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SCHOOL OF
ECONOMICS &
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UNIVERSIDADE DE LISBOA

Master

Actuarial Science

Master's Final Work

Dissertation

Risk Bounds for Unimodal Distributions under Partial Information

Rodrigue Kazzi

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Supervision:
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Maria De Lourdes Centeno
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Acknowledgments

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Abstract

Model failures that were observed in the last two decades have shown that the management of model risk is of great importance for the stability of the insurance industry and financial markets. Model risk assessment typically requires the identification of the worst case scenarios including the upper bounds of the risk measures.

In this paper, we first start off by studying the upper-bounds for the Value-at-Risk, Tail-Value-at-Risk, and Range-Value-at-Risk of unimodal distributions when only their mean and their variance upper-bound are known. In a first step, we use a simple convex ordering argument to reduce the optimization problem to a parametric optimization problem. In a second step, we solve this parametric optimization problem and obtain explicit solutions for all probability levels. Our solutions conform well with those of Li et al. (2018), but their analysis is lengthy and their solutions are limited to the case in which probabilities are in the range $[5/6; 1[$. Secondly, since the non-negativity assumption is common in actuarial studies, we study how this assumption can improve the upper bounds of the Value-at-Risk. Moreover, we utilize our two-step analysis to find the upper-bound of the Value-at-Risk in a scenario where the quantile function is fully trusted over a specific range of probability levels. Finally, we assess the model risk that a Beta model carries in a particular credit portfolio.

Results show that the addition of unimodality assumption and the full knowledge of a part of the quantile function do offer an improvement on the risk upper bounds. On the other hand, the non-negativity assumption can lead to a non-improvement in the case of a small variance or an evaluation of the Value-at-Risk on a low probability level.

Keywords: Model risk, Value-at-Risk, Tail-Value-at-Risk, Range-Value-at-Risk, Convex ordering, Unimodal distributions, Risk bounds.

Resumo

As falhas nos modelos observadas nas últimas duas décadas mostram que a gestão do risco desses modelos tem uma grande importância na estabilidade dos mercados financeiros e seguradores. A avaliação do risco do modelo requer usualmente a determinação dos piores cenários possíveis incluindo o limite superior das medidas de risco.

Neste documento, começamos por estudar os limites superiores para Value-at-Risk, Tail-Value-at-Risk e Range-Value-at-Risk de distribuições unimodais quando apenas os limites superiores da média e da variância são conhecidos. Num primeiro passo, usamos o processo do ordenamento simples convexo para reduzir o problema de otimização a um problema de otimização paramétrico. Num segundo passo, resolvemos este problema de otimização paramétrico e obtemos soluções explícitas para todos os níveis de probabilidade. As nossas soluções são consistentes com as de Li et al. (2018), mas a sua análise é longa e as suas soluções limitadas ao caso em que as probabilidades se encontram no intervalo $[5/6; 1[$. Em segundo lugar, dado que a hipótese da não negatividade é comum nos estudos atuariais, estudamos como esta hipótese pode melhorar os limites superiores do Value-at-risk. Além disso, aplicamos a análise de dois passos para encontrar o limite superior do Value-at-Risk num cenário em que a função quantil é totalmente conhecida num intervalo específico de níveis de probabilidades. Por fim, avaliamos o risco do modelo que o modelo Beta gera numa carteira específica de créditos.

Os resultados mostram que a adição da hipótese da unimodalidade e o conhecimento completo de uma parte da função quantil melhoram os limites superiores do risco. Por outro lado, a hipótese da não negatividade pode não trazer qualquer melhoria no caso de se verificar uma variância pequena ou na avaliação do Value-at-Risk a um nível de probabilidade baixo.

Palavras-chave: Risco do modelo, Value-at-Risk, Tail-Value-at-Risk, Range-Value-at-Risk, Ordenação convexa, Distribuições Unimodais, Limites de Risco.

Contents

Chapter 1

Introduction

The use of models is seen as the cornerstone of the every-day operations, planning, and decision-making in the financial world. This reliance on models has been escalating in the last decades, and this pace is expected to only augment with time. Besides, in the endeavor of building models that describe in details real-world situations, we ended-up, in many cases, with complex quantitative models raising the odds of inappropriate employment of these models and amplifying what is called 'Model Risk.' Model Risk can be defined as the potential loss that can result from the misuse of models (inspired by the definition in the Capital Requirements Directive (CRD) IV, Article 3.1.11).

Reviewing the aftermaths of past model failures, one cannot neglect the severe risk that models can present. We can recall from 1997 when LTCM, Long-Term Capital Management hedge fund, lost around 4.5 billion dollars as a consequence of lack of stress testing (Lowenstein 2008). Besides, let us look back at the famous Gaussian-copula of the actuary David X. which was, even though poorly understood by the investors yet, over-relied on. The almost-blind reliance on this copula-based correlation model constituted one of the critical implicit drivers of the 2008 financial crisis (Salmon 2009). An additional illustration can be the disaster that hit JP Morgan - The London Whale when a modeling error led to an understatement of the risks, this allowed for a fooled growth until getting hit by the European sovereign debt crisis in 2012 and causing losses of 6 billion pounds, not counting the 1 billion pounds of fines (Chase 2013)!

It should be clear by now that model risk is a serious matter with great effect on any risk measurement procedure and, hence, its quantification is of vital importance. The methodologies used in assessing the risks in the insurance or financial market are always based on a specific choice of models for the risk factors; for instance the Delta-Normal, the simulations or even the empirical methods... Therefore, the determination of the regulatory capital requirements, for example, is highly sensitive to the model choice and thus alarmingly exposed to model risk.

Model risk quantification is usually based on a comparison between the value of a risk measure given by the adopted model and the value that can occur in a worst-case scenario. This comparison can be of several types, but it always needs an evaluation of the upper bound of the risk measure that results from the application of worst case scenario. Among all the common risk measures, two of the most famous are the Value-at-Risk, which determines the amount of reserves that a company should hold in order to gain a specific level of confidence that it will not face failure, and the Tail-Value-at-Risk that describes how acute the failure would be. Moreover, one of the most recently developed risk measures is the Range-Value-at-Risk that is notable for its robustness and its practicality in describing any desired part of the quantile function.

Recent papers have achieved considerable progress in the valuation of risk bounds of portfolios. A critical point in the bounds valuation is the scenario adopted, i.e., the assumptions that are made on our knowledge towards the risk characteristics. A common ideology is to assume the knowledge of the marginal distributions of the single risks of the portfolio, and then assume either no information or partial information or total information on the dependence structures between the risks. However, it was shown by Bernard et al. (2016) that replacing the knowledge of the marginal distributions by the knowledge of the collective mean does not lead to a significant loss. Additionally, several papers neglected the information on the dependence structure and focused on the moments of the aggregate risk since these can be estimated way more accurately. Li et al. (2018) considered the case where we have information only on the mean, the variance and the shape of the collective risk and derived upper bounds for the Range-Value-at-Risk, in case of a unimodal shape of the distribution, only for probability levels in the range $[5/6; 1[$.

In this thesis, we follow the path of the researchers who only assumed knowledge of what can be accurately estimated or determined, hence we choose to base our assumption on the knowledge of the mean and the variance upper bound of the aggregate risk along with a full confidence in a unimodal shape for the distribution of the aggregate risk. We construct a two-step approach to assess the risk upper bounds under different scenarios. The first step is to use the convex ordering approach and the fact that the linear functions are limiting cases of the convex and concave functions to simplify the optimization problem to a parametric one. The second step is to perform the parametric optimization and derive formulas for the risk measures upper bounds. This two-step approach is used to recover the corresponding results in Li et al. (2018) and expand them over all the possible probability levels, i.e., we derive the upper bounds of each the Value-at-Risk, Tail-Value-at-Risk, and the Range-Value-at-Risk of unimodal aggregate risks whose mean and variance upper bounds are known for all probability levels in $]0;1[$. Additionally, it is typical to work on non-negative risks, in accordance to this fact, we studied the effect of adding the assumption of non-negativity and derived explicit new formulas for the upper bound of the Value-at-Risk. In practice, it is common to trust a central part of the distribution derived from the data and distrust the tails. Following this logic, we used the two-step approach to determine the Value-at-Risk upper bound when an additional assumption of the full knowledge of a part of the quantile function is adopted. In order to illustrate all the obtained results practically, we consider an example of a credit risk portfolio whose characteristics were chosen according to the collective point of views of credit risk researchers and practitioners. The numerical example provides a clear illustration of the extent to which our additional assumptions improve the bounds that were already derived in the literature and gives an idea of how risky is the Beta model in describing credit risk portfolios.

The structure of the thesis is as follows. In chapter 2 we present an in-depth analysis of the existing literature on risk bounds valuations. In chapter 3 an introduction to convex ordering is provided with some of the properties that are of utmost importance to solve our optimization problems. In chapter 4 we explain in detail how to apply the two-step approach in order to calculate the upper bounds of the three risk measures of interest. Chapter 5 is dedicated to the study of the effect of adding the non-negativity assumption on the upper bound of the Value-at-Risk. In chapter 6 we address the scenario of fully trusting a part of the quantile function. Chapter 7 provides a model risk assessment of a credit risk portfolio that illustrates all the findings of the paper. Finally, conclusions and future perspectives are drawn in Chapter 8.

Chapter 2

Literature review

Many studies concerning risk measure bounds preceded our paper and have offered rich contributions to the field of research in model risk assessment. The results vary with the variation in the assumptions taken, for instance, on the level of information available on the marginal distributions, the dependence among the marginals, the moments of the aggregate risk, and the shape of the aggregate risk distribution.

Risk bounds in the case where the dependence structure is unknown but information on the marginals is available have, so far, taken considerable attention by researchers. In fact, the beginning of tail bounds for a sum of two risks dates back to the eighties with Rüschendorf (1981) and Makarov (1982). Then, more recently, Embrechts and Puccetti (2006) proposed an upper bound for the distribution function of the sum of more than two risks in the absence of any dependence information. The homogeneous case, i.e., the distribution functions of the marginals are identical, offered a considerable simplification that allowed Wang and Wang (2011) to obtain sharp tail bounds in the case of monotone densities, Puccetti and Rüschendorf (2013) to establish the sharpness of dual bonds in the case of monotone and concave densities, Wang et al. (2013) to find explicit formulas for the worst Value-at-Risk when the marginal densities are monotone or tail-monotone, and Wang (2014) to study asymptotic bounds of the distribution of the sum of risks. In the inhomogeneous case, the analysis becomes more complicated, and approximations of bounds were needed. Therefore, a new algorithm, called Rearrangement Algorithm (RA), that can calculate numerically sharp bounds for the distribution of the aggregate risk was offered by Puccetti and Rüschendorf (2012b) and developed in Embrechts et al. (2013). Under the inhomogeneity assumption as well, Cai et al. (2018) studies the asymptotic equivalence of risk measures. Additionally, Bernard et al. (2014) derives a convex ordering lower bound over the admissible risk class for both homogeneous and heterogeneous risks.

On the other hand, there exist in the literature some studies of how having some information on the dependence structure, while having fixed marginals, can affect the bounds of the risk measures. We start by citing Williamson and Downs (1990) which presented a numerical representation of the probability distributions through which the dependency bounds are calculated. Then, Denuit et al. (2001) studied the effect of the positive dependence on the total risk. Embrechts et al. (2003) used the dependence information expressed in a copula function to find bounds for the Value-at-Risk measure. Rüschendorf (2005) showed how to use the stochastic ordering for bounding risks and studies the effect of the presence of a stochastic dependence on the risk functionals. A few years later, Puccetti and Rüschendorf (2012a) offered an improvement on some bounds for the distribution function and the tail probabilities of portfolios under the assumption of having full information on certain joint distributions and the assumption of having a constraint on the dependence structure. In Bignozzi et al. (2015), it is proved that an assumption of a negative dependence would mainly affect the upper bound of the Value-at-Risk but an assumption of a positive dependence would only affect the lower bound. Puccetti et al. (2016) considers the cases where the dependence information is available only in the tails, in some central part, or on a general subset of the domain of the distribution function of a risk portfolio. Still, under the setting of having fixed marginals and a partially specified dependence structure, Bernard et al. (2017) derives risk bounds (mainly of the Value-at-Risk and the Tail-Value-at-Risk) for factor models. In Puccetti et al. (2017), independence among (some) subgroups of the marginal components is assumed, this fact leads to an improvement of the Value-at-Risk bounds comparing to the case where only the marginals are known.

In practical situations, determining the dependence structure is a tough task, and in most of the cases, the results are not accurate enough. On the contrary, an estimation of the variance of the portfolio sum can be performed with high accuracy. Therefore, we can see many papers that replace the assumption on the dependence structure by a constraint on the variance as some source of dependence information. In fact, it is intuitive to see that adding variance and higher order moments constraints to a setting where only the collective mean or only the marginals are known would likely improve the risk bounds since this addition captures information that cannot be represented by either the mean or the marginals. Bernard et al. (2017) derived Value-at-Risk bounds based on the knowledge of the marginal distributions and the variance of the portfolio risk; then Bernard et al. (2017) studied these bounds after the addition of information on higher order moments (the skewness for instance). Interestingly, Bernard et al. (2016) proved the critical idea that replacing the knowledge of the marginal distributions by the knowledge of the collective mean does not cause a significant loss of information. In fact, a good number of papers have worked on risk bounds with no knowledge on either marginals or dependence structure but with information on the mean and higher order moments of the portfolio risk, we cite Kaas and Goovaerts (1986b), Hürlimann (1998), Hürlimann (2002), De Schepper and Heijnen (2010), Zymler et al. (2013) among others.

In this paper, we decided to assume, between all the above-mentioned scenarios, the setting of knowing the collective mean and a constraint on the variance. Additionally, we assume knowledge of the shape of the distribution of the aggregate risk; indeed we assume having a unimodal distribution. This assumption is undoubtedly very relevant in practice and has been considered by several papers, we cite Popescu (2005), De Schepper and Heijnen (2010), Van Parys et al. (2016), Li et al. (2018) among others. Under our settings, we firstly aim to find the upper bounds of three risk measures: Value-at-Risk, Tail-Value-at-Risk, and the Range-Value-at-Risk. The first two risk measures were extensively used in many of the papers mentioned above, but the Range-Value-at-Risk is relatively new and was defined as robust risk measure by Cont et al. (2010), and maximized in Li et al. (2018) under several scenarios. We then update our setting to include the assumption of non-negativity, an assumption that is widely used in actuarial sciences and several other areas. We can see some bounds derivation for random variables that are defined on a positive interval in many papers, for instance in Kaas and Goovaerts (1986a), Kaas and Goovaerts (1986b), and Bernard et al. (2016). Another setting that we consider is the case where we have information on the mean, the variance and we fully know the quantile function over a specific interval; this new assumption makes pretty sense since it is common, after data collection, to give confidence only for a central part of a distribution and completely distrust the tails. In fact, Bernard and Vanduffel (2015) uses the same ideology when it splits the area of the multivariate distribution into two, a "trusted area and an "untrusted" area depending on where is the data considered trustworthy enough. Finally, our analysis, as it will be clear in the following chapters, is based on the stochastic ordering and specifically the convex ordering. This ordering is highly adopted in the derivation of risk bounds or even in the approximation of different risk characteristics, for instance one can check Kaas et al. (2000), Jakobsons and Vanduffel (2015), Bernard et al. (2016), and Bernard et al. (2017) among others.

