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UNIVERSIDADE TÉCNICA DE LISBOA
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DISSERTAÇÃO

PORTFOLIO INSURANCE STRATEGIES: AN ANALYSIS OF PATH DEPENDENCIES

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PROF.ª RAQUEL M. GASPAR
PROF. JOÃO BELEZA SOUSA

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Portfolio Insurance Strategies:
An Analysis of Path-Dependencies

João Pereira Carvalho
Advisor: Professor Raquel M. Gaspar
Co-advisor: Professor João Beleza Sousa

Abstract

This thesis makes an evaluation of the path-dependency/independency of the most widespread Portfolio Insurance strategies, i.e. CPPI, OBPI and SLPI, using Monte Carlo simulations. Also, it is known that for the CPPI with multiplier higher than 1, an undesirable path-dependent behavior called 'cash-lock', can occur in some market scenarios. But in what scenarios and how often?

In this thesis we show on an empirical level, that for most of the chosen market scenarios, CPPI 3 and CPPI 5 strategies can in fact get cash-locked easily. This is a rather undesirable feature to investors, particularly if it occurs on investments whose return has to be paid at a long maturity, which is the case for many of the CPPIs offered by financing institutions. To clearly show the path dependency we assume we know the value of the underlying risky asset not only at inception but also at maturity, and study the payoff distributions for the different PI under different market conditions and product specifications. To do so, we proceed with Monte Carlo simulations of the underlying risky asset paths, all conditioned to the same final value using Gaussian Processes for Machine Learning Regression. We model the risky asset as geometric Brownian motion.

We expect that this study will contribute to reinforce the idea that CPPI products need affordable solutions to prevent cash-locked investments, which is a major drawback to investors.

Keywords: Portfolio Insurance strategies, CPPI, OBPI, SLPI, path-dependencies, cash-lock, Monte-Carlo simulations, conditioned geometric Brownian motion.

Portfolio Insurance Strategies:
An Analysis of Path-Dependencies

João Pereira Carvalho
Orientadora: Professora Raquel M. Gaspar
Co-orientador: Professor João Beleza Sousa

Resumo

Esta tese faz uma avaliação das (in)dependências do caminho das estratégias mais difundidas de *Portfolio Insurance* (PI), ou seja, CPPI, OBPI e SLPI, utilizando simulações de Monte Carlo. Além disso, sabe-se que para a estratégia CPPI com multiplicador superior a 1, um comportamento dependente do caminho e indesejável chamado '*cash-lock*', i.e bloqueio no activo sem risco, pode ocorrer em alguns cenários de mercado. Mas em que situações e com que frequência?

Neste trabalho mostramos por via de simulações, que para a maioria dos cenários de mercado escolhidos, as estratégias CPPI 3 e CPPI 5 podem facilmente ficar . Esta é uma característica muito indesejável para os investidores, especialmente se ocorrer em investimentos que não estão totalmente cobertos e cujo retorno tem que ser pago num longo prazo de vencimento, que é o caso de muitos dos CPPIs oferecidos pelas instituições financeiras. Para destacar a dependência do caminho, assumimos que se sabe o valor do activo de risco na maturidade. Estudamos, assim, as distribuições do valor na maturidade das diferentes estratégias PI sob diferentes condições de mercado e de produto. Para isso, procedemos com simulações de Monte Carlo dos caminhos do activo de risco subjacente, todos condicionados com o mesmo valor final, usando a regressão de Processos Gaussianos para Aprendizagem Automática. Neste estudo, modelou-se o activo de risco de acordo com o movimento Browniano geométrico.

Esperamos que este estudo contribua para reforçar a ideia de que os produtos CPPI com $m > 1$ precisam de soluções acessíveis para evitar que os investimentos terminem em *cash-lock*, o que é uma grande desvantagem para os investidores.

Keywords: Portfolio Insurance, CPPI, OBPI, SLPI, dependência no caminho, simulações de Monte Carlo, cash-lock, movimento Browniano geométrico condicionado.

