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MASTER

MATHEMATICAL FINANCE

MASTER'S FINAL WORK

DISSERTATION

FRACTIONAL DIFFUSION MODELS AND OPTION PRICING IN JUMP MODELS

FRANCISCO MARIA DE MATEUS E JORGE DA FONSECA

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SUPERVISION:

JOÃO GUERRA

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ABSTRACT

The problem of pricing financial derivatives has been the focal point of research within the field of Mathematical Finance since its conception. In recent years, one of the main areas of focus within the literature has been on models which assume that the dynamics of the price of the underlying asset are governed by a Lévy process (sometimes referred to as a jump process). This type of model admits the possibility of extreme events (jumps), which are not captured by classical Black-Scholes type models based on the Brownian motion.

Over the last decades, the literature has further shown that if the dynamics of the price of the underlying is governed by certain Lévy processes, such as the $CGMY$, the FMLS and the KoBoL, the price processes of European-style options satisfy a variety of fractional partial differential equations (FPDEs).

In this dissertation, we will show that if the underlying price dynamic follows a Generalized Tempered Stable process, which admits as particular cases the aforementioned $CGMY$ and KoBoL processes, prices of options satisfy an FPDE of the same type. Further, we will implement a simple finite difference scheme to solve the FPDE numerically to price European-type options.

KEYWORDS: Fractional Partial Differential Equation; Lévy Process; Tempered Stable Process; Fractional Calculus; Option Pricing; Finite Difference Scheme.

RESUMO

O problema de valorização de derivados tem sido o foco da investigação em Matemática Financeira desde a sua conceção. Mais recentemente, a literatura tem-se focado por exemplo em modelos que assumem que as dinâmicas do preço do ativo subjacente são governadas por um processo de Lévy (por vezes chamado um processo com saltos). Este tipo de modelo admite a possibilidade de eventos extremos (saltos), que não são devidamente capturados por modelos clássicos do tipo Black-Scholes, alicerçados no movimento Browniano.

Foi também demonstrado ao longo da última década que se as dinâmicas do preço do ativo subjacente seguem certos processos de Lévy, tais como o $CGMY$, o FMLS e o KoBoL, os preços das opções satisfazem uma equação diferencial parcial fracionária.

Nesta dissertação, iremos mostrar que se as dinâmicas do ativo subjacente seguem o denominado Processo Estável Temperado Generalizado, que admite como caso particular os suprareferidos processos $CGMY$ e KoBoL, então os preços das opções satisfazem igualmente uma equação diferencial parcial fracionária. Além disso, iremos implementar um método simples de diferenças finitas para resolver numericamente a equação deduzida, e valorizar opções do tipo europeu.

PALAVRAS-CHAVE: Equação Diferencial Parcial Fracionária; Processo de Lévy; Processo Estável Temperado ; Cálculo Fracionário; Valorização de Opções; Esquema de Diferenças Finitas.

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1 INTRODUCTION

The pricing of financial derivatives has long been one of the focal points of research within the field of Mathematical Finance. A derivative is, in essence, a financial instrument whose value is entirely dependent on the price of another asset, referred to as the **underlying** (usually assumed to be a stock). Hence, it is natural that the first step in the process of pricing such instruments is to model the price of the underlying asset. The pioneering and still widely utilized Black-Scholes model (BS model henceforth), proposed by Fischer Black and Myron Scholes in [4] and complemented by Robert Merton in [24] and later in [23], assumes that the logarithm of the asset price S_t has dynamics

$$\mathrm{d}(\log(S_t)) = \left(\mu - \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma \mathrm{d}B_t, \tag{1}$$

where the variables have the usual meaning: μ is the mean rate of return of S_t, σ is the volatility of S_t, and B_t is the standard Brownian motion (sometimes referred to as Wiener process).

This dissertation, as is the case for most of the literature, will focus on the pricing of options. The most well known, and simplest, examples of options are European call and put options. Recalling, a European call option is a contract which gives its holder the right to purchase a predetermined amount of the asset S_t at a certain future time T (the **maturity**) for a given price K (the **strike price**). A European put option is similar in all respects, with the exception that the holder has the right to sell the assets, rather than to buy them.

As is well known, Black, Scholes and Merton showed that if the dynamics of S_t are given by (1), then the price process of a European call option $V(S,t)$, with S_t as the underlying asset, satisfies the aptly named BS Partial Differential Equation (PDE henceforth)

$$\frac{\partial V(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2} + rS\frac{\partial V(S,t)}{\partial S} = rV(S,t), \tag{2}$$

where r is the market risk-free rate. Carrying out a change of variable $x_t = \log(S_t)$, this equation can be written as a PDE with constant coefficients

$$\frac{\partial V(x,t)}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V(x,t)}{\partial x^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial V(x,t)}{\partial x} = rV(x,t). \tag{3}$$

However, in recent years, the BS model has been shown to have numerous flaws which compromise its aptness at modelling real financial data, namely:

- Observed asset price processes seem to be discontinuous, that is, they appear to exhibit a significant number of large jumps over very small time steps, and while such large variations are possible under the BS framework, they are extremely unlikely.

- Empirical distributions of asset returns are largely found to be skewed to the left and exhibit heavy tails, characteristics which are not captured by a Gaussian process like the Brownian motion.

- The BS model produces constant implied volatilities, while empirical data seems to indicate that volatility is not constant for different strikes or different maturities - this effect is usually referred to in the literature as the volatility smile.

In order to combat these problems, a few alternatives to the BS framework have been proposed. One such alternative is to assume that asset price dynamics is driven by a Lévy process - which is in essence a stochastic process with independent and stationary increments. This type of process addresses all of the aforementioned weaknesses of the BS model. This approach gained traction with seminal works such as [18], where the authors model asset prices using the Variance-Gamma (VG) process, and [6] where the $CGMY$ model, which has become one of the most popular Lévy models in finance, was first developed and shown to be apt at modelling asset price processes. Since then, a great number of other Lévy-type models have been proposed to model asset prices (for example jump-diffusions in [14] and the Meixner process in [28]). One such process is the Generalized Tempered Stable process, as presented by [10], which generalizes the VG and $CGMY$ processes, and which will be the focus of this dissertation.

The pricing of options when the price of the underlying follows a Lévy process can be done in a variety of ways. The most common is perhaps the use of methods based on the Fast Fourier Transform ([7]), or Monte-Carlo type methods based upon the simulation of a large number of trajectories of the Lévy process ([28]). As an alternative, equations similar to (3) have been derived for a variety of Lévy-type models. However, since this type of process is inherently non-local due to the presence of jumps, the equation that is found is not a PDE but instead a partial integro-differential equation (PIDE henceforth) (a survey of this topic can be found in [10]). Furthermore, Cartea and del-Castillo-Negrete show in [8] that for certain processes - namely the $CGMY$, KoBoL and FMLS -, the PIDE can instead be written as a fractional partial differential equation (FPDE henceforth), since fractional differential operators are also non-local. The transformation of PIDEs into FPDEs opens up a wider variety of numerical methods (see [12] for example) which can be used to solve the equations, and hence price options when the underlying price dynamics follow certain Lévy processes. These methods, while conceptually more complex, are much easier to implement than methods based upon Monte-Carlo simulation. Furthermore, the use of fractional calculus in FPDEs is still relatively recent, and improvements to the numerical methods which we will present within this dissertation are being actively worked on.

Finally, and while this will not be a focus of our work, it should be noted that in [30] and [9], the authors show that under the framework of fractional calculus, it is possible to derive analytic formulas for the prices of European options when the asset dynamics are governed by a $CGMY$ and FMLS process, respectively, within the fractional calculus framework.

In this dissertation, we will derive the FPDE for a more general model, where the underlying price follows a Generalized Tempered Stable Process (GTS process henceforth), as presented by Cont and Tankov in [10]. This process includes as particular cases the aforementioned $CGMY$ and KoBoL processes. We will also present a finite difference scheme (an implicit Euler scheme) to solve the derived equations. Particular care will be taken with the description of the scheme.

The remainder of the dissertation will be organized in the following way: Section 2 is comprised of the essential definitions, propositions and theorems which are fundamental for understanding the remainder of our work: this will mainly include results regarding Lévy processes and fractional calculus, and also a very brief overview of Fourier analysis. Section 3 is dedicated to the GTS process, including a critical look at its usefulness in financial applications. Section 4 presents the FPDEs for the FMLS, KoBoL and CGMY processes, which were derived in [8], and finally we derive the FPDE for the GTS process, which is one of the main results of our work. In section 5, we will describe in detail the finite difference scheme which will be used to solve the FPDEs derived in section 4. Section 6 is comprised of the numerical results, where the CGMY FPDE is tested using data from [28] and the GTS FPDE is also shown to behave properly from the numerical point of view. Section 7 presents our conclusions and discusses possible future applications of FPDEs in finance, as well as some potential improvements to the numerical method.

2 THEORETICAL OVERVIEW

As was previously stated, this chapter will be divided into 3 subsections: the first focuses on Lévy processes, the second on fractional calculus, and the third and final one is a very brief note on some essential results of Fourier analysis. For each subsection, we will recommend textbooks and articles from which we have sourced the presented results, so that the interested reader can know where to look if they want to delve deeper into a given topic.

2.1 Lévy Processes

The following definitions and results are mainly taken from [1], [10] and [28]. We recommend these books should the reader be interested in checking proofs for the presented results.

As a preliminary we need to define what it means for a random variable to be infinitely divisible.

Definition 2.1 (Infinite Divisibility). *We say a random variable X is* ***infinitely divisible*** *if $\forall n \in \mathbb{N}$, there exists a set of i.i.d. random variables $Y_1^{(n)}, ..., Y_n^{(n)}$ such that*

$$X \overset{d}{=} Y_1^{(n)} + ... + Y_n^{(n)}. \tag{4}$$

Note that this is equivalent to saying that

$$\phi_X(u) = \left(\phi_{Y_1^{(n)}}(u)\right)^n, \tag{5}$$

where $\phi_X(u)$ is the characteristic function of X and $\phi_{Y_1^{(n)}}(u)$ is the characteristic function of $Y_1^{(n)}$ (obviously as the random variables $Y_1^{(n)}, ..., Y_n^{(n)}$ are i.i.d., they all have the same characteristic function, and as such we could substitute $Y_1^{(n)}$ by any of them).

We will now rigorously define what is a Lévy process.