Chapter 3

Convex ordering

Actuarial and financial studies have the comparison of risks at its core. A simple way to order risks is to compute a specific risk measure and then proceed to the ranking accordingly. However, decision-makers can require a risk that is preferable to another for all given risk measures; here comes what is called stochastic ordering. One type of stochastic ordering that compares the variability is the convex ordering.

In this chapter, we present and prove some of the main properties of the convex ordering that are substantially important for the proofs in the following chapters.

We define the convex ordering as in Shaked and Shanthikumar (2007),

Definition. X is said to be smaller than Y in the convex order, denoted as $X \leq_{cx} Y$, if and only if

$$E[\rho(X)] \leq E[\rho(Y)] \text{ for all convex functions } \rho : \mathbb{R} \rightarrow \mathbb{R}, \tag{3.1}$$

provided the expectations exist.

Roughly speaking, if $X \leq_{cx} Y$, then Y is more variable than X in the sense that it is more probable to Y to take extreme values.

Starting from the above definition and assuming that the necessary moments are finite, we prove multiple useful results as follows:

1. We recall that linear functions are the limiting case of convex functions and hence are considered convex. For $X \leq_{cx} Y$, if $\rho(x) = x$ then $E[X] \leq E[Y]$ and if $\rho(x) = -x$ then $E[X] \geq E[Y]$. Thus,

$$X \leq_{cx} Y \Rightarrow E[X] = E[Y] \tag{3.2}$$

2. For $X \leq_{cx} Y$, if $\rho(x)$ is equal to the convex function x^2 then $E[X^2] \leq E[Y^2]$. But $E[X] = E[Y]$ and the variance can be computed as $V[X] = E[X^2] - E[X]^2$ therefore,

$$X \leq_{cx} Y \Rightarrow V[X] \leq V[Y] \tag{3.3}$$

3. We define $\rho_d(x) = (x-d)_+ = \begin{cases} 0 & \text{for } x \in]-\infty; d[\\ x-d & \text{for } x \in [d; +\infty[\end{cases}$, with $d \in \mathbb{R}$. We can see that $\rho_d(x)$ is convex and hence $X \leq_{cx} Y \Rightarrow E[(X-d)_+] \leq E[(Y-d)_+]$, $\forall d \in \mathbb{R}$. But $E[(X-d)_+]$ can be easily shown to be equal to $\int_d^{+\infty} 1 - F_X(x)dx$ where $F_X(x)$ is the cumulative distribution function of X. Therefore,

$$X \leq_{cx} Y \Rightarrow E[(X-d)_+] \leq E[(Y-d)_+] \Leftrightarrow \int_d^{+\infty} F_X(x)dx \geq \int_d^{+\infty} F_Y(x)dx \text{ , } \forall d \in \mathbb{R} \tag{3.4}$$

4. Any convex function can be expressed as the limit of a positive linear combination of ρ_d's, while each ρ_d can have a different value of d, plus an additional linear function as expressed in Proposition 2.8.4 in Denuit et al. (2006) and presented in the Appendix A.1. Hence, we can simply deduce that if $E[X] = E[Y]$ and $E[(X-d)_+] \leq E[(Y-d)_+]$ for all $d \in \mathbb{R}$ then $E[\rho(X)] \leq$

$E[\rho(Y)]$ for all convex functions ρ and thus,

$$X \leq_{cx} Y \Leftrightarrow \begin{cases} E[X] = E[Y], \\ \text{and } \int_d^{+\infty} F_X(x)dx \geq \int_d^{+\infty} F_Y(x)dx \text{ , } \forall d \in \mathbb{R} \end{cases} \tag{3.5}$$

5. Let us consider the situation where F_X up-crosses F_Y exactly once, i.e., $\exists a \in \mathbb{R}$ such that $F_X(x) \leq F_Y(x)$ for $x < a$ and $F_X(x) \geq F_Y(x)$ for $x > a$. In this case we can easily see that $\int_d^{+\infty} F_X(x)dx \geq \int_d^{+\infty} F_Y(x)dx$ for all $d \in [a, +\infty[$ and $\int_d^{+\infty} F_X(x) - F_Y(x)dx$ is an increasing function in $d \in]-\infty, a[$. If we add the assumption of $E[X] = E[Y]$ we then have $\int_{-\infty}^{+\infty} F_X(x)dx = \int_{-\infty}^{+\infty} F_Y(x)dx$ and hence $\int_d^{+\infty} F_X(x)dx \geq \int_d^{+\infty} F_Y(x)dx$ for all $d \in \mathbb{R}$.

   We denote by F_X^{-1} and F_Y^{-1} the quantile functions of X and Y respectively. It is then straightforward that F_Y^{-1} up-crossing F_X^{-1} is equivalent to F_X up-crossing F_Y. Finally we come to the following crucial property,

$$X \leq_{cx} Y \Leftrightarrow \begin{cases} E[X] = E[Y], \\ \text{and } F_Y^{-1} \text{ up-crosses } F_X^{-1} \text{ exactly once.} \end{cases} \tag{3.6}$$

Chapter 4

Upper bounds for VaR, TVAR, and RVAR for the aggregate risk

Insurance companies and financial institutions rely on risk measures to quantify the risks they face and make strategic decisions. Two of the most popular risk measures are the Value-at-Risk and the Tail-Value-at-risk; a relatively new risk measure is the Range-Value-at-Risk.

In this paper, we take the aggregate risk as a whole in the sense that we avoid adding assumptions on the dependency between marginal risks that are likely to be inaccurate or even, in some cases, unrealistic. We derive the worst-case scenario and the upper bound for each of the VaR, TVaR, and RVaR of the aggregate risk under the assumptions that the collective mean, the upper bound of the variance and the unimodality property of this risk are known. Our assumptions are pretty reasonable from the practical point of view; in fact, the consideration of the mean and the variance is a classical framework in the distributional optimization literature (Van Parys et al. (2016)), and the unimodality assumption is the case of credit loss modelling and of most of the parametric univariate distributions for instance, Exponential, Pareto, Gamma, Normal, Log-Normal, Logistic, Beta, Weibull, and student's t-distribution...

4.1 Definitions and some notations

We start by presenting some definitions and notations that will be used extensively in the following sections.

Definition. We define unimodality similarly to Li et al. (2018); the distribution of a random variable X is considered to be unimodal if its cumulative distribution function F_X is convex-concave, i.e. $\exists m \in \mathbb{R}$ such that F_X is convex on $]-\infty; m[$ and concave on $]m; +\infty[$. Note that having a convex-concave cumulative distribution function is equivalent to having a concave-convex quantile function.

Remark 4.1. Linear functions are considered to be the limiting case of concave and convex functions. It is thus clear that, under the adopted definition, any continuous distribution composed of two consecutive linear functions is unimodal (including the uniform distribution).

We define the following classes that will be used substantially in the proofs:

$$V(\mu, s) = \{X : \ E[X] = \mu \ , V[X] \leq s^2\} \tag{4.1}$$

$$V_U(\mu, s) = \{X : \ X \text{ is unimodal} \ , X \in V(\mu, s)\} \tag{4.2}$$

$$U_R = \left\{ X : F_X^{-1}(p) = \left\{ \begin{array}{cc} a & \text{for } p \in [0; b[\\ c(p-b)+a & \text{for } p \in [b; 1] \end{array} \right. , \ a \in \mathbb{R}, \ b \in [0; 1], \ c \in \mathbb{R}^+ \right\} \tag{4.3}$$

i.e., U_R is the set of random variables whose quantile function is continuous and composed of two consecutive non-decreasing linear functions, the first one being flat.

$$U_L = \left\{ X : F_X^{-1}(p) = \left\{ \begin{array}{cl} c(p-b)+a & \text{for } p \in [0; b[\\ a & \text{for } p \in [b; 1] \end{array} \right. , a \in \mathbb{R},\, b \in [0;1],\, c \in \mathbb{R}^+ \right\} \tag{4.4}$$

i.e., U_L is the set of random variables whose quantile function is continuous and composed of two consecutive non-decreasing linear functions, the second one being flat.

Using Remark 4.1, we can deduce that each element of U_R is unimodal. Thus, it is clear that

$$U_R \cap V(\mu, s) \subset V_U(\mu, s) \tag{4.5}$$

Similarly, we have

$$U_L \cap V(\mu, s) \subset V_U(\mu, s) \tag{4.6}$$

4.2 VaR upper bound

The Value-at-Risk has become very famous in the last decades as it is chosen by the regulators as a benchmark risk measure to prevent insolvency. A Value-at-Risk at a probability level α represents the amount of capital necessary to ensure with a confidence level α that the insurance or financial institution will not be technically insolvent after a specific period.

Let us consider the aggregate risk of a portfolio and assume that we have enough information to determine its mean, its variance's upper bound, and the property of unimodality of its distribution. We denote the random variable of this aggregate risk by S. Our aim is to find the maximum value of the Value-at-Risk of S at a level α given the information on the mean, maximum variance and unimodality. We recall that the VaR is defined as

$$VaR_\alpha(S) = \inf\{x \in \mathbb{R} \mid F_S(x) \geq \alpha\},\ \alpha \in]0;1[\tag{4.7}$$

In other words, $VaR_\alpha(S)$ can be seen as the left inverse of the cumulative distribution function of S.

Therefore the problem can be stated as

Problem 1.

$$\max_{S\in V_U(\mu,s)} VaR_\alpha(S)$$

We assume that a solution exists to Problem 1. As a first step to solving Problem 1 we need to prove the following two lemmas.

In what follows, we denote by m the level corresponding to the mode x_m, i.e., the cumulative distribution function at the mode has a value of m. (i.e., $F_S(x_m) = m$).

Lemma 1.
If $\alpha \geq m$, we have that

$$\max_{S\in V_U(\mu,s)} VaR_\alpha(S) = \max_{S\in U_R \cap V(\mu,s)} VaR_\alpha(S)$$

where U_R is defined in (4.3).

Proof. Because of relation (4.5), it is straightforward that

$$\max_{S\in U_R \cap V(\mu,s)} VaR_\alpha(S) \leq \max_{S\in V_U(\mu,s)} VaR_\alpha(S) \tag{4.8}$$

In what follows, we prove the reverse inequality. Once it is shown, the equality will be automatically held.

Consider $S^* \in V_U(\mu, s)$ with

$$VaR_\alpha(S^*) = \max_{S\in V_U(\mu,s)} VaR_\alpha(S)$$

In Figure 4.1, we illustrate the quantile function of the candidate solution S^* for the Problem 1.

As a next step, we show that there exists $Y_R \in U_R \cap V(\mu, s)$ such that $VaR_\alpha(Y_R) = VaR_\alpha(S^*)$, which would directly lead to the following inequality

$$VaR_\alpha(S^*) \leq \max_{S\in U_R \cap V(\mu,s)} VaR_\alpha(S)$$

We define the random variable Y_c by its quantile function as follows,

$$F_{Y_c}^{-1}(p) = \begin{cases} F_{S^*}^{-1}(p) & \text{for } p \in [0;\alpha[\\ c(p-\alpha) + F_{S^*}^{-1}(\alpha) & \text{for } p \in [\alpha; 1] \end{cases} \tag{4.9}$$

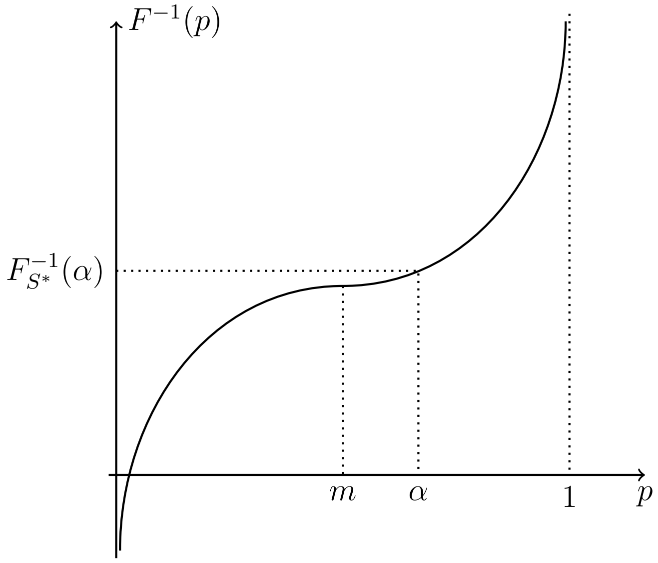

Figure 4.1: The quantile function of the candidate solution S^*

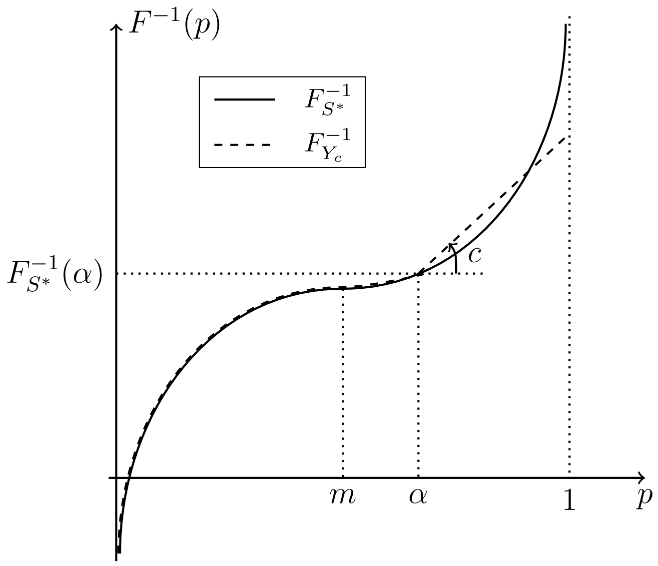

Figure 4.2: The quantile functions of S^* and Y_c

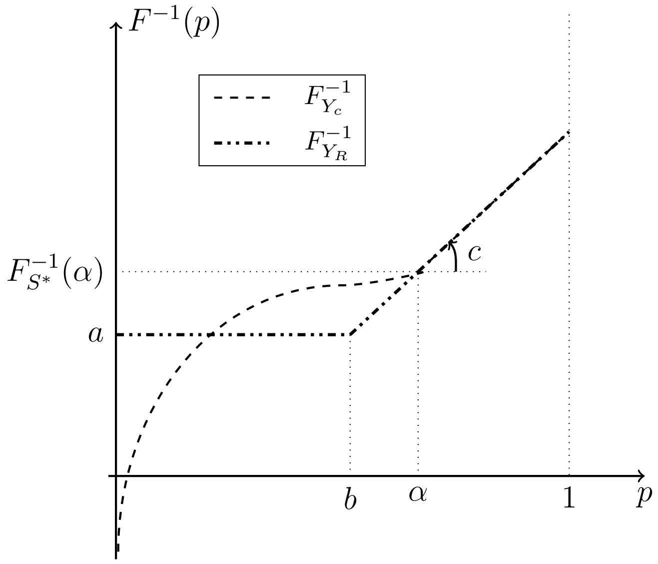

Figure 4.3: The quantile functions of Y_c and Y_R

where $c \in \mathbb{R}^+$.