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Chapter 1
Introduction

The idea of introducing insurance in investment portfolios was first proposed by Leland and Rubinstein (1976). The main motivation was to prevent the contagious divestment movements observed in the stock market crash of 1973-74, which led to the loss of significant potential gains in the subsequent 1975 rise. Therefore, a portfolio insurance (PI) would consist of an asset allocation strategy between a risk-free asset and a risky asset, so that the combination would give the investor both security and some participation in upside performance. Leland and Rubinstein (1976) developed the first PI, the Option Based Portfolio Insurance (OBPI) strategy, realizing that the risky asset, e.g. a stock or financial index, can be insured by a put option written on it and whose strike price is the amount to be insured. As listed options are not are not available for long maturities and with adequate prices for most investors, Rubinstein (1985) used the Black and Scholes (1973) (B-S) pricing model and presented a dynamic asset allocation strategy between a risky and a risk-free asset (bond) to replicate put/call options, which would synthesize the OBPI strategy with a more accessible price.[1]

Following the work of Merton (1971), Perold (1986) introduced the Constant Proportion Portfolio Insurance (CPPI) strategy as one possible solution to the Merton problem for an investor with a HARA utility function (Kingston, 1988). The strategy also consists of a dynamic asset allocation between a risky and a risk-free asset in order to guarantee a certain percentage of the investment at maturity. However, it is considerably simpler than the OBPI in its implementation which can be very appealing to a great number of investors and issuers. The term proportional derives from the fact that for every rebalancing date, the amount of the portfolio invested in the risky asset (exposure) is proportional to the cushion. The cushion is the difference between the total portfolio value at that instant and the present value (with the risk-free asset's interest rate) of the amount insured at maturity (floor). The term constant is simply because the proportionality factor is fixed at inception for the entire investment period. This constant is called the multiplier, $m \geq 1$, and in the present thesis we focus on three values, $m = 1, 3, 5$, denoting the associated strategy as CPPI m. Moreover, CPPI1 is simply a path-independent Buy and Hold plus Bond strategy, and CPPI 3 and 5 are path-dependent strategies.

Since the first appearance of the OBPI and CPPI strategies, extensive literature has sprouted on the subject with different objectives and methodologies. The first

[1] In most literature about PI, the term synthetic OBPI is shortened to OBPI, so we follow the same rule.

studies concern the properties of continuous-time PI (Bookstaber and Langsam, 2000; Black and Perold, 1992, e.g.) where CPPI is analyzed. A comparison of OBPI and CPPI strategies is also provided in this context, where Black and Rouhani (1989) conclude that OBPI outperforms under moderate market increases while CPPI has better performance when the risky asset suffers large drops or increases. Bertrand and Prigent (2002, 2005) also make a comparison using various criteria such as the first four moments of the probability distributions of both PI payoffs. They verify that the CPPI strategy performs better than OBPI when the insured amount at maturity increases, because the OBPI call is less likely to be exercised.

For empirical evaluations the discrete-time trading is a more realistic approach and Cesari and Cremonini (2003) compare an extensive variety of the most used PI strategies, arriving at the conclusion that CPPI has better performance only in bear and no-trend markets. A comparison of OBPI, CPPI and Stop-Loss Portfolio Insurance (SLPI) strategies is also studied in the papers of Annaert et al. (2009) and Zagst and Kraus (2011), using stochastic dominance criteria, but verifying no dominance between the strategies with exception of CPPI 3 dominating OBPI on third order in the second paper. In Bertrand and Prigent (2011) CPPI is concluded to be a better strategy than OBPI under the Omega performance ratio criteria. In Costa and Gaspar (2011) the OBPI, CPPI and SLPI are also compared under diverse market scenarios using statistical and performance measures. They also applied the strategies to three main world stock indices during the subprime crisis of 2008 and verified that CPPI 3 and 5 would become attached to the floor at some point in the investment period and were unable to recover from it till maturity. This behavior is usually called 'cash-lock'.

