \quad \forall (S,W,t) \in]0,+\infty[\times \mathbb{R} \times [0,+\infty[, \tag{42}$$

holds, then the equilibrium price in the illiquid market is again a smooth function $S = S(W,t)$. From now on, we will assume that $W = W_t$ is a standard Brownian motion. In this case, it is easy to derive the dynamics of the price process S_t. Notice that

$$d\chi(S,W,t) + df(S,t) = 0. \tag{43}$$

Assuming that $\frac{\partial}{\partial W}\chi \neq 0$, the inverse function theorem guarantees that W can be expressed as a function of S and t. So, applying the Itô lemma, it is possible to show that the price process solves a stochastic differential equation $dS_t = \mu(S,t)dt + \sigma(S,t)dW_t$, where

$$(44)\quad \mu(S,t) = \frac{-1}{\frac{\partial}{\partial S}\chi(S,W,t) + \frac{\partial}{\partial S}f(S,t)}\left[\frac{\partial}{\partial t}\chi(S,W,t) + \frac{\partial}{\partial t}f(S,t) + \frac{1}{2}\frac{\partial^2}{\partial W^2}\chi(S,W,t)\right.$$
$$+\frac{1}{2}\left(\frac{\frac{\partial}{\partial W}\chi(S,W,t)}{\frac{\partial}{\partial S}\chi(S,W,t) + \frac{\partial}{\partial S}f(S,t)}\right)^2\left(\frac{\partial^2}{\partial S^2}\chi(S,W,t) + \frac{\partial^2}{\partial S^2}f(S,t)\right)$$
$$\left. - \left(\frac{\frac{\partial}{\partial W}\chi(S,W,t)}{\frac{\partial}{\partial S}\chi(S,W,t) + \frac{\partial}{\partial S}f(S,t)}\right)\frac{\partial^2}{\partial S\partial W}\chi(S,W,t)\right]$$
$$(45)\quad \sigma(S,t) = -\frac{\frac{\partial}{\partial W}\chi(S,W,t)}{\frac{\partial}{\partial S}\chi(S,W,t) + \frac{\partial}{\partial S}f(S,t)}.$$

Notice that μ and σ are indeed functions of S and t alone because W can be written as a function of S and t.

On section 5 below we discuss one important case where condition (42) fails.

4.2 The value of the large trader's portfolio

In a perfectly liquid market, any amount of any asset can be converted to cash at market price, therefore the value of a given portfolio at any moment of time is a well-defined quantity.

In the Wilmot-Schönbucher market, the large trader cannot convert the amount of risky asset he holds without changing the market price. This leads to the introduction of two extreme concepts of value of portfolio: the "paper value" and the "liquidation value". The paper value is computed at current market prices

$$(46)\qquad Y_t = f_t S_t + c_t B_t,$$

where f_t is an abbreviation of $f(S_t, t)$ and c_t represents the holding in the bonds. Naturally the paper value is real only if we have the "Black-Scholes economy".

The liquidation value at time t is the amount of money the large trader would obtain if he would hold the portfolio up to time t when he converts all the risky asset he holds in numeraire. In order to compute this, we need to discuss in some detail the mechanism of transaction with the large trader.

Since the large trader seeks the best possible bargain, we assume that he gives priority to higher bidders and lower askers: suppose that the large trader wants to sell the quantity $f_{t-} - f_t$. In this case the large trader checks the quantity and the price that each small trader wants to pay. He sells to first the small traders that offer higher price and only sells to traders who offer lower prices after the first are saturated. As the equilibrium price reacts to the quantity that large trader holds, in limit the value obtained with this transaction is given by

$$-\int_{f_{t-}}^{f_t} S(x, W, t)dx. \tag{47}$$

Naturally, if the large trader wants to liquidate his position the value obtained with this mechanism of transaction is

$$\int_0^{f_t} S(x, W, t)dx, \tag{48}$$

and the liquidation value of the portfolio is

$$Y_t = \int_0^{f_t} S(x, W, t)dx + c_t B_t. \tag{49}$$

4.3 Consistency of the model

In general the models of illiquid markets have theoretical difficulties, because they predict the collapse of the market at least in some situations. The existence of large traders with capacity to influence the market price or the existence of traders with privileged information allows market manipulation. We say that there is a market manipulation when there is some trading strategy that allows to move the price to make risk free profit. If such strategies can be used without limitations, the market collapses because small traders are stripped of their wealth. We say that a model is consistent if there are neither arbitrage possibilities nor possibilities of market manipulation. There are some works as [11] and [12] studying market manipulation. Also in [16] these questions are discussed and it is shown that the Wilomott-Schönbucher model is not consistent. Here we explain the trading strategies called: market corners and the short squeezes.

We say that there is a market corner if the manipulator holds a sizeable part of the shares in the market. As we know, short sellers sell borrowed stock. Thus, the manipulator can lend his shares to short sellers who in turn sell them back (inadvertently) to the manipulator. In this way the manipulator can hold more that 100% of the shares in the market. The manipulator corners the market. He can require the short sellers his shares. If the total supply is held by the manipulator, the short sellers have to buy the shares to the manipulator in order to deliver back to the manipulator. So, the manipulator can decide the price that the short sellers have to pay. In this case we say that there is a short squeeze.

In general, when the price mechanism exhibits a delay in the adjustment the manipulator can buy the asset cheaper than the sale of this asset. In [12] it is shown that to prevent market manipulation the price mechanism cannot exhibit delay in the adjustment.

5 Collective behaviour

Despite the consistency problems mentioned above, the Wilmott-Schönbucher model is attractive to study the consequences of collective behaviour. We present a definition of self-financing strategy different from the one proposed in [16] and derive, heuristically, the Black-Scholes equation for the true value of an option in the presence of collective behaviour by a large group of traders in the market.

5.1 Collective behaviour in the Wilmott-Schönbucher model

Consider a market with a large number of small traders and no large trader. Suppose that a sizeable fraction of these small traders are following similar strategies. Since all traders in this group sell and buy in similar circumstances, they act collectively like a large trader without being aware of this fact. Situations of this kind may arise in real markets due to the widespread use of model-assisted and automatic trading when large numbers of traders use similar models or algorithms.