Definition 2.2 (Lévy Process). *Let $X = (X(t), t \geq 0)$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$. We say that X is a Lévy process if:*

1. *$X(0) = 0$, a.s.;*
2. *X has independent and stationary increments, i.e., the random variables $(X(t_{j+1}) - X(t_j), 1 \leq j \leq n)$ are independent and $X(t_{j+1}) - X(t_j) \overset{d}{=} X(t_{j+1} - t_j) - X(0)$.*
3. *X is stochastically continuous, i.e., $\forall a > 0$ and $\forall s \geq 0$,*

$$lim_{t \to s} P\left(|X(t) - X(s)| > a\right) = 0. \tag{6}$$

Lévy processes and infinite divisibility are connected through the following result:

Proposition 2.3. *If X is a Lévy process, then $X(t)$ is infinitely divisible for each $t \geq 0$.*

This connection between any Lévy process and an infinitely divisible distribution gives us a way to represent the characteristic function of any Lévy process

Theorem 2.4 (Lévy-Khintchine). *A random variable X is infinitely divisible if and only if there exists a triplet (b, σ^2, ν), with $b \in \mathbb{R}$, $\sigma^2 \geq 0$ and ν is a measure that satisfies $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} (1 \wedge x^2)\nu(\mathrm{d}x) < +\infty$, such that the characteristic function of X is given by*

$$\phi_X(u) = \mathbb{E}[e^{iuX}] = \exp\left(ibu - \frac{u^2\sigma^2}{2} + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\mathbb{1}_{\{|x|<1\}}\right)\nu(\mathrm{d}x)\right). \quad (7)$$

The function $\mathbb{1}_A$ is the usual indicator function of a set A. Usually, b and σ^2 are referred to as the drift and the diffusion coefficients respectively, and ν is called a Lévy measure. From here it is quite clear that a Lévy process can be thought of as being composed of a deterministic drift component, a diffusion or Brownian component, and a jump component.

Furthermore, it is useful to define

$$\eta(u) = \log\left(\phi_X(u)\right) = ibu - \frac{u^2\sigma^2}{2} + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\mathbb{1}_{\{|x|<1\}}\right)\nu(\mathrm{d}x), \quad (8)$$

which is usually called the characteristic exponent of X. Hence, for a given Lévy process $X = (X(t), t \geq 0)$, its characteristic function is given by

$$\phi_{X_t}(u) = \exp\left(t\eta(u)\right), \quad (9)$$

where $\eta(u)$ is the characteristic exponent of $X(1)$.

The Lévy measure $\nu(A)$ has an intuitive interpretation which will be very useful when it comes to applications: it gives us the expected number of jumps of size in A per unit of time. Furthermore, the properties of the paths and moments of the process depend on the Lévy measure in the following way.

Proposition 2.5 (Paths of a Lévy process). *For a Lévy process $X = (X(t), t \geq 0)$ with triplet (b, σ^2, ν) we have that:*

- *If $\nu(\mathbb{R}) = \infty$, then the Lévy process is said to have infinite activity, that is, there is a non-zero probability of having infinitely many small jumps occur in a given time interval.*

- *If* $\nu(\mathbb{R}) < \infty$*, then the Lévy process is said to have finite activity, and all paths have a.s. a finite number of jumps in a compact time interval.*
- *If* $\sigma^2 = 0$ *and* $\int_{|x|\leq 1} |x|\nu(\mathrm{d}x) < \infty$ *then all paths have finite variation a.s..*
- *If* $\sigma^2 \neq 0$ *or* $\int_{|x|\leq 1} |x|\nu(\mathrm{d}x) = \infty$ *then all paths have infinite variation a.s..*

From this proposition we can see that the path properties depend entirely on the diffusion (or Brownian) component of the Lévy process and on the small jumps (of size less than 1).

On the other hand, as the following proposition states, the moment properties of the process depend only on the big jumps.

Proposition 2.6 (Moments of a Lévy Process). *For a Lévy process* $X = (X(t), t \geq 0)$ *with triplet* (b, σ^2, ν) *we have that:*

- $X(t)$ *has finite moment of order* p *if and only if* $\int_{|x|\geq 1} |x|^p \nu(\mathrm{d}x) < \infty$.
- $X(t)$ *has finite exponential moment of order* p *(that is,* $\mathbb{E}[e^{pX_t}] < \infty$*) if and only if* $\int_{|x|\geq 1} e^{px}\nu(\mathrm{d}x) < \infty$.

Within the class of infinitely divisible random variables, it is relevant for us to focus on a well known sub-class: stable random variables. Generally speaking, a stable random variable is one which arises as a limit of a sum of i.i.d random variables. More rigorously, we may define it in the following way

Definition 2.7 (Stable Random Variable). *A random variable* X *is said to be* ***stable*** *if there exist real-valued sequences* $(c_n, n \in \mathbb{N})$ *and* $(d_n, n \in \mathbb{N})$ *with* $c_n > 0$ *such that*

$$X_1 + ... + X_n \overset{d}{=} c_n X + d_n, \tag{10}$$

where $X_1, ..., X_n$ *are independent copies of* X*. Furthermore, it can be shown that the only possible choices of* c_n *are of the form* $\sigma n^{1/\alpha}$*, with* $\alpha \in (0, 2]$.

The parameter α plays a key role in the study of this type of process, and is usually called the **index of stability**.

The following result tells us about the implications that the value of α has on the triplet associated to a stable random variable X:

Theorem 2.8. *If X is a stable random variable, then:*

- *If* $\alpha = 2$*, then* $\nu = 0$ *and* $X \sim N(b, \sigma^2)$*;*

- *If* $\alpha \neq 2$, *then* $\sigma^2 = 0$ *and* $\nu(\mathrm{d}x) = \frac{c_1}{x^{1+\alpha}} \mathbb{1}_{(0,\infty)}(x)\mathrm{d}x + \frac{c_2}{|x|^{1+\alpha}} \mathbb{1}_{(-\infty,0)}(x)\mathrm{d}x$, *with* $c_1 \geq 0$, $c_2 \geq 0$ *and* $c_1 + c_2 > 0$.

This class of random variables is of particular importance, since the process which our work focuses on, the GTS process, will have random variables within a class very similar to this one.

We will now focus our attention on the subject of option pricing with Lévy processes. The theory of Itô calculus for Lévy processes has been developed in the late 20th and early 21st centuries. For a comprehensive, yet largely introductory overview of this topic, we recommend [1].

A good understanding of stochastic calculus is essential for an in depth understanding of the results we will present. However, as the equations which we will derive and solve are entirely deterministic, we have opted to present only a brief overview of the way in which Lévy processes are incorporated in pricing models.

To do so, let us first recall that in the classical BS case, in order to find the arbitrage-free price of an option written on a given asset S_t, we must first write the dynamics of S_t under what is usually called the risk-neutral or equivalent martingale measure (EMM henceforth), which is usually denoted by Q.

To briefly summarize, we say that a probability measure Q is an EMM relative to a measure P if:

- Q and P are equivalent measures, that is, events which have measure 0 under P also have measure 0 under Q and vice-versa;
- The discounted asset price process $\tilde{S}_t = \exp(-rt)S_t$ is a martingale under Q.

The existence and uniqueness of an EMM in the context of financial models has been extensively studied. In fact, two of the most well known results in Mathematical Finance describe the conditions under which one can guarantee that an EMM exists and is unique. Essentially, an EMM exists if and only if our model is free of arbitrage, and is unique if and only if our model is complete.

In the case of the BS model, which is complete and free of arbitrage, we get the well known dynamics

$$\mathrm{d}(log(S_t)) = \left(r - \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma \mathrm{d}B_t^Q, \tag{11}$$

where the variables have the usual meaning, and $\mathrm{d}B_t^Q$ is the increment of a Brownian motion under the EMM, in opposition to the "real world" or probabilistic measure, usually denoted by P.

After writing the dynamics under an EMM, the price of a European call option $C(S,t)$ can be calculated by using what is usually called the **risk-neutral valuation formula** - we take the expectation of the payoff discounted by the risk-free rate

$$C(S,t) = e^{-r(T-t)}\mathbb{E}^Q\left[(S_T - K)^+\right], \tag{12}$$

where the variables have the usual meaning and the payoff is $(S_T - K)^+ = \max(S_T - K, 0)$.

Without going into much detail, what is done in order to model stock prices using Lévy processes instead, is to simply assume that in the dynamics of the asset price, the Brownian motion is replaced by a Lévy process, giving us

$$\mathrm{d}(log(S_t)) = \mu \mathrm{d}t + \mathrm{d}L_t^P, \tag{13}$$

where $\mathrm{d}L_t^P$ is the increment of a Lévy process under the real-world measure P. These dynamics must then be written under the risk-neutral measure Q as in the classical case. However, the uniqueness of an EMM which is a characteristic of the BS model, will in general be lost when dealing with a model based on a Lévy process (see [28]). To get around this issue, many different methods have been developed for choosing an EMM under which to perform the pricing of options.

In our work, we will utilize the simplest, and perhaps the most popular approach to choosing an EMM, which is usually called the **mean-correcting martingale measure method** (see once again [28]). This method assumes that under the EMM, the asset price follows a so-called geometric Lévy process (named in this way due to its similarity to the well known geometric Brownian motion). Thus, it is assumed that the asset price has risk neutral dynamics

$$\mathrm{d}\left(log(S_t)\right) = (r - \nu)\mathrm{d}t + \mathrm{d}L_t^Q, \tag{14}$$

which has solution

$$S_T = S_t \exp\left((r-\nu)(T-t) + \int_t^T dL_u\right), \tag{15}$$

ν is usually called the convexity correction, and it is chosen in such a way that S_t becomes a martingale under the measure Q, i.e., such that $\mathbb{E}^Q[S_T] = e^{r(T-t)}S_t$. One can show that $\nu = log\left(\eta(-i)\right)$ where $\eta(\cdot)$ as usual denotes the characteristic exponent of the Lévy process (the proof of this can be found in [28]).

2.2 *Fractional Calculus*

The following definitions and results are mainly taken from [26] (with some from [25], [19]). The reader should also keep in mind that the available literature within the field of fractional calculus is widely inconsistent when it comes to a proper notation. As such, we will try to clarify the notation which we will use throughout the remainder of the dissertation as we go through the definitions. Furthermore, there is a multitude of different ways to define fractional derivatives and integrals. In our case, it suffices to provide the definitions for the **Riemann-Liouville** (RL henceforth) and the **Grünwald-Letnikov** (GL henceforth) integral and derivative. Other useful definitions include the Caputo, Riesz and Weyl fractional derivatives (see [19] for details).