Cash-lock occurs when the risky asset undergoes a severe slide and as a consequence, the portfolio becomes almost uniquely invested in the risk-free asset. When this happens, the exposure to the risky asset never recovers, even if the market grows again. This behavior is extremely undesirable for an investor, especially if he has a contractualized payoff at a long maturity. This means CPPIs are subject to an additional conceptual/design risk, as the cash-lock 'mechanism' alone introduces a serious path-dependency. In this study we look deeper into the dependency generated by that mechanism alone which contradicts one of the main purposes of PI: allow participation in upside market movements.

In literature, the risky asset paths are simulated by Monte Carlo methods and modeled by stochastic processes. In this regard, the final value of the risky asset is also stochastic. The PI strategies that are calculated over all simulated paths, give rise to a distribution of the final payoffs. In this distribution, the path-dependency risk is blended in the uncertainty of the final risky asset value and thus it is not possible to analyze the frequency of cash-lock occurrences in CPPI 3 and 5. The question that arises is *When and how often do these cash-lock events happen?*. Hence, in this study we isolate the path-dependency/design risk from the risk inherent to the stochasticity of the underlying asset, assuming fixed maturity prices for the underlying. In other words, it can be understood as a 'what if' analysis which allows us to study CPPI performance in scenarios where, by construction, we know that the underlying risky asset will double or triple in price. Taking a different perspective, we try to take the point of view of an investor that 'bets' the risky asset will grow to a given level at maturity and is wondering which PI to choose. This analysis demands a different approach for simulating the risky asset in respect to previous studies. We address this issue using Gaussian Processes for Machine Learning Regression to simulate stock price sequences all ending at the same specified value. This procedure will be applied to a variety of scenarios. Regarding the business sciences which is the core discipline where the present master resides, it is important to refer that many companies, particularly in the insurance industry, have invested in CPPI $m > 1$ products. Those investments suffered cash-lock occurrences when facing more volatile periods of their underlying assets such as the subprime crisis. We expect that this study contributes to reinforce the idea that CPPI products need new improvements to prevent cash-locked investments, which is in the interest of both singular and collective investors.

In the following chapter, we expose the theoretical background necessary to comprehend the construction of the different Portfolio Insurance strategies under study. We also provide a final example depicting a situation where the CPPI 3 and 5 could become cash-locked. In Chapter 3 we present the methodology used in the simulation of the risky asset, the parameter scenarios involved and and the statistical methods. Further, in Chapter 4 the statistical results are shown and then discussed in Chapter 5. Chapter 6 has the main conclusions of this thesis and further research topics are suggested in the end.

Chapter 2
Portfolio Insurance Background

In general there are three types of investment strategies: (1) buy-and-hold which is a neutral strategy in the sense that there is no reaction if either market rises or falls, (2) *buy-falling/sell-rising stocks* corresponding to a concave payoff diagram, and (3) *buy-rising/sell-falling stocks* which supports upward performance while giving downward protection. The latter is a Portfolio Insurance (PI) strategy and therefore strategies (2) and (3) can be thought as selling and buying PI, respectively (see Perold and Sharpe (1988)).

A PI strategy can be categorized by its construction type - static vs dynamic - and by its terminal payoff behavior - path-dependent vs path-independent. Concerning the first aspect, a static PI strategy is such that the original asset allocation rule stays fixed for the whole period of the investment. Contrarily, a dynamic PI strategy bases on an algorithm which is built in such a way that it usually creates a convex payoff function by reacting to market conditions. As for the path-dependency, a PI strategy is path-independent if the payoff only depends on the terminal value of the portfolio and the corresponding parameters of the hedge, and vice-versa. Table 2.1 shows how some examples of PI strategies that integrate in the this framework. In this text we give particular emphasis to CPPI strategy due to its path-dependent nature.

	Path-independent	**Path-dependent**
Static	· OBPI with listed put	· Stop-Loss
Dynamic	· OBPI with replicated put/call	· CPPI

Table 2.1: Adapted from Köstner (2004). Portfolio Insurance classification according to its Static/Dynamic construction vs path-dependency.

