MASTER OF SCIENCE IN
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MASTER´S FINAL WORK
PROJECT

RISK PARITY APPROACH TO PORTFOLIO SELECTION

ÉMERSON BITARÃES DE MOURA FILHO

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LISBON SCHOOL OF ECONOMICS & MANAGEMENT
UNIVERSIDADE DE LISBOA

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SUPERVISOR:
RAQUEL MEDEIROS GASPAR

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Abstract

This study empirically compares the performance of *risk parity* (RP) investment strategy with other common investment strategies, resulting either from *mean variance theory* (MVT) – *tangent* and *minimum variance* portfolios – or naïve investments such as the 60/40 or the *homogeneous* (H) portfolios.

We analysed five RP-based strategies and tested their performances against four benchmark strategies, considering four different investment horizons from 2000 to 2019. We based our analysis in a 30-year data sample ended in December 2019 of five broad indexes representing different asset classes.

We concluded that RP strategies are more balanced from a purely risk point of view (risk contributions, VaR and maximum drawdown), and that some of them consistently outperformed naïve benchmark strategies in risk-adjusted returns, proving to be an effective alternative. However, RP strategies are not able to consistently outperform MVT based portfolios.

Keywords

Risk Parity; Portfolio management; Mean Variance Theory.

Resumo

Este estudo compara de forma empírica a *performance* de estratégias de investimento baseadas em paridade de risco (RP) e outras estratégias comuns, resultantes tanto da teoria de média variância – carteira tangente ou de mínima variância – ou de estratégias *naïve* como as carteiras 60/40 ou homogénea.

Analisámos a *performance* de cinco estratégias baseadas em RP face a quatro estratégias de referência durante quatro diferentes horizontes de investimento entre 2000 e 2019. Baseamos a nossa análise numa amostragem de 30 anos sobre cinco índices representantes de diferentes classes de ativos.

Concluímos que estratégias de paridade de risco são mais balanceadas de um ponto de vista de risco (contribuição de risco, VaR e máxima perda) e que algumas obtiveram resultados mais consitentes do que as carteiras *naïve* em termos de retornos ajustados, provando ser uma alternativa efetiva. Contudo, as estratégias RP não foram capazes de bater regularmente as carteias da teoria de média variância.

Palavras Chave

Paridade de risco; Gestão de portfólios; Teoria de média variância.

Acknowledgments

After two long years of studies, I take some time to look back and dedicate my accomplishments to those who at some point were part of my life and somehow contributed to this journey.

First and foremost, I would like to start by expressing my gratitude to my supervisor, professor Raquel M. Gaspar, whose demanding personality, support and guidance were essential to this work.

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Thank you all for everything.

In the memory of my grandmother Maria Lopes de Moura.

Contents

Acronyms List

ERC equal risk contribution portfolio.

H homogeneous portfolio.

LERC levered equal risk contribution

MD most-diversified portfolio.

MDD Maximum drawdown.

MRC marginal risk contribution.

MV minimum variance portfolio.

MVT mean-variance theory.

NRP naïve risk parity portfolio.

REIT real estate investment trust.

RP risk Parity.

SMA (10) 10-month simple moving average.

SR Sharpe ratio.

T tangent portfolio.

TERC trend-following equal risk contribution portfolio.

TRC total risk contribution.

List of Figures

List of Tables

Chapter 1

Introduction

Modern portfolio theory and portfolio construction are altogether based upon the leading-edge contribution of Markowitz (1952). On his framework, the author shed lights on portfolio selection and the importance of diversification. In spite of his theoretical optimality in portfolio construction, academics and practitioners easily find flaws when applying the model on *out-of-sample* data, mainly due to what is known as *estimation error*. As widely documented by Brinson *et al* (1991 and 1995), investment policy choices are the main response for long-term portfolio performance, accounting for over 90% of return variability. Thus, the relative amount of a portfolio attributed to each asset – its weight – has as definitive impact in performance.

The hurdles surrounding the implementation of *mean-variance theory* (MVT) portfolios are due to sensitivity to input estimation errors as shown, for instance by Best and Grauer (1991) or Cardoso and Gaspar (2018). As Michaud (1989) and Jobson and Korkie (1980) point out, the consequence of such errors is the estimation of portfolios which are placed far from the true optimal portfolio, causing unintentionally wrong investment decisions for investors. Jobson and Korkie (1981) and DeMiguel *et al* (2007) concluded that MVT portfolios tend to perform better when (i) the input sample is large and (ii) the number of risky assets involved on the optimization is small.

Although these remarks on the workability of MVT optimization are not related to any intrinsic error of Markowitz's framework, the untrustworthiness of the estimates ultimately imposes a barrier on empirical implementation of the theory. The challenge is to define an appealing strategy not only theoretically, but also empirically. Thus, given the importance of efficient allocation of funds across securities, the question on how an investor should allocate his/her wealth in a practical context remains open both in the academy and industry.

Risk parity (RP) portfolios emerged through the contribution of authors such as Qian (2005) as an alternative to common portfolio allocation approaches. The RP

reasoning is a simple one: diversification through risk, such that no security is held in disproportionate amount and it does not contribute to losses more than its peers. Besides the theoretical appeal of balancing risk between the portfolio's constituents, another compelling advantage is that its inputs do not require estimation of expected returns, and therefore, are less exposed to input estimation risk than MVT portfolios.

On this work we aim to understand the potential benefits of RP as a criterion approach for portfolio selection. For this purpose, we analyse five RP-based approaches for portfolio construction. We analyse a *naïve version* of a risk-based (NRP) portfolio, where we assigned weights according to the inverse of volatilities. We then follow the steps of Maillard *et al* (2009) in building an *equal risk contribution* (ERC) portfolio. From this portfolio we added two features. Firstly, we levereged the ERC weights in order to match 60/40 portfolio's *ex-post* volatility ($LERC$). Secondly, we used a price-based, trend-following method introduced by Faber (2007). The last RP-based approach was introduced by Choueifaty and Coignard (2008) where we build the *most-diversified* (MD) portfolio on an *ex-ante* basis.

We then compare RP portfolios to MVT portfolios, such as the *minimum variance* (MV) and *tangent* (T), as well as to the naïve portfolios such as the *homogeneous* (H) and 60/40 portfolios. In this analysis we consider 5 different asset classes (bonds, stocks, high yield bonds, real estate and commodities) and 4 different investment horizons, ranging from 1 to 20 years. Furthermore, we account for trading costs incurred by each strategy, namely turnover and leverage costs in order to stress test our results.

Our study contributes to existent literature by applying RP allocations to a global diversified portfolio and testing *out-of-sample* results against benchmark strategies, analysing different risk and return metrics.

The remaining of this work is structured as follows. In Chapter 2 we briefly summarize the relevant literature. In Chapters 3 and 4 we introduce the indices that compose our portfolios and detail the analytical composition and theoretical properties of the different portfolios. Chapter 5 presents and discusses the main results. Finally, Chapter 6 concludes, debating the main limitations of the analysis and suggesting further research.

Chapter 2

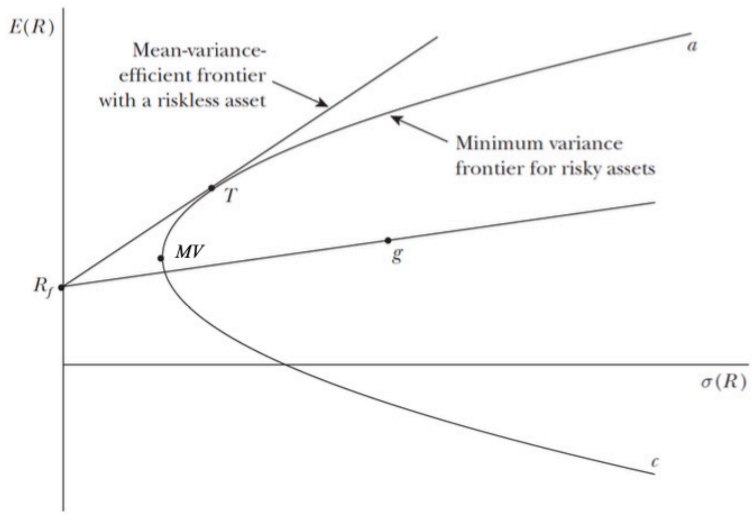
Literature Review

In this section we review the most important topics in literature surrounding *mean variance theory* (MVT) and *risk parity* (RP) approaches to portfolio selection. We start by making a review of MVT empirical results and the need for alternative allocation methods. We then follow to review the reasoning logic underlying RP.

2.1 Mean Variance Theory

MVT assumes that concerning an investment selection that will produce a stochastic return in the future, investors only care about mean and variance of future returns. Investors willing to act according to MVT, should then select the so-called efficient portfolios that for a given level of return, have the lowest possible volatility or that for a given level of volatility, have the highest possible level of expected return. The resulting efficient frontier is hyperbola-shaped and comprises all the efficient combinations of risky assets.

FIGURE 1: EFFICIENT FRONTIER



Source: Fama and French (2004)

As shown in Figure 1, in the case of no riskless asset, only portfolios above the *minimum variance* (MV) portfolio (i.e. in the upper part of the hyperbola) are efficient. Portfolios laying on the lower bound or inside the parabola display lower returns for a given level of volatility, thus becoming non-efficient. Under the assumption that a riskless asset exists and can be used for both lending and borrowing, the efficient opportunity set becomes a straight line connecting the risk-free rate to the so-called *tangent* (T) portfolio. In addition, Sharpe (1964) and Lintner (1965) developed the *capital asset pricing model* ($CAPM$) equilibrium model, showing that a logical linear relationship between risk and expected return holds in equilibrium. Consequently, in equilibrium every investor holds portfolio T and risk-free asset (R_f) in different proportions. Therefore, T must be the *value-weighted market portfolio* ($VWMP$).

2.1.1 Is MVT optimization optimal?

MVT requires input estimation of moments of the future distribution of portfolio returns, which are unknown. Lack of accuracy in estimating these parameters easily guide the optimization to misleading results, nowhere close to the also unknown optimal portfolio as shown by Jobson and Korkie (1981). According to their research, for a set of 20 stocks and a 60-month period of data, the expected optimal risk and return parameters could be overestimated up to 8 times when compared to the realized values. This factor drops to 4 if 100 months are used as input. The researchers conclude that in order to unbiasedly estimate risk and return it is required at least 200 monthly observations of returns.

DeMiguel *et al* (2007), tested out-of-sample results of the sample-based MVT portfolios and its extensions versus a *homogeneous* (H) portfolio. The research concludes that there is no evidence that MVT portfolios are consistently better than H in terms of Sharpe ratio (SR), certainty-equivalent return, or turnover. The research provides further information on the critical estimation window needed for the sample based MVT portfolios outperform portfolio H. The window length is a function of the number of assets. As the latter increases, so does the former. For a 25-asset set, the critical estimation window is 3,000 months. This number more than doubles for a 50-asset set. Another conclusion is that MVT optimization is more likely to outperform H strategy if: (i) estimation window is lengthily enough; (ii) *ex-ante* Sharpe ratio of MVT portfolio is significantly superior than that of the H; and (iii) the number of assets is small. This imposes a hurdle for MVT optimization application on out-of-sample context, i.e. in practice. As noted by Michaud (1989), MVT process significantly overweights (underweights) securities that provide large (small) estimated returns, negative (positive) correlations and small (large) variances, which often are the cases where estimation errors most likely occurred.

2.2 The RP approach

The RP approach to portfolio construction is relatively recent concept that has been drawing attention from practitioners on newspapers and news since the global financial crisis in the post 2008. The RP reasoning is a simple one: it reduces the problem to risk diversification, so that no security is held in disproportionate amount nor it contributes to losses more than its peers. As Qian (2011) illustrates, considering stock and bond's volatility to be 15% and 5%, respectively, a 60/40 portfolio allocates 60% of funds to stocks which contributes to 92%[1] of the portfolio risk, whereas bonds only account for roughly 8% of risk. Furthermore, the return correlation of the mentioned portfolio to stock's return is extremely high.

The RP approach relies on an implicit assumption about returns, especially regarding higher-risk assets: the risk premium of such securities is not high enough in order to deservedly earn a disproportionate risk allocation. Therefore, the intuition behind equal risk allocation can only be correct under the assumption that all assets are expected to provide equal risk-adjusted returns.

2.2.1 Leverage aversion theory

Asness *et al* (2012) found empirical evidence of RP superiority over the $VWMP$, which according to the $CAPM$ should be the optimal combination of risky assets, i.e. the *tangent* portfolio (T). In this sense, $VWMP$ investors seem to be taking disproportionate risk on high-beta assets, which in turn does not provide superior risk premium over low-beta assets.