In this case, the issues related to market manipulation do not apply because the individual traders are unaware of their mutual synchronization and are competitors. The Wilmott-Schönbucher model is attractive to study the consequences of such collective behaviour, due to its relative simplicity and "first principles" approach.

To simplify, we take $\chi(S_t, W_t, t) = \alpha(S_t^* - S_t)$, where $\alpha > 0$ is a constant and S_t^* verifies $dS_t^* = \theta S_t^* dt + \nu S_t^* dW_t$. We assume that the synchronized small traders are hedgers, who try to replicate an European put option using the Black-Scholes strategy. So, this group of hedgers acts like a large trader with delta strategy $f(S_t, t) = N(d_1) - 1$, where $N(d_1) = \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{d_1} \exp\left(-\frac{z^2}{2}\right)$ and $d_1 = \frac{\log(\frac{S_t}{K}) + (r + \frac{\nu^2}{2})(T-t)}{\nu\sqrt{T-t}}$.

5.2 The price process

It is important to understand the price mechanism in our example to deduce the Black-Scholes equation for this market. The geometric shape of the strategy of the large trader varies as the time approaches to maturity. Near the maturity the strategy of the large trader approaches a step function with step from -1 to 0 as shown in Figure 1. On the other hand, the function excess of demand is linear. So, near the maturity, the function $S \mapsto \chi(S, W, t) + f(S, t)$ is S-shaped as shown in Figure 2. This function has one local minimizer and one local maximizer which we denote by $x_1(t)$ and $x_2(t)$, respectively. We also define the points $\Sigma_1(t)$, $\Sigma_2(t)$ as the unique solutions to

$$\begin{cases} \chi(S, W, t) + f(S, t) = \chi(x_i(t), W, t) + f(x_i(t), t) \\ S \neq x_i(t) \end{cases} \tag{50}$$

$i = 1, 2$. Notice that in our case $x_1(t), x_2(t), \Sigma_1(t), \Sigma_2(t)$ do not depend on W. Now we discuss the price equilibria.

The sequence of Figures 3,4 and 5 illustrates one possible sequence for the price equilibrium. In Figure 3, there is a unique equilibrium price, $x(t)$ and any small perturbation caused by the Brownian motion moves only a little the equilibrium price. In the Figure 4 there are three possibles equilibria, $x(t), y(t)$ and $z(t)$. The middle equilibrium, $y(t)$, is unstable. Notice that around $y(t)$ the slope of $\chi(S, W, t) + f(S, t)$ is positive. This means that the bigger the positive price variation is, the greater the positive excess of demand variation gets. So, at price $y(t)$ any perturbation moves the equilibrium price to $x(t)$, or to $z(t)$, which are stable equilibria. Finally consider the case when there are two equilibria (Figures 5 and 6). In figure 5 the equilibria are $x_1(t)$ and $\Sigma_1(t)$. $\Sigma_1(t)$, is stable and acts as the equilibrium price in the Figure 3. The equilibrium, $x_1(t)$, is stable with respect to negative price perturbation but any positive price perturbation moves the equilibrium price to a price in the positive slope and consequently to $\Sigma_1(t)$. So we consider that in the first moment t, where the market price reaches $x_1(t)$ from the left, there is a jump in the price process from $S_{t-} = x_1(t)$ to $S_t = \Sigma_1(t)$. The case in figure 6 is analogous: in the first moment the price reaches $x_2(t)$ from the right there is a price jump from $S_{t-} = x_2(t)$ to $S_t = \Sigma_2(t)$.

5.3 Self-financing strategies

A self-financing strategy is roughly a dynamic strategy where, after the initial moment, a purchase of more shares of risky asset is only financed by the sale of riskless asset, and vice-versa. Therefore, after implementing the strategy, there are not any cash inflows or outflows.

Wilmott and Schönbucher in [16] state that a strategy is self-financed it satisfies:

$$dY_t = f_{t-}dS_t + c_{t-}dB_t. \tag{51}$$

We can obtain this dynamic using the self-financing condition:

$$(f_t - f_{t-})S_t + (c_t - c_{t-})B_t = 0,$$

but this is conceptually unsatisfactory, because it assumes that the large trader trades in block, but as we saw in section 4.2 this is not the optimal trading procedure. If we consider that the large trader uses the optimal form of trade, then, the self-financing condition is

$$\int_{f_{t-}}^{f_t} S(x, W_t, t)dx + (c_t - c_{t-})B_t = 0.$$

Now, we want to obtain the dynamic of the self-financing strategy to the collective behaviour model. To simplify our task we set $B_t \equiv 1$. As seen below, the equilibrium price jumps when it hits the minimizer or the maximizer, so we can formalize the portfolio value with the following processes. Let $\Gamma_i(W,t)$, with $i=1,2$, be the functions:

$$\Gamma_1(W,t) = \min\{S : \alpha(S^* - S) + f(S,t) = 0\} \tag{52}$$

$$\Gamma_2(W,t) = \max\{S : \alpha(S^* - S) + f(S,t) = 0\} \tag{53}$$

So, we can define the cadlag process (S_t, I_t), where:

$$I_t = \begin{cases} 1 & \text{if} \quad S_{t-} < x_1(t) \vee S_{t-} = x_2(t) \\ 2 & \text{if} \quad S_{t-} > x_2(t) \vee S_{t-} = x_1(t) \end{cases} \tag{54}$$

$$S_t = \Gamma_{I_t}(W_t, t) \tag{55}$$

The strategy of the big portion of the small investors is $\phi(W_t,t) = f\left(\Gamma_{I_t}(W_t,t),t\right)$. In consequence, the value of the portfolio is

$$Y_t = \phi(W_t,t)\Gamma_{I_t}(W_t,t) + c_t, \tag{56}$$

and the dynamics of Y_t is obtained by the Itô Lemma:

$$\begin{aligned} dY_t =& d\left(\phi(W_t,t)\Gamma_{I_t}(W_t,t)\right) + dc_t \\ =& \Gamma_{I_t}(W_t,t)d\phi(W_t,t) + \phi(W_t,t)d\Gamma_{I_t}(W_t,t) + d\phi(W_t,t)d\Gamma_{I_t}(W_t,t) + dc_t \end{aligned} \tag{57}$$