In this text we consider the usual choice of the assets that compose a portfolio p: the risky or performance asset, S (e.g. stock), which is modeled by a stochastic process, and the risk-free asset, B (e.g. bond), i.e. riskless cash accounts that are used to assure the minimum payoff the investor has initially negotiated. Thus, PI can be represented for every $t \in [0,T]$ by the pair (ν^B, ν^S), which are the risky asset number and the number of bonds, respectively. The portfolio value $(V_t^p)_{t\in[0,T]}$ is hence given by $V_t^p = \nu_t^B B_t + \nu_t^S S_t$ (Balder and Mahayni, 2010). In this thesis we consider only self-financing PI, i.e. with no exogenous injection or withdrawal of money during $]0,T[$. Naturally, the CPPI, OBPI and SLPI strategies correspond to this type of strategies which can only purchase more assets if they have previously sold others. This self-financing property implies that (see e.g. Bjork (2009))

$$dV_t^p = \nu_t^B dB_t + \nu_t^S dS_t, \tag{2.1}$$

and typically the insured component of the investment can be translated into the expression

$$B_T = \eta V_0^p, \tag{2.2}$$

where η (typically ranging from 80% to 100%) is the percentage of the initial invested capital to be insured. Assuming non-arbitrage, at $t = 0$ we have $\nu_0^S > 0$ and hence $V_0^p > \nu_0^B B_0$, which mean η is limited to the future value of the initial portfolio investment, and therefore $0 \leq \eta < e^{rT}$ (Zagst and Kraus, 2011). In other words, an investor can never insure more than the present value of its investment. This percentage is one of the scenario parameters used in the results of this thesis, because of it influence on PI performance.

2.1 Stop-Loss Portfolio Insurance (SLPI)

As its name suggests, this simple strategy consists on the portfolio being entirely invested into the risky asset, and if it falls below the investor's pre-established floor F_t, the portfolio is automatically rebalanced into the risk-free asset. The floor is a representation of the bond with continuously compound interest in $[t, T]$:

$$F_t = F_T e^{-r(T-t)}. \tag{2.3}$$

Thus, the portfolio value of the SLPI strategy can be formally defined by

$$V_t^{SLPI} = \frac{V_0}{S_0} S_t 1_{\{\tau > t\}} + F_t 1_{\{\tau \leq t\}}, \tag{2.4}$$

where $\tau = \inf\{t > 0 : V_t^{SLPI} = F_t\}$ is the first instant that the portfolio 'touches' the floor barrier, if it exists.[1] The indicator functions $(1_{\{\tau > t\}}, 1_{\{\tau \leq t\}})$ are respectively $(1, 0)$ if $\tau \notin]0, t]$ - i.e., in this period the portfolio never touched the barrier - and $(0, 1)$, otherwise. Therefore, SLPI is clearly a path-dependent strategy because its value at t depends on wether the risky asset path dropped to F before t. In other words, if we imagine two stock paths leading to the same ending, if one has reached the floor barrier and the other hasn't, the final portfolio value will be F_T and S_T, respectively.

[1] Note that we do not include $t = 0$ because it would mean deliberate investment in the risk-free asset. Also note that if $\{t > 0 : V_t^{SLPI} = F_t\} = \emptyset$, then its infimum is ∞ and the definition also holds.

2.2 Option Based Portfolio Insurance (OBPI)

An option is a contingent claim where the investor purchases the right, but not the obligation, to buy (call) or sell (put) the underlying asset S at a specified strike price K.[2] For the sake of simplicity and because OBPI strategy is not the main focus of this thesis, we consider only the European call option which restricts the exercise only at expiration date T. The exercise payoff can be defined by the following contract function (Lundvik, 2005):

$$\Phi(S_T) = \max[(S_T - K), 0]. \tag{2.5}$$

To exemplify, let us assume for a certain stock that $S_0 = 100$. The main idea is to pay at $t = 0$ a certain amount, say $P = 10$, that gives the right to buy the stock at $t = T$ for the price of, suppose $K = 120$. The option buyer is thus expecting (or wishing) that at expiration date T, at least $S_T > K + P$ to profit.[3] In this case, e.g. $S_T = 150$, the simplest operation the investor can do is to buy S by its rightful price of 120 and sell it immediately after by its market value 150, yielding a profit of $(-10 - 120 + 150) = 20$.[4] But of course $S_T < K$ could also happen and if so, the investor does nothing and the contract expires out of the money, leaving a loss of 10.