The variable c is chosen such that the following mean condition is satisfied: $E[Y_c] = \mu$. In particular, by construction, we have that $VaR_\alpha(Y_c) = VaR_\alpha(S^*) = F_{S^*}^{-1}(\alpha)$.

In other words, as shown in Figure 4.2, the quantile function of Y_c is identical to the one of S^* until level α, after which it continues as a linear function in a way that equates the mean of Y_c to the one of S^*.

In Figure 4.2, we illustrate how the equality of means, the linearity of $F_{Y_c}^{-1}$ on $[\alpha; 1]$, and the convexity of $F_{S^*}^{-1}$ on this same interval ensures that $F_{S^*}^{-1}$ up-crosses $F_{Y_c}^{-1}$. As a result, $F_{S^*}^{-1}$ up-crosses $F_{Y_c}^{-1}$ once and $E[Y_c] = E[S^*]$ which directly leads, using Property (3.6), to a convex order relationship between Y_c and S^*, i.e. $Y_c \leq_{cx} S^*$.

Having the convexity order we can deduce that $V[Y_c] \leq V[S^*] \leq s^2$, thus $Y_c \in V(\mu, s)$.

We define the random variable Y_R by its quantile function as follows,

$$F_{Y_R}^{-1}(p) = \begin{cases} c(b-\alpha) + F_{S^*}^{-1}(\alpha) & \text{for } p \in [0; b[\\ c(p-\alpha) + F_{S^*}^{-1}(\alpha) & \text{for } p \in [b; \alpha[\\ F_{Y_c}^{-1}(p) & \text{for } p \in [\alpha; 1] \end{cases} \tag{4.10}$$

where $b \in [0; \alpha]$ and c is as evaluated while finding Y_c.

The variable b is determined such that $E[Y_R] = E[Y_c]$. In particular, by construction, we have that $VaR_\alpha(Y_R) = VaR_\alpha(Y_c) = VaR_\alpha(S^*)$.

In other words, as illustrated in Figure 4.3, the quantile function of Y_R is identical to the one of Y_c for the part of $[\alpha; 1]$, then we extend this same linear function over a larger interval limited on the left at $p = b$, and then the function becomes flat over all the rest of the interval, where the b was chosen such that the total mean is conserved.

In Figure 4.3, we illustrate how $F_{Y_c}^{-1}$ will eventually up-cross $F_{Y_R}^{-1}$ once (because of the equality of means, the linearity of $F_{Y_R}^{-1}$ and the concavity of $F_{Y_c}^{-1}$). Adding the fact that $E[Y_R] = E[Y_c]$ leads to $Y_R \leq_{cx} Y_c \Rightarrow V[Y_R] \leq V[Y_c] \Rightarrow Y_R \in V(\mu, s)$.

But looking at the quantile function of Y_R we can clearly notice that $Y_R \in U_R$, hence $Y_R \in U_R \cap V(\mu, s)$. Finally we get

$$VaR_\alpha(S^*) = VaR_\alpha(Y_R) \leq \max_{S \in U_R \cap V(\mu,s)} VaR_\alpha(S)$$

□

Lemma 2.
If $\alpha < m$, we have that

$$\max_{S \in V_U(\mu,s)} VaR_\alpha(S) = \max_{S \in U_L \cap V(\mu,s)} VaR_\alpha(S)$$

where U_L is defined in (4.4).

Proof. Relation (4.6) implies that

$$\max_{S \in U_L \cap V(\mu,s)} VaR_\alpha(S) \leq \max_{S \in V_U(\mu,s)} VaR_\alpha(S) \tag{4.11}$$

We still have to prove the reverse inequality.

Consider S^* a candidate solution for the Problem 1. And we define the random variable Y_c by its quantile function as shown in Figure 4.4, i.e., the quantile function of Y_c is linear up to level α after which it continues identically to the one of S^*, Y_c is chosen such that its mean is equal to the one of S^*.

Since $F_{S^*}^{-1}$ up-crosses $F_{Y_c}^{-1}$ (as shown in Figure 4.4) and the mean of Y_c is equal to the one of S^*, then $Y_c \leq_{cx} S^*$ which implies that $V[Y_c] \leq V[S^*] \leq s^2 \Rightarrow Y_c \in V(\mu, s)$.

We define Y_L by its quantile function which is represented in Figure 4.5, i.e. the quantile function of Y_L is identical to the one of Y_c for $p \in [0; \alpha]$, this same linear function is extended until $p = b$ after which the quantile function becomes flat, where b was chosen such that $E[Y_L] = E[Y_c]$.

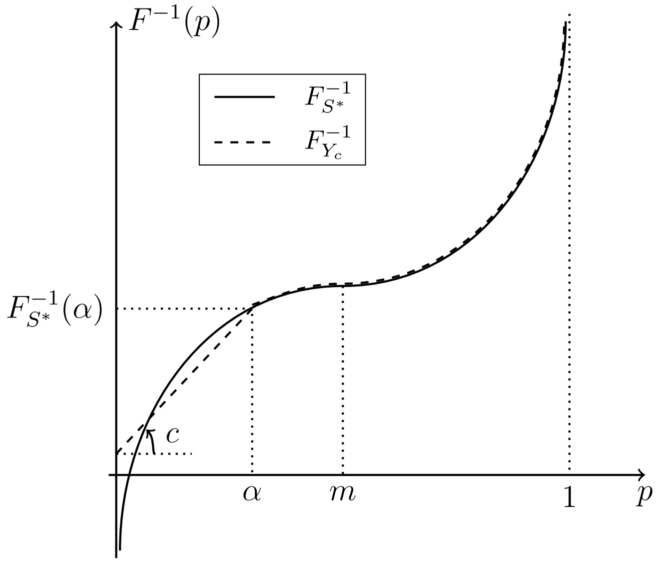

Figure 4.4: The quantile functions of S^* and Y_c

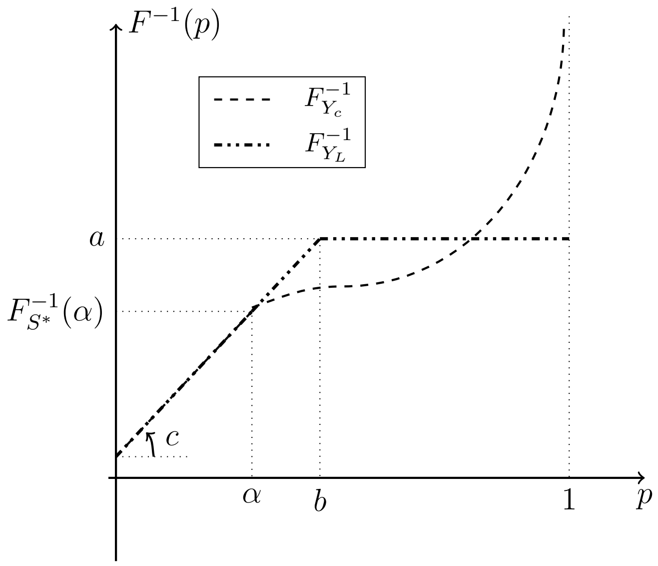

Figure 4.5: The quantile functions of Y_c and Y_L

As illustrated in Figure 4.5, $F_{Y_c}^{-1}$ does necessarily up-cross $F_{Y_L}^{-1}$ exactly once. Having $E[Y_R] = E[Y_c]$ we deduce that $Y_L \leq_{cx} Y_c \Rightarrow V[Y_L] \leq V[Y_c] \Rightarrow Y_L \in V(\mu, s)$. Hence $Y_R \in U_R \cap V(\mu, s)$. Finally we get

$$VaR_\alpha(S^*) = VaR_\alpha(Y_L) \leq \max_{S \in U_L \cap V(\mu,s)} VaR_\alpha(S)$$

□

We define S^* as a solution of Problem 1, i.e. the maximum is realized for S^* with a value of $VaR_\alpha(S^*)$.

Proposition 1.

- If $\alpha \in [5/6; 1[$, $VaR_\alpha(S^*) = \mu + s\sqrt{\frac{4}{9(1-\alpha)} - 1}$
- If $\alpha \in]0; 5/6[$, $VaR_\alpha(S^*) = \mu + s\sqrt{\frac{3\alpha}{4-3\alpha}}$

We underline the fact that the above proposition recovers, in a new method, the same result as in Theorem 1 in Li et al. (2018) for $\alpha \geq 5/6$ (where we tend β to α in their formula of $RVaR_{\alpha,\beta}$ to get the VaR_α), and extends it by solving the problem for the rest of the domain.

Remark 4.2. Very interestingly, we can see that the difference between the maximum median $VaR_{0.5}(S^*)$ and the mean μ is $s\sqrt{\frac{3}{5}}$ which is the exact general upper bound of the absolute difference between the median of a unimodal distribution and its mean as derived by Basu and DasGupta (1997).

Proof. The proof of Proposition 1 will be split into two cases; the first one is the case of having the level α on the right of the level m which indicates the mode, the second case considers the level α being lower than the level m of the mode. We denote by Y_R^* and Y_L^* the optimal solution in each of the two cases respectively. Lacking the knowledge of the position of the mode, we derive the solution to Problem 1 by comparing Y_R^* and Y_L^* and taking the one that maximizes the Value-at-Risk.

Case 1.1. Evaluation of VaR on the right of the mode

As a first step, we show that Problem 1 can be reduced to a simpler problem by reducing the class of variables used in the optimization. This reduction, in this case, is expressed in Lemma 1.

Using Lemma 1 the maximization problem over $V_U(\mu, s)$ can now be reduced to a maximization problem over $U_R \cap V(\mu, s)$. Denoting by Y_R a random variable that belongs to $U_R \cap V(\mu, s)$, $F_{Y_R}^{-1}$ would be expressed as in (4.3):

$$F_{Y_R}^{-1}(p) = \begin{cases} a & \text{for } p \in [0; b[\\ c(p-b)+a & \text{for } p \in [b; 1] \end{cases}, \; a \in \mathbb{R},\; b \in [0;1],\; c \in \mathbb{R}^+ \tag{4.12}$$

The optimization will be split into two steps, in the first we optimize over the three parameters a, b and c and then in the second we optimize over the variance σ^2 given in $[0; s^2]$(i.e. over the standard deviation σ given in $[0; s]$).

Firstly, we use the equations $E[Y_R] = \mu$ and $V[Y_R] = \sigma^2$ to express a and c as functions of b, which leads to the following expression of the quantile function:

$$F_{Y_R}^{-1}(p) = \begin{cases} \mu - \sigma \sqrt{\frac{1-b}{\frac{1}{3}+b}} & \text{for } p \in [0; b[\\ \mu + \sigma \frac{(2p-1-b^2)}{\sqrt{(1-b)^3(1/3+b)}} & \text{for } p \in [b; 1] \end{cases} \tag{4.13}$$

Respecting that $b \in [0; \alpha]$, we maximize $F_{Y_R}^{-1}(\alpha)$ in terms of b, and then we maximize the results in terms of σ over the interval $[0; s]$ to get $Y_R^* \in U_R \cap V(\mu, s)$ where

$$VaR_\alpha(Y_R^*) = \max_{Y_R \in U_R \cap V(\mu,s)} VaR_\alpha(Y_R)$$

The results are as follows,

- $\forall \alpha \in]2/3; 1[$, Y_R^* is obtained for $b = 3\alpha - 2$ and $\sigma = s$, and

$$VaR_\alpha(Y_R^*) = \mu + s\sqrt{\frac{4}{9(1-\alpha)} - 1} \tag{4.14}$$

- $\forall \alpha \in]1/2; 2/3]$, Y_R^* is obtained for $b = 0$ and $\sigma = s$, and

$$VaR_\alpha(Y_R^*) = \mu + s\sqrt{3}(2\alpha - 1) \tag{4.15}$$

- $\forall \alpha \in]0;1/2]$, Y_R^* is obtained for $b = 0$ and $\sigma = 0$, and

$$VaR_\alpha(Y_R^*) = \mu \tag{4.16}$$

Case 1.2. Evaluation of VaR on the left of the mode

As shown in Lemma 2, the maximization problem over $V_U(\mu, s)$ can be reduced to a maximization over $U_L \cap V(\mu, s)$. Let Y_L and Y_L^* two random variables of the class $U_L \cap V(\mu, s)$, with

$$VaR_\alpha(Y_L^*) = \max_{Y_L \in U_L \cap V(\mu,s)} VaR_\alpha(Y_L)$$

Similarly to the first case, we use the conditions $E[Y_L] = \mu$ and $V[Y_L] = \sigma^2$ to get the following expression of the quantile function of Y_L:

$$F_{Y_L}^{-1}(p) = \begin{cases} \mu + \sigma\sqrt{3}\,\frac{(2p-2b+b^2)}{\sqrt{b^3(4-3b)}} & \text{for } p \in [0;b[\\ \mu + \sigma\sqrt{\frac{3b}{4-3b}} & \text{for } p \in [b;1] \end{cases} \tag{4.17}$$

We maximize $F_{Y_L}^{-1}(\alpha)$ first in terms of b for $b \in [\alpha, 1]$ and then in terms of σ over the interval $[0;s]$. We get the following result:

$\forall \alpha \in]0;1[$, Y_L^* is obtained for $b = \alpha$ and $\sigma = s$, and

$$VaR_\alpha(Y_L^*) = \mu + s\sqrt{\frac{3\alpha}{4-3\alpha}} \tag{4.18}$$

Finally, we compare $VaR_\alpha(Y_R^*)$ and $VaR_\alpha(Y_L^*)$ to find the optimum which would hold the value of $VaR_\alpha(S^*)$.

As a result we get,

$$VaR_\alpha(S^*) = max[VaR_\alpha(Y_R^*); VaR_\alpha(Y_L^*)] = \begin{cases} \mu + s\sqrt{\frac{4}{9(1-\alpha)} - 1} & \text{for } \alpha \in [5/6;1[\\ \mu + s\sqrt{\frac{3\alpha}{4-3\alpha}} & \text{for } \alpha \in]0;5/6[\end{cases} \tag{4.19}$$

□

4.3 TVaR upper bound

One of the drawbacks of the Value-at-Risk is that a single VaR does not indicate the severity of the default, i.e., VaR does not present any information about the upper tail of the distribution and hence hides the extent to which the default is severe. Therefore, it is common to complement the risk assessment by the evaluation of an additional risk measure, the Tail-Value-at-Risk that indicates how fat is the upper tail. Indeed, $TVaR_\alpha$ can be seen as the average of the quantiles from α on.

In this chapter, we are interested in finding the maximum value of the Tail-Value-at-Risk of the aggregate risk S at specific level α. The $TVaR$ is defined as

$$TVaR_\alpha(S) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_u(S)du,\ \alpha \in]0;1[$$

where $VaR_\alpha(S)$ is the left inverse of the cumulative distribution function of S as defined before.