As a preliminary, we must recall the definition of the Gamma function, since it is central to the way in which we will define fractional integrals and derivatives:

Definition 2.9 (Gamma Function). *The gamma function, $\Gamma(z)$, is defined as*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \mathrm{d}t. \tag{16}$$

The gamma function is an holomorphic function $\forall z \in \mathbb{C} \setminus \{0, -1, -2, ...\}$, i.e. for all the complex numbers except the non-positive integers.

For our purposes, the most relevant property of the gamma function is that it is, in essence, a generalization of the factorial function, in the sense that

$$\Gamma(n+1) = n!. \tag{17}$$

Now, recall that the usual n-fold integral can be represented as

$$\mathrm{D}^{-n} f(t) = \frac{1}{(n-1)!} \int_a^t (t-\tau)^{n-1} f(\tau) \mathrm{d}\tau. \tag{18}$$

Therefore, it appears natural to simply replace $(n-1)!$ by $\Gamma(n)$, such that the definition becomes valid for any positive integer $\alpha > 0$. This yields the first, and most widely used, definition of the fractional integral:

Definition 2.10 (Riemann-Liouville Fractional Integral). *The Riemann-Liouville fractional integral of order $\alpha > 0$ is defined by*

$$\boldsymbol{D}_{(a,t)}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) \mathrm{d}\tau. \tag{19}$$

The definition of the fractional derivative is then built using the fractional integral in the following way:

Definition 2.11 (**Riemann-Liouville Fractional Derivative**)**.** *The Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined by*

$$\boldsymbol{D}^{\alpha}_{(a,t)}f(t) = \frac{1}{\Gamma(k-\alpha)}\frac{d^k}{dt^k}\int_a^t (t-\tau)^{k-\alpha-1}f(\tau)\mathrm{d}\tau, \tag{20}$$

where k is the smallest integer greater than α.

Note that this can be written as

$$\boldsymbol{D}^{\alpha}_{(a,t)}f(t) = \frac{d^k}{dt^k}\left(\boldsymbol{D}^{-(k-\alpha)}_{(a,t)}f(t)\right), \tag{21}$$

so in essence, all we are doing is computing a fractional integral, followed by an ordinary derivative. The RL operators will always be denoted by a bold face **D**. Aside from the RL definition, we also have the following definitions for the fractional integral and derivative, which will be particularly useful when we develop our numerical scheme:

Definition 2.12 (Grünwald-Letnikov fractional integral)**.** *The Grünwald-Letnikov fractional integral of order $\alpha > 0$ is defined by*

$$\mathrm{D}^{-\alpha}_{(a,t)}f(t) = \lim_{h\to 0} h^{\alpha}\sum_{k=0}^{n}(-1)^k\binom{-\alpha}{k}f(t-kh), \tag{22}$$

where $n = \frac{[t-a]}{h}$ and $[t-a]$ is the integer part of $t-a$.
The fractional derivative is defined analogously.

Definition 2.13 (Grünwald-Letnikov fractional derivative)**.** *The Grünwald-Letnikov fractional derivative of order $\alpha > 0$ is defined by*

$$\mathrm{D}^{-\alpha}_{(a,t)}f(t) = \lim_{h\to 0} h^{-\alpha}\sum_{k=0}^{n}(-1)^k\binom{\alpha}{k}f(t-kh). \tag{23}$$

Note that in both these cases, $\binom{\alpha}{k} = \frac{(\alpha)(\alpha-1)\ldots(\alpha-k+1)}{k!}$, not to be confused with the usual binomial coefficients. It is important to note that the GL operators will always be denoted by a non-bold D. From both of these definitions, one should immediately note the main difference between fractional derivatives and ordinary derivatives. Suppose we are differentiating a given function f at a certain point $x \in [a, b]$. The ordinary derivative is a local operator, i.e., the value of the derivative at x depends only on the values of f in a neighbourhood of x. On the other hand, the fractional derivative is non-local, i.e., the value of the left fractional derivative at x depends on the behaviour of f in the interval $[a, x]$ and the value of the right fractional derivative depends on the behaviour of f in the interval $[x, b]$. The definitions we have given above were all for left-sided derivatives. The analogous right-sided RL derivative is given by

$$\boldsymbol{D}^{\alpha}_{(t,b)}f(t) = \frac{(-1)^k}{\Gamma(k-\alpha)}\frac{d^k}{dt^k}\int_t^b (\tau - t)^{k-\alpha-1}f(\tau)\mathrm{d}\tau. \tag{24}$$

And later on we will also give the definition for the right-sided GL derivative.

This is the main reason why fractional derivatives are of great interest in physics and finance: it allows for the introduction of spatial non-locality into our models. This is why it is natural that they should appear within the context of a Lévy-type model, which is inherently non-local due to the possibility of jumps.

Unfortunately, we will not be able to obtain closed form solutions for the vast majority of fractional differential equations. Hence, the approach is usually to write down the problem as a fractional differential equation using RL type derivatives, and then discretize the setting by using the GL definition so that numerical schemes can be applied in order to solve it.

The operators which we have discussed enjoy a great many properties (see [26]), some of which differ from the ones we know very well for their integer order counterparts. For our purposes, these properties are in general not necessary, and as such we will skip over most of them. However, we will present the following result, which justifies the fact that we can approximate the RL derivative by the GL derivative in most practical cases.

Proposition 2.14. *If $f(t)$ is sufficiently smooth, i.e., it admits $k+1$ continuous derivatives, then:*

$$\boldsymbol{D}^{\alpha}_{(a,t)}f(t) = \mathrm{D}^{\alpha}_{(a,t)}f(t), \tag{25}$$

where k is the smallest integer greater than α.

In most applications, it is fairly justified to assume that the function to be differentiated should be sufficiently smooth. This is in fact one of the central assumptions within the classical BS framework (see [4]).

2.3 Fourier Analysis

This section will very briefly introduce the concept of the Fourier transform, as well as some of its properties which will play a key part in the derivation of the FPDEs in section 4. The definitions and results presented can be found in [3] and [2] (and in most fourier analysis books). We begin with the definition of the Fourier transform.

Definition 2.15 (Fourier Transform). *Given a function $f(t) \in L_1(\mathbb{R})$ which is at least piece-wise continuous, its Fourier transform is defined as:*

$$\mathcal{F}\{f(t)\}(x) = \hat{f}(x) = \int_{-\infty}^{+\infty} f(t)e^{-ixt}\mathrm{d}t. \tag{26}$$

Analogously, one can prove the following inversion theorem.

Theorem 2.16 (Fourier Inversion Theorem). *Suppose the Fourier transform exists, and $\hat{f}(x) \in L_1(\mathbb{R})$. Then $f(t)$ is a continuous function and*

$$\mathcal{F}^{-1}\{\hat{f}(x)\}(t) = f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(x)e^{itx}\mathrm{d}x. \tag{27}$$

Furthermore, the following 3 properties will be of use to us.

Proposition 2.17 (Properties of the Fourier Transform). *Let f and g be sufficiently smooth functions, $\hat{f}$ and $\hat{g}$ their Fourier transforms. Then the following hold:*

1. *(Linearity) For a and b real constants, it holds that $h = af + bg \implies \hat{h} = a\hat{f} + b\hat{g}$.*
2. *(Derivative) It holds that $\widehat{f'}(x) = -ix\hat{f}(x)$.*
3. *(Fractional derivative) It holds that $\mathcal{F}\{\boldsymbol{D}^{\alpha}_{(-\infty,t)}f(t)\}(x) = (-ix)^{\alpha}\hat{f}(x)$ and $\mathcal{F}\{\boldsymbol{D}^{\alpha}_{(t,\infty)}f(t)\}(x) = (ix)^{\alpha}\hat{f}(x)$.*

Property 3 is perhaps the only one which will not show up in most texts that deal with Fourier analysis. The proof for this property can be found in [26].

The Fourier transform is an extensively utilized tool in the theory of differential equations, as, in general, it will significantly simplify the problem at hand. However, this simplification often comes at a steep cost: after solving the transformed equation in Fourier space, one must invert the transform in order to obtain a solution for the original problem. This inversion is in general very difficult to carry out analytically, which slightly limits the usefulness of this method in practice.

In our case, the Fourier transform will be utilized in a different way: the pricing equation is derived in Fourier space and then the inverse transform is carried outin order to get a pricing equation in the original space (see [8]).

Fourier transforms appear naturally and are used extensively in the context of option pricing (see, for example, [27], [11], [7]). As was already stated and is well known, the price of an option is given by the discounted value of the expected payoff under the risk-neutral measure Q: $C(S,t) = e^{-rT}\mathbb{E}^{Q}[\Pi(S,T)]$, where $\Pi(S,T)$ is the value of the payoff function at maturity. In order to evaluate this expected value, one will in general have to integrate an expression which involves the density function of S_t, and as we stated in section 2.1, most Lévy processes do not admit a closed-form expression for their density function. However, their characteristic function is always available via the Lévy-Khintchine theorem. For this reason, it becomes desirable to connect the density function to the characteristic function. The Fourier transform allows us to do so, since it can easily be seen that the characteristic function is the Fourier transform of the density function. This can be rigorously stated in the following way.

Proposition 2.18. *Let $f_X(x)$ be the density function of a random variable X. Furthermore, let $\phi_X(z)$ be its characteristic function. The characteristic function is the Fourier transform of the density function*

$$\phi_X(z) = \mathcal{F}\{f_X(x)\}(z) = \int_{-\infty}^{+\infty} e^{izx} f_X(x)\mathrm{d}x. \tag{28}$$

Analogously, the density function can be recovered from the characteristic function via inverting the transform

$$f_X(x) = \mathcal{F}^{-1}\{\phi_X(z)\}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-izx} \phi_X(z)\mathrm{d}z. \tag{29}$$

3 THE GENERALIZED TEMPERED STABLE PROCESS

As has been stated, one of the main focuses of this work is the derivation of the pricing FPDE for a model which assumes that the price dynamics of the underlying asset is governed by a GTS process. The GTS process is, in essence, a generalization of the well known CGMY process (first introduced in [6]) and of the lesser known but still widely utilized Tempered Stable Process (TS process henceforth). Furthermore, the GTS process also admits as particular cases many other well known processes, including the KoBoL ([5]), the truncated Lévy flight ([13]), and the Variance Gamma ([18]) and bilateral Gamma processes ([15]). For this reason, the FPDE which we will derive in section 4 will in fact admit as particular cases the FPDEs for any of the aforementioned processes. This class of processes has been utilized in a multitude of applications in physics, biology, and of course in finance. Our description of this process will follow [10] (with some details taken from [16]).