Although empirical studies in literature conclude that the positive relationship between beta (systematic risk) and average return is true, this relationship seems loose. This means that the *capital market line* is flatter than predicted by the $CAPM$. Fama and French (2004) early empirical tests suggests that returns on low-beta portfolios are too high, whereas

[1] Assuming a 20% correlation between stocks and bonds, the risk contribution from stocks is $\frac{(0.6^2 \cdot 15\%^2 + 0.6 \cdot 0.4 \cdot 0.2 \cdot 15\% \cdot 5\%)}{(0.6^2 \cdot 15\%^2 + 0.4^2 \cdot 5\%^2 + 2 \cdot 0.6 \cdot 0.4 \cdot 0.2 \cdot 15\% \cdot 5\%)} = 92\%$. Bond´s risk contribution is 8%.

high-beta portfolio's returns are too low. For a universe of 200 random portfolios drawn from 788 stocks through the period of January 1960 to June 1968, Friend and Blume (1970) arrive at noticeable conclusions. Namely, risk-adjusted returns are dependent upon risk, as one could expect. However, this relationship is inverse, differently from what $CAPM$ advocates. The author's findings also suggest that the optimal portfolio should not be proxied by the market portfolio as suggested by $CAPM$. The latter seems to be riskier than the risky portfolio involved in the optimality condition.

In order to exploit this inconsistency, Frazzini and Pedersen (2014) assembled an investment strategy going long on safer assets and short on riskier assets, which provided significant positive risk-adjusted returns.

An apparently explanation for these findings is the *theory of leverage aversion* as proposed by Asness *et al* (2012). Many investors are either constrained or unwilling to take on leverage in order to achieve long term return goals, thus they are forced to overweight riskier assets in their portfolios. Therefore, investors bid up riskier asset's prices and through the relative pricing in the market, the returns associated with these securities shrink. The inverse is true for safer assets.

Finally, according to this theory, investors who are less leverage averse or unconstrained can archive positive risk-adjusted returns by tilting portfolios towards safer assets and leveraging according to volatility or long-term return goals.

Chapter 3

Data

In order to examine the potential benefits of the *risk parity* (*RP*) approach proposed by Qian (2006) we use a portfolio of 5 different classes[2]: bonds, high yield, commodities, real estate and stocks. We take as our basic assets, indices that closely track these markets on a global scale.

TABLE I: INDEX DESCRIPTION

Name (Ticker)	Class	Description	Top 3 Holdings
Bloomberg Barclays Global High Yield Total Return Index Value Unhedged (LG30TRUU)	High Yield	Measure of high-yield debt market. Gathers US, Pan-European and Emerging Markets (EM) HY indices.	Industrial Gov-Related Utility
Bloomberg Barclays Global-Aggregate Total Return Index Value Unhedged USD (LEGATRUU)	Bond	Measure of global debt from 24 markets. Includes treasury, government-related, corporate and securitized fixed-rate bonds from both developed (DM) and EM markets.	Treasury Gov-Related Corporate
Bloomberg Commodity Index (BOMTR)	Commodity	Composed of 23 exchanged-traded futures on physical commodities.	Energy Agriculture Industry Metals
FTSE Nareit All Equity REITs Total Return Index (FNRETR)	Real Estate	Comprehensive family of REIT performance indexes that spans commercial real estate.	American Tower Corp Prologis Crown Castle Intl Corp
MSCI World Index (NDDUWI)	Stock	Large and mid-cap representation across 23 DM countries.	Apple Microsoft Corp Amazon

Source: Bloomberg.

[2] Bekkers *et al* (2009) finds evidence that real estate, commodities and high yield bonds add the most value to the traditional Bond, Stock and Cash mix, providing significant reduction in volatility.

Portfolios of these asset classes allow the common investor to build sufficiently well-diversified, long-term oriented investment portfolios. Table I summarizes the selected indices characteristics and its main component exposures.

Monthly closing quotes were gathered from Bloomberg. We collected data from the first quarter of 1990 until the last quarter of 2019. Given the global Coronavirus outbreak and its consequent impact on the financial markets, data from 2020 onwards was not considered due to the high volatility some asset classes have experienced, namely US stocks and Oil, both indirect object of analysis in our work.

Furthermore, in order to avoid currency fluctuation, we analyse prices quoted in US Dollars, hence currency impact in our research is inexistent. Table II exhibits basic descriptive statistics of the five asset classes. Figure 2 and Table III display monthly return frequencies and correlation coefficients, respectively, for the full sample.

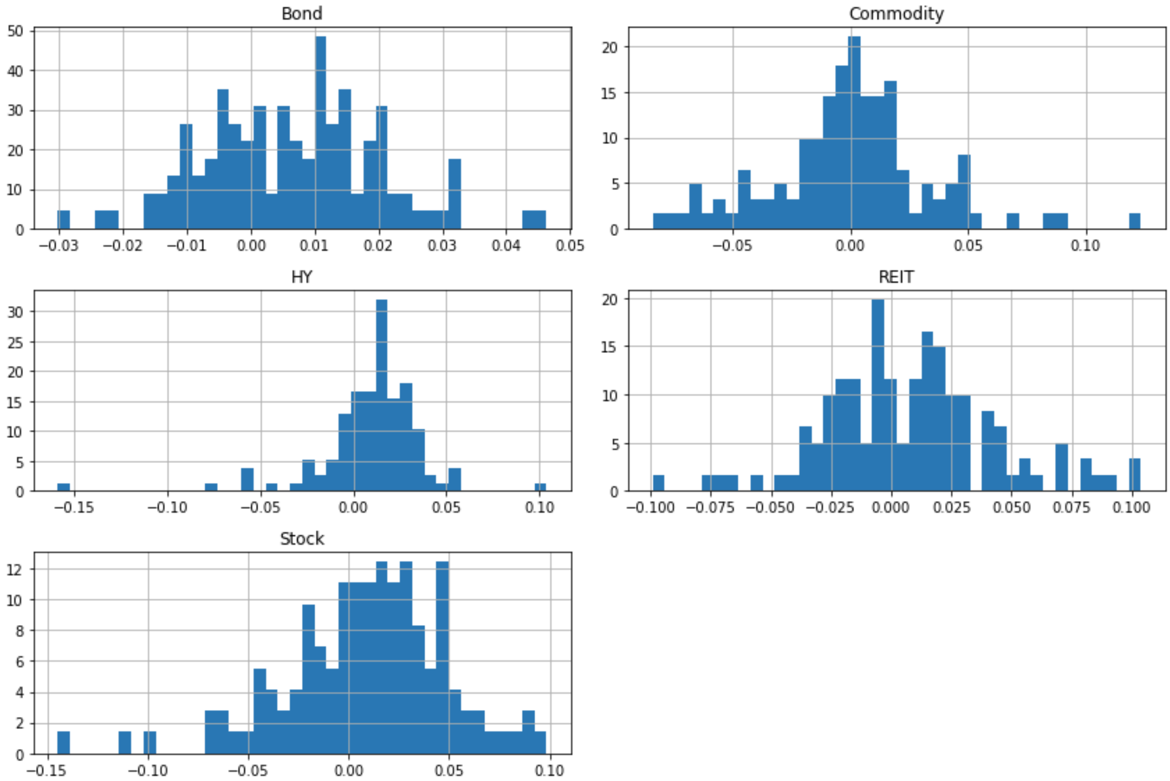
TABLE II: RETURN DESCRIPTIVE STATISTICS 1990Q1 TO 2019Q4

	HY	Bond	Commodity	REIT	Equity
Mean Return	8.75%	5.5%	-0,45%	10.38%	4.93%
Excess Return	0.25%	-3.00%	-8.95%	1.88%	-3.57%
Volatility	9.69%	5.29%	14.59%	18.50%	14.78%
Sharpe Ratio	0.026	-0.567	-0.613	0.102	-0.242
Max Drawdown	33.37%	10.08%	54.52%	62.31%	50.79%
Skewness	-1.978	0.149	0.260	0.171	-0.728
Kurtosis	13.759	3.061	4.407	3.584	4.462

Source: Bloomberg data and own calculations. Excess returns based on 30-year US Treasury yield of 8.5% as of January 1990.

For the 360-month period ended in 31st December 2019, HY and REIT were the only classes to display positive excess returns. REIT exhibits the highest excess return at cost of recordkeeping both the highest volatility and annual loss among all classes. As shown on Table II, on a risk-adjusted basis, high yield and real estate are the best performers of the period.

FIGURE 2: MONTHLY RETURN DENSITIES



Return frequencies based upon monthly data from January 1990 to December 2019 on the indices: Bloomberg Barclays Global High Yield Total Return Index Value Unhedged (high yield bonds), Bloomberg Barclays Global-Aggregate Total Return Index Value Unhedged USD (bonds), Bloomberg Commodity Index (commodities), FTSE Nareit All Equity REITs Total Return Index (real estate) and MSCI World Index (stocks).

The five classes exhibit low correlation among them, with Commodities exhibiting the lowest correlation to all other assets (see Table III). Despite negative excess return, the low correlation displayed between the Commodities and the remaining classes is an important feature to consider in terms of risk diversification.

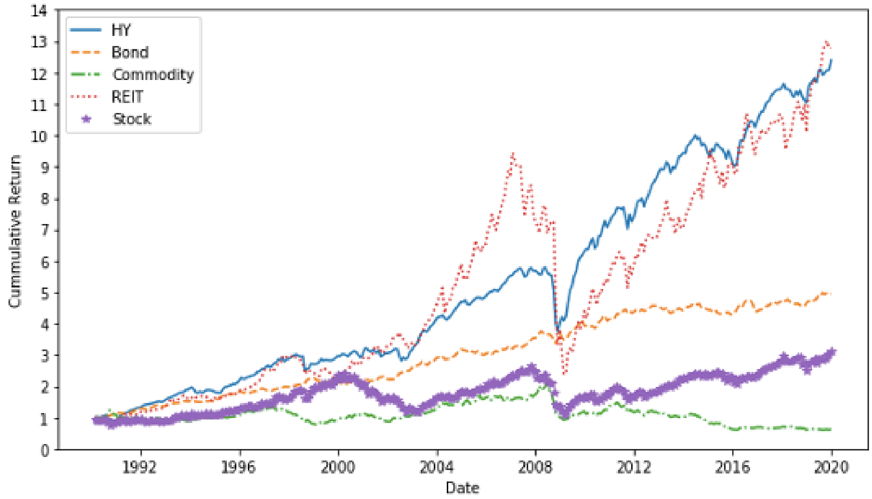
Figure 3 demonstrates both the cumulative returns for each asset class as well as mean-variance diagram for the full period considering a risk-free rate of 8.5%. Despite the staggering drawdown experienced in the aftermath of the 2008 global financial crisis, both stocks and high yield were able to swiftly recover in the following twelve years.

TABLE III: CORRELATION COEFFICIENTS, 1990Q1 TO 2019Q4

	HY	**Bond**	**Commodity**	**REIT**	**Equity**
HY	1.00				
Bond	0.04	1.00			
Commodity	0.05	-0.04	1.00		
REIT	0.50	0.14	-0.01	1.00	
Equity	0.54	0.32	-0.05	0.36	1.00

FIGURE 3: CUMULATIVE RETURNS AND MEAN VARIANCE PLANE

(a)



(b)

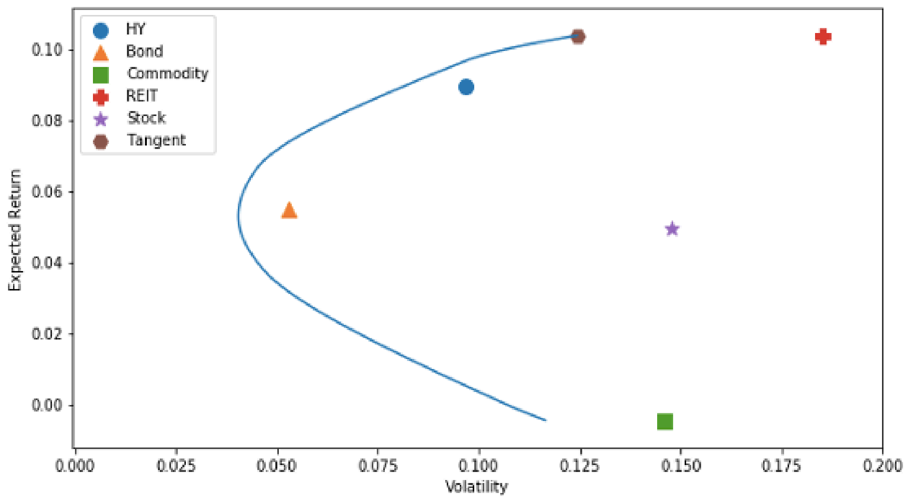


Figure 3 (a) exhibits normalized returns for the five different asset classes. Figure 3 (b) exhibits mean-variance frontier, considering a risk-free asset rate of 8,5%.

Chapter 4

Methodology

This section presents the implementation of the various investment strategies and is divided as follows. Firstly, we introduce the notation and explain the allocation strategy of each portfolio. We then move to explain our estimation procedure and subsequent *out-of-sample* implementation based on a rolling-sample approach. All portfolio strategies and their optimization procedures are implemented using Python programming language.