Now as for the 10 price of the call option, it must be generated by a certain function that in some way measures the probability that S_t may end up above or bellow the strike price K. The Black-Scholes (B-S) model deals with this issue by assuming completeness of the financial market, that there are no arbitrage possibilities, among other assumptions that make the model an approximation to the real financial market. But if we can buy options in the market we need not bother with this for now.

The main idea of OBPI, as introduced by Leland and Rubinstein (1976) consists in buying q call options and investing the amount $F_0 = F_T e^{-rT}$ in Zero-Coupon (Z-C) bonds, that will insure the principal investment if the call expires out of the money. Therefore, the proportion q and exercise price K are such that:

$$V_0^{OBPI} = qCall(0, S_0) + F_T e^{-rT}, \tag{2.6a}$$

$$F_T = qK, \tag{2.6b}$$

[2]Contingent claim is a financial asset (claim) whose future payoff depends (contingent) on the uncertain value of another risky asset.

[3]More accurately, an investor may want to take into account the present value of K, but let us simplify.

[4]It might seem that a 20 profit is not very attractive comparing to the $150 - 100 = 50$ profit of the simple buy-and-hold strategy. However, one should note that in this case the leverage effect of the option (200%) is largely superior than of the buy-and-hold approach (50%), so if the buyer had purchased 10 calls on the same stock, he would pay 100 and profit 200.

where $F_T = \eta V_0^{OBPI}$ - recall eq.(2.2) - and thus, both can be found numerically. Since the listed call + bond OBPI is a static PI strategy (table 2.1), once we have all parameters set to begin the strategy, no further calculations are required until $t = T$ where the exercise of the contract takes place and the value of the portfolio is given by

$$V_T^{OBPI} = q\max[(S_T - K), 0] + F_T = \begin{cases} qS_T & \text{if } S_T > K \\ qK & \text{if } S_T < K \end{cases} \tag{2.7}$$

As it was mentioned before, listed options are usually unreachable to most investors. An alternative solution to this problem is to replicate (hedge) options by creating a dynamic PI strategy consisting of risky-assets and bonds only. In other words, we want (ν^B, ν^S) such that it matches the performance of an OBPI strategy for every $t \in [0, T[$. Again, since we consider only the simple case of European call option, the model can deliver a closed-form price function for this type of option.

In the B-S framework asset B is also considered a continuously compound Z-C bond at risk-free interest rate r and S_t follows a geometric Brownian motion (GBM), i.e.

$$dB_t = rB_t dt \qquad \Rightarrow \qquad B_t = B_T e^{-r(T-t)}, \tag{2.8a}$$

$$dS_t = S_t(\mu dt + \sigma dW_t) \qquad \Rightarrow \qquad S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}, \tag{2.8b}$$

where $(W_t)_{t\in[0,T]}$ is a Brownian motion and $\mu > r \geq 0$ and $\sigma > 0$ are commonly referred to as the *drift* and *volatility parameters*, respectively. With this model, the call function can be obtained by introducing the contract function of eq.(2.5) as a boundary condition to the B-S partial differential equation, yielding the following solution (for derivation of the B-S equation see, e.g., Hull (2008)):

$$Call(t, S_t) = S_t\,\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2), \tag{2.9}$$

where $\mathcal{N}(.) \equiv \mathcal{N}(0, 1; .)$ is the cumulative distribution function for the standard normal distribution and

$$d_1 \equiv d_1(t, S_t) = \frac{1}{\sigma\sqrt{T-t}}\left[\log\frac{S_t}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right] \tag{2.10a}$$

$$d_2 \equiv d_2(t, S_t) = d_1 - \sigma\sqrt{T-t}. \tag{2.10b}$$

Hence, the replicated portfolio is given in terms of asset numbers by $\nu_t^B = q[1 - \mathcal{N}(d_2)]$ and $\nu_t^S = q\mathcal{N}(d_1)$ for $t \in [0, T[$ and for $t = T$, eq.(2.7) also holds.[5] Note that this