We will begin by defining the GTS process.

Definition 3.1 (GTS process). *A Lévy process is said to be a GTS process if it has Lévy triplet given by* $(b, 0, \nu)$ *with* ν *given by*

$$\nu(x) = \frac{c_-}{|x|^{1+\alpha_-}} e^{-\lambda_- |x|} \mathbb{1}_{x<0} + \frac{c_+}{x^{1+\alpha_+}} e^{-\lambda_+ x} \mathbb{1}_{x>0}, \tag{30}$$

where the parameters satisfy: $c_-, c_+, \lambda_-, \lambda_+ > 0$ *and* $\alpha_+, \alpha_- < 2$.

Hence, the GTS process is a pure jump process, i.e., it has no Brownian component. If we recall the previously mentioned stable Lévy processes (see 2.8) , we can see that the GTS process is in essence just a stable process with an exponential dampening factor on either side of the distribution (hence the denomination 'tempered stable'). This dampening effect ensures that the small jumps (of size less than 1) preserve their stable behaviour, while preventing the magnitude of large jumps from growing too quickly, which means the distribution will have finite moments of all orders, and will also mean that the conditions for the existence of exponential moments will also not be too steep.

All 6 parameters of the GTS process have a simple interpretation in practice:

- c_+ and c_- determine the overall frequency of jumps. As we will see in a moment, the process may be one of infinite activity (i.e., the frequency of jumps will be infinite), but in such a case, these parameters are still highly relevant as they give us an idea of the relative frequency of jumps larger than a given size.

- λ_+ and λ_- control the tail behaviour of the measure. These parameters are particularly important in financial applications, since they allow us to gauge the frequency with which large losses may occur, and risk management is usually particularly concerned with the probability of extreme loss events.

- α_+ and α_- determine the fine structure of the process, which in essence describes how the small jumps behave. If these parameters are large, i.e., close to 2, then the process will behave very similarly to a Brownian motion. If they are small, i.e., close to 0, then changes are mostly due to the large jumps.

The ability of the process to capture Brownian-like behaviour is a very appealing one in the context of applications, as it allows us to utilize a pure jump process to model the asset price dynamics, simplifying the model without worry of losing accuracy by not including a Brownian component (see for example [6]).

As we stated, the fine structure of the process is governed by α_+, α_-. To briefly summarize, we have that:

- The GTS process is a Compound Poisson process for $\alpha_+, \alpha_- < 0$.
- The GTS process has trajectories of infinite activity if $\alpha_+, \alpha_- > 0$.
- The GTS process has trajectories of infinite variation if $\alpha_+, \alpha_- > 1$.

It is also of note that in the case of the GTS process, contrary to what happens for regular stable processes, we have no lower bound for the values of α_+, α_-. When these parameters assume negative values, the process becomes of the Compound Poisson type (see [1] for more on Compound Poisson processes). However, in order to preserve the stable-like nature of the small jumps, we will in general assume $\alpha_+, \alpha_- > 0$, since this is usually a desirable property in the context of financial applications. We will also see when we discuss the characteristic function of the GTS process that it becomes slightly simplified if we assume $\alpha_+, \alpha_- \in (0, 1)$.

It has been argued (see for example [10]) that the GTS process does not provide a very significant improvement in term of explanatory power over a more popular model like the $CGMY$. The big addition that the GTS process provides over the $CGMY$ process is the ability to capture asymmetry of small jumps, and also asymmetry in terms of activity level on either side of the distribution. The argument provided for not allowing for this asymmetry in the case of financial models is that, as we have previously stated, risk management is usually concerned with large movements in the price, losses in particular, and as such it is argued that option prices will mostly depend on large jumps, and as such the modelling of the small jumps should be of lesser importance.

However, the importance of the small jumps is far from negligible, as is made obvious by the fact that most financial models are based upon processes which do allow for an infinite amount of small jumps, and not upon something like a Compound Poisson process which would only take into account large jumps. Furthermore, the complexity of the model and of the numerical methods needed to price options do not increase significantly by using the GTS process over the $CGMY$, and as such one could easily argue that the benefits of doing so would outweigh the effort of utilizing a slightly more complex model.

As we said, the most important tool in the derivation of the GTS FPDE is the characteristic exponent.

Proposition 3.2 (GTS characteristic function). *If $X = (X(t), t > 0)$ is a GTS process with triplet $(b, 0, \nu)$, then its characteristic exponent is given by*

$$\begin{aligned}\eta(u) = iub &+ \Gamma(-\alpha_+)c_+\left((\lambda_+ - iu)^{\alpha_+} - \lambda_+^{\alpha_+} + iu\alpha_+\lambda_+^{\alpha_+-1}\right)\\ &+ \Gamma(-\alpha_-)c_-\left((\lambda_- + iu)^{\alpha_-} - \lambda_-^{\alpha_-} - iu\alpha_-\lambda_-^{\alpha_--1}\right), \text{ if } \alpha_+, \alpha_- \neq 0, 1,\end{aligned}$$

$$\begin{aligned}\eta(u) = iu(b + c_+ - c_-) &+ c_+(\lambda_+ - iu)\log\left(1 - \frac{iu}{\lambda_+}\right)\\ &+ c_-(\lambda_- + iu)\log\left(1 + \frac{iu}{\lambda_-}\right), \text{ if } \alpha_+ = \alpha_- = 1,\end{aligned} \tag{31}$$

$$\begin{aligned}\eta(u) = iub &- c_+\left(\frac{iu}{\lambda_+} + \log\left((1 - \frac{iu}{\lambda_+}\right)\right)\\ &- c_-\left(-\frac{iu}{\lambda_-} + \log\left(1 + \frac{iu}{\lambda_-}\right)\right), \text{ if } \alpha_+ = \alpha_- = 0.\end{aligned}$$

Obviously we may mix up the cases, and the characteristic exponent will be a combination of the above expressions.

As a particular case, the authors in [16] show that if $\alpha_+, \alpha_- \in (0, 1)$ then the characteristic exponent becomes

$$\begin{aligned}\eta(u) = iub &+ \Gamma(-\alpha_+)c_+\left((\lambda_+ - iu)^{\alpha_+} - \lambda_+^{\alpha_+}\right)\\ &+ \Gamma(-\alpha_-)c_-\left((\lambda_- + iu)^{\alpha_-} - \lambda_-^{\alpha_-}\right).\end{aligned} \tag{32}$$

As is well known, if we take derivatives of the characteristic exponent at $u = 0$ then we can obtain the cumulants of the distribution, which allow us to compute the usual statistical measures, i.e., expectation, variance, kurtosis and skewness. The expressions for the cumulants are quite long, and as such we will skip ahead and only present the measures.

Proposition 3.3 (Moments of the GTS process). *If $X = (X(t), t > 0)$ is a GTS process with triplet $(b, 0, \nu)$, then:*

- $\mathbb{E}[X(t)] = bt$,
- $Var(X(t)) = t\Gamma(2-\alpha_+)c_+\lambda_+^{\alpha_+-2} + t\Gamma(2-\alpha_-)c_-\lambda_-^{\alpha_--2}$,
- *Skewness* $= \frac{t\Gamma(3-\alpha_+)c_+\lambda_+^{\alpha_+-3} - t\Gamma(3-\alpha_-)c_-\lambda_-^{\alpha_--3}}{Var(X(t))^{3/2}}$,
- *Kurtosis* $= \frac{t\Gamma(4-\alpha_+)c_+\lambda_+^{\alpha_+-4} - t\Gamma(4-\alpha_-)c_-\lambda_-^{\alpha_--4}}{Var(X(t))^{2}}$.

As for the exponential moments, we have the following result.

Proposition 3.4. *If $X = (X(t), t > 0)$ is a GTS process with triplet $(b, 0, \nu)$, then it has finite exponential moments of order $p \in [-\lambda_-, \lambda_+]$.*

A proof of this proposition can be found in appendix A.1.

Finally, before we move on to the actual model, it is useful to summarize the conditions on the parameters which yield each of the particular cases of the GTS process:

- If $\alpha_+ = \alpha_-$ we have a TS or KoBoL distribution,
- If $\alpha_+ = \alpha_-$ and $c_+ = c_-$ we have a CGMY distribution,
- If $\alpha_+ = \alpha_-$ and $\lambda_+ = \lambda_-$ we have a truncated Lévy flight,
- If $\alpha_+ = \alpha_- = 0$ we have a bilateral Gamma distribution,
- If $\alpha_+ = \alpha_- = 0$ and $c_+ = c_-$ we have a Variance Gamma distribution,

Moving onto the actual model, and following the process which we detailed in section 2.1, the real-world price process will be given by

$$\mathrm{d}(log(S_t)) = \mu \mathrm{d}t + \mathrm{d}X_{GTS}^P, \tag{33}$$

where $\mathrm{d}X_{GTS}^P$ represents the increment of the GTS process.

The convexity correction which was also previously discussed is thus given by $\nu = \eta(-i)$ where $\eta(\cdot)$ is the characteristic exponent of the GTS process. Thus, in our case:

$$\begin{aligned} \nu = b &+ \Gamma(-\alpha_+)c_+\left((\lambda_+ - 1)^{\alpha_+} - \lambda_+^{\alpha_+} + \alpha_+\lambda_+^{\alpha_+-1}\right) \\ &+ \Gamma(-\alpha_-)c_-\left((\lambda_- + 1)^{\alpha_-} - \lambda_-^{\alpha_-} - \alpha_-\lambda_-^{\alpha_--1}\right). \end{aligned} \tag{34}$$

Thus, the asset price is assumed to have risk-neutral dynamics given by $\mathrm{d}(log(S_t)) = (r - \nu)\mathrm{d}t + \mathrm{d}X^Q_{GTS}$.

It should be noted that the parameters for the real-world and risk-netural processes are different, hence the distinction between X^P_{GTS} and X^Q_{GTS}.

4 THE FPDE APPROACH TO OPTION PRICING

As we have stated previously, the fractional calculus approach to option pricing was pioneered by Cartea and del-Castillo-Negrete in [8], and this is the paper from which we take the results which will allow us to derive the FPDE for the GTS process.