We express the problem as follows,

Problem 2.

$$\max_{S\in V_U(\mu,s)} TVaR_\alpha(S)$$

Assuming that a solution exists, the structure of solving Problem 2 will be similar to the one of Problem 1, i.e., we will start by two lemmas and then split the proof into two cases, each case studies the interval on one side of the mode which occurs at level m.

Lemma 3.
If $\alpha \geq m$, we have that

$$\max_{S\in V_U(\mu,s)} TVaR_\alpha(S) = \max_{S\in U_R\cap V(\mu,s)} TVaR_\alpha(S)$$

where U_R is defined in (4.3).

Proof. From relation (4.5) we deduce that

$$\max_{S\in U_R\cap V(\mu,s)} TVaR_\alpha(S) \leq \max_{S\in V_U(\mu,s)} TVaR_\alpha(S) \tag{4.20}$$

We consider that S^* is a candidate solution for Problem 2. In the following, we prove the reverse inequality, i.e. we demonstrate that $\exists Y_R \in U_R \cap V(\mu, s)$ such that $TVaR_\alpha(Y_R) = TVaR_\alpha(S^*)$ which would imply that

$$TVaR_\alpha(S^*) \leq \max_{S \in U_R \cap V(\mu,s)} TVaR_\alpha(S)$$

We consider a random variable Y_c similar to the one defined in the proof of Lemma 1 and presented in Figure 4.2 but relative to the candidate solution S^* of Problem 2.

Thus we have that $Y_c \leq_{cx} S^*$ and $Y_c \in V(\mu, s)$. Moreover, we can easily see that,

$$\begin{aligned} TVaR_\alpha(S^*) &= \frac{1}{1-\alpha} \int_\alpha^1 VaR_u(S^*)du = \frac{1}{1-\alpha} \left[E[S^*] - \int_0^\alpha VaR_u(S^*)du \right] \\ &= \frac{1}{1-\alpha} \left[E[Y_c] - \int_0^\alpha VaR_u(Y_c)du \right] = TVaR_\alpha(Y_c) \end{aligned}$$

We then consider a random variable Y_R similar to the one defined in the proof of Lemma 1 and presented in Figure 4.3. Therefore, $Y_R \leq_{cx} Y_c$ and $Y_R \in U_R \cap V(\mu, s)$. Having $F_{Y_R}^{-1}(p) = F_{Y_c}^{-1}(p)$ for $p \in [\alpha; 1]$ implies that $TVaR_\alpha(Y_R) = TVaR_\alpha(Y_c) = TVaR_\alpha(S^*)$. Hence,

$$TVaR_\alpha(S^*) \leq \max_{S \in U_R \cap V(\mu,s)} TVaR_\alpha(S)$$

.

$\square$

Lemma 4.
If $\alpha < m$, we have that

$$\max_{S \in V_U(\mu,s)} TVaR_\alpha(S) = \max_{S \in U_L \cap V(\mu,s)} TVaR_\alpha(S)$$

where U_L is defined in (4.4).

The proof of Lemma 4 does not present any additional challenge comparing to the proofs of the previous lemmas. However we explain it in Appendix A.2.

We define S^* as a solution of Problem 1, i.e. the maximum is realized for S^* with a value of $VaR_\alpha(S^*)$.

Proposition 2.

- If $\alpha \in [1/2; 1[$, $TVaR_\alpha(S^*) = \mu + s\sqrt{\frac{8}{9(1-\alpha)} - 1}$

- If $\alpha \in]0; 1/2[$, $TVaR_\alpha(S^*) = \mu + \frac{s}{3}\,\frac{\sqrt{\alpha(8-9\alpha)}}{1-\alpha}$

We point out that the above proposition recovers, in a new method, the same result as in Theorem 1 in Li et al. (2018) for $\alpha \geq 5/6$ (when we replace β by 1 in their formula of $RVaR_{\alpha,\beta}$), and extends it to the rest of the interval.

Proof. As in the case of the proof of Problem 1, we split the proof into two cases, an evaluation on each the right and the left of the mode. We denote by Y_R^* and Y_L^* the optimal solution in each of the two cases respectively. Then we compare Y_R^* and Y_L^* to find the one that maximizes the $TVaR_\alpha$.

Case 2.1. Evaluation of $TVaR$ on the right of the mode

In this part of the proof we can use Lemma 3 to reduce the problem over $V_U(\mu, s)$ to a maximization over $U_R \cap V(\mu, s)$. We denote by Y_R a random variable that belongs to $U_R \cap V(\mu, s)$. Respecting that $E[Y_R] = \mu$ and $V[Y_R] = \sigma^2$, $F_{Y_R}^{-1}$ would be expressed as in Equation (4.13). We then derive the function of $TVaR_\alpha$ that can be expressed as

$$TVaR_\alpha(Y_R) = \mu + \sigma\,\frac{\alpha - b^2}{\sqrt{(1-b)^3(1/3+b)}},\ b \in [0; \alpha] \tag{4.21}$$

We proceed to the optimization of the above function, first in terms of b over the interval $[0; \alpha]$, and then in terms of σ over the interval $[0; s]$. The optimization leads to $Y_R^* \in U_R \cap V(\mu, s)$ where

$$TVaR_\alpha(Y_R^*) = \max_{Y_R \in U_R \cap V(\mu,s)} TVaR_\alpha(Y_R)$$

The results are as follows,

- $\forall \alpha \in]1/3; 1[$, Y_R^* is obtained for $b = \frac{3}{2}\alpha - \frac{1}{2}$ and $\sigma = s$, with

$$TVaR_\alpha(Y_R^*) = \mu + s\sqrt{\frac{8}{9(1-\alpha)} - 1} \tag{4.22}$$

- $\forall \alpha \in]0;1/3]$, Y_R^* is obtained for $b=0$ and $\sigma = s$, with

$$TVaR_\alpha(Y_R^*) = \mu + s\alpha\sqrt{3} \tag{4.23}$$

Case 2.2. Evaluation of $TVaR$ on the left of the mode

Based on Lemma 4, the maximization problem over $V_U(\mu, s)$ can be reduced to a maximization over $U_L \cap V(\mu, s)$. Let Y_L and Y_L^* two random variables of the class $U_L \cap V(\mu, s)$, with

$$TVaR_\alpha(Y_L^*) = \max_{Y_L \in U_L \cap V(\mu,s)} TVaR_\alpha(Y_L)$$

Respecting that $E[Y_L] = \mu$ and $V[Y_L] = \sigma^2$ we get the $F_{Y_L}^{-1}$ expressed in Equation (4.17).We then derive the following function of $TVaR_\alpha$,

$$TVaR_\alpha(Y_L) = \mu + \frac{\sigma\sqrt{3}}{1-\alpha} \frac{2b\alpha - b^2\alpha - \alpha^2}{\sqrt{b^3(4-3b)}},\ b \in [\alpha; 1] \tag{4.24}$$

We maximize $TVaR_\alpha(Y_L)$ first in terms of b for $b \in [\alpha, 1]$ and then in terms of σ over the interval $[0; s]$. We get the following results:

- $\forall \alpha \in]2/3;1[$, Y_L^* is obtained for $b=1$ and $\sigma = s$, with

$$TVaR_\alpha(Y_L^*) = \mu + s\alpha\sqrt{3} \tag{4.25}$$

- $\forall \alpha \in]0;2/3]$, Y_L^* is obtained for $b = \frac{3}{2}\alpha$ and $\sigma = s$, with

$$TVaR_\alpha(Y_L^*) = \mu + \frac{s}{3}\frac{\sqrt{\alpha(8-9\alpha)}}{1-\alpha} \tag{4.26}$$

At a last step, we compare $TVaR_\alpha(Y_R^*)$ and $TVaR_\alpha(Y_L^*)$ to find $TVaR_\alpha(S^*)$ as follows,

$$TVaR_\alpha(S^*) = max[TVaR_\alpha(Y_R^*); TVaR_\alpha(Y_L^*)] = \begin{cases} \mu + s\sqrt{\frac{8}{9(1-\alpha)} - 1} & \text{for } \alpha \in [1/2; 1[\\ \mu + \frac{s}{3}\frac{\sqrt{\alpha(8-9\alpha)}}{1-\alpha} & \text{for } \alpha \in]0; 1/2[\end{cases} \quad (4.27)$$

□

4.4 RVaR upper bound

In Cont et al. (2010), a new risk measure was introduced as a robust risk measure that includes the Value-at-Risk and the Tail-Value-at-Risk as limiting cases; this risk measure is called the Range-Value-at-Risk. This new measure can be seen as the average of the quantiles between two specific probability levels.

In this section, our objective is to find the upper bound of the Range-Value-at-Risk of the aggregate risk S at specific levels α and β. The $RVaR$ can be defined as

$$RVaR_{\alpha,\beta}(S) = \frac{1}{\beta - \alpha}\int_\alpha^\beta VaR_u(S)du,\ 0 < \alpha < \beta < 1$$

where $VaR_\alpha(S)$ is the left inverse of the cumulative distribution function of S as defined before.

Hence, we present the problem as follows,

Problem 3.

$$\max_{S \in V_U(\mu,s)} RVaR_{\alpha,\beta}(S)$$

We assume that a solution exists to Problem 3. We start by presenting the following two lemmas. As previously, m represents the value of the cumulative distribution function of the aggregate risk at the mode.

Lemma 5.
If $\alpha \geq m$, we have that

$$\max_{S \in V_U(\mu,s)} RVaR_{\alpha,\beta}(S) = \max_{S \in U_R \cap V(\mu,s)} RVaR_{\alpha,\beta}(S)$$

where U_R is defined in (4.3).

Proof. Relation (4.5) directly implies that

$$\max_{S \in U_R \cap V(\mu,s)} RVaR_{\alpha,\beta}(S) \leq \max_{S \in V_U(\mu,s)} RVaR_{\alpha,\beta}(S) \tag{4.28}$$

Let us prove the opposite inequality. We denote by S^* a candidate solution for Problem 3, and we define a random variable Y_c similarly to the one defined in the proof of Lemma 1 and presented in Figure 4.2 but relative to the candidate solution of Problem 3. We note that, because of the equality of means between Y_c and S^* and the convexity of F_{S*}^{-1} on $[\alpha, 1]$, the slope of the linear part of $F_{Y_c}^{-1}$ is necessarily higher that the derivative of F_{S*}^{-1} at α which implies that $F_{Y_c}^{-1}$ is concave-convex. Thus, $Y_c \in V_U(\mu, s)$ and,

$$RVaR_{\alpha,\beta}(Y_c) \leq RVaR_{\alpha,\beta}(S^*) \tag{4.29}$$

We start with the assumption that $F_{Y_c}^{-1}$ and F_{S*}^{-1} are not identical, and we denote by i the abscissa of the intersection point.

Based on the assumption above and respecting the convexity property on $[\alpha, 1]$,we can observe the following results:

- In the case of $\beta \leq i$, $\forall u \in [\alpha, \beta]$ we have $VaR_u(S^*) \leq VaR_u(Y_c)$ (with the equality being only for $\beta = i$), hence $RVaR_{\alpha,\beta}(S^*) < RVaR_{\alpha,\beta}(Y_c)$

- In case $\beta > i$,$\forall u \in [\beta, 1]$ we have $VaR_u(S^*) > VaR_u(Y_c)$,

$$\begin{aligned}
RVaR_{\alpha,\beta}(S^*) &= \frac{1}{\beta - \alpha}\left[E[S^*] - \int_0^{\alpha} VaR_u(S^*)du - \int_{\beta}^{1} VaR_u(S^*)du\right] \\
&= \frac{1}{\beta - \alpha}\left[E[Y_c] - \int_0^{\alpha} VaR_u(Y_c)du - \int_{\beta}^{1} VaR_u(S^*)du\right] \\
&< RVaR_{\alpha,\beta}(Y_c)
\end{aligned}$$

Hence, if $F_{Y_c}^{-1}$ and F_{S*}^{-1} were not identical we would definitely obtain $RVaR_{\alpha,\beta}(S^*) < RVaR_{\alpha,\beta}(Y_c)$ which contradicts inequation (4.29), i.e., if the two quantile functions are not identical, we can find for any $S^* \in V_U(\mu, s)$ a $Y_c \in V_U(\mu, s)$ that strictly improves the $RVaR$ which contradicts the optimality of S^*. We cannot but conclude that $F_{Y_c}^{-1} = F_{S*}^{-1}$ which directly leads to $RVaR_{\alpha,\beta}(S^*) = RVaR_{\alpha,\beta}(Y_c)$.

We now recall Y_R defined in the proof of Lemma 1 and presented in Figure 4.3. We know that $Y_R \in U_R \cap V(\mu, s)$.

Having $F_{Y_R}^{-1}(p) = F_{Y_c}^{-1}(p)$ for $p \in [\alpha; 1]$ leads to $RVaR_{\alpha,\beta}(Y_R) = RVaR_{\alpha,\beta}(Y_c) = RVaR_{\alpha,\beta}(S^*)$. Hence,

$$RVaR_{\alpha,\beta}(S^*) \leq \max_{S \in U_R \cap V(\mu,s)} RVaR_{\alpha,\beta}(S)$$

.

□

Lemma 6.
If $\alpha < m$, we have that

$$\max_{S \in V_U(\mu,s)} RVaR_{\alpha,\beta}(S) = \max_{S \in U_L \cap V(\mu,s)} RVaR_{\alpha,\beta}(S)$$

where U_L is defined in (4.4).

Proof. Relation (4.6) implies that

$$\max_{S \in U_L \cap V(\mu,s)} RVaR_{\alpha,\beta}(S) \leq \max_{S \in V_U(\mu,s)} RVaR_{\alpha,\beta}(S) \tag{4.30}$$

We now demonstrate the opposite inequality.

Let us consider a candidate solution S^* for Problem 3 and a random variable Y_c defined similarly to the one in the proof of Lemma 2 and presented in Figure 4.4; therefore $Y_c \in V(\mu, s)$. And since $F_{Y_c}^{-1}(p) = F_{S^*}^{-1}(p)$ for $p \in [\alpha; 1]$ we necessarily have $RVaR_{\alpha,\beta}(Y_c) = RVaR_{\alpha,\beta}(S^*)$.

We then consider Y_L defined similarly to the one in the proof of Lemma 2 and presented in Figure 4.5. Which means that, as proven previously, $Y_L \in U_L \cap V(\mu, s) \subset V_U(\mu, s)$. Hence,

$$RVaR_{\alpha,\beta}(Y_L) \leq RVaR_{\alpha,\beta}(S^*) = RVaR_{\alpha,\beta}(Y_c) \tag{4.31}$$

If we pretend that $F_{Y_c}^{-1}$ and $F_{Y_L}^{-1}$ are not identical, and we respect the equality of means and the concave-convex form of $F_{Y_c}^{-1}$, we will necessarily arrive to the following relation: $RVaR_{\alpha,\beta}(Y_c) < RVaR_{\alpha,\beta}(Y_L)$, when $0 < \alpha < \beta < 1$. The last result contradicts the inequation (4.31); therefore, we can conclude that $F_{Y_c}^{-1}$ can only be identical to $F_{Y_L}^{-1}$. Thus $RVaR_{\alpha,\beta}(Y_L) = RVaR_{\alpha,\beta}(Y_c) = RVaR_{\alpha,\beta}(S^*)$. Finally we can say that,

$$RVaR_{\alpha,\beta}(S^*) \leq \max_{S\in U_L\cap V(\mu,s)} RVaR_{\alpha,\beta}(S)$$

□

Proposition 3.