[5]Parallelly, the put-call parity relation $Put(t, S_t) + S_t = Call(t, S_t) + Ke^{-r(T-t)}$ tells us that there are many other ways to build an OBPI strategy, namely to replicate a call based PI strategy using

strategy is model dependent, as it is necessary to estimate a proper value for σ, usually based on historical data of the stock or index. Fig.2.1 illustrates a synthetic OBPI path.

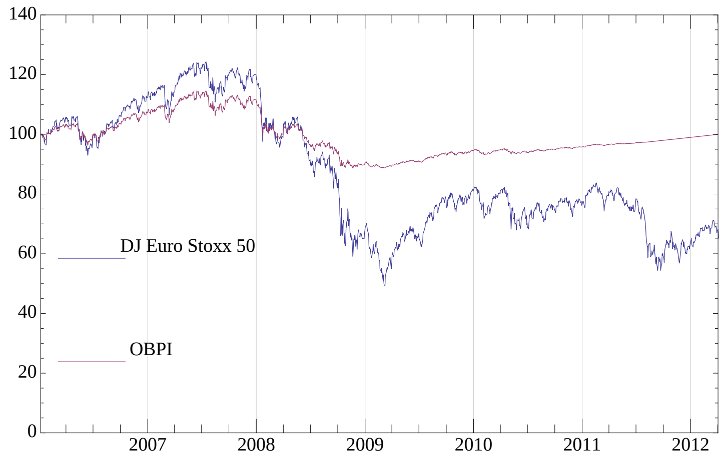


Figure 2.1: Synthetic OBPI strategy applied over DJ Euro Stoxx 50 index: $V_0 = 100$; $r = 4\%$; $\eta = 100\%$.

2.3 Constant Proportion Portfolio Insurance (CPPI)

The Constant Proportion Portfolio Insurance is a dynamic asset allocation strategy witch leverages the participation in the risky asset movements and simultaneously ensures the purchase of risk-free bond that matures at the predefined floor value F_T required by the investor.

In general, the literature refers to the amount invested in the risky asset as the *exposure* of the portfolio V to the risky asset S and we shall denote it by $(E_t)_{t\in[0,T]}$. The amount invested in the risk-free asset B is simply the rest of the portfolio $V-E$. Thus, we can verify that the relation $V = E+(V-E)$ obviously holds, which becomes much more enlightening if we rewrite it as $V = \frac{E}{S}S + \frac{V-E}{B}B$ because we now can compare it with eq.(2.1). In fact, we can see that $\nu^S = \frac{E}{S}$ and $\nu^B = \frac{V-E}{B}$. In the case of the CPPI strategy, the exposure is given by the difference between the portfolio value and the floor, namely the cushion C, leveraged by a constant m (typically $1 \geq m \geq 5$), i.e

$$E_t = mC_t = m(V_t^{cppi} - F_t). \tag{2.11}$$

As an example, let us suppose at $t = 0$ an investor invests 100 in this CPPI strategy with $m = 3$ and with full guarantee ($\eta = 100\%$), i.e $F_T = 100$. Taking $r = 4\%$ and

put options. We focus, however, only on the PI strategy with the call option.

$T = 5$ years we have $F_0 = 100e^{-0.04\times5} \approx 81.87$ and $C_0 = 100-81.87 = 18.13$. Therefore, initially we have $E_0 = 3C_0 = 54.39$ invested in the risky asset and the rest $100 - E_0 = 45.61$ in bonds. Now the evolution of both S and B assets play its role. Assume, to simplify, that the next trading day takes place exactly a year after ($t = 1$) and S has risen 10%. We have $F_1 = 85.21$ and hence $V_1 = 54.39(1 + 10\%) + 45.61\frac{F_1}{F_0} = 107.3$, because B evolves at the same rate of F_t, r. Therefore, $C_1 = 107.3 - 85.21 = 22.09$, $E_1 = 66.27$ and $V_1 - E_1 = 41.03$. An example of CPPI 3 applied to the DJ Euro Stoxx index index is shown at three different dates in Fig. 2.2. We can see in this case that the exposure proportion becomes very short which mean the investment ended practically cash-locked.