As a preface, note that unless stated otherwise, the Fourier transforms which follow within this chapter are **always** done with respect to the space variable x which turns into u in Fourier space.

The following result was shown both by the aforementioned authors in [8] and by the authors in [10] in two different ways.

Proposition 4.1. *Suppose the log-price process of a given asset has dynamics given by* $\mathrm{d}(\log(S_t)) = (r-\nu)\mathrm{d}t + \mathrm{d}L_t$*, where* L_t *is a Lévy process with characteristic exponent* $\eta(u)$*. Then, the Fourier transform of the value of a European style option with underlying* S_t *(with respect to the variable* $x_t = \log(S_t)$*), denoted by* $\mathcal{F}\{V(x,t)\}(u,t) = \hat{V}(u,t)$*, satisfies the equation*

$$\frac{\partial \hat{V}(u,t)}{\partial t} = (r + iu(r-\nu) - \eta(-u))\,\hat{V}(u,t). \tag{35}$$

It should be noted that this result is only valid for Lévy processes which have finite exponential moments. The derivation of the pricing FPDEs is then performed by computing the inverse Fourier transform of this equation for a variety of different Lévy processes.

Recalling the results from Proposition 2.17, we can move onto the derivation of the pricing FPDE for the FMLS, CGMY and KoBoL processes, which are presented in [8].

Proposition 4.2 (The FMLS FPDE). *Let* L_t *be an FMLS process, with characteristic exponent given by*

$$\eta(u) = -\frac{1}{2}\sigma^\alpha \sec\left(\frac{\alpha\pi}{2}\right)(iu)^\alpha. \tag{36}$$

Then, the price of a European style option $V(x,t)$ *satisfies*

$$\begin{aligned}\frac{\partial V(x,t)}{\partial t} + \left(r + \frac{1}{2}\sigma^\alpha \sec\left(\frac{\alpha\pi}{2}\right)\right)\frac{\partial V(x,t)}{\partial x} - \frac{1}{2}\sigma^\alpha \sec\left(\frac{\alpha\pi}{2}\right)\boldsymbol{D}^\alpha_{(-\infty,x)}(V(x,t)) \\ = rV(x,t).\end{aligned} \tag{37}$$

Proposition 4.3 (The CGMY FPDE). *Let L_t be a CGMY process, with characteristic exponent given by*

$$\eta(u) = C\Gamma(-Y)\left((M-iu)^Y - M^Y + (G+iu)^Y - G^Y\right). \tag{38}$$

Then the price of a European style option V(x,t) satisfies

$$\begin{aligned}\frac{\partial V(x,t)}{\partial t} + (r-\nu)\frac{\partial V(x,t)}{\partial x} + C\Gamma(-Y)\Big[e^{Mx}\boldsymbol{D}^Y_{(x,\infty)}\left(e^{-Mx}V(x,t)\right) \\ + e^{-Gx}\boldsymbol{D}^Y_{(-\infty,x)}\left(e^{Gx}V(x,t)\right)\Big] = (r + C\Gamma(-Y)(M^Y+G^Y))V(x,t),\end{aligned} \tag{39}$$

with $\nu = C\Gamma(-Y)\left((M-1)^Y - M^Y + (G+1)^Y - G^Y\right)$.

Proposition 4.4 (The KoBoL FPDE). *Let L_t be a KoBoL process, with characteristic exponent given by*

$$\eta(u) = \frac{1}{2}\sigma^\alpha\left(p(\lambda-iu)^\alpha + q(\lambda+iu)^\alpha - \lambda^\alpha\right), \text{if } \alpha \in (0,1), \tag{40}$$

$$\eta(u) = \frac{1}{2}\sigma^\alpha\left(p(\lambda-iu)^\alpha + q(\lambda+iu)^\alpha - \lambda^\alpha - iu\alpha\lambda^{\alpha-1}(q-p)\right), \text{if } \alpha \in (1,2]. \tag{41}$$

Then, the price of a European style option $V(x,t)$ satisfies

$$\begin{aligned}&\frac{\partial V(x,t)}{\partial t} + (r-\nu)\frac{\partial V(x,t)}{\partial x} \\ &\quad + \frac{1}{2}\sigma^\alpha\left(pe^{\lambda x}\boldsymbol{D}^\alpha_{(x,\infty)}(e^{-\lambda x}V(x,t)) + qe^{-\lambda x}\boldsymbol{D}^\alpha_{(-\infty,x)}(e^{\lambda x}V(x,t))\right) \\ &\quad = \left(r + \frac{1}{2}\sigma^\alpha\lambda^\alpha\right)V(x,t), \text{ if } \alpha \in (0,1),\end{aligned} \tag{42}$$

$$\begin{aligned}&\frac{\partial V(x,t)}{\partial t} + \left(r - \nu + \frac{1}{2}\sigma^\alpha\alpha\lambda^{\alpha-1}(q-p)\right)\frac{\partial V(x,t)}{\partial x} \\ &\quad + \frac{1}{2}\sigma^\alpha\left(pe^{\lambda x}\boldsymbol{D}^\alpha_{(x,\infty)}(e^{-\lambda x}V(x,t)) + qe^{-\lambda x}\boldsymbol{D}^\alpha_{(-\infty,x)}(e^{\lambda x}V(x,t))\right) \\ &\quad = \left(r + \frac{1}{2}\sigma^\alpha\lambda^\alpha\right)V(x,t), \text{ if } \alpha \in (1,2].\end{aligned} \tag{43}$$

Proposition 4.5 (The GTS FPDE). *Let L_t be a GTS process, with characteristic exponent given by*

$$\begin{aligned}\eta(u) = iub + \Gamma(-\alpha_+)c_+\left((\lambda_+ - iu)^{\alpha_+} - \lambda_+^{\alpha_+}\right) \\ + \Gamma(-\alpha_-)c_-\left((\lambda_- + iu)^{\alpha_-} - \lambda_-^{\alpha_-}\right), \text{ if } \alpha \in (0,1),\end{aligned} \tag{44}$$

$$\begin{aligned}\eta(u) = iub + \Gamma(-\alpha_+)c_+\Big((\lambda_+ - iu)^{\alpha_+} - \lambda_+^{\alpha_+} + iu\alpha_+\lambda_+^{\alpha_+-1}\Big) \\ + \Gamma(-\alpha_-)c_-\Big((\lambda_- + iu)^{\alpha_-} - \lambda_-^{\alpha_-} - iu\alpha_-\lambda_-^{\alpha_--1}\Big), \text{ if } \alpha \in (1,2].\end{aligned} \tag{45}$$

Then, the price of a European style option $V(x,t)$ satisfies

$$\begin{aligned}&\frac{\partial V(x,t)}{\partial t} + (r - \nu + b)\frac{\partial V(x,t)}{\partial x} \\ &+ \Gamma(-\alpha_+)c_+e^{\lambda_+x}\boldsymbol{D}^{\alpha_+}_{(x,\infty)}(e^{-\lambda_+x}V(x,t)) \\ &+ \Gamma(-\alpha_-)c_-e^{-\lambda_-x}\boldsymbol{D}^{\alpha_-}_{(-\infty,x)}(e^{\lambda_-x}V(x,t)) \\ &= AV(x,t), \text{ if } \alpha \in (0,1),\end{aligned} \tag{46}$$

$$\begin{aligned}&\frac{\partial V(x,t)}{\partial t} + B\frac{\partial V(x,t)}{\partial x} \\ &+ \Gamma(-\alpha_+)c_+e^{\lambda_+x}\boldsymbol{D}^{\alpha_+}_{(x,\infty)}\left(e^{-\lambda_+x}V(x,t)\right) \\ &+ \Gamma(-\alpha_-)c_-e^{-\lambda_-x}\boldsymbol{D}^{\alpha_-}_{(-\infty,x)}\left(e^{\lambda_-x}V(x,t)\right) \\ &= AV(x,t), \text{ if } \alpha \in (1,2].\end{aligned} \tag{47}$$

where
$A = (r + \Gamma(-\alpha_+)c_+\lambda_+^{\alpha_+} + \Gamma(-\alpha_-)c_-\lambda^{\alpha_-})$
and
$B = \Big(r - \nu + b + \Gamma(-\alpha_+)c_+\alpha_+\lambda_+^{\alpha_+-1} - \Gamma(-\alpha_-)c_-\alpha_-\lambda_-^{\alpha_--1}\Big).$

As was already stated, all of these processes fall within the umbrella of exponentially damped processes, except for the FMLS. However, both the exponentially damped processes and the FMLS share a common characteristic: they both have finite exponential moments. In fact, these processes are built with the specific intent of ensuring finiteness of exponential moments.

5 NUMERICAL METHOD

The use of fractional calculus in any applied sense, and subsequently the need for reliable methods which can be used to solve fractional differential equations, is a relatively recent topic. As we have previously stated, most (if not all) of the FPDEs which have any practical use do not have an explicit solution. Hence, we must make use of numerical algorithms to solve them.

It should come as no surprise that, given the variety of definitions that exist for fractional integrals and derivatives, there also exist many different numerical methods which have been proposed to solve FDEs and FPDEs. Most of these methods, as is the case with ours, are based upon the GL definition, which lends itself naturally to discretization. Examples of these methods include the G, D, R and L-algorithms. Spectral methods, mesh-free methods, and methods based upon the Adomian decomposition have also been developed. It should be noted that for many of these methods, there is a lack of rigorous results concerning convergence and stability. A comprehensive description of most of these methods can be found in [12], [17] and references therein. The method we will describe and utilize is a variation of the G-algorithm, and is based upon the method developed in [22], and implemented in [20].