There are 4 possibles scenarios:

$$I_{t-} = 1; I_t = 1 \longrightarrow S_{t-} < x_1(t) \tag{58}$$

$$I_{t-} = 2; I_t = 1 \longrightarrow S_{t-} = x_2(t) \tag{59}$$

$$I_{t-} = 1; I_t = 2 \longrightarrow S_{t-} = x_1(t) \tag{60}$$

$$I_{t-} = 2; I_t = 2 \longrightarrow S_{t-} > x_2(t) \tag{61}$$

To guarantee that our self-financing condition is verified we do:

$$
\begin{aligned}
dc_t = c_t - c_{t-} &= -\int_{f_{t-}}^{f_t} S(x, W_t, t)dx = -\int_{f_{t-}}^{f_t} \left(S^*(W_t, t) + \frac{x}{\alpha}\right) dx \\
&= -\left(S^*(W_t, t) + \frac{f_t + f_{t-}}{2\alpha}\right)(f_t - f_{t-}) \qquad (62)\\
&= -\left(S^*(W_{t-}, t-) + \frac{f_{t-}}{\alpha} + S^*(W_t, t) - S^*(W_{t-}, t-) + \frac{f_t - f_{t-}}{2\alpha}\right)(f_t - f_{t-}) \\
&= -\left(S\left(f_{t-}, W_{t-}, t-\right) + S^*(W_t, t) - S^*(W_{t-}, t-) + \frac{f_t - f_{t-}}{2\alpha}\right)(f_t - f_{t-}) \\
&= -\left(S\left(f_{t-}, W_{t-}, t-\right) + dS_t^* + \frac{1}{2\alpha} df_t\right) df_t.
\end{aligned}
$$

We can rewrite this result as

$$
dc_t = -\left(\Gamma_{I_{t-}}(W_{t-}, t-) + dS_t^* + \frac{1}{2\alpha} d\phi_t\right) d\phi_t \qquad (63)
$$

To calculate the dynamics of $\Gamma_{I_t}(W_t, t)$ we notice that:

$$
\begin{aligned}
d\Gamma_{I_t} =& \Gamma_{I_t}(W_t, t) - \Gamma_{I_{t-}}(W_{t-}, t-) \\
=& \Gamma_{I_t}(W_t, t) - \Gamma_{I_{t-}}(W_t, t) + \Gamma_{I_{t-}}(W_t, t) - \Gamma_{I_{t-}}(W_{t-}, t-) \qquad (64)\\
=& \Gamma_{I_t}(W_t, t) - \Gamma_{I_{t-}}(W_t, t) + \left(\frac{\partial \Gamma_{I_{t-}}}{\partial t} + \frac{\partial^2 \Gamma_{I_{t-}}}{\partial W^2}\right) dt + \frac{\partial \Gamma_{I_{t-}}}{\partial W} dW_t
\end{aligned}
$$

where the difference of two first terms can be simplified in:

$$
\Gamma_{I_t}(W_t, t) - \Gamma_{I_{t-}}(W_t, t) = \begin{cases} 0 & \text{if } \ I_{t-} = I_t \\ \Gamma_{I_t}(W_t, t) - x_2(t) & \text{if } \ I_{t-} = 2 \ \text{ and } \ I_t = 1 \\ \Gamma_{I_t}(W_t, t) - x_1(t) & \text{if } \ I_{t-} = 1 \ \text{ and } \ I_t = 2. \end{cases} \qquad (65)
$$

To derive the dynamics of ϕ_t we need to observe the 4 different scenarios:

$$d\phi_t = \phi_t - \phi_{t-} = \begin{cases} f\left(\Gamma_1(W_t,t),t\right) - f\left(\Gamma_1(W_{t-},t-),t-\right) & \text{if} \quad S_{t-} < x_1(t) \\ f\left(\Gamma_1(W_t,t),t\right) - f\left(\Gamma_2(W_{t-},t-),t-\right) & \text{if} \quad S_{t-} = x_2(t) \\ f\left(\Gamma_2(W_t,t),t\right) - f\left(\Gamma_1(W_{t-},t-),t-\right) & \text{if} \quad S_{t-} = x_1(t) \\ f\left(\Gamma_2(W_t,t),t\right) - f\left(\Gamma_2(W_{t-},t-),t-\right) & \text{if} \quad S_{t-} > x_2(t) \end{cases}. \tag{66}$$

In the scenario (58) and (61) we have that:

$$\begin{aligned} f\left(\Gamma_i(W_t,t),t\right) - f\left(\Gamma_i(W_{t-},t-),t-\right) &= \frac{\partial f}{\partial \Gamma_i} d\Gamma_i + \frac{\partial f}{\partial t} dt + \frac{1}{2}\frac{\partial^2 f}{\partial \Gamma_i^2}\left(d\Gamma_i\right)^2 \\ &= \frac{\partial f}{\partial \Gamma_i}\left(\frac{\partial \Gamma_i}{\partial W} dW_t + \frac{\partial \Gamma_i}{\partial t} dt + \frac{1}{2}\frac{\partial^2 \Gamma_i}{\partial W^2} dt\right) + \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial \Gamma_i^2}\left(\frac{\partial \Gamma_i}{\partial W}\right)^2\right) dt \\ &= \left(\frac{\partial f}{\partial \Gamma_i}\frac{\partial \Gamma_i}{\partial t} + \frac{1}{2}\left(\frac{\partial f}{\partial \Gamma_i}\frac{\partial^2 \Gamma_i}{\partial W^2} + \frac{\partial^2 f}{\partial \Gamma_i^2}\left(\frac{\partial \Gamma_i}{\partial W}\right)^2\right) + \frac{\partial f}{\partial t}\right) dt + \frac{\partial f}{\partial \Gamma_i}\frac{\partial \Gamma_i}{\partial W} dW_t \\ &= \mu_i(t-, S_{t-})dt + \sigma_i(t-, S_{t-})dW_t \end{aligned} \tag{67}$$