For $0 < \alpha < \beta < 1$,

- If $2\alpha + \beta = 1$, $RVaR_{\alpha,\beta}(S^*) = \mu + \frac{s}{3}\sqrt{\alpha(3\alpha+8)}$
- If $2\alpha + \beta \neq 1$,

$$RVaR_{\alpha,\beta}(S^*) = \begin{cases} \mu + s\sqrt{\frac{8}{9(2-\alpha-\beta)} - 1} & \text{for } \alpha \in \left[\frac{5}{6};1\right[, \\ max\left[\ \mu + s\sqrt{\frac{8}{9(2-\alpha-\beta)} - 1}\ ; Q\right] & \text{for } \alpha \in \left]\frac{1}{2};\frac{5}{6}\right[\\ & \text{and } \ \beta \in \left]f(\alpha);1\right[, \\ Q & \text{otherwise,} \end{cases}$$

where

$\beta = f(\alpha)$ is, for $\alpha \in]1/3;5/6[$, the solution of the equation[1]

$$27\alpha^3+54\alpha^2\beta^2-27\alpha^2\beta-54\alpha^2+36\alpha\beta^3-135\alpha\beta^2+108\alpha\beta-42\beta^4+95\beta^3-54\beta^2 = 0$$

and

$$Q = \mu + \frac{s\sqrt{3}}{\beta-\alpha}\ \frac{b^2(\beta-\alpha-1)+2b\alpha-\alpha^2}{\sqrt{b^3(4-3b)}}$$

for

$$b = \alpha\ \frac{3\alpha+2-\sqrt{(3\alpha-2)^2+12(1-\beta)}}{2(2\alpha+\beta-1)}$$

We note that Proposition 3 recovers Theorem 1 in Li et al. (2018) and extends it by solving the problem for the rest of the domain.

Remark 4.3. Clearly, $VaR_\alpha(S^*)$ and $TVaR_\alpha(S^*)$ are the limiting cases of $RVaR_{\alpha,\beta}(S^*)$ when β tends to α and 1 respectively. This fact is obviously respected since, as can be shown easily, Proposition 1 and Proposition 2 conform with Proposition 3.

Proof. Similarly to what was done previously, we will consider the two cases, when α is on each the right and the left of the mode. We then perform a comparison to find

[1]In fact, for $\alpha \in]1/3;5/6[$, $\beta > f(\alpha)$ is equivalent to having $27\alpha^3 + 54\alpha^2\beta^2 - 27\alpha^2\beta - 54\alpha^2 + 36\alpha\beta^3 - 135\alpha\beta^2 + 108\alpha\beta - 42\beta^4 + 95\beta^3 - 54\beta^2 > 0$

the upper bounds presented in Proposition 3.

Case 3.1. Evaluation of RVaR when α is on the right of the mode

Using Lemma 5, the maximization problem over $V_U(\mu, s)$ is again reduced to a maximization over $U_R \cap V(\mu, s)$. We denote by Y_R and Y_R^* two random variables of $U_R \cap V(\mu, s)$ such that,

$$RVaR_{\alpha,\beta}(Y_R^*) = \max_{Y_R \in U_R \cap V(\mu,s)} RVaR_{\alpha,\beta}(Y_R)$$

Applying the two relations $E[Y_R] = \mu$ and $V[Y_R] = \sigma^2$, $F_{Y_R}^{-1}$ would be expressed as in Equation (4.13). We then derive the function of $RVaR_{\alpha,\beta}$ that can be expressed as

$$RVaR_{\alpha,\beta}(Y_R) = \mu + \sigma \, \frac{\alpha + \beta - 1 - b^2}{\sqrt{(1-b)^3(1/3+b)}}, \; 0 \leq b \leq \alpha < \beta < 1 \tag{4.32}$$

We optimize the above function in terms of b and σ over the intervals $[0; \alpha]$ and $[0; s]$ respectively. The optimization leads to Y_R^* with the following characteristics,

- If $4/3 < \alpha + \beta < 2$, Y_R^* is obtained for $b = \frac{3}{2}(\alpha + \beta) - 2$ and $\sigma = s$, with

$$RVaR_{\alpha,\beta}(Y_R^*) = \mu + s\sqrt{\frac{8}{9(2-\alpha-\beta)} - 1} \tag{4.33}$$

- If $1 < \alpha + \beta < 4/3$, Y_R^* is obtained for $b = 0$ and $\sigma = s$, with

$$RVaR_{\alpha,\beta}(Y_R^*) = \mu + s\sqrt{3}(\alpha + \beta - 1) \tag{4.34}$$

- If $0 < \alpha + \beta < 1$, Y_R^* is obtained for $b = 0$ and $\sigma = 0$, with

$$RVaR_{\alpha,\beta}(Y_R^*) = \mu \tag{4.35}$$

Case 3.2. Evaluation of RVaR When α is on the left of the mode

Based on Lemma 6, the maximization problem over $V_U(\mu, s)$ can be reduced to a maximization over $U_L \cap V(\mu, s)$. Let Y_L and a random variable of the class $U_L \cap V(\mu, s)$.

In this case, we have two possibilities for the position of b, either between α and β or between β and 1. Since we need to maximize $RVaR_{\alpha,\beta}(Y_L)$ in terms of b over the full domain $[\alpha, 1]$, we will have to study each possibility alone and then compare to find the maximum.

Let us start by start by the case where $0 < \alpha < \beta \leq b \leq 1$. Applying $E[Y_L] = \mu$ and $V[Y_L] = \sigma^2$ we get the $F_{Y_L}^{-1}$ expressed in Equation (4.17).We then derive the following function of $RVaR_{\alpha,\beta}$,

$$RVaR_{\alpha,\beta}(Y_L) = \mu + \sigma\sqrt{3}\ \frac{\alpha + \beta - 2b + b^2}{\sqrt{b^3(4 - 3b)}},\ b \in [\beta; 1] \tag{4.36}$$

We maximize $RVaR_{\alpha,\beta}(Y_L)$ first in terms of b for $b \in [\beta, 1]$, the maximum in terms of b is obtained at $b = 1$ in case $\alpha < \beta(1-\beta)$ and $3\alpha^2\beta^2 + 2\alpha^2\beta + \alpha^2 + 6\alpha\beta^3 - 2\alpha\beta^2 + 3\beta^4 - 4\beta^3 + \beta^2 > 0$, and at $b = \beta$ otherwise. We then maximize in terms of σ over the interval $[0; s]$, if the function was maximized at $b = 1$ then the maximum in terms of σ would be at $\sigma = 0$, if the function was maximized at $b = \beta$ then the maximum in terms of σ would be at $\sigma = 0$ in case $\alpha < \beta(1 - \beta)$ and at $\sigma = s$ otherwise
If we denote $Y_{L,1}^*$ a random variable with

$$RVaR_{\alpha,\beta}(Y_{L,1}^*) = \max_{Y_L \in U_L \cap V(\mu,s)} RVaR_{\alpha,\beta}(Y_L)\ ,\ 0 < \alpha < \beta \leq b \leq 1$$

we then express the results as follows,

$$RVaR_{\alpha,\beta}(Y_{L,1}^*) = \begin{cases} \mu & \text{if } \alpha < \beta(1-\beta), \\ \mu + s\sqrt{3}\ \frac{\alpha-\beta+\beta^2}{\sqrt{\beta^3(4-3\beta)}} & \text{otherwise.} \end{cases} \tag{4.37}$$

Now we consider the case where $0 < \alpha \leq b < \beta < 1$, respecting the condition on the mean and the variance we can derive the following function for $RVaR_{\alpha,\beta}$,

$$RVaR_{\alpha,\beta}(Y_L) = \mu + \frac{\sigma\sqrt{3}}{\beta - \alpha}\ \frac{b^2(\beta - \alpha - 1) + 2b\alpha - \alpha^2}{\sqrt{b^3(4 - 3b)}},\ b \in [\alpha; \beta[\tag{4.38}$$

If we denote $Y^*_{L,2}$ a random variable with

$$RVaR_{\alpha,\beta}(Y^*_{L,2}) = \max_{Y_L \in U_L \cap V(\mu,s)} RVaR_{\alpha,\beta}(Y_L) \ , \ 0 < \alpha \leq b < \beta < 1$$

Then, after maximizing in terms of b over $[\alpha; \beta[$ and in terms of σ over $[0, s]$, we get the followings results:

- If $2\alpha + \beta = 1$, then $Y^*_{L,2}$ is obtained at $b = \frac{3\alpha}{3\alpha+2}$ and $\sigma = s$ with

$$RVaR_{\alpha,\beta}(Y^*_{L,2}) = \mu + \frac{s}{3}\sqrt{\alpha(3\alpha+8)} \tag{4.39}$$

- If $2\alpha + \beta \neq 1$, then $Y^*_{L,2}$ is obtained at

$$b = \alpha \ \frac{3\alpha + 2 - \sqrt{(3\alpha-2)^2 + 12(1-\beta)}}{2(2\alpha+\beta-1)} \text{ and } \sigma = s$$

If we denote Y^*_L the random variable such that

$$RVaR_{\alpha,\beta}(Y^*_L) = \max_{Y_L \in U_L \cap V(\mu,s)} RVaR_{\alpha,\beta}(Y_L)$$

Then $RVaR_{\alpha,\beta}(Y^*_L)$ is calculated as follows,

$$RVaR_{\alpha,\beta}(Y^*_L) = max[RVaR_{\alpha,\beta}(Y^*_{L,1}); RVaR_{\alpha,\beta}(Y^*_{L,2})]$$

Finally, we compare $RVaR_{\alpha,\beta}(Y^*_R)$ and $RVaR_{\alpha,\beta}(Y^*_L)$ to find $RVaR_{\alpha,\beta}(S^*)$ presented in Proposition 3, i.e.,

$$RVaR_{\alpha,\beta}(S^*) = max[RVaR_{\alpha,\beta}(Y^*_R); RVaR_{\alpha,\beta}(Y^*_L)] \tag{4.40}$$

□

Chapter 5

VaR upper bound for non-negative aggregate risks

Many studies, mainly in actuarial sciences, consider non-negative random variables. And since it is intuitive that two random variables can have the same mean and variance even if only one of them is non-negative, then it appears to be meaningful to add the assumption of non-negativity when it is the case.

In this chapter, we consider how the non-negativity assumption affects the upper bound of the Value-at-Risk of the portfolio sum.

We recall the notation for the coordinates of the mode represented by $F_S(x_m) = m$, where F_S is the cumulative distribution function of the portfolio sum S.

For simplification, we only consider the part of the quantile function that is higher than m; i.e., we study the upper bounds of $VaR_\alpha(S)$ only for $\alpha \geq m$. This choice is reasonable since it constitutes the common case confronted in practice and it helps to avoid some heavy optimization complications. It is worth keeping in mind that the coordinates of the mode are considered unknown.

We define the new set:

$$V_U^+(\mu, s) = \{X : \ X \text{ is unimodal }, E[X] = \mu \ , V[X] \leq s^2 \ , X \text{ is non-negative}\} \tag{5.1}$$

Using $VaR_\alpha(S)$ as defined previously, our problem would be expressed as follows:

Problem 4.

$$\max_{S \in V_U^+(\mu, s)} VaR_\alpha(S)$$

We assume that a solution S^* exists for Problem 4, a solution to this problem can be expressed in the following proposition:

Proposition 4.

For $0 \leq \alpha < 1$ and $s^2 \in \left[0; \frac{\alpha+1/3}{1-\alpha}\mu^2\right]$,

$$VaR_\alpha(S^*) = \begin{cases} \frac{\mu}{2(1-\alpha)} & \text{for } \alpha \in \left]\frac{1}{2}; 1\right[\\ & \text{and } s^2 \in \left[\frac{\alpha-1/3}{1-\alpha}\mu^2; \frac{\alpha+1/3}{1-\alpha}\mu^2\right], \\ \mu + \frac{9}{8\mu^3}[\alpha(s^2+\mu^2)^2 - (s^4 + \frac{5}{9}\mu^4 + \frac{2}{3}s^2\mu^2)] & \text{for } \alpha \in \left]\frac{2}{3}; 1\right[\\ & \text{and } s^2 \in \left]\frac{\alpha-5/9}{1-\alpha}\mu^2; \frac{\alpha-1/3}{1-\alpha}\mu^2\right[, \\ & \text{or } \alpha \in \left]\frac{1}{2}; \frac{2}{3}\right] \\ & \text{and } s^2 \in \left]\frac{\mu^2}{3}; \frac{\alpha-1/3}{1-\alpha}\mu^2\right[, \\ \mu + s\sqrt{\frac{4}{9(1-\alpha)} - 1} & \text{for } \alpha \in \left]\frac{2}{3}; 1\right[\\ & \text{and } s^2 \in \left[0; \frac{\alpha-5/9}{1-\alpha}\mu^2\right], \\ \mu + s\sqrt{3}(2\alpha - 1) & \text{for } \alpha \in \left]\frac{1}{2}; \frac{2}{3}\right] \text{ and } s^2 \in \left[0; \frac{\mu^2}{3}\right], \\ \mu & \text{for } \alpha \in \left]0; \frac{1}{2}\right], \end{cases}$$

Proof. As mentioned formerly, we only consider in this chapter the case where the probability level α at which the VaR is evaluated is higher than the probability level m of the mode. Hence, using Lemma 1, we can reduce our maximization Problem 4 to a maximization over a set $U_R^+ \cap V(\mu, s)$ with

$$U_R^+ = \left\{ X : F_X^{-1}(p) = \begin{cases} a & \text{for } p \in [0; b[\\ c(p-b) + a & \text{for } p \in [b; 1] \end{cases}, a \in \mathbb{R}^+,\, b \in [0;1],\, c \in \mathbb{R}^+ \right\} \tag{5.2}$$

In other words, adding the assumption of non-negativity implies a non-negative quantile function, i.e. $F_S^{-1}(0) \geq 0$.

We denote by Y_R^+ a random variable that belongs to $U_R^+ \cap V(\mu, s)$. The quantile function of Y_R^+ can be expressed similarly to the quantile in Equation (4.13):

$$F_{Y_R^+}^{-1}(p) = \begin{cases} \mu - \sigma\sqrt{\frac{1-b}{\frac{1}{3}+b}} & \text{for } p \in [0; b[\\ \mu + \sigma\frac{(2p-1-b^2)}{\sqrt{(1-b)^3(1/3+b)}} & \text{for } p \in [b; 1] \end{cases} \tag{5.3}$$

but only under the condition

$$\mu - \sigma \sqrt{\frac{1-b}{\frac{1}{3}+b}} \geq 0$$

Thus, the maximization of $F^{-1}_{Y_R^+}(\alpha)$ will be in terms of the variables b and σ over the intervals $\left[\max\left(\frac{\sigma^2-\mu^2/3}{\sigma^2+\mu^2};0\right);\alpha\right]$ and $[0,s]$ respectively. To ensure that $\frac{\sigma^2-\mu^2/3}{\sigma^2+\mu^2} \leq \alpha$ for all possible values of α we should have $s^2 \leq \frac{\alpha+1/3}{1-\alpha}\mu^2$; in fact, this upper bound for the possible values of the variance is pretty reasonable specially for large values of α.