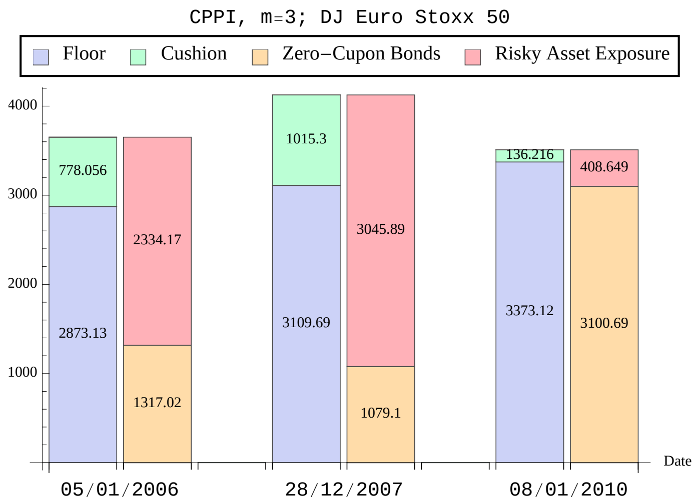


Figure 2.2: Bar chart of CPPI 3 structure at three different dates. Underlying Asset: DJ Euro Stoxx 50 index.

If we are considering continuous-time trade, the pair (ν^S, ν^B) defined in the second paragraph of section 2.3 can be introduced in eq.(2.1), yielding

$$dV_t = \frac{E_t}{S_t}dS_t + \frac{V_t - E_t}{B_t}dB_t. \tag{2.12}$$

Again, the model for both risky and risk-free assets in eq. 2.8 is useful to formulate a continuous-time closed-form solution of a CPPI path. Taking this model, the CPPI exposure definition in eq.(2.11) and introducing both in eq.(2.12) we obtain (after some simple algebra - see Appendix A), a log-normal type stochastic differential equation for the cushion. Hence, the CPPI value at any given instant follows immediately by $V_t = C_t + F_t$:

$$V_t^{cppi} = V_0^{cppi}\left[\eta e^{-r(T-t)} + (1 - \eta e^{-rT})e^{\lambda t}\left(\frac{S_t}{S_0}\right)^m\right], \tag{2.13}$$

where $\lambda = (1-m)(r + m\frac{\sigma^2}{2})$ and η is still as defined in eq.(2.2) (see e.g. Cont and Tankov (2009)). However, a continuous-time based CPPI is not representative of the discrete trading day reality. The discrete rebalancing algorithm which is applied in real cases is given in the following chapter.

Chapter 3
Methodology

In this thesis we propose to study how the different portfolio insurance strategies vary with the value of the stock at maturity T. In other words, we seek to simulate N stock paths S with initial value S_0 all ending at the same fixed S_T. In particular, this analysis is suggested as a way to analyze path-dependencies of PI, specifically CPPI 3 and 5 strategies. We model stock trajectories using geometric Brownian motion for its wide usage in the literature. However in this case the geometric Brownian motion paths are tied to S_T at maturity.

3.1 Gaussian Processes For Machine Learning Regression

To generate the conditioned GBM paths, we use gaussian processes for machine learning regression (GPR), which is given by Rasmussen and Williams (2005). Following this work, applications to different stochastic processes are provided by Sousa et al. (2012), in particular for the GBM. The GBM follows a lognormal distribution, which means its logarithm is a gaussian process. Therefore, we generate a process y_t which is a Brownian Motion with drift and conditioned to y_n (Brownian Bridge) and obtain the GBM by exponentiation i.e. $S_t = S_0 e^{y_t}$. Furthermore, this choice is motivated by the fact that this is essentially a vectorial procedure, a desirable aspect because the code was implemented in Mathematica which is a favorable language for matrix operations.