When $S_{t-} = x_2(t)$ the dynamics of the self-financing strategy is:

$$\begin{aligned} f\left(\Gamma_1(W_t,t),t\right) - f\left(\Gamma_2(W_{t-},t-),t-\right) &= f\left(\Gamma_1(W_t,t),t\right) - f\left(\Gamma_2(W_t,t),t\right) \\ &+ f\left(\Gamma_2(W_t,t),t\right) - f\left(\Gamma_2(W_{t-},t-),t-\right) \\ &= f\left(\Gamma_1(W_t,t),t\right) - f\left(x_2(t),t-\right) + \mu_2(t-, S_{t-})dt + \sigma_2(t-, S_{t-})dW_t \end{aligned} \tag{68}$$

Finally we have the situation $S_{t-} = x_1(t)$ and the dynamics is:

$$\begin{aligned} f\left(\Gamma_2(W_t,t),t\right) - f\left(\Gamma_1(W_{t-},t-),t-\right) &= f\left(\Gamma_2(W_t,t),t\right) - f\left(\Gamma_1(W_t,t),t\right) \\ &+ f\left(\Gamma_1(W_t,t),t\right) - f\left(\Gamma_1(W_{t-},t-),t-\right) \\ &= f\left(\Gamma_2(W_t,t),t\right) - f\left(x_1(t),t-\right) + \mu_1(t-, S_{t-})dt + \sigma_1(t-, S_{t-})dW_t \end{aligned} \tag{69}$$

Notice that $\frac{\partial \Gamma_i}{\partial W}$ is unbounded. We have the equilibrium condition (41) from which we can write $S(W,t) = g(W,t)$. So, the implicit function theorem guarantees that:

$$\frac{\partial \Gamma_i}{\partial W} = -\frac{\partial\left(\chi(S,W,t)+f(S,t)\right)}{\partial W}\left(\frac{\partial\left(\chi(S,W,t)+f(S,t)\right)}{\partial S}\right)^{-1}. \tag{70}$$

When $S \to x_1(t)$ or $S \to x_2(t)$ the denominator tends to 0, then $\frac{\partial \Gamma_i}{\partial W}$ is unbounded. For each scenario we can specify a little more the dynamics of c_t. When we have $S_{t-} < x_1(t)$ or $S_{t-} > x_2(t)$ the dynamics of c_t is:

$$\begin{aligned} dc_t &= -\left(S\left(f_{t-}, W_{t-}, t-\right) + dS_t^* + \frac{1}{2\alpha}d\phi_t\right)d\phi_t \\ &= -\left(S_{t-} + \theta S^* dt + \nu S^* dW_t + \frac{1}{2\alpha}\left(\mu_i(t-,S_{t-})dt + \sigma_i(t-,S_{t-})dW_t\right)\right) \qquad (71)\\ &\quad \times \left(\mu_i(t-,S_{t-})dt + \sigma_i(t-,S_{t-})dW_t\right) \\ &= -\left(S_{t-}\mu_i(t-,S_{t-}) + (\nu S^*)\sigma_i(t-,S_{t-}) + \frac{\sigma_i^2(t-,S_{t-})}{2\alpha}\right)dt - S_{t-}\sigma_i(t-,S_{t-})dW_t, \end{aligned}$$

for $i = 1$ or $i = 2$ respectively. There are two other cases $S_{t-} = x_1(t)$ and $S_{t-} = x_2(t)$. We derive the dynamics of c_t for the $S_{t-} = x_1(t)$:

$$\begin{aligned} dc_t = &-\Big(S_{t-} + \theta S^* dt + \nu S^* dW_t \\ &+ \frac{1}{2\alpha}\left(f(\Gamma_2(W_t,t),t)) - f(x_1(t),t) + \mu_1(t-,S_{t-})dt + \sigma_1(t-,S_{t-})dW_t\right)\Big) \\ &\times \left(f(\Gamma_2(W_t,t),t)) - f(x_1(t),t) + \mu_1(t-,S_{t-})dt + \sigma_1(t-,S_{t-})dW_t\right) \\ = &-\left(S_{t-} + \frac{1}{2\alpha}(f(\Gamma_2(W_t,t),t)) - f(x_1(t),t))\right)\left(f(\Gamma_2(W_t,t),t)) - f(x_1(t),t)\right) \\ &-\left(\left(\theta S^* + \frac{\mu_1(t-,S_{t-})}{\alpha}\right)dt + \left(\nu S^* + \frac{\sigma_1(t-,S_{t-})}{\alpha}\right)\right)\left(f(\Gamma_2(W_t,t),t)) - f(x_1(t),t)\right) \\ &-\left(S_{t-}\mu_1(t-,S_{t-}) + (\nu S^*)\sigma_1(t-,S_{t-}) + \sigma_1^2(t-,S_{t-})\right)dt - S_{t-}\sigma_1(t-,S_{t-})dW_t \end{aligned}$$

for the case $S_{t-} = x_2(t)$ the derivation of the dynamics of c_t is similar.

5.4 The Black-Scholes equation

In this section we want to derive the the Black-Scholes equation for the collective behaviour in the Wilmott-Schönbucher model.

First we present the dynamic of the price process:

$$\sigma(S_t,t) = \frac{\alpha\nu S^*}{\alpha - \frac{\partial}{\partial S}f(S_t,t)} \tag{72}$$

$$\mu(S_t.t) = \frac{1}{\alpha - \frac{\partial}{\partial S}f(S_t,t)}\left(\alpha\theta S^* + \frac{\partial}{\partial t}f(S_t,t) + \frac{1}{2}\sigma^2(S_t,t)\frac{\partial^2}{\partial S^2}f(S_t,t)\right) \tag{73}$$

We notice that the authors write the drift of the diffusion as a function of S_t and t. Indeed, we can write the Brownian Motion, W_t, as a function of S_t and t. So, it's easy to show that the drift and the volatility of the diffusion is given by:

$$\sigma(S_t,t) = \frac{\alpha\nu\left(S - f(S_t,t)\right)}{\alpha - \frac{\partial}{\partial S}f(S_t,t)} \tag{74}$$

$$\mu(S_t.t) = \frac{1}{\alpha - \frac{\partial}{\partial S}f(S_t,t)}\left(\alpha\theta\left(S - f(S_t,t)\right) + \frac{\partial}{\partial t}f(S_t,t) + \frac{1}{2}\sigma^2(S_t,t)\frac{\partial^2}{\partial S^2}f(S_t,t)\right) \tag{75}$$