Before we move on to the method itself, we should first recall the definition for the left and right-handed GL derivative

$$D^{-\alpha}_{(a,t)} f(t) = \lim_{\substack{h\to 0 \\ n\to\infty}} h^{-\alpha} \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} f(t-kh), \tag{48}$$

and

$$D^{-\alpha}_{(t,b)} f(t) = \lim_{\substack{h\to 0 \\ n\to\infty}} h^{-\alpha} \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} f(t+kh). \tag{49}$$

In order to simplify notation slightly we will define

$$(-1)^k \binom{\alpha}{k} = g_{\alpha,k}\ ,\ k = 1, 2, ...\ \text{ and }\ g_{\alpha,0} = 1. \tag{50}$$

However, in [21] the authors show that if we utilize the standard GL approximations, the finite difference schemes which can be derived are unstable. This is due to the behavior of the fractional derivatives near the boundaries. For this reason, the most utilized numerical schemes rely instead on the shifted GL approximation, given by

$$D^{-\alpha}_{(a,t)} f(t) = \lim_{\substack{h\to 0 \\ n\to\infty}} h^{-\alpha} \sum_{k=0}^{n} g_{\alpha,k} f(t-(k-1)h). \tag{51}$$

5.1 Implicit Euler Method for a two-sided space-fractional PDE

All of the FPDEs which we have derived are of the form

$$\begin{aligned} &\frac{\partial V(x,t)}{\partial t} + A\frac{\partial V(x,t)}{\partial x} + a(x)\mathbf{D}^{\alpha+}_{(x,\infty)}(b(x)V(x,t)) + c(x)\mathbf{D}^{\alpha-}_{(-\infty,x)}(d(x)V(x,t)) \\ &= BV(x,t), \end{aligned} \tag{52}$$

subject to a terminal condition and two boundary conditions, and where the fractional derivatives are defined in the RL sense.

In order to work the problem numerically, we obviously must truncate the space domain somehow. In [8], the truncation is done through the use of the Caputo derivative, since the authors show that if we approximate $\mathbf{D}^{\alpha-}_{(-\infty,x)}(d(x)V(x,t)) \approx \mathbf{D}^{\alpha-}_{(a,x)}(d(x)V(x,t))$ then, in general, the integral included within the expression for the fractional derivative will be singular at the lower bound a. In our case, we have opted for a simple truncation, without making use of the Caputo derivative, to make it as simple as possible. As we will see, this will imply that our results have some problems near the boundaries. Nonetheless, the results we achieve are still quite promising as a first approach.

As usual, we will be working with the spatial variable $x = \log(S)$, with truncated domain $[-\log(4K), \log(4K)]$ where K is the strike price. The time variable has domain $[0, T]$. Both grids are equally spaced. The space and time steps will as usual be denoted as Δx and Δt respectively. The number of space and time steps will be denoted as N_x and N_t respectively.

We will be working with the notation $V_i^n = V(x_{min} + i\Delta x, n\Delta t)$ to make things more readable, with $i = 0, ..., N_x$ and $n = 1, ..., N_t$. Note that this notation is also used for the remaining functions.

We will be using an implicit Euler scheme. In [22] the authors show that this scheme is unconditionally stable for two-sided FPDEs of order $\alpha \in [1, 2]$. The only difference between this type of equation and our own is that in our case a first-order derivative with respect to the space variable is also present. However, in [12], the authors show that the implicit Euler scheme is also unconditionally stable for a space-fractional equation in the presence of a first-order space derivative. Hence, combining these two results, we will also have unconditional stability of the implicit Euler method in our case. Note that both of these schemes have first-order accuracy both in time and in space.

For the discretization of the first order derivative, we will be using a centered difference. As we have already stated, to discretize the fractional derivative terms, we will use a shifted GL approximation. Hence, our scheme becomes

$$\begin{aligned}\frac{V_i^{n+1}-V_i^n}{-\Delta t}+A\frac{V_{i+1}^{n+1}-V_{i-1}^{n+1}}{2\Delta x}+\frac{a_i}{(\Delta x)^{\alpha_+}}\sum_{k=0}^{Nx-i+1}g_{\alpha,k}b_{i+k-1}V_{i+k-1}^{n+1}\\ +\frac{c_i}{(\Delta x)^{\alpha_-}}\sum_{k=0}^{i+1}g_{\alpha,k}d_{i-k+1}V_{i-k+1}^{n+1}=BV_i^{n+1}.\end{aligned} \tag{53}$$

Rewriting, and grouping up the similar terms:

$$\begin{aligned}(1+\Delta tB)V_i^{n+1}-\frac{A\Delta t}{2\Delta x}V_{i+1}^{n+1}+\frac{A\Delta t}{2\Delta x}V_{i-1}^{n+1}-\frac{a_i\Delta t}{(\Delta x)^{\alpha_+}}\sum_{k=0}^{Nx-i+1}g_{\alpha_+,k}b_{i+k-1}V_{i+k-1}^{n+1}\\ -\frac{c_i\Delta t}{(\Delta x)^{\alpha_-}}\sum_{k=0}^{i+1}g_{\alpha_-,k}d_{i-k+1}V_{i-k+1}^{n+1}=V_i^n.\end{aligned} \tag{54}$$

We must now write this equation in matrix form $FV^{n+1}=V^n$. The first thing to note is that, unlike in integer order PDEs, the coefficient matrix F is a square matrix of size Nx, and will not be tridiagonal, which will slow down our numerical algorithm quite significantly. This is due to the non-local nature of the fractional derivatives. We can summarily describe the structure of F in the following way, for $i=1,...N_x-2$ and $j=1,...,N_x-2$:

$$F_{i,j}=\begin{cases}-\xi_i d_j g_{\alpha_-,i-j+1}\text{ , if } j<i-1\\ \frac{A\Delta t}{2\Delta x}-(\xi_i d_j g_{\alpha_-,2}+\eta_i b_j g_{\alpha_+,0})\text{ , if } j=i-1\\ 1+\Delta tB-(\xi_i d_j g_{\alpha_-,1}+\eta_i b_j g_{\alpha_+,1})\text{ , if } j=i\\ -\frac{A\Delta t}{2\Delta x}-(\xi_i d_j g_{\alpha_-,0}+\eta_i b_j g_{\alpha_+,2});\text{ , if } j=i+1\\ -\eta_i b_j g_{\alpha_+,j-i+1}\text{ , if } j>i+1,\end{cases} \tag{55}$$

with $\xi_i=\frac{c_i\Delta t}{(\Delta x)^{\alpha_-}}$ and $\eta_i=\frac{a_i\Delta t}{(\Delta x)^{\alpha_+}}$. As usual, to comply with the boundary conditions, we have that $F_{0,0}=1$, $F_{0,j}=0$ for $j=1,...,N_x-1$, $F_{N_x-1,N_x-1}=1$ and $F_{N_x-1,j}=0$ for $j=1,...,N_x-1$ (i.e., the top and bottom rows of the matrix only have one non-zero element).

One can note that the coefficients associated with the left and right-sided fractional derivatives appear respectively on the lower and upper parts of the matrix, which is quite natural. One can also think of this matrix as the sum of a tridiagonal matrix associated to the integer part of the equation, and a full matrix associated to the fractional part of the equation.

Note that one could adapt this into an explicit scheme quite easily. However, since the explicit scheme is shown in [22] to be only conditionally stable, it does not provide much of an advantage over the fully implicit scheme. A Crank-Nicolson type scheme is also implemented in [20], although the proof of stability is not presented.

Now, all we have to deal with are the boundary conditions. Usually, for European type options, the behavior of numerical methods is similar regardless of if we are dealing with call or put options. However, in the context of fractional calculus, this is not the case. Hence, we will have to deal with each case separately.

In the case of a European call option, the payoff is given by $\Pi_T = (S_T - K)^+$. Since we must truncate the spatial domain in order to make it workable from a numerical point of view, we will make use of the usual boundary conditions $V(x_{min}, t) = 0$ and $V(x_{max}, t) = \exp(x_{max}) - K\exp(-r(T-t))$. However, the imposition of the upper boundary condition becomes problematic due to the fractional derivatives. Since they are non-local, and since the true domain of the space variable should be semi-infinite, the fractional derivatives contribute to the solution from beyond the boundary, and truncating the domain and imposing the usual boundary condition disregards this contribution completely.

This problem can be handled by extending the space grid and imposing a boundary matrix instead of a simple boundary vector. However, over the course of our numerical trials, we found that despite the error incurred by simply using the standard boundary conditions for the call option case being clearly visible in the graphical representation, the results obtained are still very good.

In the case of a European put option, the payoff is given by $\Pi_T = (K - S_T)^+$ and we will make use of the usual boundary conditions $V(x_{min}, t) = K\exp(-r(T-t)) - \exp(x_{min})$ and $V(x_{max}, t) = 0$. In the put option case, the previous problem does not arise, as the value of the option decays very rapidly towards zero for very large values of the asset price, making the contributions of the fractional derivative largely irrelevant. Hence, we can take the approach of a boundary vector, emulating the integer order PDE case. However, as we will see in the following section, there is still some error around the left boundary, the source of which we have failed to identify, and which seems to affect the solution far more than in the call option case.

Finally, in [8] the authors deal with the ill-behaviour of the fractional derivatives near the boundaries by utilizing the Caputo definition of the fractional derivative instead of the Riemann-Liouville definition which we have used in this work, since this approach regularizes the behaviour of the derivative near the boundaries. This could be a good choice for future research within this topic, since it is also shown in [29] that this type of scheme can achieve second-order accuracy in space.

5.2 The discretized GTS FPDE

Having discussed the framework, we may now just adapt the general discretization to the GTS FPDE. Recalling proposition 4.5, in the case of $\alpha \in (1, 2]$, and following the notation in (52), we have that $A = \left(r - \nu + b + \Gamma(-\alpha_+)c_+\alpha_+\lambda_+^{\alpha_+-1} - \Gamma(-\alpha_-)c_-\alpha_-\lambda_-^{\alpha_--1}\right)$; $a(x) = \Gamma(-\alpha_+)c_+e^{\lambda_+ x}$; $b(x) = \left(e^{-\lambda_+ x}\right)$; $c(x) = \Gamma(-\alpha_-)c_-e^{-\lambda_- x}$; $d(x) = \left(e^{\lambda_- x}\right)$; $B = (r + \Gamma(-\alpha_+)c_+\lambda_+^{\alpha_+} + \Gamma(-\alpha_-)c_-\lambda^{\alpha_-})$ (the expression for ν can be found in (34)).

For the GTS process, we once again run into a particular issue. Due to the low popularity of this model within finance, we have failed to find any source for parameter estimation with a basis on real world option prices. Hence, it becomes quite hard to quantify the error that our method produces.

To test our algorithm, we will instead use it to solve the $CGMY$ FPDE, for which [28] provides parameters for both the BS and $CGMY$ models.