4.1 Portfolio Terminology

We consider N assets with returns, R_i, for $i = 1, \dots, N$, and a riskless asset with return R_f. The individual weight of asset i on a given portfolio P is given by ω_i^P, whereas the Nx1 vector of weights is represented by $W^P = \{\omega_i, \dots, \omega_N\}$. Every asset i can be then characterized by its expected return, $\bar{R}_i$, and volatility, σ_i. Furthermore, the correlation between the returns of asset i and j is defined as $\rho_{ij} \in [-1,1]$. Finally, the NxN dimension variance-covariance matrix is defined as Ω, where the diagonal elements denote assets' variances, σ_i^2, and the off-diagonal elements designate the respective covariances, $\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$. For the benchmark strategies, we consider only portfolios without short-selling positions, i.e. every portfolio must verify the following restrictions:

$$\sum_{i=1}^{N} \omega_i = 1$$

$$0 \leq \omega_i \leq 1 \qquad (1)$$

We allow leverage for one risk parity portfolio since both expected returns and volatilities of these portfolios are usually lower than the benchmark strategies.

4.2 Risk Measures

From an individual asset risk measure such as volatility, we can infer about the general portfolio risk.

Once the vector of weights and the variance matrix is defined, the overall portfolio volatility becomes:

$$\sigma^P = \sqrt{W^{P\prime}\Omega W^P} \tag{2}$$

4.2.1 Marginal, Total and Percentage Risk Contribution

In order to understand the *risk parity* (RP) strategies, we must in first place introduce concepts *of marginal risk contribution* (MRC) and *total risk contribution* (TRC), as defined by Qian (2006) and Maillard *et al* (2009). From the first derivative in Equation (2), it is possible to conclude about the marginal contribution to volatility by a marginal change in individual weights.

$$MRC_P = \frac{\partial \sigma^P}{\partial W^P} = \frac{\Omega W^P}{\sigma^P} \tag{3}$$

The vector originated by Equation (3) provides the MRC of each of the individual assets:

$$MRC_i = \frac{\partial \sigma^P}{\partial \omega_i} = \frac{\omega_i \sigma_i^2 + \sum_{j \neq i} \omega_j \rho_{ij} \sigma_i \sigma_j}{\sigma^P} \tag{4}$$

From MRC we can arrive to TRC. The sum of the contributions of each asset is equal to the total portfolio volatility:

$$TRC_i = \omega_i \frac{\partial \sigma^P}{\partial \omega_i} = \frac{\omega_i^2 \sigma_i^2 + \sum_{j \neq i} \omega_j \rho_{ij} \sigma_i \sigma_j}{\sigma^P}$$

$$\sigma^P = \sum_{i=1}^{N} TRC_i \tag{5}$$

Equations (4) and (5) can be interpreted as volatility "elasticity" w.r.t the weight of a given asset. As Qian (2006) mentions, from previous equations we can define the percentage contribution to risk, p_i^P, through the division product of TRC by σ^P:

$$p_i^P = \frac{TRC_i}{\sigma^P} = \frac{\omega_i^2 \sigma_i^2 + \sum_{j \neq i} \omega_j \rho_{ij} \sigma_i \sigma_j}{(\sigma^P)^2}$$

$$\sum_{i=1}^{n} p_i^P = 1 \tag{6}$$

The percentage contribution to risk is the ratio of the covariance between component return of asset i and the overall portfolio return, to the portfolio volatility, with the sum of percentage contributions adding up to one. Qian (2006) concludes that when portfolio losses are significantly large, percentage contributions to risk are highly related to and can be interpreted as the actual percentage contribution to portfolio loss of a given asset. In this sense, if the percentage contribution to risk of a given asset i is 30%, when the portfolio experiences a severe loss, we can expect that 30% of the loss is provided by asset i.

4.2.2 Diversification Ratio

Choueifaty and Coignard (2008) introduced a criterion for portfolio selection that can also be interpreted as a risk measure, the so-called Diversification Ratio. Let the vector of asset volatilities be $\sum = \{\sigma_1, \dots, \sigma_N\}$, for any portfolio $W^P = \{\omega_i^P, \dots, \omega_N^P\}$, the Diversification Ratio of portfolio P is defined as follows:

$$D^P = \frac{W^{P\prime}\Sigma}{\sqrt{W^{P\prime}\Omega W^P}} \quad (7)$$

The diversification ratio stands for the weighted average of volatilities divided by the overall portfolio volatility. One the theoretical property of this measure is that the ratio is strictly equal to one only in the case of a portfolio composed exclusively by a single asset. In all other cases, this ratio will be higher than one, thus the higher the ratio, the more diversified the portfolio is said to be.

4.3 Return Measures

A major problem on portfolio construction lies on the requirement of accessing expectations regarding future returns and the covariances (see Best and Grauer (1991) and Chopra and Ziemba (1993)). There is vast evidence in literature on the shortcomings of parameter misestimation (estimation risk), or from incorrect assumptions (model risk). As Michaud (1989) and Jobson and Korkie (1980) point out, the consequences are the estimation of portfolios which are placed far from the true optimal portfolio, causing unintentionally wrong investment decisions. Since the true values of these moments are not known, one way to estimate them is by recurring to historical data. Considering the closing monthly price of asset i at time t, P_t, we can calculate the monthly return yielded, $r_{i,t}$, through logarithmic application:

$$r_{i,t} = \log\left(\frac{P_t}{P_{t-1}}\right) \quad (8)$$

From the vector of logarithmic returns $R_i = \{r_{i,1}, \dots, r_{i,T}\}$ with T observations, we arrive at asset's i annual mean return, $\bar{R}_i$:

$$\bar{R}_i = \frac{\sum_{t=1}^{T} r_t}{T} \times 12 \tag{9}$$

Subsequently, given that portfolio's expected return, $\mathrm{E}[R_P]$, is a linear product of the individual assets' weights $\omega_i, \dots, \omega_N$, and mean returns, we can define it as:

$$\bar{R}^P = \mathrm{E}[R^P] = \sum_{i=1}^{N} \omega_i \bar{R}_i \tag{10}$$

One of the metrics used on this work in order to evaluate portfolios performance, is the widely used measure introduced by Sharpe (1966), where risk adjusted returns provided by different strategies are compared. The so-called *Sharpe ratio* (SR) requires two inputs: excess returns and volatilities. Excess returns are drawn from mean returns, $\bar{R}^P$, from which the riskless rate of return, R_f, is subtracted:

$$SR^P = \frac{\bar{R}^P - R_f}{\sigma^P} \tag{11}$$

4.4 Risk Parity Investment Strategies

After defining the essential terminology required for our analysis, we move on to analytically define the RP portfolios and other benchmark strategies.

TABLE IV: INVESTMENT STRATEGIES SUMMARY

Active Investment Strategies	**Passive Investment Strategies**
Naïve Risk Parity (NRP)	Minimum Variance (MV)
Equal Risk Contribution (ERC)	Tangent (T)
Trend-following ERC (TERC)	Homogeneous (H)
Levered ERC (LERC)	60/40
Most-diversified (MD)	

There are several versions of RP portfolios, however, for the scope of this work, we will analyse five versions: *naïve risk parity* (NRP); *equal risk contribution* (ERC) and its trend-following ($TERC$) and levered ($TERC$) extensions; and the *most-diversified* (MD) portfolio. In order to test RP soundness, we compare it to common approaches in literature such as *minimum variance* (MV) and *tangent* (T) portfolios as proposed by Markowitz (1952), as well as with other less time-consuming approaches such as the *homogeneous* (H) for the 60/40 portfolios. Table IV summarizes the investment strategies under consideration.

We consider RP-based strategies as active investment strategies since the optimization process takes places at each month end, with the weights being recalculated based on a rolling sample approach (Qian (2005), Choueifaty and Coignard (2008), Maillard *et al* (2009), Asness *et al* (2012) all use a similar approaches). For the passive investment strategies (benchmark strategies), the weights are defined at the beginning of the investment period and rebalancing to the initial weights takes place at each month end.

4.4.1 NRP - Naïve Risk Parity Portfolio

The first risk-diversified portfolio to be introduced is the so-called NRP proposed by Asness *et* al (2012) that assigns to each asset i a weight that is inversely related to its volatility. There is no objective function to be optimized and the portfolio weights are defined as follows:

$$W^{NRP} = \frac{1/\sigma_i}{\sum_{i=1}^{N} 1/\sigma_i} \qquad (12)$$

$$s.t.\ W'\mathbb{1} = 1$$

where $\mathbb{1}$ is a vector of ones. Although the NRP portfolio takes into consideration risk diversification by penalizing highly volatile assets, its main drawback stands for the lack of consideration regarding assets' return correlations. Therefore, low correlations between assets, which potentially decrease the overall portfolio volatility can be disregarded simply because of one asset's excessive volatility. The weight assigned to each asset is equal to the inverse of its volatility divided by the harmonic average of volatilities. Therefore, the higher (lower) the volatility, the lower (higher) the component's weight on the NRP portfolio.

4.4.2 ERC - Equal Risk Contribution Portfolio

The main shortcoming of the NRP strategy is the lack of consideration for assets' interrelationships and how the overall portfolio could benefit from this. The ERC overcomes this hurdle by considering the covariance matrix and consequently the correlation between assets. The ERC portfolio refers to the concepts of TRC and risk budgeting introduced earlier (recall Equation (5)). We used the framework developed by Maillard *et al* (2009) and Bruder and Roncalli (2012). Considering that risk budgets of different assets must be equal, we have that $TRC_i = TRC_j$, thus we face the following optimization problem:

$$W^{ERC} = \min_{W} f(W)$$
$$f(W) = \sum_{i=1}^{N} \sum_{j=1}^{N} [\omega_i (\Omega\omega)_i - \omega_j (\Omega\omega)_j]^2 \quad (13)$$
$$s.t. W'\mathbb{1} = 1$$

Assuming pairwise equal correlations $\rho_{i,j} = \rho$ for all i, j, the covariance between return of security i and the portfolio P is $\sigma_{iP} = cov(r_i, \sum_j \omega_j^P r_j) = \sum_j \omega_j^P \sigma_{ij}$. We have $TRC_i = \omega_j^P \sigma_{iP} / \sigma^P$. By definition, $\beta_i^P = \sigma_{iP} / (\sigma^P)^2$, such that $TRC_i = \omega_i \beta_i^P \sigma^P$. Moreover, by construction the ERC portfolio requires that $TRC_i = TRC_j = \sigma^P / N$, therefore, for all i, j, we have:

$$\omega_i^P = \frac{\beta_i^{P-1}}{\sum_{j=1}^{N} \beta_j^{P-1}} = \frac{\beta_i^{P-1}}{N} \quad (14)$$

From the previous equation, one can see that the weight attributed to security i is inversely proportional to its beta component. In this case, the beta indicates the sensitivity of security i to the risk of portfolio P. Therefore, the higher (lower) the beta the lower (higher) the weight. The beta interpretation is more appealing and differently from the NRP strategy, not only assets with high volatilities are penalized but also those with high correlation to other assets.

As shown by Maillard *et* al (2010), in a universe of constant correlations, the MV and ERC portfolios provide the same solution when cross-diversification effect is at the its highest level, i.e. when we consider a constant correlation of $\rho = -1$. Therefore, ERC portfolios stand as robust risk-balanced alternative. Finally, ERC portfolio's volatility lies between the MV and H portfolios, such that: $\sigma^{MV} \leq \sigma^{ERC} \leq \sigma^{H}$. Therefore, this portfolio is positioned as an intermediary alternative to the latter, being a form of MV portfolio under a diversification constrain regarding weights.

4.4.3 TERC - Trend-following ERC Portfolio

Departing from the original ERC portfolio, a new feature can be introduced in an attempt to reduce portfolio risk and improve risk adjusted return measures. Trend-following strategies have been widely used in different assets, especially on future markets. As Hurst *et al* (2010) notes, the effectiveness of such methods can be explained by a variety of factors essentially related to behavioral biases such as underreaction to news or tendency to exhibit herding behavior.

We consider a trend-following strategy as suggested by Faber (2007), who demonstrated the effectiveness of this method by achieving equity-like returns with bond-like volatilities. See also Clare *et al* (2016) who looks at the potential benefits of RP and trend-following strategies. There are many trend-following methods, such as breakouts or moving average crossovers, among others. Taylor and Allen (1992) and Lui and Mole (1998), found that moving average based systems are the most popular among practitioners. We focus on moving averages and assets prices as our main couriers for trend. Faber (2007) defines a positive trend for security i when the closing monthly price, P_t, is above the 10-month simple moving average of price, $SMA(10)_{t-1}$. Portfolio $TERC$ consists on assuming long positions as stablished in first place by the ERC strategy only when trend is positive. Otherwise, the trend is considered to be negative, thus the fraction attributed to security i, is assigned to cash. Let ω_C be weight invested in cash, we have:

$$\omega_i^{TERC} = \begin{cases} \omega_i^{ERC} \text{ if } P_t > SMA(10)_{t-1} \\ 0 \text{ and } \omega_C = 1 - \sum_{i=1}^{N} \omega_i \,, \qquad otherwise \end{cases} \tag{15}$$

$$s.t.\ \omega_C + \sum_{i=1}^{N} \omega_i = 1$$

4.4.4 LERC - Levered ERC

Risk parity antagonists identify as main drawback of this strategy the fact that, when compared to others, namely the 60/40, RP portfolios lack expected returns, since the main component tend to be safer assets, such as bonds. On the other hand, RP defenders suggest the employment of leverage to exploit security returns inconsistencies as identified by Frazzini and Pedersen (2014) (see also Asness *et al* (2012)) and increase the expected portfolio return to the desired levels. Although leverage entails it owns concerns specially in the short-term, investors have been open to its long-term benefits.