If we consider that there are no jumps between t^- and t then, we can obtain the dynamics of the self-financing strategy value such as in the last section. This dynamics can be simplified by considering

$$dY_t = a(S_t,t)dt + b(S_t,t)dW_t \tag{76}$$

Then, the HJB equation and the usually boundary condition are is given by

$$V_t(t,S,Y) + \mathcal{L}V(t,S,Y) = 0, \quad \forall(t,S,Y) \in [0,T[\times\mathbb{R}^+\times\mathbb{R} \tag{77}$$

$$V(T,S) = \Phi(S,Y), \quad \forall(S,Y) \in \mathbb{R}^+\times\mathbb{R} \tag{78}$$

where $\mathcal{L}$ is the infinitesimal generator, i.e., the operator defined as:

$$\begin{aligned}\tilde{\mathcal{L}}g =& \lim_{h\to 0^+} \frac{E_t^{s,y}\left[g(S_{t+h},Y_{t+h})\right]-g(s,y)}{h}\\ =&\frac{\partial g}{\partial y}(S_t,Y_t)a(S_t,t)+\frac{\partial g}{\partial s}(S_t,Y_t)\mu(S_t,t)+\frac{\partial^2 g}{\partial y\partial s}(S_t,Y_t)\sigma(S_t,t)b(S,t)\\ &+\frac{1}{2}\left(\frac{\partial^2 g}{\partial y^2}(Y_t,S_t)b^2(S_t,t)+\frac{\partial^2 g}{\partial s^2}(Y_t,S_t)\sigma^2(S_t,t)\right)\end{aligned} \tag{79}$$

for g smooth and bounded.

In the scenario (60) we have a jump when the price reaches the minimum price $x_1(t)$. So, in the moment t, the value of S attains the value $x_1(t)$ and jumps to the value $\Sigma_1(t)$. We need to compute the jump in the self-financing strategy value:

$$\begin{aligned}Y_t =&Y_{t-}-\left(x_1(t)+\frac{f(\Sigma_1(t),t)-f(x_1(t),t)}{2\alpha}\right)\big(f(\Sigma_1(t),t)-f(x_1(t),t)\big)\\ &+f(\Sigma_1(t),t)\Sigma_1(t)-f(x_1(t),t)x_1(t)\\ =&Y_{t-}-\alpha\left(x_1(t)+\frac{\Sigma_1(t)-x_1(t)}{2}\right)\big(\Sigma_1(t)-x_1(t)\big)\\ &+f(\Sigma_1(t),t)\Sigma_1(t)-f(x_1(t),t)x_1(t)\\ =&Y_{t-}-\alpha\frac{\Sigma_1(t)+x_1(t)}{2}\big(\Sigma_1(t)-x_1(t)\big)+f(\Sigma_1(t),t)\Sigma_1(t)-f(x_1(t),t)x_1(t)\end{aligned} \tag{80}$$

On the other hand, in the scenario (61), in the moment t the price process S_t attains $x_2(t)$ and jumps to the value $\Sigma_2(t)$.

The jump in the self-financing strategy value is given by:

$$Y_t = Y_{t-}-\alpha\frac{\Sigma_2(t)+x_2(t)}{2}\big(\Sigma_2(t)-x_2(t)\big)+f(\Sigma_2(t),t)\Sigma_2(t)-f(x_2(t),t)x_2(t) \tag{81}$$

Then the Black-Scholes equation is (77) and (78) adding the conditions:

$$\begin{aligned} V(t,x_1(t),y) =&V\Bigg(t,\Sigma_1(t),y-\alpha\frac{\Sigma_1(t)+x_1(t)}{2}\big(\Sigma_1(t)-x_1(t)\big) \\ &+f(\Sigma_1(t),t)\Sigma_1(t)-f(x_1(t),t)x_1(t)\Bigg),\ \forall(t,Y)\in[0,T[\times\mathbb{R} \end{aligned} \tag{82}$$

$$\begin{aligned} V(t,x_2(t),y) =&V\Bigg(t,\Sigma_2(t),y-\alpha\frac{\Sigma_2(t)+x_2(t)}{2}\big(\Sigma_2(t)-x_2(t)\big) \\ &+f(\Sigma_2(t),t)\Sigma_2(t)-f(x_2(t),t)x_2(t)\Bigg),\ \forall(t,Y)\in[0,T[\times\mathbb{R} \end{aligned} \tag{83}$$

The jumps in the price process suggest that the solution, if it exists, is not continuous. Observe the Figure 6 where we try to illustrate the boundary conditions (78), (82) and (83). Suppose that there is a continuous solution, then $\lim_{S\to x_1(t)^-}\lim_{t\to T^-}V(t,S,Y)=\lim_{t\to T^-}\lim_{S\to x_1(t)^-}V(t,S,Y)$. However this is not verified. Notice that,

$$\begin{aligned} &\lim_{S\to x_1(t)^-}\lim_{t\to T^-}V(t,S,Y)=V(T,K,Y) \\ &\lim_{t\to T^-}\lim_{S\to x_1(t)^-}V(t,S,Y)=V(T,K+\frac{1}{\alpha},Y-\frac{1}{2\alpha}) \end{aligned}$$

where K is the strike price. Naturally, $V(T,K,Y)=V(T,K+\frac{1}{\alpha},Y-\frac{1}{2\alpha})$ for all $Y\in\mathbb{R}$ is not verified.

6 Conclusion and open questions

The existence of large traders and collective behaviours in financial markets justifies the importance of models taking into account market illiquidity. The Black-Scholes equations derived from these models are non-linear, so the concept of viscosity solutions is useful in this setting. For a problem proposed by Frey and Polte in [7], we show the existence and uniqueness of viscosity solutions and we discuss also some other properties.

We also presented a model proposed by Wilmott and Schönbucher and some problems related to these models are discussed. In many cases these models allow the collapse of the market, a problem that we avoid by considering the particular case of collective behaviour. In the context of this model, we deduce the concept of self-financing strategy.

In this work we presented some relevant questions to be further developed that we list:

1. Prove that the process (S_t, I_t) is Markov;

2. Study the consequences of the fact that the function $\frac{\partial \Gamma_i}{\partial W}$ is unbounded and the possible solutions for this potential problem;

3. Verify the dynamic programming principle in the derivation of the HJB equation for the collective behaviour case;

4. The Black-Scholes equation for our model of collective behaviour has unusual boundary conditions. Due to this, it is not clear if the concept of viscosity solution applies to this problem.