Previously, when we maximized the quantile function of Y_R, we got three different functions depending on the domain in which α lays. In our current case, we study how the restricted domains of b and σ will affect the results. Let us denote the fraction $\frac{\sigma^2-\mu^2/3}{\sigma^2+\mu^2}$ by $\underline{b}$.

The study is as follows:

- For $\alpha \in]2/3;1[$,
  We compare $\underline{b}$ vs $b = 3\alpha - 2$,
  - If $\underline{b} \leq 3\alpha - 2$, which is equivalent to $\sigma^2 \leq \frac{\alpha-5/9}{1-\alpha}\mu^2$, the maximum would still be satisfied at $b = 3\alpha - 2$,
    * If $s^2 \leq \frac{\alpha-5/9}{1-\alpha}\mu^2$ then the maximum would still be satisfied at $\sigma = s$ as well and the upper bound $\mu + s\sqrt{\frac{4}{9(1-\alpha)} - 1}$ would remain.
    * If $s^2 > \frac{\alpha-5/9}{1-\alpha}\mu^2$ then the maximum would be satisfied at $\sigma = \sqrt{\frac{\alpha-5/9}{1-\alpha}}\mu$ and the upper bound would be $\frac{4\mu}{9(1-\alpha)}$.
  - If $\underline{b} > 3\alpha - 2$, which is equivalent to $\sigma^2 > \frac{\alpha-5/9}{1-\alpha}\mu^2$ (clearly, this can only be satisfied for $s^2 > \frac{\alpha-5/9}{1-\alpha}\mu^2$), the maximum would be attained at $b = \underline{b}$, replacing in Equation 5.3 for $\alpha > b$ we get the quantile function $F^{-1}_{Y_R^+}(\alpha)\big|_{b=\underline{b}} = \mu + \frac{9}{8\mu^3}\left[\alpha(\sigma^2+\mu^2)^2 - (\sigma^4 + \frac{5}{9}\mu^4 + \frac{2}{3}\sigma^2\mu^2)\right]$
    * If $\frac{\alpha-5/9}{1-\alpha}\mu^2 < s^2 < \frac{\alpha-1/3}{1-\alpha}\mu^2$ then the maximum would be realized at $\sigma = s$, with an upper bound $\mu + \frac{9}{8\mu^3}\left[\alpha(s^2+\mu^2)^2 - (s^4 + \frac{5}{9}\mu^4 + \frac{2}{3}s^2\mu^2)\right]$.
    * If $s^2 \geq \frac{\alpha-1/3}{1-\alpha}\mu^2$ then the maximum would be realized at $\sigma = \sqrt{\frac{\alpha-1/3}{1-\alpha}}\mu$, and the upper bound would become $\frac{\mu}{2(1-\alpha)}$.

- For $\alpha \in [0; 2/3]$,
  We compare $\underline{b}$ vs $b = 0$,
  - If $\underline{b} < 0$, which is equivalent to $\sigma^2 < \frac{\mu^2}{3}$, the maximum would still be satisfied at $b = 0$,
    * If $s^2 < \frac{\mu^2}{3}$ then the maximum would still be attained at $\sigma = s$ for $\alpha > 1/2$ and $\sigma = 0$ otherwise, and the upper bounds would be $\mu + s\sqrt{3}(2\alpha - 1)$ and μ respectively, which replicates previous results.
    * If $s^2 \geq \frac{\mu^2}{3}$ then the maximum would be attained at $\sigma = \frac{\mu}{\sqrt{3}}$ for $\alpha > 1/2$ and $\sigma = 0$ otherwise, and the upper bounds would be $2\mu\alpha$ and μ respectively.
  - If $\underline{b} \geq 0$, which is equivalent to $\sigma^2 > \frac{\mu^2}{3}$ (which is only satisfied for $s^2 > \frac{\mu^2}{3}$), the maximum would be attained at $b = \underline{b}$,
    * For $\alpha \in]1/2; 2/3]$,
      · If $\frac{\mu^2}{3} < s^2 < \frac{\alpha-1/3}{1-\alpha}\mu^2$ then the maximum would be realized at $\sigma = s$, with an upper bound $\mu + \frac{9}{8\mu^3}\left[\alpha(s^2+\mu^2)^2 - (s^4 + \frac{5}{9}\mu^4 + \frac{2}{3}s^2\mu^2)\right]$.
      · If $s^2 \geq \frac{\alpha-1/3}{1-\alpha}\mu^2$ then the maximum would be realized at $\sigma = \sqrt{\frac{\alpha-1/3}{1-\alpha}}\mu$, and the upper bound would become $\frac{\mu}{2(1-\alpha)}$.
    * For $\alpha \in]0; 1/2]$, then $\sigma^2 > \frac{\mu^2}{3} \geq \frac{\alpha-1/3}{1-\alpha}\mu^2$. Thus, the maximum would be realized at $\sigma = \frac{\mu}{\sqrt{3}}$, and the upper bound would become $2\mu\alpha$.

To complete the analysis, we need to finish the maximization in terms of σ over $[0; s]$. Firstly, we can observe that for $\alpha \in]2/3; 1[$ and $s^2 \in \left[\frac{\alpha-1/3}{1-\alpha}\mu^2; \frac{\alpha+1/3}{1-\alpha}\mu^2\right]$, we have maximized in terms of σ over each of the regions $\left[0; \mu\sqrt{\frac{\alpha-5/9}{1-\alpha}}\right]$ and $\left]\mu\sqrt{\frac{\alpha-5/9}{1-\alpha}}; s\right]$ and got the two maximums $\frac{4\mu}{9(1-\alpha)}$ and $\frac{\mu}{2(1-\alpha)}$ respectively. What is left to do is to compare these two maximums for the specific regions of α and s and get their maximum, which is clearly $\frac{\mu}{2(1-\alpha)}$ in this case. Thus,

$$VaR_\alpha(S^*) = \frac{\mu}{2(1-\alpha)} \text{ for } \alpha \in \left]\frac{2}{3}; 1\right[\text{ and } s^2 \in \left[\frac{\alpha - 1/3}{1-\alpha}\mu^2; \frac{\alpha + 1/3}{1-\alpha}\mu^2\right]$$

We proceed with the comparisons within all the possible combinations of α and s to get the final result presented in Proposition 4.

□

Chapter 6

VaR upper bound when a part of the quantile function is fully known

In chapter 4 we have derived the upper bound of the Value-at-Risk in case we have some information about the mean, the variance and the shape of the distribution of the aggregate risk. What if we, additionally, do trust a part of the distribution but distrust the rest that includes the tails? This situation is pretty typical since the tails reflect extreme scenarios and the probabilities to experience them are inherently hard to establish.

In this chapter, we present a methodology that leads to the derivation of the upper bound of the Value-at-Risk of the aggregate risk under the assumption of knowing the mean, the upper bound of the variance, the unimodality property and having the full knowledge of the quantile function over a specific range that includes the mode. We point out that our knowledge of the quantile function over a specific region includes the knowledge of its derivative at the extremities of the interval. The valuation of the Value-at-Risk, in this case, is only considered at probability levels that are higher than the one at the upper extremity of the trusted interval.

Let us introduce some notations that will be used in the following proofs. The coordinates of the mode, the lower extremity of the trusted interval and the upper extremity of the trusted interval are represented by $F_S(x_m) = m$, $F_S(x_k) = k$ and $F_S(x_l) = l$ respectively, where F_S is the cumulative distribution function of the portfolio sum S. And let us denote by g a function that is defined and differentiable over $]0;1[$ and covers the trusted part of the quantile function (i.e., $g(p) = F_S^{-1}(p)$ for $p \in [k, l]$, $\frac{\partial g(p)}{\partial p}\big|_{p=k} = \frac{\partial F_S^{-1}(p)}{\partial p}\big|_{p=k}$, and $\frac{\partial g(p)}{\partial p}\big|_{p=l} = \frac{\partial F_S^{-1}(p)}{\partial p}\big|_{p=l}$).

Remark 6.1. A differentiable function $h(x)$ is concave over $[a; b]$ if and only if $\frac{\partial h}{\partial x}$ is non-increasing over]a;b[. On the other hand, $h(x)$ is convex over $[a; b]$ if and only if $\frac{\partial h}{\partial x}$ is non-decreasing over]a;b[.

We recall the definition (4.1) of $V(\mu, s)$ in chapter 4 and we define the new sets U_g and $V_U^g(\mu, s)$ as follows:

$$U_g = \left\{ X : F_X^{-1}(p) = \begin{cases} d(p-k) + g(k) & \text{for } p \in [0; k[\\ g(p) & \text{for } p \in [k; l[\\ e(p-l) + g(l) & \text{for } p \in [l; b[\\ c(p-b) + e(b-l) + g(l) & \text{for } p \in [b; 1] \end{cases} \begin{matrix} , d \in \left[\frac{\partial g(p)}{\partial p}\Big|_{p=k}; +\infty \right[\\ , 0 \le k \le l < b \le 1 \\ , \frac{\partial g(p)}{\partial p}\Big|_{p=l} \le e \le c \le +\infty \end{matrix} \right\} \tag{6.1}$$

where $g(p)$ is defined and differentiable over $]0; 1[$ and concave-convex over $[k; l]$ i.e., U_g is the set of random variables whose quantile function is continuous, non-decreasing and composed of, in a consecutive order, a linear function, a predefined concave-convex function g, and two linear functions in a way that the quantile function is concave until a point in g after which it continues as convex.

$$V_U^g(\mu, s) = \{X : \ X \text{ is unimodal }, E[X] = \mu \ , V[X] \le s^2 \ , F_X^{-1}(p) = g(p) \text{ for } p \in [k; l]\} \tag{6.2}$$

Using $VaR_\alpha(S)$ as defined previously in (4.7), our problem would be expressed as follows:

Problem 5.

$$\max_{S \in V_U^g(\mu, s)} VaR_\alpha(S)$$

In order to solve the problem, a critical first step is to prove the following lemma,

Lemma 7.

$$\max_{S \in V_U^g(\mu, s)} VaR_\alpha(S) = \max_{S \in U_g \cap V(\mu, s)} VaR_\alpha(S)$$

Proof. Firstly, when we look at remark (6.1) and the definitions of $V(\mu, s)$, U_g, and $V_U^g(\mu, s)$, we can straightforwardly see that

$$U_g \cap V(\mu, s) \subset V_U^g(\mu, s) \tag{6.3}$$

which directly implies that

$$\max_{S \in U_g \cap V(\mu, s)} VaR_\alpha(S) \le \max_{S \in V_U^g(\mu, s)} VaR_\alpha(S) \tag{6.4}$$

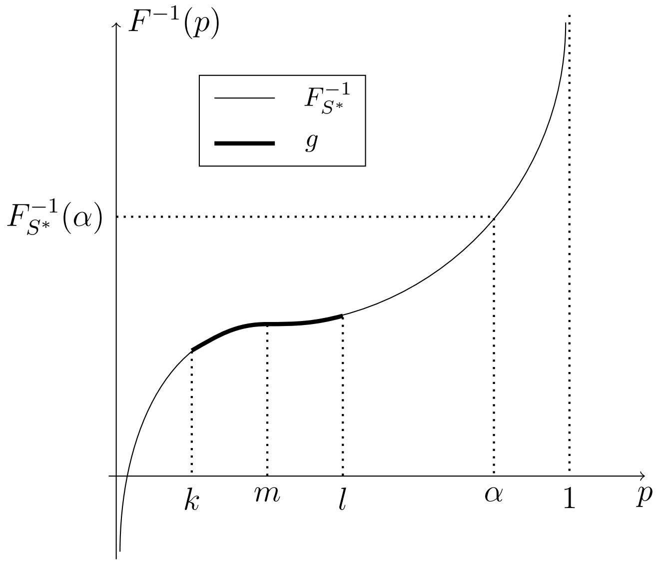

Figure 6.1: The quantile function of the candidate solution S^* when F_S^{-1} is known over $[k, l]$.

The next step is to prove the reverse inequality.

Let us call S^* a random variable that belongs to $V_U^g(\mu, s)$ such that

$$VaR_\alpha(S^*) = \max_{S \in V_U^g(\mu,s)} VaR_\alpha(S)$$

We illustrate in Figure 6.1 the quantile distribution of S^* and a part of the function g that represents the exact trusted part of the quantile function.

We define Y_c similarly to the one in (4.9) and present its quantile function in Figure 6.2. Thus $Y_c \in V(\mu, s)$ and $VaR_\alpha(Y_c) = VaR_\alpha(S^*) = F_{S^*}^{-1}(\alpha)$.

We then define the random variable Y_b by its quantile function as follows,

$$F_{Y_b}^{-1}(p) = \begin{cases} F_{Y_c}^{-1}(p) & \text{for } p \in [0; l[\\ \frac{\left[c(b-\alpha)+F_{S^*}^{-1}(\alpha)\right](p-l)-g(l)(p-b)}{b-l} & \text{for } p \in [l; b[\\ c(p-\alpha) + F_{S^*}^{-1}(\alpha) & \text{for } p \in [b; \alpha[\\ F_{Y_c}^{-1}(p) & \text{for } p \in [\alpha; 1] \end{cases} \tag{6.5}$$

where $b \in]l; \alpha]$ and c is as evaluated while finding Y_c.

The variable b is calculated such that $E[Y_b] = E[Y_c]$. And, obviously, we have that $VaR_\alpha(Y_b) = VaR_\alpha(Y_c) = VaR_\alpha(S^*)$.

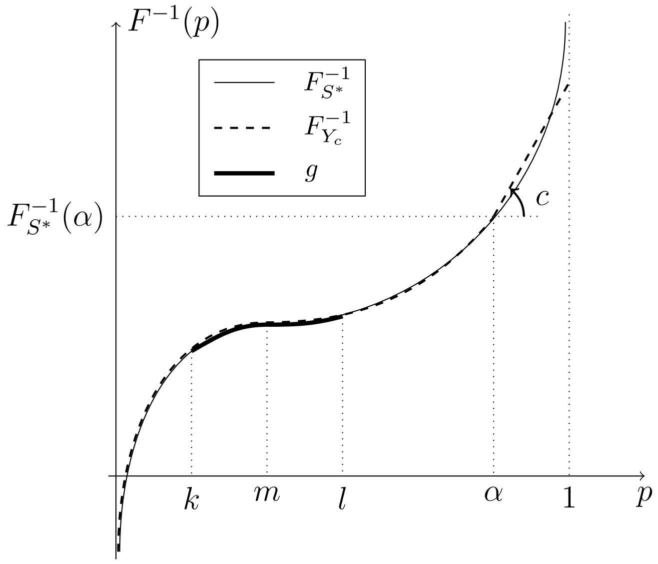

Figure 6.2: The quantile function of S^* and Y_c when F_S^{-1} is known over $[k, l]$.