Let λ be the leverage ratio. As Anderson *et al* (2014) defines, for a given amount of capital, L, an investor chooses λ, borrows $(\lambda - 1)L$, and invests λL in the source portfolio. We define $\lambda = \sigma^{60/40}/\sigma^{ERC}$ in order to match the ERC's and 60/40 portfolio's *ex-post* volatilities. The *levered equal risk contribution* ($LERC$) portfolio's weights are:

$$\omega_i^{LERC} = \omega_i^{ERC}\lambda \tag{16}$$

The single period return of $LERC$, R^{LERC}, and its source portfolio, R^{ERC} is given by $R^{LERC} = \lambda R^{ERC} - (1 - \lambda)R^b$, where R^b is the cost of borrowing. In our study, we use R^b qual to cash rate[3]. This is the only portfolio to which we apply leverage.

4.4.5 MD - Most-diversified Portfolio

The last approach under the RP characteristics to be analysed in the *most-diversified portfolio* (MD). This portfolio attempts to maximize the ratio exposed on Equation (7) between the weighted average individual volatilities and total portfolio volatility. We face the following optimization problem:

$$W^{MD} = \max_{W} f(W)$$

$$f(W) = \frac{\sum_{i=1}^{N} \omega_i \sigma_i}{\sigma^P} = \frac{W^{P\prime}\Sigma}{\sqrt{\omega'\Omega\omega}} \tag{17}$$

$$s.t. W'\mathbb{1} = 1$$

where $\Sigma = \{\sigma_i, \ldots, \sigma_N\}$ is the vector of asset's volatilities and ω is the vector of weights. We can also define the MD portfolio as (see Choueifaty and Coignard (2008)):

$$\omega^{MD} = \kappa \Sigma^{-1}\Omega^{-1}1 \tag{18}$$

where κ is a constant. The difference between the numerator and denominator of Equation (17) roughly resides on correlations. With correlations considered on the denominator and considering that we are facing a maximization problem, this in turn means that this portfolio attributes higher weights to securities with lower correlations w.r.t. other securities.

Choueifaty and Coignard (2008) shed lights on the theoretical properties of this portfolio. Firstly, they conclude that in a universe where all the assets have expected excess returns equal to their volatilities, maximizing the Diversification ratio is equal to maximizing

[3] EURUSD deposit rate.

Sharpe ratio. Secondly, if one denotes the correlation between a given portfolio $W^P = \{\omega_i^P, \dots, \omega_N^P\}$ and the MD portfolio as:

$$\rho_{P,MD} = \frac{W^{P'}\Sigma\Omega\Sigma W^{MD}}{\sigma^P \sigma^{MD}} = \frac{W^{P'}\Sigma\Omega\Sigma\kappa\Sigma^{-1}C^{-1}1}{\sigma^P \sigma^{MD}} = \frac{\sum_i \omega_i \sigma_i}{\sigma^P}\frac{\kappa}{\sigma^{MD}} = D^P \frac{\kappa}{\sigma^{MD}} \tag{19}$$

The correlation between portfolio a P and MD is proportional to the diversification ratio of P. By applying Equation (19) to the correlation between security i and the MD portfolio, one can see that $\rho_{i,MD} = {}^{\kappa}/_{\sigma_{MD}}$, since $D^i = 1$. This is true for all securities in the portfolio. Therefore, all the component securities have the same positive correlation to the MD portfolio. For securities with $\omega > 0$, the correlation terms to the MD portfolio are equal. All the remaining assets with $\omega = 0$ have correlation terms to the MD portfolio higher than the correlation of non-zero-weight assets. Lastly, the authors also show that in a hypothetical situation where all the assets have equal volatility, the MD becomes the MV portfolio.

4.5 Benchmark Strategies

In this section we introduce the benchmark approaches that will be tested against the RP strategies. RP strategies must withstand against strategies such as *mean-variance theory* (MVT) related portfolios or naïve portfolios in order to be regarded as a sound allocation criterion.

4.5.1 Minimum Variance Portfolio

We start looking at the MVT portfolios – MV and T portfolios. The ground-breaking work developed by Markowitz (1952), developed guidelines on effective wealth allocation across risky assets under the assumption that investors only care about mean and variance of a portfolio is one long-lasting approach. According to the framework, by developing the so-called efficient frontier, investors should select portfolios laying on it through return targeting and variance minimization.

The MV portfolio is unique and does not require the estimation of expected returns. Hence, we face the following minimization problem:

$$\begin{gathered} W^{MV} = \min_W f(W) \\ f(W) = W'\Omega W \\ s.t. W'\mathbb{1} = 1 \end{gathered} \tag{20}$$

Whose solution is $W^{MV} = \frac{1}{A}\Omega^{-1}\,\mathbb{1}$, where $A = \mathbb{1}^{'}\Omega^{-1}\,\mathbb{1}$. *Ex-ante*, this is the portfolio with the lowest possible level of risk, but also the one with the lowest expected return. However, *ex-post*, these characteristics might change and under some environments, the MV portfolio can outperform other strategies.

4.5.2 Tangent Portfolio

Likewise, portfolio T also belongs to the efficient frontier and exhibits a unique feature, it is the one that maximizes the Sharpe ratio, i.e. the portfolio excess return to volatility ratio:

$$W^{T} = \max_{W^{P}} f(W)$$

$$f(W^{P}) = \frac{\bar{R}^{P} - R_{f}}{\sigma^{P}} \tag{21}$$

$$s.t.\, \omega'\mathbb{1} = 1$$

The solution to the above problem is $W^{T} = \Omega^{-1}\left[\bar{R}^{P} - R_{f}\right]$. Here we consider short selling restrictions, so we need to rely on numerical solutions to obtain the tangent portfolio. Despite the intuition behind the framework, *out-of-sample* applicability of the method is far from flawless due to possible estimation errors.

4.5.3 Homogeneous Portfolio

Often investors have preference for easy-to-understand and more computationally friendly strategies which require no complex financial understanding but still are able to deliver consistent performance.

Probably the simplest portfolio allocation strategy is to divide one's wealth equally across the available assets, the so-called *homogeneous* (H) strategy. Differently from all the approaches so far analysed, this requires no input in terms of excess return nor covariance matrix, which implies that there is no estimation error on this method. Let N be the number of assets, the 1/N weights are such that:

$$\omega_{i}^{H} = \frac{1}{N} \tag{22}$$

Despite its simplicity, is often used as benchmark in the financial literature as has shown to "beat" more sophisticated allocation strategies, *out-of-sample*.

4.5.4 60/40 Portfolio

The last approach to be implemented on this thesis is the so-called 60/40 Strategy, which invests 60% of the wealth on Equity and 40% on Bonds. As Chaves *et al* (2011) reminds, in practice, large institutional investors adopt this approach with alternative investments being assigned only modest weights. Brinson *et al* (1986), Qian (2011) and Hurst *et al* (2013) also use the 60/40 portfolio as a benchmark strategy to measure relative portfolio performance.

4.6 Implementation Method

Having introduced all the portfolios, their theoretical properties and analytical solutions, we move forward to our implementation framework. Our analysis is based upon replication of all investment strategies – RP and benchmarks – for multiple investment periods, covered by our sample, using a rolling-sample approach.

Let the estimation window, M, be of length $M = 120$ (10 years). In the first iteration, the vector of expected returns and the covariance matrix is estimated from $t = 1$ to $t = M$. In the second iteration, the same parameters will be estimated based on data ranging from $t = 2$ to $t = M + 1$ and so on.

For the passive benchmark strategies, initial weights are computed at the beginning of the investment period and are kept constant throughout, which is equivalent to assuming monthly rebalancing of these strategies. For the RP-based strategies, optimal weights are computed on a monthly basis. In the determination of the target portfolio weights we consider the risk-free rate that is consistent with the investment period. For all strategies we analyse investment periods of 1, 5, 10 and 20 years.

4.7 Evaluating Portfolio Performance

Our performance measures consider not only realized excess returns and level of wealth attained at the end the period, but also the level of risk and diversification. The measures we use can be listed as:

- Diversification ratio (D), as in Equation (7).
- Sharpe (1966) ratio (SR), as in Equation (11).
- Maximum drawdown (MDD) is defined as:

$$MDD_i = \frac{P - max_{0<s<t}(P_t)}{max_{0<s<t}(P_t)} \tag{23}$$

- Turnover ratio, proposed by DeMiguel *et al* (2007) and defined as follows:

$$Turnover_P = \frac{1}{T-M}\sum_{t=1}^{T-M}\sum_{j=1}^{N}\left(\left|\omega_{P,j,t+1} - \omega_{P,j,t^+}\right|\right) \tag{24}$$

where $\omega_{P,j,t}$ be the weight of asset j at time t under portfolio P, ω_{P,j,t^+} be the portfolio weight at the moment before the rebalancing and $\omega_{P,j,t+1}$ be the intended weight at time $t+1$, after rebalancing. For portfolio H, we have that $\omega_{H,j,t} = \omega_{H,j,t+1}$. However, ω_{H,j,t^+} can be different given returns provided by the portfolio constituents.

The MDD expresses the maximum cumulative losses experienced given a time period and we continuously evaluate this measure on an annual basis.

Finally, when considering trading costs, we deduct them from the wealth development of each strategy. Carhart (1997) estimate that round-trip turnover cost is 95 basis points (bps) with a standard error of 40 bps and Balduzzi and Lynch (1999) use 50 bps as proxy for transaction costs. We use 50 bps as reference trading cost. Let $R_P = \sum_{j=1}^{N} R_{j,t+1}\omega_{P,j,t}$ be the return yielded by portfolio P before the rebalancing, when the rebalancing occurs at $t+1$, the changes in individual assets weight's is provided by $\left|\omega_{P,j,t+1} - \omega_{P,j,t^+}\right|$. Denoting $c \times \sum_{j=1}^{N}\left|\omega_{P,j,t+1} - \omega_{P,j,t^+}\right|$ as the transaction cost incurred, we can denote the wealth evolution, W_P, as follows:

$$W_{P,t+1} = W_{P,t}(1 + R_P)\left(1 - c \times \sum_{j=1}^{N}\left|\omega_{P,j,t+1} - \omega_{P,j,t^+}\right|\right) \tag{25}$$

We evaluate the various investment strategies performances across several investment periods, namely $I = 240$ (20 years), $I = 120$ (10 years), $I = 60$ (5 years) and $I = 12$ (1 year), starting at each possible month until $T - I$. For a data set of 360 months (recall $M = 120$), we arrive at 120, 180 and 228 different starting dates for 10-, 5- and 1-year investments, respectively.

Chapter 5

Results

In this chapter we present our main findings archived by both *risk parity* (RP) portfolios and benchmark strategies highlighted in Chapter 4. We start by analysing performances over the 20-year period ending in 31st December 2019. For this investment horizon, we simulate one portfolio per strategy.

We then move to shorter investment periods, where for each strategy we have various portfolios with a given investment horizon starting at each month from January 2000.

5.1 Performance Analysis

The results for the 20-year investment starting at January 2000 are summarized in Table V where the values are accounted before trading costs. All portfolios realized positive annualized returns varying from 3.50% to 6.27% for the 60/40 and *tangent* (T) portfolios, respectively. Among the RP strategies, the *leveraged equal risk contribution* ($LERC$) recorded the highest annual return of 6.22%, followed by the *naïve risk parity* (NRP) with 5.02%. Regarding excess returns, the results shrink significantly given the high level of the risk-free return in the early 2000's, with all portfolios recording negative excess returns.

TABLE V: PORTFOLIO STATISTICS BEFORE TRADING COSTS

	Ann. Return	Excess Return	Ann. Volatility	Sharpe Ratio	Skewness	Kurtosis	Historic VaR (5%)	Historic CVaR (5%)	Max Drawdown	Divers. Ratio
MV	4.30%	-2.24%	5.99%	-0.373	-0.748	6.586	0.025	0.037	-17.68%	1.343
T	6.27%	-0.27%	6.73%	-0.040	-1.324	13.215	0.024	0.041	-22.99%	1.199
H	5.01%	-1.53%	10.37%	-0.147	-1.901	15.120	0.038	0.077	-44.64%	1.379
60/40	3.50%	-3.04%	9.88%	-0.308	-0.851	5.654	0.050	0.070	-39.71%	1.146
NRP	5.02%	-1.52%	8.13%	-0.187	-1.700	13.719	0.027	0.057	-33.46%	1.429
ERC	4.91%	-1.63%	7.79%	-0.209	-1.621	12.937	0.027	0.055	-31.75%	1.449
MD	4.12%	-2.42%	7.41%	-0.326	-1.384	10.204	0.029	0.052	-30.46%	1.478
TERC	4.26%	-2.28%	4.66%	-0.489	-0.311	4.412	0.019	0.029	-9.59%	1.327
LERC	6.22%	-0.32%	9.88%	-0.032	-1.621	12.937	0.034	0.069	-38.85%	1.449

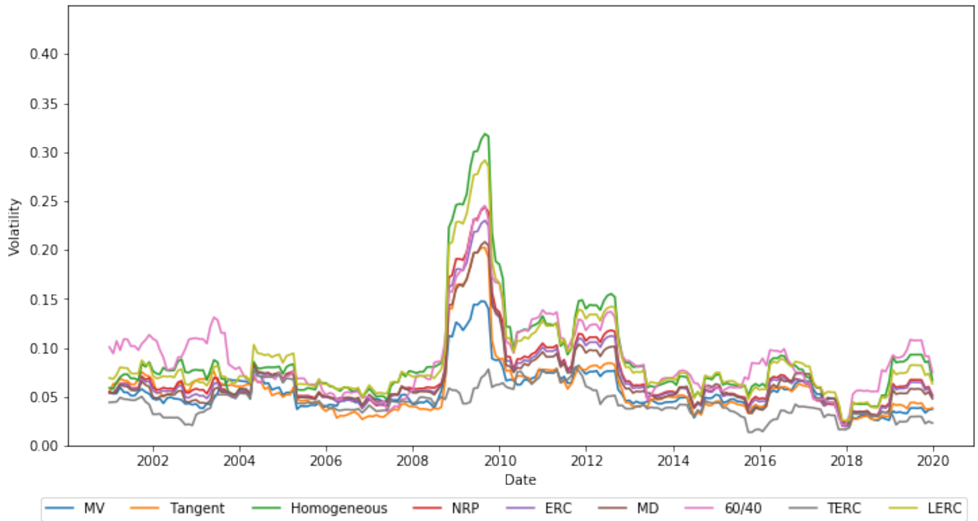
Excess returns calculated based on US 20-year yield rate of 6.54% as of January 2000. Source: Federal Reserve Economic Data, Bloomberg and own calculations.