A Complementary results about viscosity solutions

We present some theory about viscosity solutions. We give the results without proofs, which can be found in many books such as [6] or other publications such as [4]. Here we consider $F : \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}^d \mapsto \mathbb{R}$, the equation

$$F(x, u(x), Du(x), D^2u(x)) = 0, \tag{84}$$

and we define below the concept of viscosity solution. We present an alternative definition based in the second order Taylor form. Let ψ be a lower semi-continuous function. We consider $(x_0, \phi) \in \mathcal{O} \times C^2(\mathcal{O})$ where x_0 is a local maximum point of the difference $(\overline{\psi} - \phi)$ on $\mathcal{O}$. To simplify we set $p = D\phi$ and $A = D^2\phi$. Motivated by the second order Taylor form we have that:

$$\psi(x) \geq \psi(x_0) + p.(x - x_0)^T + \frac{1}{2}(x - x_0)A(x - x_0)^T + o\left(|x - x_0|^2\right). \tag{85}$$

For the upper semi-continuous function, φ, and the pair $(x_0, \phi) \in \mathcal{O} \times C^2(\mathcal{O})$ where x_0 is a local minimum point of the difference $(\overline{\varphi} - \phi)$ on $\mathcal{O}$ we have

$$\varphi(x) \leq \varphi(x_0) + p.(x - x_0)^T + \frac{1}{2}(x - x_0)A(x - x_0)^T + o\left(|x - x_0|^2\right). \tag{86}$$

Definition A.1. The subjet of the function ψ at the point x_0 is defined as

$$J^{-}_{\mathcal{O}}\psi(x_0) = \{(p, A) \in R^d \times \mathcal{S}^d : (x_0, p, A) \ \text{ verify } \ (85)\} \tag{87}$$

The subjet of the function φ at the point x_0 is defined as

$$J^+_{\mathcal{O}}\varphi(x_0) = \{(p, A) \in R^d \times \mathcal{S}^d : (x_0, p, A) \text{ verify } (86)\} \tag{88}$$

Proposition A.1. *Let φ be an upper semi-continuous function. φ is a viscosity subsolution of (84) if*

$$F\left(x, \overline{u}(x), p, A\right) \leq 0 \tag{89}$$

for all $(p, A) \in J^+_{\mathcal{O}}\varphi(x)$

Similarly, let ψ be an lower semi-continuous function. ψ is a viscosity supersolution of (84) if

$$F\left(x, \overline{u}(x), p, A\right) \geq 0 \tag{90}$$

for all $(p, A) \in J^-_{\mathcal{O}}\psi(x)$

Now we present some results on the stability of viscosity solutions.

Lemma A.1. *Consider an upper semi-continuous function, $\varphi : \mathcal{O} \mapsto \mathbb{R}$, and a triplet $(x_0, p, A) \in \mathcal{O} \times \mathbb{R}^d \times \mathcal{S}^d$ such that $(p, A) \in J^+_{\mathcal{O}}\varphi(x_0)$. Suppose there is a sequence of upper semi-continuous functions $\{\varphi_i : \mathcal{O} \mapsto \mathbb{R}\}_{i\in\mathbb{N}}$ satisfying the conditions:*

1. *There is a sequence $\{x_i \in A\}_{i\in\mathbb{N}}$ such that $\lim(x_i, \varphi_i(x_i)) = (x_0, \varphi(x_0))$;*
2. *Any sequence $\{x_i \in A\}_{i\in\mathbb{N}}$ satisfies $\limsup \varphi(x_i) \leq \varphi(\lim x_i)$.*

In this case there is a sequence $\{(x_0^i, p_i, A_i) \in \mathcal{O} \times \mathbb{R}^d \times \mathcal{S}^d\}_{i\in\mathbb{N}}$ such that:

1. *$(p_i, A_i) \in J^+_{\mathcal{O}}\varphi(x_0^i)$ for every $i \in \mathbb{N}$;*

2. $\lim(x_0^i, \varphi_i(x_0^i), p_i, A_i) = (x_0, \varphi(x_0), p, A)$.

Theorem A.2. *Consider an elliptic continuous function* $F : \mathcal{O}\times\mathbb{R}\times\mathbb{R}^d\times\mathcal{S}^d \mapsto \mathbb{R}$*, and let* $\mathcal{F}$ *be a non-empty set of viscosity subsolutions of (84). If the function* $\phi = \sup\{\varphi : \varphi \in \mathcal{F}\}$ *with* $x \in \mathcal{O}$ *is locally bounded, then it is a viscosity subsolution of (84).*

Theorem A.3. *Consider a sequence of elliptic functions* $\{F_i : \mathcal{O}\times\mathbb{R}\times\mathbb{R}^d\times\mathcal{S}^d \mapsto \mathbb{R}\}_{i\in\mathbb{N}}$*, and fix an elliptic function* $F : \mathcal{O}\times\mathbb{R}\times\mathbb{R}^d\times\mathcal{S}^d \mapsto \mathbb{R}$ *satisfying*

$$F(x, r, p, A) \leq \liminf_{i\to\infty} \left\{ F_j(\tilde{x}, \tilde{r}, \tilde{p}), \tilde{A}) : j > i, (\tilde{x}, \tilde{r}, \tilde{p}) \in B_{\frac{1}{i}}(x, r, p, A) \right\} \tag{91}$$

for every $(x, r, p, A) \in \mathcal{O}\times\mathbb{R}\times\mathbb{R}^d\times\mathcal{S}^d$.

For each $i \in \mathbb{N}$*, let* $\phi_i : \mathcal{O} \mapsto \mathbb{R}$ *be a viscosity subsolution of (84).*

If the function $\phi(x) = \limsup_{i\to\infty} \{\phi_j(z) : j > i, z \in B_{frac1i}(x)\}$ *with* $x \in \mathcal{O}$ *is locally bounded, then it is a viscosity subsolution of (84).*

There are some results concerning Dirichlet problems.