6 NUMERICAL RESULTS AND DISCUSSION

The parameters we will utilize, as well as the real call option prices, can all be found in [28] and concern call options on the S&P500 index, on the 18th of April 2002. For this day, the market closed on a price of $S_0 = 1124.47$. The registered risk-free rate and dividend rates were respectively $r = 1.9\%$ and $q = 1.2\%$. The estimated BS volatility has a value of $\sigma = 0.1812$. The strike price and maturity which we have chosen for the graphical representation are $K = 1025$ and $T = 0.3726$ (from 18 April 2002 to September 1 2002). The estimated $CGMY$ parameters are $C = 0.0244$, $G = 0.0765$, $M = 7.5515$, $Y = 1.2945$. The number of space steps used is $N_x = 1200$ and the number of time steps is $N_t = 1000$.

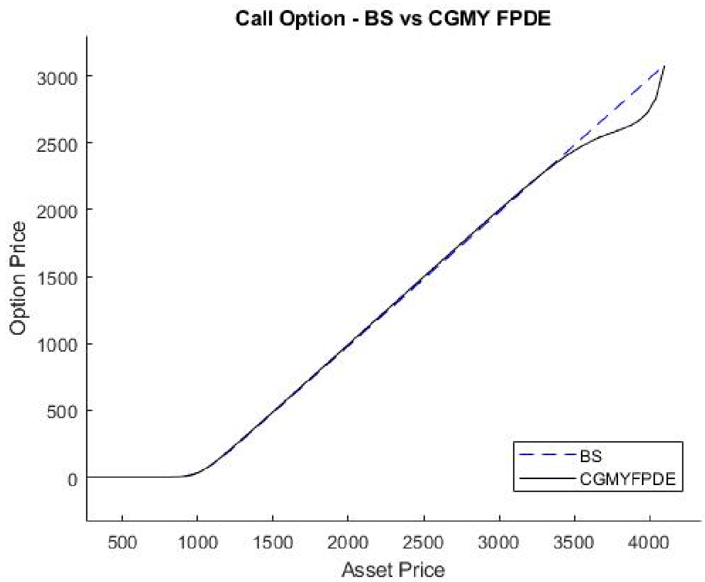

FIGURE 1: Call Option Price - Black-Scholes versus CGMY FPDE ($K = 1025$) - September 2002

The calculated call options prices are $C = 118.12$ for the BS model and $C = 119.25$ for the CGMY model. The observed price is $C = 120.10$. So, for this case, even with the visible error near the boundary, the method still produces results which are significantly better than the BS model.

We will now present tables concerning 3 different maturities:

Strike	975	995	1025	1050	1075	1135	1150	1175	1200	1225
Real	161.6	144.8	120.1	100.7	82.5	45.5	38.1	27.7	19.6	13.2
CGMY	163.2	145.2	119.0	98.4	79.5	42.7	35.6	25.8	18.4	12.9
BS	160.6	142.9	118.1	99.1	81.7	48.11	41.5	32.0	24.2	18.0

TABLE I: Comparison between real prices and prices obtained through the BS equation and the CGMY FPDE - September 2002

The errors, quantified via the root-mean square error as defined in [28]

$$RMSE = \sqrt{\sum \frac{(\text{market price} - \text{model price})^2}{\text{number of options}}} \tag{56}$$

for each of the models in this case are $\mathrm{RMSE}_{\mathrm{CGMY}} = 2.162$ and $\mathrm{RMSE}_{\mathrm{BS}} = 2.835$.

Strike	975	995	1025	1050	1075	1140	1150	1175	1200	1225
Real	173.3	157.0	133.1	114.8	97.6	58.9	53.9	42.5	33	24.9
CGMY	173.9	156.8	132.4	113.3	95.5	57.1	52.3	41.5	32.4	25.1
BS	171.0	154.7	131.7	114.0	97.8	62.7	58.2	48.1	39.3	31.9

TABLE II: Comparison between real prices and prices obtained through the BS equation and the CGMY FPDE - December 2002

The errors for each of the models in this case are $\mathrm{RMSE}_{\mathrm{CGMY}} = 1.221$ and $\mathrm{RMSE}_{\mathrm{BS}} = 4.069$.

Strike	1025	1100	1125	1150	1175	1200	1225	1250	1275
Real	146.5	96.2	81.7	68.3	56.6	46.1	36.9	29.3	22.5
CGMY	144.5	93.7	79.5	66.8	55.5	45.7	37.4	30.3	24.4
BS	144.0	97.1	84.1	72.3	61.9	52.6	44.4	37.3	31.1

TABLE III: Comparison between real prices and prices obtained through the BS equation and the CGMY FPDE - March 2003

The errors for each of the models in this case are $\mathrm{RMSE}_{\mathrm{CGMY}} = 1.621$ and $\mathrm{RMSE}_{\mathrm{BS}} = 5.714$.

It is clear that in all cases, the FPDE outperforms the BS model, sometimes by a very large margin. From observation, it seems that while the BS model performs almost as well, sometimes better, than the FPDE for options which are near the money (i.e. when $K \approx S_0$), the FPDE performs significantly better in the remaining cases, particularly for options which are out of the money (i.e. when $K > S_0$).

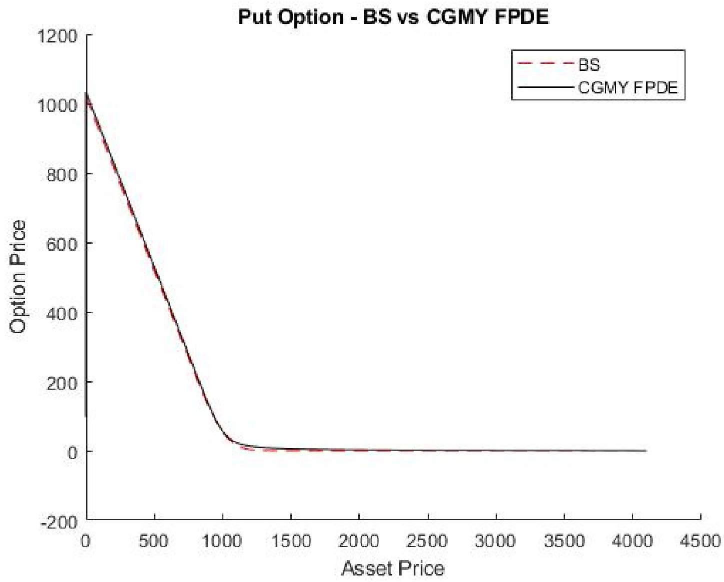

FIGURE 2: Put Option Price - Black-Scholes versus CGMY FPDE

The calculated put option prices are $P = 12.72$ for the BS model and $P = 19.82$ for the CGMY model. Via put-call parity, the real price should be $P = 13.9$, so clearly in this case, the bad behavior of the solver near the boundary clearly has a bigger influence.

Hence, the results seem to indicate that for call options, our model seems to produce very good results, even though the behavior near the boundary is not the best. For put options, the use of the put-call parity after pricing the corresponding call option could be a better choice.

For the GTS FPDE, as we have said, we didn't find any articles or papers which carried out an estimation of parameters based on real option prices. Hence, it is difficult for us to quantify the accuracy of our method in a similar way to what we did with the $CGMY$ FPDE. For these reasons we have chosen the following approach - since the GTS process is in essence a generalization of the $CGMY$ process, we have taken the same parameters which we used before, and introduced some small perturbations, while maintaining the values for the market variables (i.e. $S_0 = 1124.47$, $K = 1025$, $q = 1.2\%$, $r = 1.9\%$ and $T = 0.3726$). As for the model parameters, we use $c_+ = 0.02$, $c_- = 0.03$, $\lambda_+ = 7.65$, $\lambda_- = 0.1$, $\alpha_+ = 1.25$, $\alpha_- = 1.35$. We will use $N_x = 1200$ and $N_t = 1000$ just as before. Since we only made very small changes in the parameters, the option prices should be fairly close to those which we reached in the $CGMY$ case, which, recalling, were $C = 119.25$ and $P = 19.82$ for the call and put option respectively. In fact, given that we increased λ_- and λ_+, which makes the tails of the process heavier, the prices should be slightly higher, given the extra probability for extreme events.

For a put option:

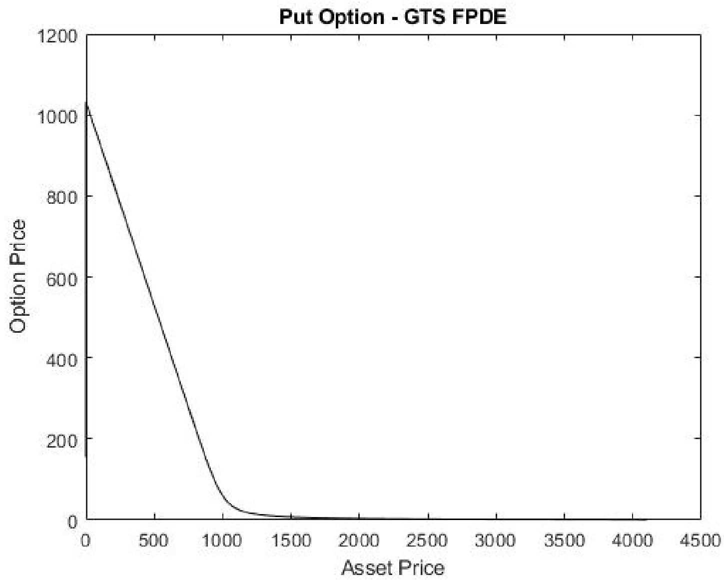

FIGURE 3: Put Option Price - GTS FPDE

For a call option:

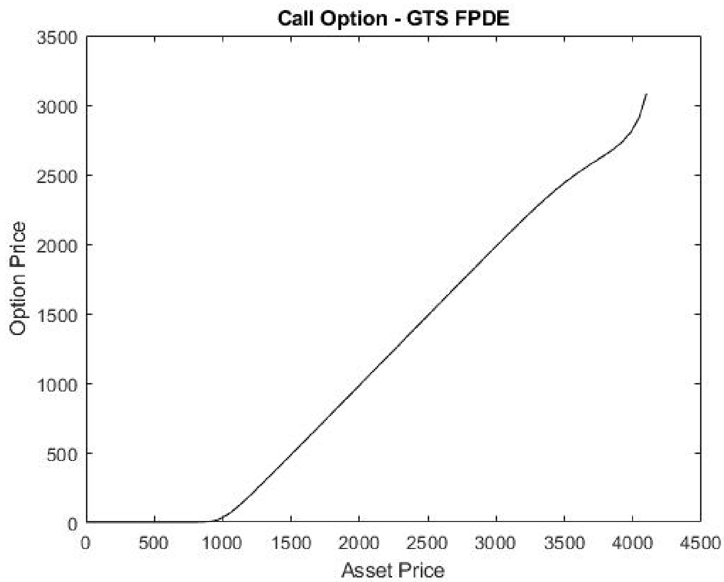

FIGURE 4: Call Option Price - GTS FPDE

The calculated prices are $P = 23.88$ and $C = 124.13$ for the put and call options respectively, which is consistent with our expectations, giving us some confidence that our model is working properly for the GTS FPDE.