Figure 4 exhibits rolling 12-month returns. Returns in the early 2000's were extremely penalized specially by equity's performance. Given its high exposure to stocks, 60/40 portfolio lagged behind the others. As one can see, returns suffered a sharp decline followed by an equally steep recovery during the 2008 to 2010 period. Portfolio *homogeneous* (H) experienced the most extreme variations in return when compared to the other strategies.

FIGURE 4: ANNUAL RETURNS

Rolling 12-months returns of different investment strategies, from January 2000 to December 2019.

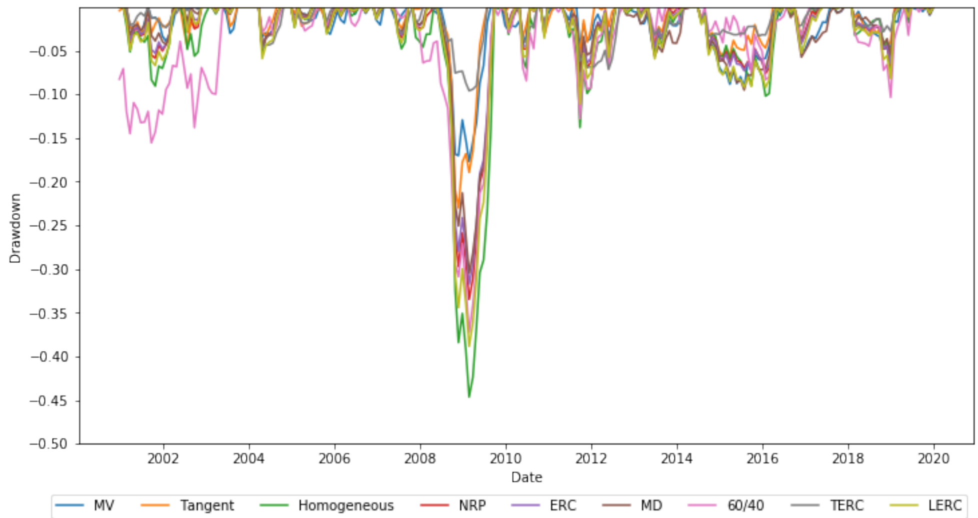
FIGURE 5: YEARLY VOLATILITIES



Annual volatilities of each strategy, from January 2000 to December 2019.

Figures 5 and 6 exhibit the volatilities and the monthly drawdowns incurred by each strategy over a 12-month period. During the 2008 global financial crisis, all the portfolio's volatilities spiked the most, with the biggest drawdowns being registered on this period. Portfolios $LERC$ and H recorded volatilities around 30% and experienced severe losses with drawdowns of around 40%. *Trend-following equal risk contribution* ($TERC$) and *minimum variance* (MV) recorded volatilities of 8.03% and 14.78%, respectively. $TERC$ strategy outperformed all the remaining portfolios, including the MV, in terms of volatility (4.66% vs 5.99%) with low drawdown recorded (9.59% vs 17.68% for MV). The implementation of a trend-following system allowed a shrinkage in annual volatility of 3.13% (or around 40%) when compared to the initial *equal risk contribution* (ERC) strategy at cost of only 0.65% reduction of annual return (or around 13%). Portfolio H is the most volatile (10.37%) with 44.64% of drawdown.

FIGURE 6: MAXIMUM DRAWDOWN

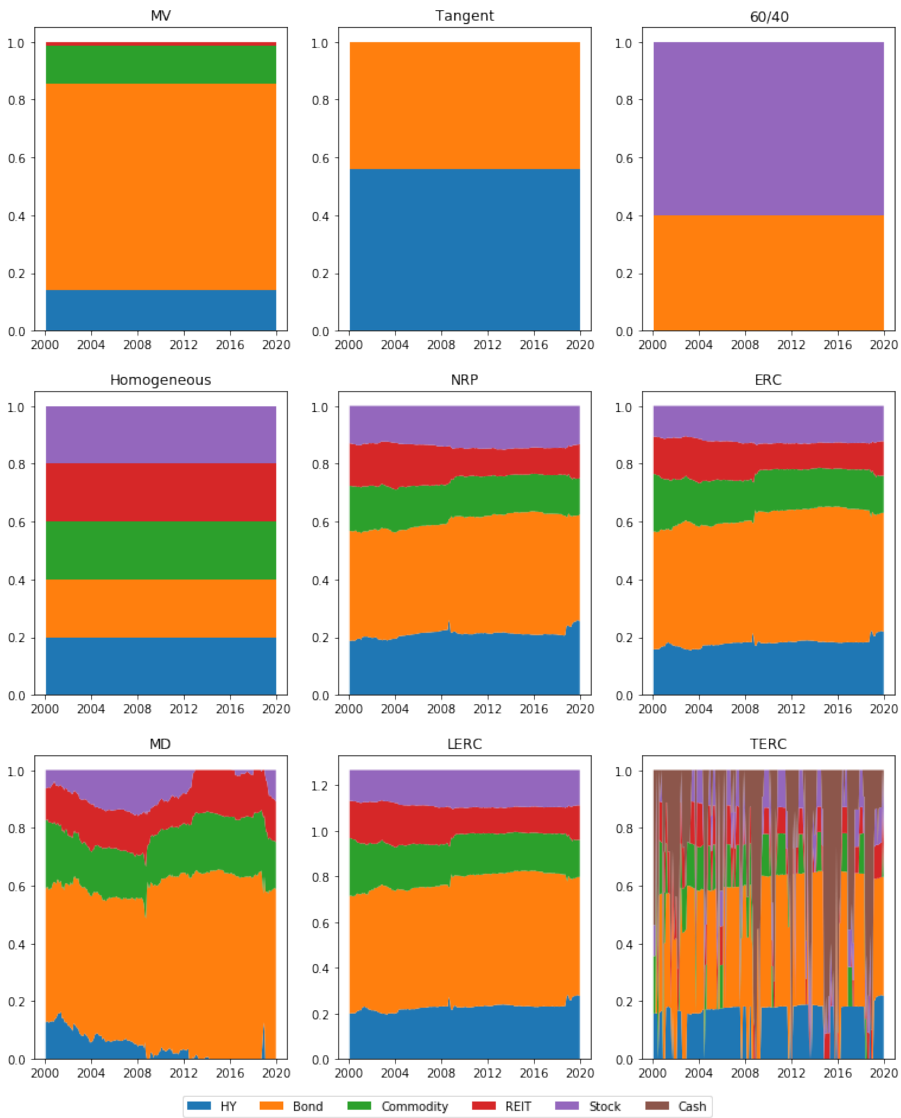


Peak to through drawdown incurred by each strategy over 12-month period, from January 2000 to December 2019.

RP strategies recorded lower volatilities and drawdowns than portfolios H and 60/40. For the same volatility level as the 60/40 portfolio, the $LERC$ portfolio was able to record an annualized return 2.72% higher than the former. This accounts for a 78% gain in annual return for the same level of volatility.

Additionally, $LERC$ portfolio recorded higher diversification and slightly lower drawdown, VaR and CVaR. As displayed in column 6 of Table V, $TERC$ is the least skewed portfolio. In this sense, this portfolio displays more frequent small losses and fewer extreme gains (see Figure 20 in the Appendix for the portfolio's return densities). Therefore, the portfolio is exposed to frequent though limited losses, instead of less frequent but unlimited downside risk. RP portfolios in general exhibit not only lower volatilities but also lower maximum drawdowns, CVaR or VaR when compared to the H and 60/40 strategies. Figure 7 displays the weighting composition of each strategy throughout the investment period. As many RP criticizers argue, these portfolios seem to be indeed tilted towards safer assets, in this case, bonds. However, both the MV and T portfolios are also extremely positioned towards bonds.

FIGURE 7: PORTFOLIO ALLOCATIONS AFTER MONTHLY REBALANCING

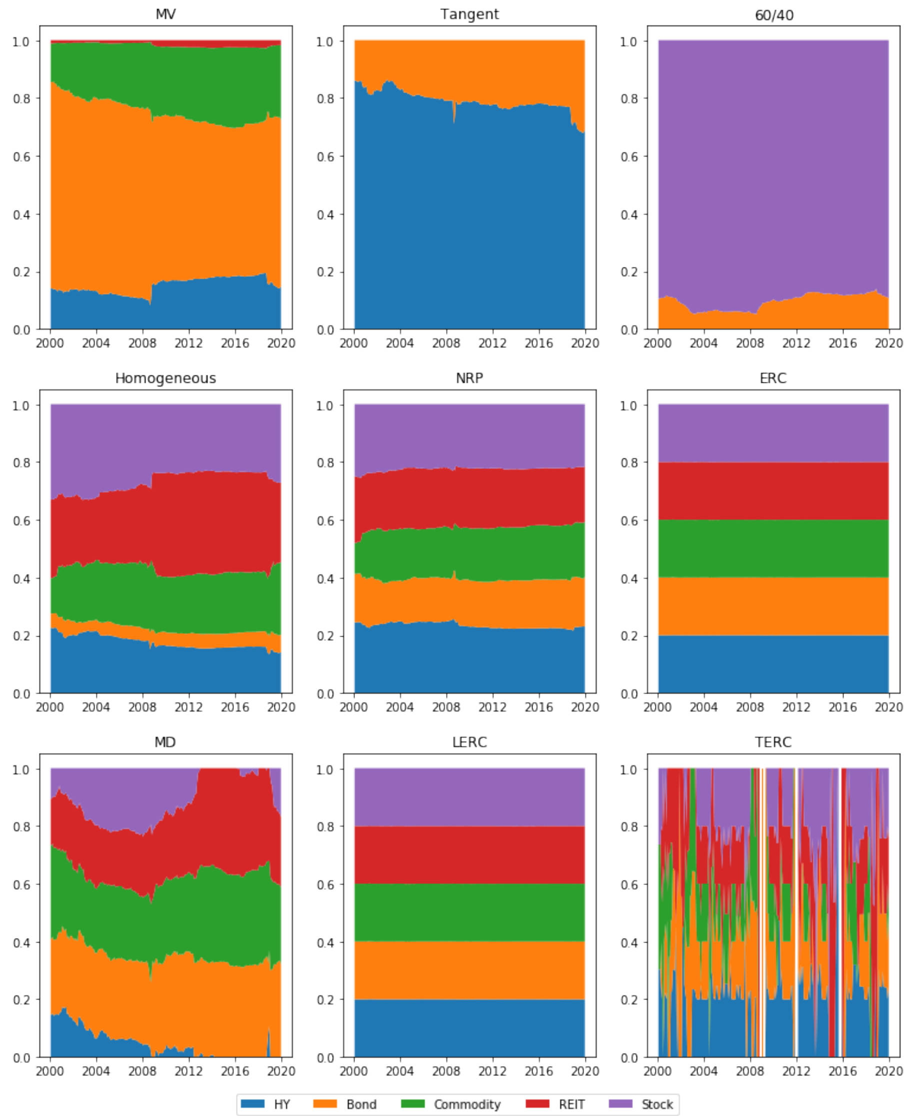


Investment strategies allocations to each asset class after monthly rebalance to optimal weights. Horizontal axis refers to end of the month dates.

Due to its inherent features, the $TERC$ strategy often suffers considerable swings in allocation, a characteristic that penalizes the performance. In at least two moments during our analysis, the $TERC$ portfolio was completely allocated to cash given the relationship between price and the 10-month price moving average (see Figure 19 in the Appendix). Figure 8 denotes total risk contributions from individual assets. RP portfolios are more balanced from the risk point of view than the benchmark strategies. Nevertheless, $TERC$

risk contributions are not as balanced as other RP strategies due to high trading activity and exposure to cash.