Lemma A.2. *Suppose* F *is continuous and pick* $\phi : \mathcal{O} \mapsto \mathbb{R}$*, a viscosity subsolution of the Dirichelet problem. Suppose there is a triplet* $(x_0, p, A) \in \mathcal{O}\times\mathbb{R}^d\times\mathcal{S}^d$ *such that:*

$$(p, A) \in J_{\mathcal{O}}^{-}\phi(x_0) \tag{92}$$

$$F(x_0, \phi(x_0), p, A) < 0. \tag{93}$$

There exist $\epsilon > 0$ and a viscosity solution, $\phi_\epsilon : \mathcal{O} \mapsto \mathbb{R}$, of the Dirichlet problem such that

$$\sup_{x \in \mathcal{O}} (\phi_\epsilon(x) - \phi(x)) > 0, \tag{94}$$

$$\phi_\epsilon(x) \geq \phi(x), \quad \forall x \in \mathcal{O}, \tag{95}$$

$$\phi_\epsilon(x) = \phi(x), \quad \forall x \in \mathcal{O} \setminus B_\epsilon(x) \tag{96}$$

Theorem A.4 (Ishii)**.** *Suppose that F is continuous and a Dirichlet problem verifies the comparison principle and admits a viscosity subsolution $\phi : \mathcal{O} \mapsto \mathbb{R}$ and a viscosity supersolution $\varphi : \mathcal{O} \mapsto \mathbb{R}$ such that*

$$\liminf_{z \to x, z \in \mathcal{O}} \phi = \limsup_{z \to x, z \in \mathcal{O}} \varphi = h(x), \qquad \forall x \in \partial \mathcal{O} \tag{97}$$

Then, the function

$$u(x) = \sup \{\psi(x) : \psi \ \ \text{is a viscosity subsolution and} \ \ \phi \leq \psi \leq \varphi(x)\} \tag{98}$$

is a viscosity solution of the Dirichlet problem

Finally we present some results for the HJB equation:

$$-\frac{\partial}{\partial t} u(t,x) + F\left(t, x, u(t,x), \frac{\partial}{\partial x} u(t,x), \frac{\partial^2}{\partial t^2} u(t,x)\right) = 0 \tag{99}$$

Lemma A.3. *Let $\phi : \mathcal{O} \mapsto \mathbb{R}$ and $\psi : \mathcal{O} \mapsto \mathbb{R}$ be two locally bounded upper semi-continuous functions. Fix a real function $\varphi \in C^2(\mathcal{O} \times \mathcal{O})$, and let (x_0, y_0) be a maximum point of the function*

$$(x, y) \mapsto \phi(x) + \psi(y) - \varphi(x, y). \tag{100}$$

For every $\epsilon > 0$ such that $Id - \epsilon D^2 \varphi(x_0, y_0) > 0$ there are sequences $\left\{(A_i, B_i) \in \mathcal{S}^d \times \mathcal{S}^d\right\}_{i \in \mathbb{N}}$,

$\{(x_i, y_i) \in \mathcal{O} \times \mathcal{O}\}_{i\in\mathbb{N}}$ *and* $\left\{(p_i, w_i) \in \mathbb{R}^d \times \mathbb{R}^d\right\}_{i\in\mathbb{N}}$ *such that*

1. $\lim(x_i, y_i) = (x_0, y_0)$, $\lim \phi(x_i) = \phi(x_0)$, $\lim \psi(y_i) = \psi(y_0)$, $\lim(p_i, w_i) = \left(\frac{\partial}{\partial x}\varphi(x_0, y_0), \frac{\partial}{\partial y}\varphi(x_0, y_0)\right)$ *and* $\lim(A_i, B_i) = (A, B)$ *for some* $(A, B) \in \mathcal{S}^d \times \mathcal{S}^d$*;*

2.

$$(p_i, A_i) \in J^+_{\mathcal{O}}\phi(x_i) \tag{101}$$

$$(w_i, B_i) \in J^+_{\mathcal{O}}\psi(y_i) \tag{102}$$

3.

$$-\frac{1}{\epsilon} Id \le \begin{pmatrix} X & \mathbf{0} \\ \mathbf{0} & Y \end{pmatrix} \le \left(Id - \epsilon D^2\varphi(x_0, y_0)\right)^{-1} D^2\varphi(x_0, y_0). \tag{103}$$

B Graphics

The following figures complement the explanation of the price process developed in section 5.2. We consider t fixed, close to the maturity.

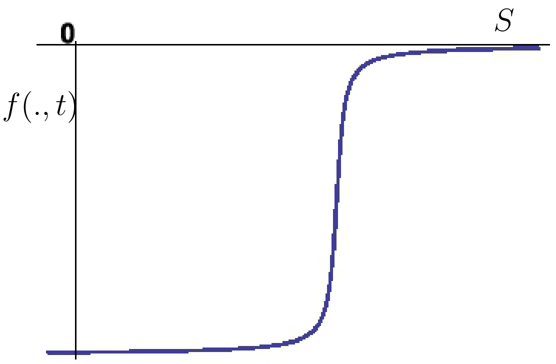

Figure 1: The large trader strategy.

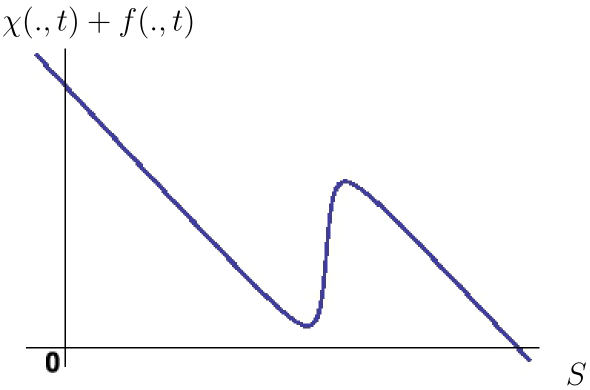

Figure 2: The excess demand function.

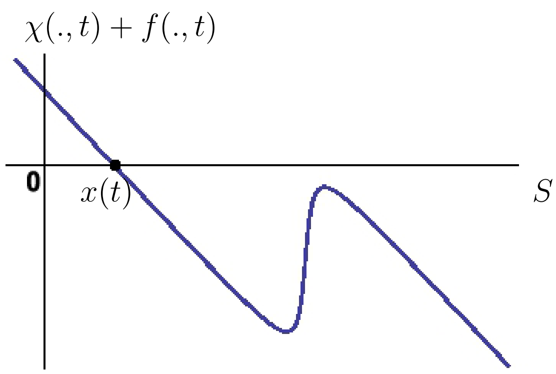

Figure 3: A stable equilibrium price.