As a final note, one should note that an increase in any of the parameters will, *ceteris paribus* produce an increase in the price of the option, since it increases the uncertainty within the market (in an analogous way as what happens with increases in volatility in the BS model). One should also note that the generalization of the $CGMY$ model into the GTS framework is quite effortless, and hence we believe that this model has potential to be more widely utilized within the field of Mathematical Finance. For the reasons we stated above, we don't have a way of fully testing its capabilities, but the added versatility should, at least in theory, give it an edge over the $CGMY$ model.

All the necessary functions and code can be found on this GitHub page. (with URL `https://github.com/franciscofonseca96/thesis`).

7 CONCLUSION

The use of Lévy processes to model asset prices is becoming increasingly popular within the world of financial mathematics. While theoretically more complex than Brownian motion, they make up for it by being much better at capturing real world market movements. By connecting markets driven by Lévy processes to fractional partial differential equations, we gain access to an ever developing world of analytic and numerical methods for solving FPDEs, and consequently for pricing options.

In this dissertation, we have shown that when asset prices follow a Generalized Tempered Stable process, perhaps the most general Lévy process which has thus far been used within the field of financial mathematics, the prices of options follow a two-sided space fractional PDE. We then used a simple finite difference scheme to solve the FPDE and price European options. It was found that the implemented algorithm produced very good results for call options using the $CGMY$ FPDE, although there is some instability near the upper boundary, and we also observed that the scheme had some issues dealing with put options. We believe that these issues are due to the characteristically poor behaviour of Riemann-Liouville type fractional derivatives near the boundaries of their domain, and as such, future research should perhaps consider utilizing Caputo derivatives instead.

There is also a great potential for this method to be used to price more complex instruments, such as exotic options - Barrier, Lookback, Asian - and American type options, after appropriate boundary treatment.

The numerical method we have developed, as well as nearly all of the methods which are found in the literature, only work for equations which have fractional derivatives of order $\alpha \in (1, 2]$. However, in [6] the estimated values for Y, which determine the order of the fractional derivatives in the $CGMY$ FPDE, are almost all in the interval $(0, 1]$. Hence, the development of methods which are stable for this case would be highly desireable, and could be the focus of future research.

Finally, the field of numerical analysis for fractional differential equations is expanding rapidly, and as such the scheme which we used for solving our equations can probably be greatly improved. Different types of methods can also be used to provide a greater order of accuracy and faster computation times. Good candidates would be the use of the aforementioned Caputo fractional derivative, or the use of entirely different methods (Adomian decomposition, mesh-free, among others).

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A PROOFS AND DERIVATIONS

A.1 Proof of 3.4

Proof. Recalling proposition 2.6, and computing the expression for the GTS measure, we have that

$$\int_{|x|\geq 1} \exp(px) \left(\frac{c_-}{|x|^{1+\alpha_-}} \exp(-\lambda_-|x|) \mathbb{1}_{x<1} + \frac{c_+}{|x|^{1+\alpha_+}} \exp(-\lambda_+|x|) \mathbb{1}_{x>1} \right) \mathrm{d}x. \quad (57)$$

Splitting this into a positive and negative part, and simplifying the exponentials, we end up with

$$-c_- \int_{-\infty}^{-1} \frac{\exp((p+\lambda_-)x)}{x^{1+\alpha_-}} \mathrm{d}x + c_+ \int_1^{\infty} \frac{\exp(p-\lambda_+)x}{x^{1+\alpha_+}} \mathrm{d}x. \quad (58)$$

The integral on the left converges for $p + \lambda_- > 0$ and the integral on the right converges for $p - \lambda_+ < 0$, and hence we arrive at the desired result. □

A.2 Derivation of the CGMY FPDE

Proof. In this case we have

$$\eta(-u) = C\Gamma(-Y) \left((M + iu)^Y - M^Y + (G - iu)^Y - G^Y \right), \quad (59)$$

and

$$\nu = C\Gamma(-Y) \left((M - 1)^Y - M^Y + (G + 1)^Y - G^Y \right). \quad (60)$$

Substituting both into (35) and rearranging into the most convenient form:

$$\begin{aligned} \frac{\partial \hat{V}(u,t)}{\partial t} &= \left(r + C\Gamma(-Y)(M^Y + G^Y) \right) \hat{V}(u,t) \\ &+ (r - \nu) iu \hat{V}(u,t) \\ &- C\Gamma(-Y)(M + iu)^Y \hat{V}(u,t) \\ &- C\Gamma(-Y)(G - iu)^Y \hat{V}(u,t). \end{aligned} \quad (61)$$

From here we can perform the inversion term by term:

$$\begin{aligned} &\left(r + C\Gamma(-Y)(M^Y + G^Y) \right) \mathcal{F}^{-1}\{\hat{V}(u,t)\}(x,t) \\ &= \left(r + C\Gamma(-Y)(M^Y + G^Y) \right) V(x,t), \end{aligned}$$

$$(r-\nu)\,\mathcal{F}^{-1}\{iu\hat{V}(u,t)\}(x,t) = -(r-\nu)\frac{\partial V(x,t)}{\partial x}, \tag{62}$$

$$\begin{aligned} &- C\Gamma(-Y)\mathcal{F}^{-1}\{(M+iu)^Y\hat{V}(u,t)\}(x,t)\\ &= -C\Gamma(-Y)e^{Mx}\mathbf{D}^Y_{(x,\infty)}(e^{-Mx}V(x,t)), \end{aligned}$$

$$\begin{aligned} &- C\Gamma(-Y)\mathcal{F}^{-1}\{(G-iu)^Y\hat{V}(u,t)\}(x,t)\\ &= -C\Gamma(-Y)e^{-Gx}\mathbf{D}^Y_{(-\infty,x)}(e^{Gx}V(x,t)). \end{aligned}$$

Putting all of this together and rearranging, we arrive at the desired expression.

□

A.3 Derivation of the GTS FPDE

Proof. The proof for both cases is very similar, so to keep it short we will only present the derivation for the more general case $\alpha \in (1,2]$.

In this case we have

$$\begin{aligned} \eta(u) = iub &+ \Gamma(-\alpha_+)c_+\left((\lambda_+ - iu)^{\alpha_+} - \lambda_+^{\alpha_+} + iu\alpha_+\lambda_+^{\alpha_+-1}\right)\\ &+ \Gamma(-\alpha_-)c_-\left((\lambda_- + iu)^{\alpha_-} - \lambda_-^{\alpha_-} - iu\alpha_-\lambda_-^{\alpha_--1}\right), \end{aligned} \tag{63}$$

and

$$\begin{aligned} \nu = b &+ \Gamma(-\alpha_+)c_+\left((\lambda_+ - 1)^{\alpha_+} - \lambda_+^{\alpha_+} + \alpha_+\lambda_+^{\alpha_+-1}\right)\\ &+ \Gamma(-\alpha_-)c_-\left((\lambda_- + 1)^{\alpha_-} - \lambda_-^{\alpha_-} - \alpha_-\lambda_-^{\alpha_--1}\right). \end{aligned} \tag{64}$$

Substituting both into (35) and putting it into the most convenient form:

$$\begin{aligned} \frac{\partial\hat{V}(u,t)}{\partial t} &= \left(r + \Gamma(-\alpha_+)c_+\lambda_+^{\alpha_+} + \Gamma(-\alpha_-)c_-\lambda_-^{\alpha_-}\right)\hat{V}(u,t)\\ &+ \left(r - \nu + b + \Gamma(-\alpha_+)c_+\alpha_+\lambda_+^{\alpha_+-1} - \Gamma(-\alpha_-)c_-\alpha_-\lambda_-^{\alpha_--1}\right) iu\hat{V}(u,t)\\ &- \Gamma(-\alpha_+)c_+(\lambda_+ + iu)^{\alpha_+}\hat{V}(u,t) - \Gamma(-\alpha_-)c_-(\lambda_- - iu)^{\alpha_-}\hat{V}(u,t). \end{aligned} \tag{65}$$

From here we can perform the inversion term by term:

$$(r + \Gamma(-\alpha_+)c_+\lambda_+^{\alpha_+} + \Gamma(-\alpha_-)c_-\lambda_-^{\alpha_-})\,\mathcal{F}^{-1}\{\hat{V}(u,t)\}(x,t)$$
$$= (r + \Gamma(-\alpha_+)c_+\lambda_+^{\alpha_+} + \Gamma(-\alpha_-)c_-\lambda_-^{\alpha_-})\,V(x,t)$$

$$\left(r - \nu + b + \Gamma(-\alpha_+)c_+\alpha_+\lambda_+^{\alpha_+-1} - \Gamma(-\alpha_-)c_-\alpha_-\lambda_-^{\alpha_--1}\right)\mathcal{F}^{-1}\{iu\hat{V}(u,t)\}(x,t)$$
$$= -\left(r - \nu + b + \Gamma(-\alpha_+)c_+\alpha_+\lambda_+^{\alpha_+-1} - \Gamma(-\alpha_-)c_-\alpha_-\lambda_-^{\alpha_--1}\right)\frac{\partial V(x,t)}{\partial x} \tag{66}$$

$$-\Gamma(-\alpha_+)c_+\mathcal{F}^{-1}\{(\lambda_+ + iu)^{\alpha_+}\hat{V}(u,t)\}(x,t)$$
$$= -\Gamma(-\alpha_+)c_+e^{\lambda_+ x}\mathbf{D}^{\alpha_+}_{(x,\infty)}(e^{-\lambda_+ x}V(x,t))$$

$$-\Gamma(-\alpha_-)c_-\mathcal{F}^{-1}\{\lambda_- - iu)^{\alpha_-}\hat{V}(u,t)\}(x,t)$$
$$= -\Gamma(-\alpha_-)c_-e^{-\lambda_- x}\mathbf{D}^{\alpha_-}_{(-\infty,x)}(e^{\lambda_- x}V(x,t)).$$

Putting all of this together and rearranging, we arrive at the desired expression. □