FIGURE 8: TOTAL RISK CONTRIBUTIONS

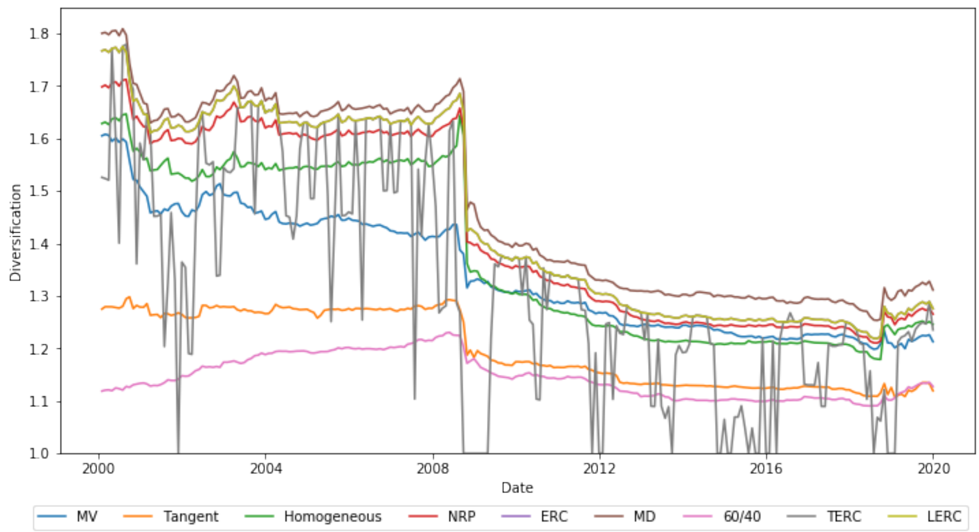


Portfolio`s ex-ante total risk contribution after monthly rebalance. Horizontal axis refers to end of the month dates.

At least one of the asset classes outweighs the remaining ones regarding total risk contribution in MV, T and 60/40 portfolios. In the portfolio H, although money-wise allocations are qual, total risk contributions are far from similar. Likewise, the 60/40 and T portfolio`s risk contributions are almost exclusively dominated by equity risk and high yield respectively, with negligible bond contribution.

Figure 9 portraits the progress in diversification ratio on a monthly basis. One can see the general shrinkage in this metric for all portfolios after 2008 due to an increase in the asset's correlations. In the last column of Table V, we have average diversification ratios for the full period.

FIGURE 9: DIVERSIFICATION RATIO



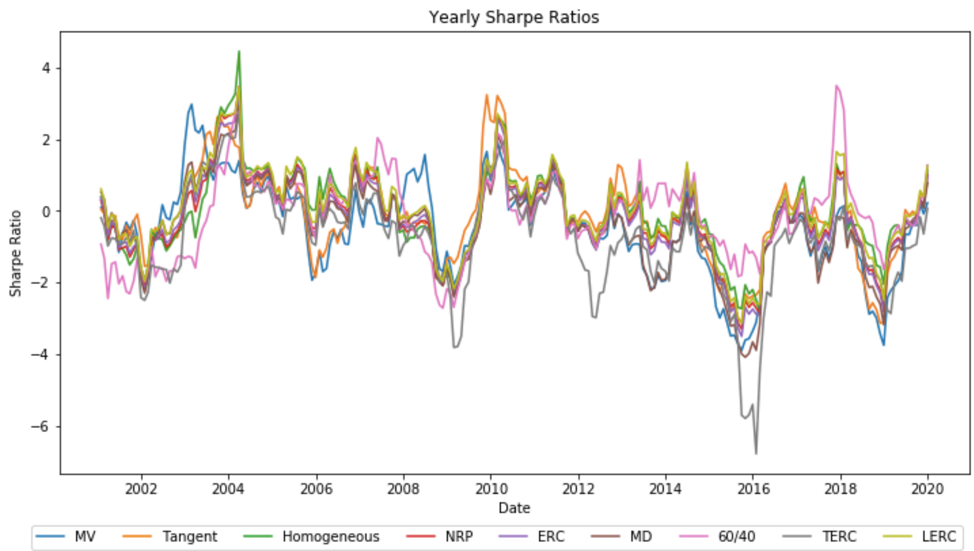
Monthly diversification ratios for each strategy after monthly rebalancing to optimal weights. Horizontal axis refers to end of the month dates.

Four out of five RP portfolios outperform all the benchmark peers in diversification, with the *most diversified* (MD) portfolio leading. MD's diversification ratio starts as high as 1.8 and drops to as low as 1.25. $TERC$ portfolio in some periods demonstrates a ratio equal to one, meaning that the portfolio is composed solely by one asset. The 60/40 portfolio exhibits the lowest diversification ratio, with portfolio T following closely, since these two portfolios are essentially composed by two assets.

Regarding *Sharpe ratio* (SR) performance, T portfolio offsets all others followed by $LERC$. At exception of MD, RP strategies are able to outperform benchmarks, specially the NRP and ERC strategies. Figure 10 tracks the SR development over a 12-month window. One can see that SR were particularly low for most portfolios during three periods: Dot-com bubble in the early 2000's; the aftermath of the global financial crisis in 2008; and at the end of 2015. During these 3 periods, returns deteriorated the most and volatilities spiked.

Figure 11 exhibits the wealth evolution of a hypothetical \$ 100,000 initial investment in each of the strategies before trading cots. Portfolios T and $LERC$ more than tripled the initial investment. The initial investment would have turned into \$ 309,724.37 under the $LERC$ strategy and into \$ 328,634.54 under T strategy – Figure 18 in the Appendix provides a comparison between the wealth development of the two strategies. The 60/40 portfolio delivered only an accumulated 74% growth (initial investment would have turned into $ 174,062.83).

FIGURE 10: YEARLY SHARPE RATIOS



Rolling 12-month Sharpe ratio development for each strategy. Horizontal axis refers to end of the month dates.

FIGURE 11: Wealth Plot, 2000Q1 to 2019Q4

Wealth plot development of an initial $ 100,000 investment according to each strategy over a 20-year investment period. Trading costs are not incorporated.

5.1.1. Other investment horizons

We move to analyse the performance of the different strategies in shorter periods of investment. We start with 10-year investment horizon simulations. Given 240 return observations, it is possible to simulate 120 different 10-year outcomes starting at each month end from January 2000 until December 2009[4]. Figure 12 illustrates this process. We demonstrate the wealth plot for 6 out of the 120 simulations we performed for the MV portfolio with an initial \$ 100,000 investment. We proceeded similarly for other portfolios and for the different horizons. We calculated the measures introduced earlier for each simulation and averaged the results for each investment horizon.

FIGURE 12: WEALTH PLOT MV PORTFOLIO 10-YEAR INVESTMENTS

Different 10-year investments based on MV strategy. Investments starting at each possible date from January 2000.

Table VI summarizes the average statistics for all three different investment horizons and Figure 13 exhibits the development of annual returns, volatilities and Sharpe ratios on a yearly basis.

Panel (a) of Table VI summarizes the average of key measures for the 120 different 10-year investments. All the strategies exhibit positive returns on average, from 3.65% to 6.42% for MD and T, respectively. Although volatility increased on average when compared to the 20-year investment, all the portfolios exhibited higher returns when we decreased the investment period. We also observe that most portfolios now exhibit positive Sharpe ratios. $LERC$ portfolio outperformed all the benchmark strategies apart from T in risk adjusted returns. NRP and ERC were only able to outperform H and 60/40 in Sharpe ratio terms.

[4] See Figures 15 to 17 for the weighting composition of portfolio T and MV for the different investment horizons.

As the left chart in the first row of Figure 13 exhibits, annualized returns started to decrease after 2012. This means that 10-year investments starting after 2003 yielded a lower annual return than the ones started before. This is partially explained by the severe losses most asset classes suffered between 2008 and 2010. In terms of risk measures, RP portfolios demonstrate lower volatilities than most of benchmarks and lower drawdowns.

TABLE VI: PORTFOLIO STATISTICS BEFORE TRADING COSTS

Panel (a): 10-year investments

	Ann. Return	Excess Return	Ann. Volatility	Sharpe Ratio	Skewness	Kurtosis	Historic VaR (5%)	Historic CVaR (5%)	Max Drawdown
MV	5.10%	0.67%	6.48%	0.103	-0.835	6.515	0.028	0.043	-15.17%
T	6.42%	1.99%	11.96%	0.166	-1.403	10.601	0.043	0.091	-34.75%
H	4.76%	0.33%	12.41%	0.026	-1.735	11.559	0.055	0.097	-40.02%
60/40	3.93%	-0.51%	10.88%	-0.047	-0.961	5.742	0.057	0.081	-33.44%
NRP	4.90%	0.47%	9.62%	0.049	-1.618	10.999	0.043	0.071	-29.79%
ERC	4.74%	0.31%	9.19%	0.034	-1.564	10.536	0.041	0.067	-28.23%
MD	3.65%	-0.79%	8.64%	-0.091	-1.373	8.616	0.037	0.065	-27.26%
TERC	4.18%	-0.25%	5.16%	-0.049	-0.324	3.777	0.021	0.032	-9.19%
LERC	5.66%	1.23%	10.90%	0.113	-1.564	10.536	0.049	0.080	-33.68%

Panel (b): 5-year investments

	Ann. Return	Excess Return	Ann. Volatility	Sharpe Ratio	Skewness	Kurtosis	Historic VaR (5%)	Historic CVaR (5%)	Max Drawdown
MV	4.81%	1.76%	5.81%	0.303	-0.607	4.619	0.023	0.037	-8.87%
T	5.24%	2.19%	9.29%	0.235	-0.805	5.440	0.040	0.068	-17.63%
H	4.96%	1.91%	10.53%	0.182	-0.910	5.947	0.047	0.075	-30.28%
60/40	4.97%	1.92%	8.32%	0.231	-0.587	4.150	0.045	0.066	-22.51%
NRP	4.20%	1.15%	9.78%	0.118	-0.878	5.867	0.035	0.057	-25.87%
ERC	4.80%	1.75%	7.99%	0.219	-0.856	5.766	0.033	0.055	-21.34%
MD	3.78%	0.73%	7.63%	0.095	-0.783	5.279	0.036	0.053	-21.29%
TERC	4.18%	1.13%	4.84%	0.234	-0.453	4.030	0.020	0.030	-7.65%
LERC	6.22%	3.17%	9.79%	0.323	-0.856	5.766	0.040	0.066	-18.88%