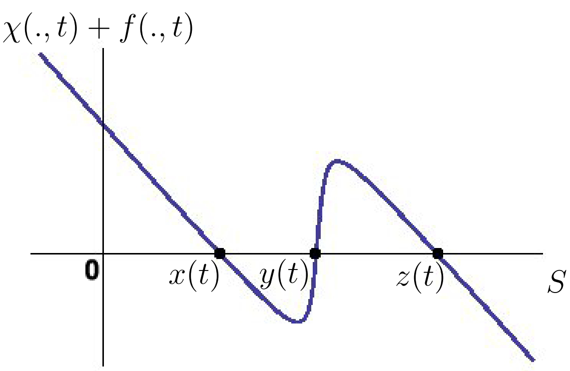

Figure 4: Two stables equilibrium prices.

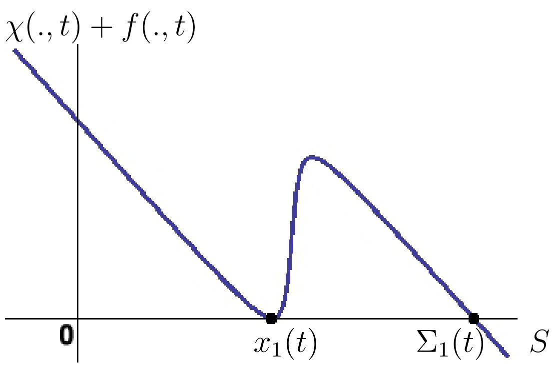

Figure 5: Jump 1 in the price process.

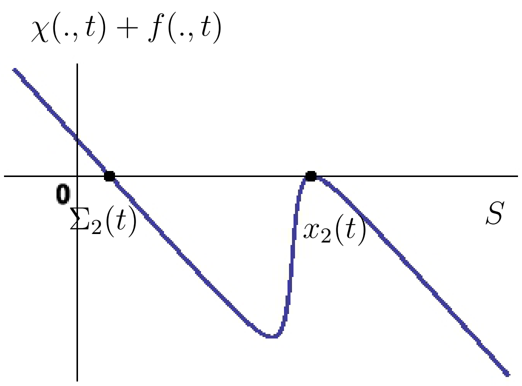

Figure 6: Jump 2 in the price process.

This last figure complements the explanation about the difficulty in obtaining a classical solution for the Black-Scholes equation developed in section 5.4.

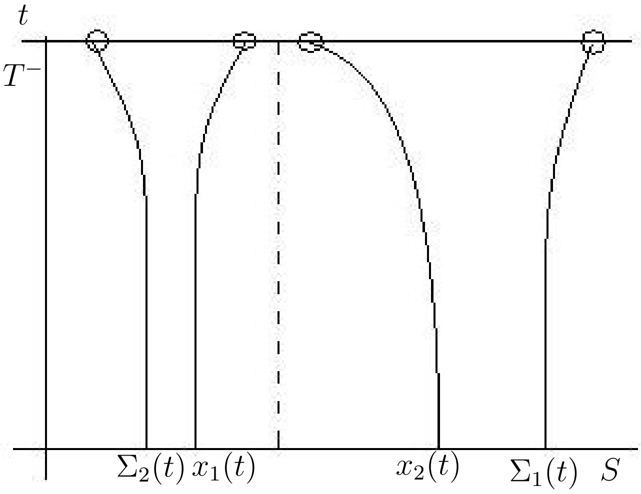

Figure 5: The boundary conditions (78), (82), (83).

C Calculation of the maximum and minimum points in collective behaviour case

When the variables W and t are fixed we can find the extreme points using the usual tools for one variable function. We start with the calculation of critical points.

$$\begin{aligned}\frac{\partial}{\partial S}\left\{\chi(S_t,W_t,t)+f(S_t,t)\right\} &= 0\\ -\alpha+\frac{1}{\sqrt{2\Pi}}\exp\left(-\frac{d_1^2}{2}\right)\frac{\partial}{\partial S}d_1 &= 0\\ \frac{1}{S_t\nu\sqrt{2\Pi(T-t)}}\exp\left(-\frac{d_1^2}{2}\right) &= \alpha.\end{aligned}$$

In order to simplify we set $A(t)=\nu\sqrt{2\Pi(T-t)}$. Therefore,

$$-\frac{d_1^2}{2} = \log(\alpha A(t))+\log(S_t) \tag{104}$$

$$-\frac{1}{2}\left(\frac{\log(S_t)}{B(t)}+C(t)\right)^2 = \log(\alpha A(t))+\log(S_t), \tag{105}$$

where $B(t)=\nu\sqrt{T-t}$ and $C(t)=\frac{(r+\frac{\nu^2}{2})(T-t)-\log(K)}{B(t)}$. If we use the substitution $\log(S_t)=y$, we will have a second order equation:

$$-\frac{1}{2B^2(t)}y^2-(1+\frac{C(t)}{B(t)})y-(\frac{C^2(t)}{2}+\log(\alpha A))=0$$

So, we have $y=\frac{-\left(1+\frac{C(t)}{B(t)}\right)\pm\sqrt{\left(1+\frac{C(t)}{B(t)}\right)^2-\frac{C^2(t)}{B^2(t)}-\frac{2\log(\alpha A)}{B^2(t)}}}{\frac{1}{B^2(t)}}$. After some simplifications we can obtain $y=-E(t)\pm\sqrt{B^4(t)D(t)-2B^2(t)\log(\alpha A)}$. Here $E(t)=(r+\frac{3}{2}\nu^2)(T-t)-\log(K)$

and $D(t) = \frac{2}{\nu^2}(\frac{\log(K)}{T-t} + r + \nu^2)$. Then the minimum and maximum points are given by,

$$S_t = \exp\left(-E(t) \pm \sqrt{B^4(t)D(t) - 2B^2(t)\log(\alpha A)}\right). \tag{106}$$

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