In Figure 6.3 we illustrate how the quantile function of Y_b is identical to the one of Y_c for $p \in [\alpha; 1]$, then this same linear is extended to the left until $p = b$ at which the function, while remaining continuous, changes to another linear function and joins the function g at $p = l$ after which it becomes again identical to F_{Y_c}.

We can see that $F_{Y_c}^{-1}$ necessarily up-crosses $F_{Y_b}^{-1}$ exactly once if the equality of mean and the convexity property is to be respected. Hence, $Y_b \leq_{cx} Y_c \Rightarrow V[Y_b] \leq V[Y_c] \leq s^2 \Rightarrow Y_b \in V(\mu, s)$.

Finally, we define the random variable Y_d by its quantile function as well,

$$F_{Y_d}^{-1}(p) = \begin{cases} d(p-k) + g(k) & \text{for } p \in [0; k[\\ F_{Y_b}^{-1}(p) & \text{for } p \in [k; 1] \end{cases} \tag{6.6}$$

where $d \in \mathbb{R}^+$.

We calculate d such that $E[Y_d] = E[Y_b]$, and we illustrate $F_{Y_d}^{-1}$ in Figure 6.4. Clearly, $VaR_\alpha(Y_d) = VaR_\alpha(Y_b) = VaR_\alpha(S^*)$.

Respecting the equality of means and the concavity property, we can clearly deduce that a single up-crossing is necessary implying that $Y_d \leq_{cx} Y_b$ and hence $V[Y_d] \leq V[Y_b]$, thus $Y_d \in V(\mu, s)$.

$F_{Y_d}^{-1}$ is clearly continuous, non-decreasing and composed of, in a consecutive order, a linear function, the concave-convex function g, and two linear functions. Additionally, $F_{Y_d}^{-1}$ is necessarily concave on $[0, m[$ and convex on $]m; 1]$. Therefore $Y_d \in U_g$.

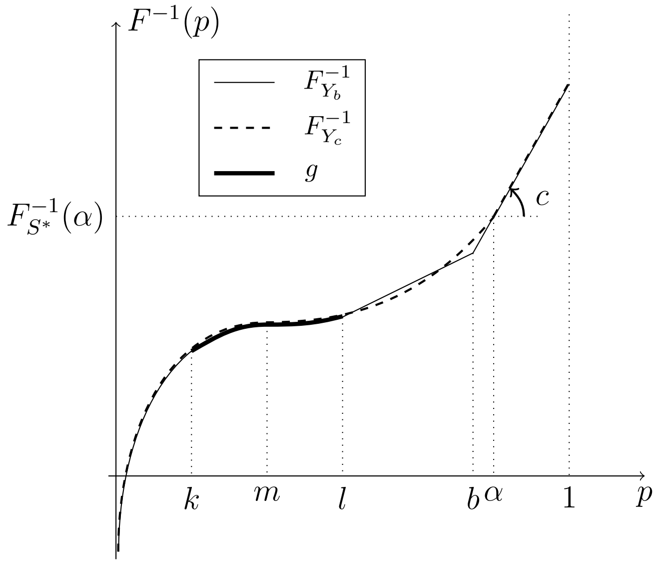

Figure 6.3: The quantile function of Y_c and Y_b when F_S^{-1} is known over $[k, l]$.

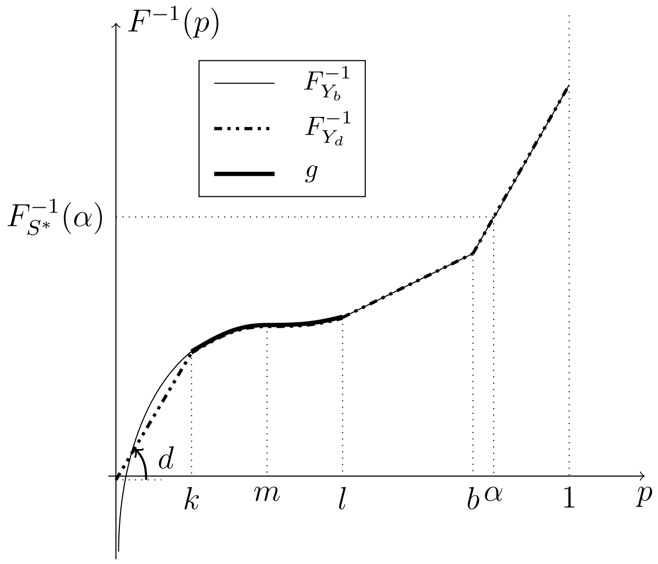

Figure 6.4: The quantile function of Y_b and Y_d when F_S^{-1} is known over $[k, l]$.

Finally we can conclude that $\exists Y_d \in U_g \cap V(\mu, s)$ such that $VaR_\alpha(Y_d) = VaR_\alpha(S^*)$, which implies that

$$\max_{S \in V_U^g(\mu,s)} VaR_\alpha(S) \leq \max_{S \in U_g \cap V(\mu,s)} VaR_\alpha(S)$$

□

Now we can use Lemma 7 to reduce Problem 5 from an optimization over $V_U^g(\mu, s)$ to an optimization over $U_g \cap V(\mu, s)$. Let us call Y_g a random variable that belongs to $U_g \cap V(\mu, s)$, hence its quantile function would be similar to the one in (6.1). Since we assume that we have full knowledge of the quantile function over an interval $[k; l]$ then we can derive from the data the following inputs: $k, l, g(k), g(l), \frac{\partial g}{\partial p}\big|_{p=k}$, and $\frac{\partial g}{\partial p}\big|_{p=l}$. And we have as variables: $b, c, d,$ and e . We then equate $E[Y_g]$ to μ and $V[Y_g]$ to σ^2. Thus, we reduce the number of variables by 1 to get a quantile function that is dependent, for instance, on the variables $b, d,$ and σ. We then maximize the function $VaR_\alpha(Y_g) = F_{Y_g}(\alpha)$ for $\alpha \in [b; 1[$ in terms of $b, d,$ and σ over $]l; \alpha]$, $\left[\frac{\partial g(p)}{\partial p}\big|_{p=k}; +\infty\right[$, and $[0; s^2]$ respectively, under the constraint $\frac{\partial g(p)}{\partial p}\big|_{p=l} \leq e \leq c \leq +\infty$.

Assuming that a solution S^* to Problem 5 exists and the maximization lead to $b = b^*$, $d = d^*$, and $\sigma = \sigma^*$ and hence the values c^* and e^* for c and e respectively, we can then express the upper bound of the Value-at-Risk as

$$\max_{S \in V_U^g(\mu,s)} VaR_\alpha(S) = VaR_\alpha(S^*) = c^*(\alpha - b^*) + e^*(b^* - l) + g(l) \text{ for } \alpha \in]l; 1[\quad (6.7)$$

Remark 6.2. It is worth noting that our results cover several common cases:

- Trusting the quantile function up to a level l, this can be easily found by tending k to 0 .
- Trusting the coordinates of the mode solely, i.e m and $F^{-1}(m)$, this can be solved by tending k and l to m.

Chapter 7

Numerical application to a credit risk portfolio

In this chapter, we apply all the results obtained so far to a credit risk portfolio. We adopt one of the principal models that are used in the industry to evaluate the risk measures of credit risk portfolios, namely the Beta model.

7.1 Model description

We consider a portfolio of n loans given by a bank to n companies that are subject to default. Let us denote the probability of default of a company i by p_i, the maximum amount of loss that can occur due to the default of company i by EAD_i (which stands for Exposure-At-Default), and the percentage of the loss on loan i resulting from the default of the relative company by LGD_i (which stands for Loss-Given-Default). We define an indicator I_i which takes either the value 1 in case of default of the company i or 0 otherwise, i.e.,

$$I_i = \begin{cases} 1 & \text{with probability } p_i \\ 0 & \text{otherwise} \end{cases}$$

Hence, the aggregate portfolio loss, S, can be expressed as

$$S = \sum_{i=1}^{n} I_i \; EAD_i \; LGD_i$$

Another random variable of interest is the aggregate portfolio loss as a percentage of the aggregate exposure at default and can be expressed as

$$\frac{S}{\sum_{i=1}^{n} \; EAD_i \; LGD_i} = \frac{\sum_{i=1}^{n} I_i \; EAD_i \; LGD_i}{\sum_{i=1}^{n} \; EAD_i \; LGD_i}$$

In the actuarial science framework, the CreditRisk+ model is frequently used for modeling credit risk. However, Dhaene et al. (2003) showed that, in case of homogeneous portfolio where each of the Exposure-At-Default, the Loss-Given-Default, and the probability of default has the same value for all the risks of the portfolio, the single factor CreditRisk+ model (CreditRisk+ model with the assumption that the credit quality is driven by one single factor, the global economy for instance) can be replaced by a Beta model when n is large enough and p is small enough since both models would give close results when they have the same values for the first two moments. Based on this result we decided to adopt the Beta model which would simplify the analytical computations as compared to the CreditRisk+ model.

In the Beta model, the aggregate portfolio loss as a percentage of the aggregate exposure at default behaves like a Beta distribution. We assume homogeneity among the risks and we assume that $EAD = v$ and $LGD = 1$, therefore

$$\frac{S}{nv} \sim \text{ Beta}(a, b) \text{ , for } a > 0 \text{ and } b > 0$$

Details on the Beta distribution can be found in Appendix A.3.

7.2 Numerical example

In this chapter, we consider a portfolio of 10000 loans of amount 1 million Euros each, these loans are given by a bank to companies whose rating is high enough to get a probability of default on the loan of 0.1% (rating of A for instance). This probability of default can be taken as the expected value of the aggregate loss as a percentage of the aggregate exposure. In practice, the ratio of the standard deviation of the aggregate loss as a percentage of the aggregate exposure over the relative mean is often around 1 in credit risk portfolios, we choose to equate this ratio to 1.3 in our numerical example. Hence,

$$E\left[\frac{S}{nv}\right] = 0.1\% \text{ and } V\left[\frac{S}{nv}\right] = (0.13\%)^2$$

$$\Rightarrow E[S] = 10 \text{ million Euros and} \sqrt{V[S]} = 13 \text{ million Euros}$$

The parameters a and b can be directly computed by moment matching, in our case we get

$$a = \frac{99731}{169000} \text{ and } b = \frac{9963126}{169000}$$

Because of the positive homogeneity property of each of the Value-at-Risk, Tail-

Value-at-Risk, and the Range-Value-at-Risk (check Appendix A.4 for details) we have that $VaR_\alpha(S) = nv\ VaR_\alpha\left(\frac{S}{nv}\right)$, $TVaR_\alpha(S) = nv\ TVaR_\alpha\left(\frac{S}{nv}\right)$, and $RVaR_{\alpha,\beta}(S) = nv\ RVaR_{\alpha,\beta}\left(\frac{S}{nv}\right)$.

We denote by $\overline{VaR_\alpha}$, $\overline{TVaR_\alpha}$, and $\overline{RVaR_{\alpha,\beta}}$ the upper bounds for the Value-at-Risk, the Tail-Value-at-Risk, and the Range-Value-at-Risk under the assumption of having information on the mean, variance and unimodal shape of the aggregate risk and presented in Propositions 1, 2 ,and 3 respectively. Moreover, we denote by $\overline{VaR_\alpha^+}$ the Value-at-Risk in the case of adding the assumption of the non-negativity of the risks and having α higher than the probability level of the mode, and by $\overline{VaR_\alpha^p}$ the upper bounds of the Value-at-Risk under the assumption of the full knowledge of the quantile function up to a probability level 75%, these upper bounds are presented in Proposition 4 and Equation (6.7) (with $k = 0$ and $l = 0.75$) respectively. To illustrate the effect of adding the unimodality assumption, we will calculate for each case the upper bound so-called Cantelli bound presented in Barrieu and Scandolo (2015) and Bernard et al. (2017). These bounds are derived under the assumption of having information on the mean (μ) and the upper bound of the variance (s^2) solely. We denote the Cantelli bound by $\overline{VaR_\alpha^c}$, this bound was proven to be equal to $\mu + s\sqrt{\frac{\alpha}{1-\alpha}}, \forall\alpha \in]0;1[$. In fact, in Li et al. (2018), it is shown that under the assumption of having information only on the mean and the variance, the upper bounds of the Tail-Value-at-Risk and the Range-Value-at-Risk are equal to the Cantelli upper bound of the Value-at-Risk, i.e., $\overline{TVaR_\alpha^c} = \overline{RVaR_{\alpha,\beta}^c} = \overline{VaR_\alpha^c}$.

α	$VaR_\alpha(S)$	$\overline{VaR_\alpha^c}$	$\overline{VaR_\alpha}$	$\overline{VaR_\alpha^+}$	$\overline{VaR_\alpha^p}$
75%	13.546	32.517	24.741	21.465	13.546
90%	26.106	49	34.127	34.127	30.85
95%	36.182	66.666	46.513	46.513	42.94
99.5%	71.290	193.388	131.874	131.874	89.232

Table 7.1: Upper bounds of the Value-at-Risk under different scenarios regarding the distributional information that is available. The first column depicts the "true" risk measure assuming complete information. All figures are in million Euros.

It is worth noting that, in optimization problems, an equality constraint can be replaced by an inequality for mathematical convenience and hence the assumption of having the exact variance would be replaced by an inequation making the value of the variance a maximum value.This fact renders the formulation of the problem where the exact value of the variance is known to the same formulation used when we assume having the value of the upper bound of the variance.

We present the results of our calculations in Table 7.1 and Table 7.2. In Table 7.1, we can notice that the addition of the unimodality assumption had a great effect on

α	β	$TVaR_\alpha(S)$	$\overline{TVaR^c_\alpha}$	$\overline{TVaR_\alpha}$	$RVaR_{\alpha,\beta}(S)$	$\overline{RVaR^c_{\alpha,\beta}}$	$\overline{RVaR_{\alpha,\beta}}$
75%	90%	27.648	32.517	30.782	18.785	32.517	26.131
90%	95%	40.943	49	46.513	30.538	49	38.853
95%	99.5%	51.348	66.666	63.249	47.385	66.666	60.619
99.5%	99.9%	87.01	193.388	182.845	80.646	193.388	167.696

Table 7.2: Upper bounds of the Tail-Value-at-Risk and the Range-Value-at-Risk under different scenarios regarding the distributional information that is available. The first and the third columns depict the "true" risk measure assuming complete information. All figures are in million Euros.

the upper bounds, the assumption of knowing the first three-quarter of the quantile improved the upper bounds significantly as well especially for high probability levels. On the contrary, the addition of the non-negativity assumption in this example, and probably in typical credit risk portfolios as well, makes no significant improvement if any. In fact, the improvement made at $\alpha = 75\%$ is the result of assuming that the probability level of the mode preceded 75% and not the result of the non-negativity assumption. On the other hand, in Table 7.2, the unimodality assumption improved the bounds of the Tail-Value-at-Risk and the Range-Value-at-Risk but not to the same extent as in the case of the Value-at-Risk.