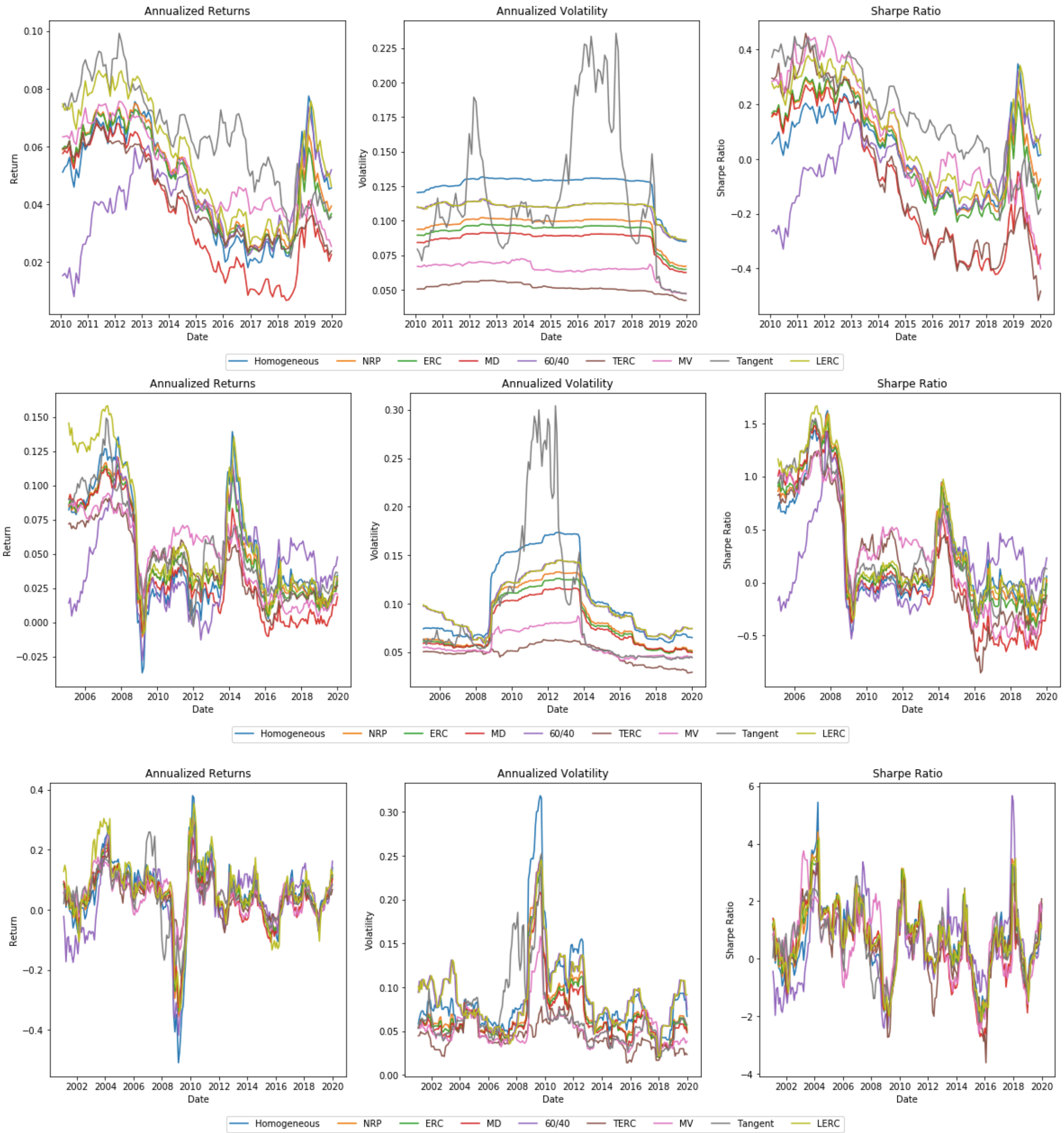
Panel (c): 1-year investments

	Ann. Return	Excess Return	Ann. Volatility	Sharpe Ratio	Skewness	Kurtosis	Historic VaR (5%)	Historic CVaR (5%)	Max Drawdown
MV	4.60%	2.75%	5.25%	0.524	-0.352	2.662	0.019	0.025	-3.61%
T	4.95%	3.10%	6.76%	0.458	-0.345	2.842	0.025	0.033	-4.87%
H	4.81%	2.96%	8.90%	0.333	-0.449	3.182	0.034	0.048	-3.69%
60/40	3.36%	1.51%	8.89%	0.169	-0.200	2.611	0.035	0.044	-4.19%
NRP	4.89%	3.04%	7.12%	0.427	-0.404	3.060	0.027	0.037	-2.74%
ERC	4.77%	2.92%	6.87%	0.425	-0.394	3.033	0.026	0.036	-2.63%
MD	3.96%	2.11%	6.64%	0.319	-0.373	2.951	0.025	0.034	-2.80%
TERC	4.19%	2.34%	4.42%	0.530	-0.441	3.137	0.016	0.022	-1.55%
LERC	6.47%	4.62%	8.91%	0.519	-0.394	3.033	0.033	0.045	-6.27%

Average values based on 120, 180 and 228 different 10-, 5- and 1-year investments, respectively. Source: Bloomberg, Federal Reserve Economic Data and own computations.

Panel (b) of Table VIII exhibits the average metrics for the 180 investments comprising of 5-year investments. Excess returns have increased for all strategies and volatilities decreased. One can observe positive excess returns and an increase in SR when compared to both 20- and 10-year investments. We have now excess returns between 0.73% (MD) and 3.17% ($LERC$). $LERC$ portfolio outperforms all the remaining in risk adjusted returns, followed by MV. Portfolio $TERC$ and T exhibit similar SR.

FIGURE 13: DEVELOPMENT OF KEY METRICS, 10-, 5- AND 1-YEAR INVESTMENTS



Yearly development of annualized returns, volatilities and Sharpe ratios for the different investment horizons. First row, 10-year investments. Second row, 5-year investment. Third row, 1-year investments.

Finally, Panel (c) summarizes the average metrics for the 228 1-year investment simulations starting from January 2000 until December 2018. Likewise, one can see again an increase in risk adjusted returns. Volatilities lay between 4.42% ($TERC$) and 8.91% ($LERC$ and 60/40). $TERC$ and MV remain as the least volatile portfolios. As one could expect given the shorter investment period, drawdowns reduced considerably. $TERC$ portfolio incurred on average drawdowns of 1.55%, whereas for $LERC$ portfolio, this mounts to 6.27%.

Given the decrease in volatilities and an increase in excess returns, this results in an increase of risk adjusted returns. We have now $TERC$ leading with this regard, closely followed by MV and $TERC$ portfolios. However, when we account for risk measures, $TERC$ outperforms by a wide margin.

5.2. Robustness Analysis – Trading Costs

Often, investment strategies which undertake a high trading frequency seem very appealing before trading costs are considered. In this sense, we calculated monthly turnover for each strategy for the 20-year investment as exhibited on Figure 14. We considered trading costs of 50 *basis points* (bps) *per* 100% turnover. Table VII summarizes our findings.

As one can note, $TERC$ strategy requires a considerably high amount of turnover in order to be implemented. This strategy exhibits the highest turnover ratio of 3.898, resulting in a holding period of only 3 months. Both MD and $LERC$ strategies distantly follow with holding periods around 30 months. Portfolio H has the lowest average holding period of 35 months among the benchmark strategies as one could expect since it is the only strategy that invests in all 5 available asset classes.

The high level of trading activity experienced by the $TERC$ fallouts in a drawback to annual return of 1.95%. For the remaining strategies, the trading costs lie between 0.10% to 0.20% annually.

For the $LERC$ portfolio, we must consider additionally the cost of leveraging the portfolio. We estimated that the $LERC$ strategy is penalized by 0.52% annually due to the leverage requirements. Therefore, the total cost incurred by the latter strategy mounts to 0.72% annually.

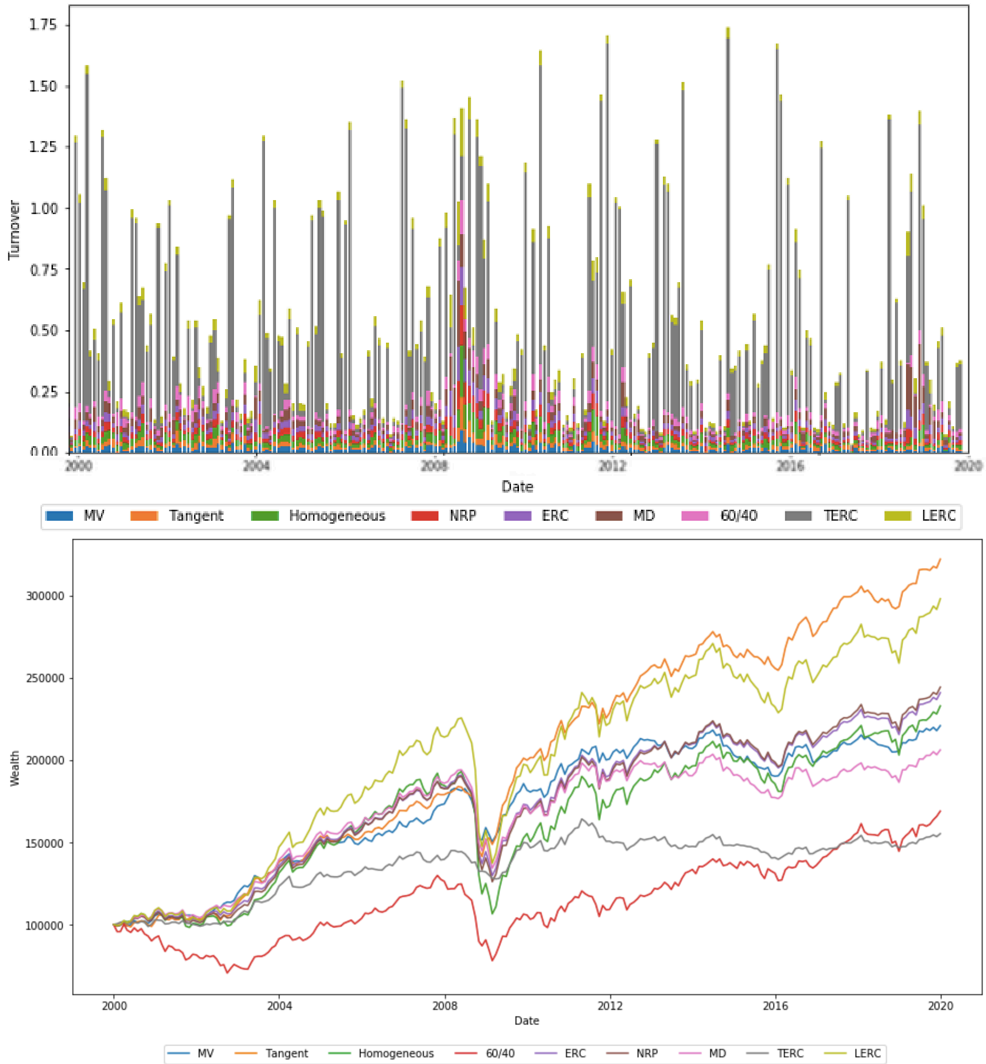
TABLE VII: PORTFOLIO'S STATISTICS AFTER TRADING COSTS, 2000Q1 TO 2019Q4

	MV	T	H	60/40	NRP	ERC	MD	TERC	LERC
Turnover Ratio	0.203	0.199	0.338	0.297	0.297	0.302	0.403	3.898	0.383
Avg. Holding Period (Months)	59	60	35	40	40	40	30	3	31
Ann. Turnover Cost	0.10%	0.10%	0.17%	0.15%	0.15%	0.15%	0.20%	1.95%	0.19%
Ann. Leverage Cost	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.52%
Total Trading Cost	0.10%	0.10%	0.17%	0.15%	0.15%	0.15%	0.20%	1.96%	0.72%
Excess Return After Cost	-2.34%	-0.37%	-1.69%	-3.19%	-1.67%	-1.78%	-2.62%	-4.24%	-1.04%
Sharpe Ratio After Cost	-0.390	-0.055	-0.163	-0.323	-0.205	-0.229	-0.353	-0.908	-0.105

Turnover costs of 50 bps and leverage cost equal to EURUSD deposit rate. Source: Federal Reserve Economic Data, Bloomberg and own computations.

Portfolio T still outperforms its peers in risk-adjusted returns, this time by a wider margin. Portfolio T and $LERC$'s excess return adjusted for trading costs are now -0.055% and -0.105%, respectively. The $TERC$ portfolio is extremely penalized such that its excess return adjusted for trading costs is -4.24%.

FIGURE 14: Monthly turnover and Wealth Plot after Trading Costs



Upper figure exhibits the monthly turnover of each figure. Lower figure depicts the wealth development when accounting for leverage and turnover costs.

Considering trading costs, an initial investment for \$ 100,000 in the $TERC$ portfolio would have turned into \$ 155,233.26 instead of \$ 229,401.01 if trading costs were not considered. For the $LERC$ strategy, the initial investment would have turned into \$ 298,076.2 as opposed to \$ 309,724.37 in the neither trading nor leverage cost case.

Chapter 6

Conclusion

This study empirically compares the performance of RP investment strategy with other common investment strategies, resulting either from MVT – T and MV portfolios – or naïve investments such as the 60/40 or H portfolios. We analysed 5 different RP-based strategies and tested their performances against 4 passive benchmark strategies in 4 different investment horizons. We concluded that RP portfolios are indeed much more balanced from a purely risk point of view. All the benchmark approaches are tilted to one asset class when we analyse risk contributions. The trend-following method applied to ERC portfolio enabled us to arrive at a portfolio whose *ex-post* volatility is inferior to the MV portfolio. By applying leverage to the ERC portfolio, the resulting strategy arrived at considerably superior risk-adjusted return than the 60/40 portfolio for the same level of risk.

We concluded that most RP portfolios consistently outperformed the H and 60/40 portfolios in risk-adjusted returns, VaR and maximum drawdown, proving to be an effective alternative. However, RP were not able to outperform in a regular manner MVT based portfolios, specially the T portfolio on the 20-year investment case. When we decreased the investment horizon, RP portfolios outperformed in 2 out 3 cases in risk adjusted terms. As Jobson and Korkie (1981) and DeMiguel *et al* (2007) documented, the higher the input length and the lower the number of risky assets, the better MV portfolios tend to perform. Having this said, further analysis with a considerably higher amount of risky assets within the proposed asset classes and with shorter input length should be conducted in order to conclude about a possible superiority of RP over MVT portfolios.

Regardless of the selected investment strategy, we found both benefits and drawbacks and periods where one or another strategy performed better. Therefore, the challenge for private and institutional investors remains the same: from one side, the development of models and tools that allow investors to develop assumptions about input parameters is very much needed; on the other, the enhancement of strategies such as RP where input risk is reduced is likewise necessary.

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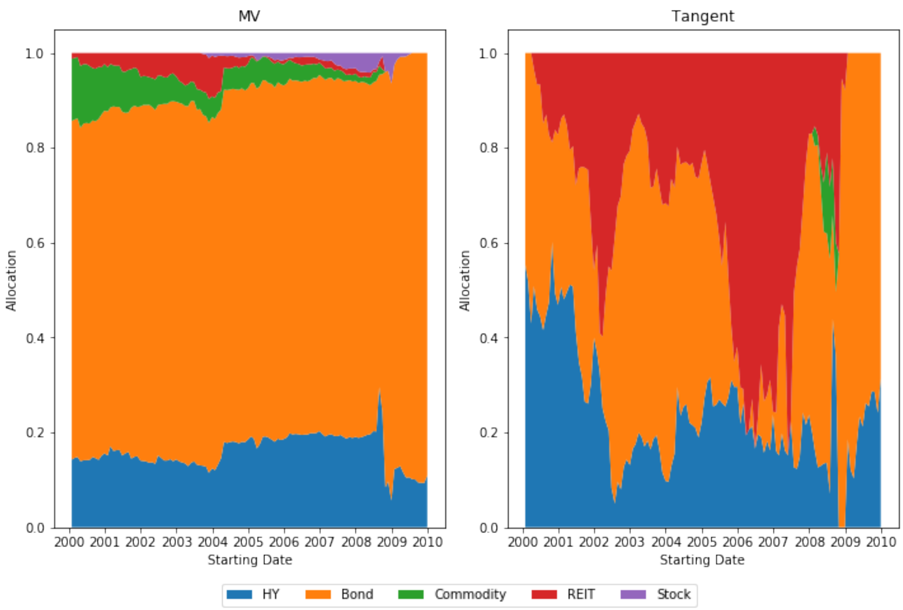
Appendix A

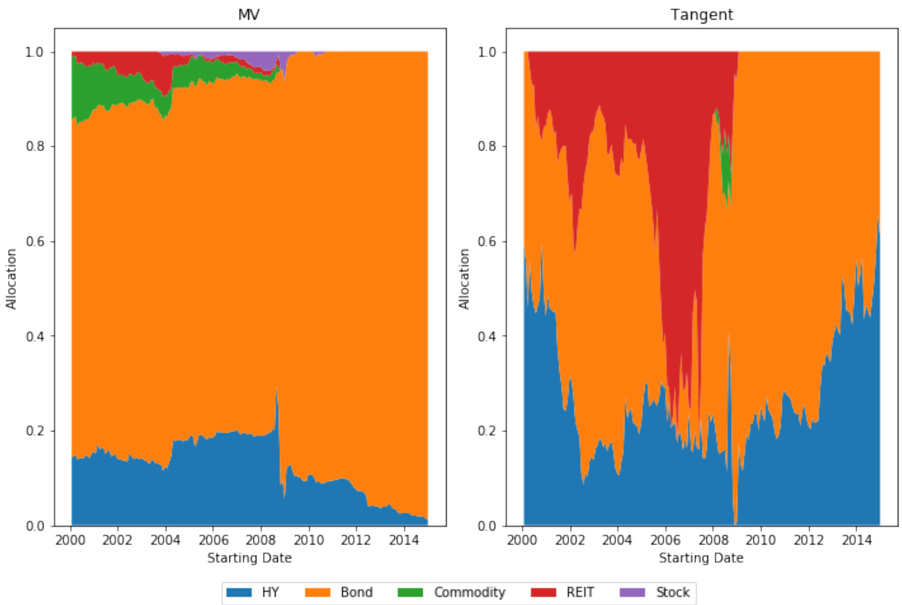
Output Figures

FIGURE 15: MV AND T ALLOCATIONS - 10-YEAR INVESTMENTS



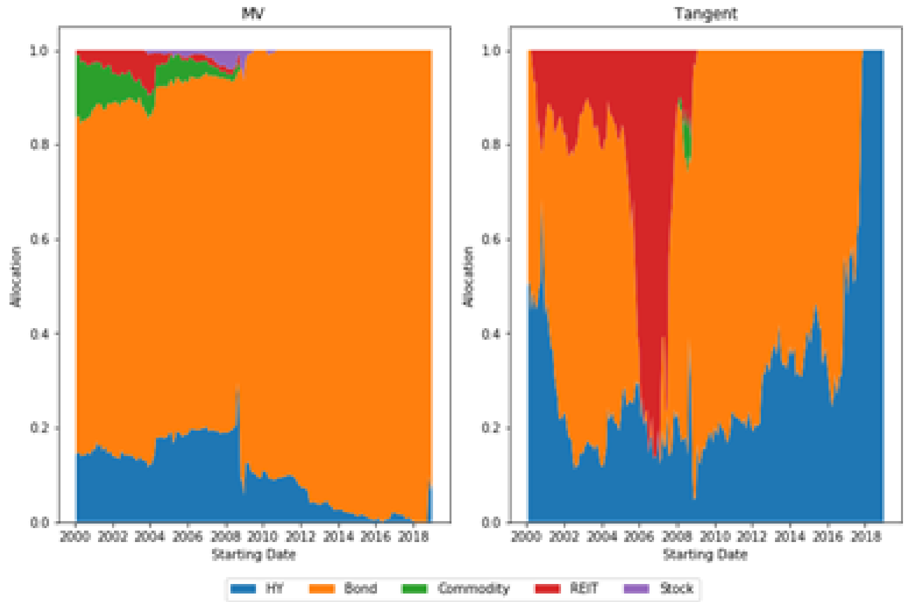
MV and T portfolio’s composition for 10-year investments at each optimization date, from January 2000 to December 2010.

FIGURE 16: MV AND T ALLOCATIONS - 5-YEAR INVESTMENTS



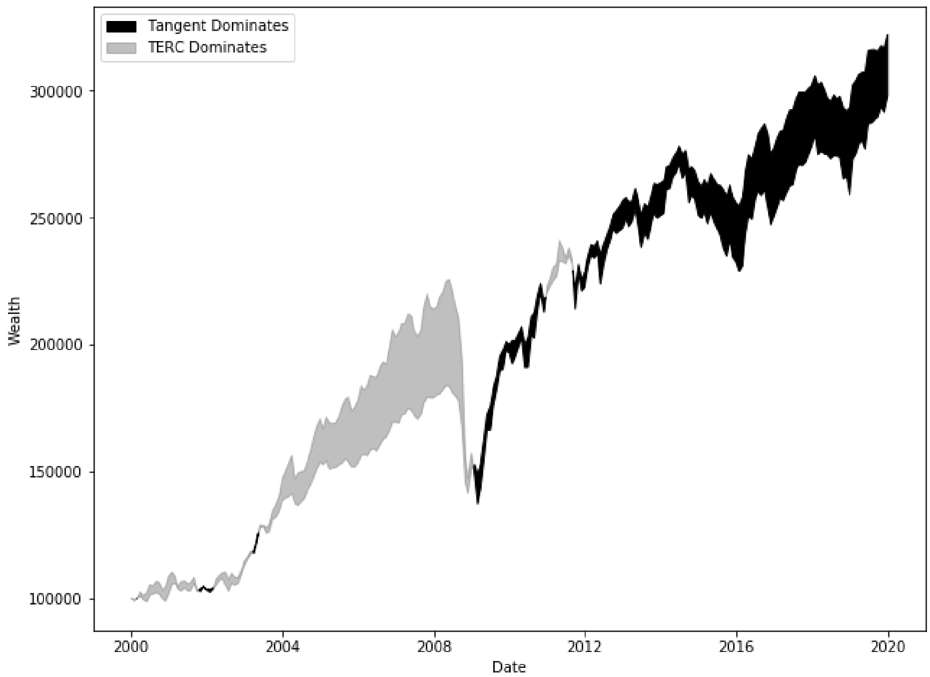
MV and T portfolio’s composition for 5-year investments at each optimization date, from January 2000 to December 2015.

FIGURE 17: MV AND T ALLOCATIONS - 1-YEAR INVESTMENTS



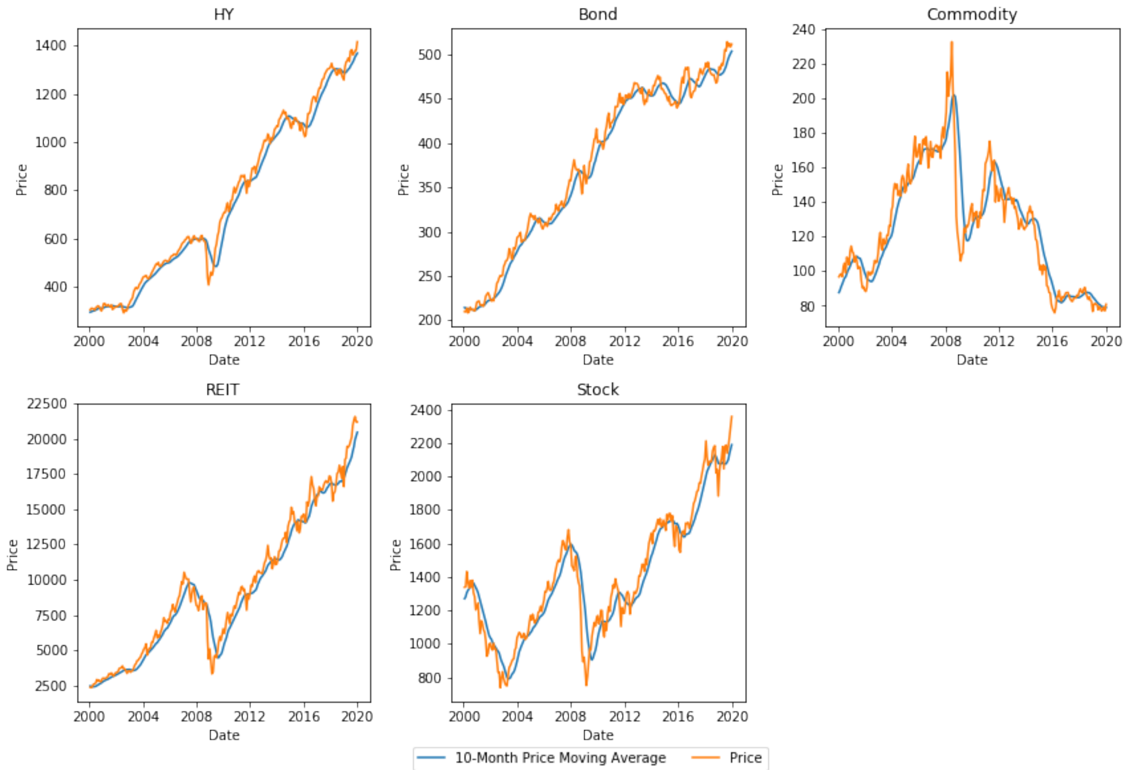
MV and T portfolio's composition for 1-year investments at each optimization date, from January 2000 to December 2019.

FIGURE 18: LERC VS T PORTFOLIO WEALTH EVOLUTION FOR 20-YEAR INVESTMENTS



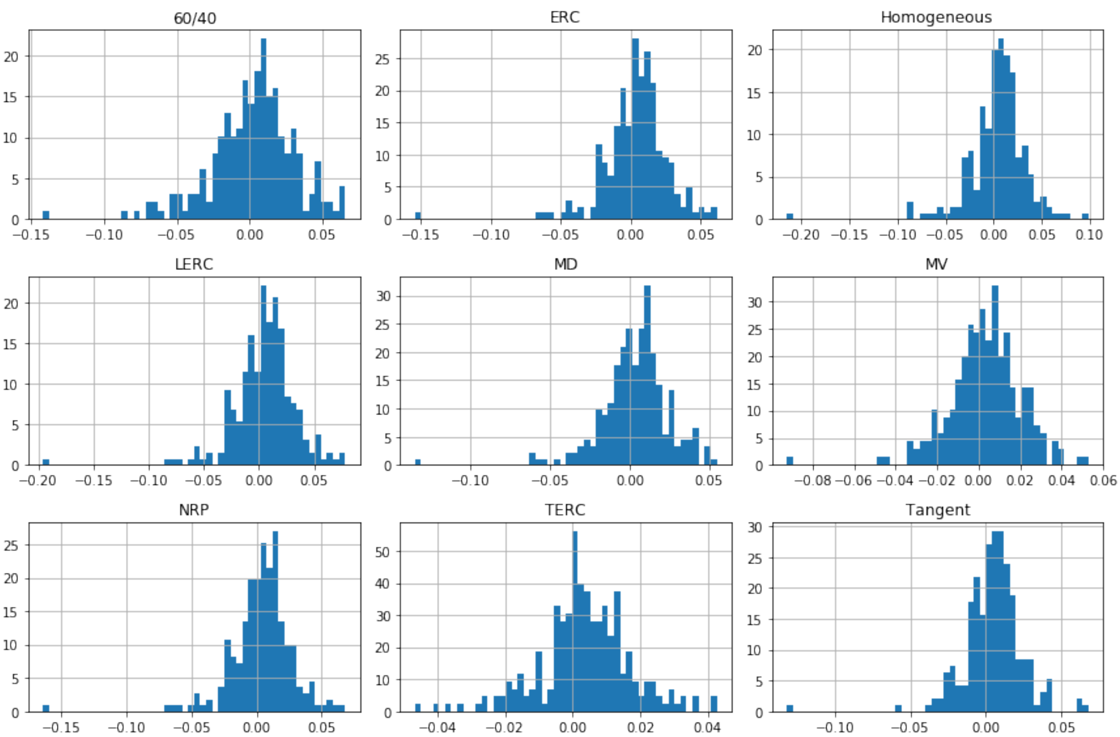
Grey (black) area represents periods of dominance from TERC (T) portfolio over portfolio T (LERC) for a 20-investment horizon. Period of analysis from January 2000 to December 2019.

FIGURE 19: INDEX PRICES AND 10-MONTH PRICE MOVING AVERAGE



Relationship between the index price and the corresponding 10-month price moving average. When the price is above the moving average, this triggers a buy signal for the index on the TERC portfolio. Otherwise, the index weight is assigned to cash.

FIGURE 20: PORTFOLIO'S RETURNS DENSITIES



Return frequencies of 20-year investments based upon monthly returns from January 2000 to December 2019 for each portfolio.