The model risk can be assessed by calculating the difference between the actual value of the risk measure and the correspondent upper bound (Barrieu and Scandolo (2015)). Looking at the results, we can clearly see that the model risk increases with the probability level; this fact makes the reserving regulations very susceptible to model risk. The Beta model, in this case, presents serious model risk at high probability levels if the quantile function is not trusted over the interval $[0; 0, 75]$. A suggestion would be to try different credit risk models and compare the relative model risks.

Chapter 8

Conclusions

The determination of the risk upper bounds is fundamental in model risk assessment. Indeed, the upper bounds depend heavily on the adopted assumptions. In the existing literature, specifically in Li et al. (2018), upper bounds of the Range-Value-at-Risk (and implicitly the upper bounds of the Value-at-Risk and the Tail-Value-at-Risk) in the setting of knowing the mean, the variance, and the unimodal shape of the risk were found for probability levels higher than 5/6.

This thesis offers several contributions to the field of model risk assessment. Firstly, it extends the results of Li et al. (2018) to cover the full domain of probability levels. In fact, the analysis in Li et al. (2018) is lengthy and not very straightforward in its approach. In contrast, our results are more general and our proofs are based on well-known properties on convex ordering, which greatly simplifies the optimization problem. Interestingly, using the same approach, we could provide further contributions. An explicit upper bound of the Value-at-Risk for non-negative unimodal aggregate risk is derived. Not very surprisingly, the non-negativity assumption lead to improvement only when we had a combination of a probability level that is higher than 50% and a sufficiently large variance. Moreover, the scenario where we fully trust a part of the quantile function is considered. In this last scenario, the risk optimization problem is simplified, and a direct methodology is provided to complete the optimization numerically. Finally, an example of a credit risk portfolio was presented to illustrate the upper bounds derived in this thesis and an example of model risk assessment for the Beta model is performed. This example clarified to what extent a model can be subject to model risk and how the additional assumptions can either effectively improve the upper bounds (the unimodality assumption and the assumption of knowing part of the quantile distribution) or have no significant effect (the non-negativity assumption in this particular case).

Starting with this thesis, we can embark on multiple new interesting research ideas. A directly linked idea would be to use the practical advantage of our two-step approach to test the effect of adding new assumptions on the upper bounds of the risk measures; for instance, we can consider higher order moments like the skewness and check whether the knowledge of the skewness would improve the risk upper bounds or not. A more extensive thought process would be to get inspired by the Remark 4.2 to think whether we can use the approach developed in this thesis to recover the characteristics of unimodal distributions found in Basu and DasGupta (1997) and maybe discover new ones or even expand the work to other types of distributions. Another extensive idea would be to consider the relationship between systemic risk and model risk assessment; this is, in fact, a very actual topic since the International Association of Insurance Supervisors (IAIS) is in the stage of developing new approach called the activity-based-approach (ABA) after having used an entity-based-approach (EBA) for the last few years. The activity-based approach focuses on identifying the activities that can create a systemic risk rather than the entities that can do so. Knowing that similar activities tend to use similar models, a model failure can then generate a failure in an activity on a large scale and stimulate a systemic problem. Hence, we can bet that the model risk assessment would be an essential criterion in identifying the systemically important activities. Additionally, systemic risk assessment and model risk assessment are perceived to have some similarities like the study of worst-case scenarios for instance; this analogy opens our eyes to more reliance and trust in using the findings of this thesis as a building block in establishing the link between the two risks. I strongly believe that there are many areas that are yet to be investigated when it comes to the incidence of risk and the studies behind risk preservation and assessment, mathematical development can be drawn from this paper and ideas can be extended for future reference.

Appendix A

A.1 Proposition 2.8.4 from Denuit et al. (2006)

1. If $\{f_n, n = 1, 2, 3, ...\}$ is a sequence of convex functions $f_n : I \to \mathbb{R}$ converging to a finite limit function f on I, then f is convex. Moreover, the convergence is the uniform on any closed subinterval of I.

2. Every continuous function f convex on $[a, b]$ is the uniform limit of the sequence

$$f_n(x) = \alpha_1^{(n)} + \alpha_2^{(n)} x + \sum_{j=0}^{n} \beta_j^{(n)} (x - t_j^{(n)})_+$$

   with $\beta_j^{(n)} \geq 0$, knots $t_j^{(n)} \in [a, b]$ for $j = 0, 1, ..., n$ and real constants α_1, α_2.

A.2 Proof of Lemma 4

Relation (4.6) implies that

$$\max_{S \in U_L \cap V(\mu,s)} TVaR_\alpha(S) \leq \max_{S \in V_U(\mu,s)} TVaR_\alpha(S) \tag{A.1}$$

We still have to prove the reverse inequality.

We consider a candidate solution S^* and a random variable Y_c defined similarly as in the proof of Lemma 2 and presented in Figure 4.4. Hence $Y_c \in V(\mu, s)$ and having $F_{Y_c}^{-1}(p) = F_{S*}^{-1}(p)$ for $p \in [\alpha; 1]$ implies that $TVaR_\alpha(Y_c) = TVaR_\alpha(S^*)$.

We then consider $Y_L \in U_L \cap V(\mu, s)$ defined similarly as in the proof of Lemma 2 and presented in Figure 4.5. Furthermore, we can easily deduce that

$$\begin{aligned} TVaR_\alpha(Y_c) &= \frac{1}{1-\alpha}\int\limits_\alpha^1 VaR_u(Y_c)du = \frac{1}{1-\alpha}\left[E[Y_c] - \int\limits_0^\alpha VaR_u(Y_c)du\right] \\ &= \frac{1}{1-\alpha}\left[E[Y_L] - \int\limits_0^\alpha VaR_u(Y_L)du\right] = TVaR_\alpha(Y_L) \end{aligned}$$

Finally we get

$$TVaR_\alpha(S^*) = TVaR_\alpha(Y_c) = TVaR_\alpha(Y_L) \leq \max_{S\in U_L\cap V(\mu,s)} TVaR_\alpha(S)$$

This ends the proof.

A.3 Beta distribution

If $X \sim \text{Beta}(a,b)$ with $a > 0$ and $b > 0$, then the density function of X is given by

$$f_X(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} \text{ for } 0 < x < 1,$$

where

$$\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}dt \text{ for } x > 0$$

The first two central moments of X are:

$$E[X] = \frac{a}{a+b} \text{ and } V[X] = \frac{ab}{(a+b)^2(a+b+1)}$$

A.4 Positive homogeneity property of risk measures

A risk measure R is said positive homogeneous if, for any random variable X,

$$\forall c \in \mathbb{R}^+,\ R(cX) = cR(X)$$

Proof that VaR, $TVAR$, and $RVAR$ are positive homogeneous:

For $c \in \mathbb{R}^+$, $x \in \mathbb{R}$, $\alpha \in]0;1[$ and X a random variable,

$$\begin{aligned} VaR_\alpha(cX) = F_{cX}^{-1}(\alpha) \leq x &\Leftrightarrow \alpha \leq F_{cX}(x) \\ &\Leftrightarrow \alpha \leq P(cX \leq x) \\ &\Leftrightarrow \alpha \leq F_X(x/c) \\ &\Leftrightarrow F_X^{-1}(\alpha) \leq x/c \\ &\Leftrightarrow c\ VaR_\alpha(X) \leq x \end{aligned}$$

Thus, $VaR_\alpha(cX) = c\ VaR_\alpha(X)$ and VaR is positive homogeneous.

$$\begin{aligned} TVaR_\alpha(cX) &= \frac{1}{1-\alpha} \int_\alpha^1 VaR_u(cX) du \\ &= \frac{c}{1-\alpha} \int_\alpha^1 VaR_u(X) du \\ &= c\ TVaR_\alpha(X) \end{aligned}$$

Hence TVaR is positive homogeneous. Same reasoning can be applied to prove that RVaR is positive homogeneous.

Bibliography

Barrieu, P. and G. Scandolo (2015). Assessing financial model risk. *European Journal of Operational Research 242*(2), 546–556.

Basu, S. and A. DasGupta (1997). The mean, median, and mode of unimodal distributions: a characterization. *Theory of Probability & Its Applications 41*(2), 210–223.

Bernard, C., M. Denuit, and S. Vanduffel (2016). Measuring portfolio risk under partial dependence information. *Journal of Risk and Insurance*.

Bernard, C., X. Jiang, and R. Wang (2014). Risk aggregation with dependence uncertainty. *Insurance: Mathematics and Economics 54*, 93–108.

Bernard, C., L. Rüschendorf, and S. Vanduffel (2017). Value-at-risk bounds with variance constraints. *Journal of Risk and Insurance 84*(3), 923–959.

Bernard, C., L. Rüschendorf, S. Vanduffel, and R. Wang (2017). Risk bounds for factor models. *Finance and Stochastics 21*(3), 631–659.

Bernard, C., L. Rüschendorf, S. Vanduffel, and J. Yao (2017). How robust is the value-at-risk of credit risk portfolios? *The European Journal of Finance 23*(6), 507–534.

Bernard, C. and S. Vanduffel (2015). A new approach to assessing model risk in high dimensions. *Journal of Banking & Finance 58*, 166–178.

Bignozzi, V., G. Puccetti, and L. Rüschendorf (2015). Reducing model risk via positive and negative dependence assumptions. *Insurance: Mathematics and Economics 61*, 17–26.

Cai, J., H. Liu, and R. Wang (2018). Asymptotic equivalence of risk measures under dependence uncertainty. *Mathematical Finance 28*(1), 29–49.

Chase, J. M. (2013). Report of jpmorgan chase & co. management task force regarding 2012 cio losses.

Cont, R., R. Deguest, and G. Scandolo (2010). Robustness and sensitivity analysis of risk measurement procedures. *Quantitative finance 10*(6), 593–606.

De Schepper, A. and B. Heijnen (2010). How to estimate the value at risk under incomplete information. *Journal of Computational and Applied Mathematics 233*(9), 2213–2226.

Denuit, M., J. Dhaene, M. Goovaerts, and R. Kaas (2006). *Actuarial theory for dependent risks: measures, orders and models*. John Wiley & Sons.

Denuit, M., J. Dhaene, and C. Ribas (2001). Does positive dependence between individual risks increase stop-loss premiums? *Insurance: Mathematics and Economics 28*(3), 305–308.

Dhaene, J., S. Vanduffel, M. Goovaerts, R. Olieslagers, R. Koch, et al. (2003). On the computation of the capital multiplier in the fortis credit economic capital model. *Belgian Actuarial Bulletin 3*(1), 50–57.

Embrechts, P., A. Höing, and A. Juri (2003). Using copulae to bound the value-at-risk for functions of dependent risks. *Finance and Stochastics 7*(2), 145–167.

Embrechts, P. and G. Puccetti (2006). Bounds for functions of dependent risks. *Finance and Stochastics 10*(3), 341–352.

Embrechts, P., G. Puccetti, and L. Rüschendorf (2013). Model uncertainty and var aggregation. *Journal of Banking & Finance 37*(8), 2750–2764.

Hürlimann, W. (1998). Extremal moment methods and stochastic orders. *Application in Actuarial Science. Monograph manuscript (available from the author)*.

Hürlimann, W. (2002). Analytical bounds for two value-at-risk functionals. *ASTIN Bulletin: The Journal of the IAA 32*(2), 235–265.

Jakobsons, E. and S. Vanduffel (2015). Dependence uncertainty bounds for the expectile of a portfolio. *Risks 3*(4), 599–623.

Kaas, R., J. Dhaene, and M. J. Goovaerts (2000). Upper and lower bounds for sums of random variables. *Insurance: Mathematics and Economics 27*(2), 151–168.

Kaas, R. and M. Goovaerts (1986a). Best bounds for positive distributions with fixed moments. *Insurance: Mathematics and Economics 5*(1), 87–92.

Kaas, R. and M. J. Goovaerts (1986b). Extremal values of stop-loss premiums under moment constraints. *Insurance: Mathematics and Economics 5*(4), 279–283.

Li, L., H. Shao, R. Wang, and J. Yang (2018). Worst-case range value-at-risk with partial information. *SIAM Journal on Financial Mathematics 9*(1), 190–218.

Lowenstein, R. (2008). Long-term capital management: It's a short-term memory.

Makarov, G. (1982). Estimates for the distribution function of a sum of two random variables when the marginal distributions are fixed. *Theory of Probability & its Applications 26*(4), 803–806.

Popescu, I. (2005). A semidefinite programming approach to optimal-moment bounds for convex classes of distributions. *Mathematics of Operations Research 30*(3), 632–657.

Puccetti, G. and L. Rüschendorf (2012a). Bounds for joint portfolios of dependent risks. *Statistics & Risk Modeling with Applications in Finance and Insurance 29*(2), 107–132.

Puccetti, G. and L. Rüschendorf (2012b). Computation of sharp bounds on the distribution of a function of dependent risks. *Journal of Computational and Applied Mathematics 236*(7), 1833–1840.

Puccetti, G. and L. Rüschendorf (2013). Sharp bounds for sums of dependent risks. *Journal of Applied Probability 50*(1), 42–53.

Puccetti, G., L. Rüschendorf, and D. Manko (2016). Var bounds for joint portfolios with dependence constraints. *Dependence Modeling 4*(1).

Puccetti, G., L. Rüschendorf, D. Small, and S. Vanduffel (2017). Reduction of value-at-risk bounds via independence and variance information. *Scandinavian Actuarial Journal 2017*(3), 245–266.

Rüschendorf, L. (1981). Sharpness of fréchet-bounds. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete 57*(2), 293–302.

Rüschendorf, L. (2005). Stochastic ordering of risks, influence of dependence and as constructions. *Advances on Models, Characterizations and Applications*, 19–56.

Salmon, F. (2009). Recipe for disaster: The formula that killed wall street.

Shaked, M. and J. G. Shanthikumar (2007). *Stochastic orders*. Springer Science & Business Media.

Van Parys, B. P., P. J. Goulart, and D. Kuhn (2016). Generalized gauss inequalities via semidefinite programming. *Mathematical Programming 156*(1-2), 271–302.

Wang, B. and R. Wang (2011). The complete mixability and convex minimization problems with monotone marginal densities. *Journal of Multivariate Analysis 102*(10), 1344–1360.

Wang, R. (2014). Asymptotic bounds for the distribution of the sum of dependent random variables. *Journal of Applied Probability 51*(3), 780–798.

Wang, R., L. Peng, and J. Yang (2013). Bounds for the sum of dependent risks and worst value-at-risk with monotone marginal densities. *Finance and Stochastics 17*(2), 395–417.

Williamson, R. C. and T. Downs (1990). Probabilistic arithmetic. i. numerical methods for calculating convolutions and dependency bounds. *International journal of approximate reasoning 4*(2), 89–158.

Zymler, S., D. Kuhn, and B. Rustem (2013). Worst-case value at risk of nonlinear portfolios. *Management Science 59*(1), 172–188.