In the general case, the purpose of GPR is to obtain the non-linear regression function $y = f(\vec{x})$ that maps the data $(\mathbf{X}, \vec{y})$ called the training set, assuming a specific prior gaussian process, i.e $\mathcal{GP} \sim \mathcal{N}(m(\vec{x}, cov(\vec{x}_1, \vec{x}_2))$. The matrix $\mathbf{X}$ gathers the n vectors $\vec{x}_i = x_i^1, \ldots, x_i^d$ which contain the d parameters that originate the corresponding n observations $y_i = f(\vec{x}_i)$ with $i = 0, \ldots, n$. In the present case however, this setting is much more simplified because $\vec{x} = t$ and the training set reduces to the single observation $(t_n = T, y_n = \log \frac{S_T}{S_0})$. The remaining time steps $t_0, t_1, \ldots, t_{n-1}$ are collected in the vector $\mathbf{t}^*$ called the test set and represent the instants where $y_i^* = f(t_i^*)$, $i = 0, \ldots, n-1$ was not observed.[1] The regression process is also gaussian and it is obtained by the mean and covariance functions of the process defined by all the trajectories of the prior process that passes through the training set. Since the process is gaussian,

[1]The arrow representation was only used to represent the general case of a vector with $i = 1, \ldots, d$ different parameters. In this case we use only one parameter, t. Different points $k = 0, \ldots, n$ are collected in vectors represented by bold font, e.g. $\mathbf{t} = (t_0, t_1, \ldots, t_n)$

we have

$$\begin{bmatrix} y_n \\ \mathbf{y}^* \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} m(T) \\ \mathbf{m}^* \end{bmatrix}, \begin{bmatrix} cov(T,T) & \mathbf{cov}^{*\top} \\ \mathbf{cov}^* & \mathbf{cov}^{**} \end{bmatrix} \right) \tag{3.1}$$

where $\mathbf{m}^* = \big(m(0), m(t_1), \dots, m(t_{n-1})\big)$, $\mathbf{cov}^* = \big(cov(0,T), cov(t_1,T), \dots, cov(t_{n-1},T)\big)$ and the matrix elements $(cov^{**})_{ij} = cov(t_i^*, t_j^*)$, with $i, j = 0, 1, \dots, n-1$. The conditional distribution is given by

$$p(\mathbf{y}^* \mid \mathbf{t}^*, T, y_n) \sim \mathcal{N}\left(\mathbf{m}^* + \frac{y_n - m(T)}{cov(T,T)} \mathbf{cov}^*, \mathbf{cov}^{**} - \frac{1}{cov(T,T)} \mathbf{cov}^* \mathbf{cov}^{*\top} \right), \tag{3.2}$$

where one should note that $\mathbf{cov}^*\mathbf{cov}^{*\top}$ must be read as an outer product resulting in a $n \times n$ matrix with elements $cov(t_i, T) \cdot cov(t_j, T)$, $i, j = 0, \dots, n-1$. The mean and covariance of this process are used to build respectively the regression function and regression confidence, by extending to the whole $\mathbf{t}$ set. Therefore, the posterior process on the data has the following mean and covariance functions

$$m_{\mathcal{D}}(t) = m(t) + \frac{1}{cov(T,T)} cov(t,T)(y_n - m(T)), \tag{3.3a}$$

$$cov_{\mathcal{D}}(s,t) = cov(s,t) - \frac{1}{cov(T,T)} cov(s,T) cov(t,T), \tag{3.3b}$$

Hence, using eqs. (3.3a) we can simulate any path of a gaussian process with mean m and covariance cov that passes through (in this case end at) (T, y_n). In our particular framework, we deal with a Brownian motion with mean and covariance given by

$$m(t) = \left(\mu - \frac{\sigma^2}{2}\right) t, \tag{3.4a}$$

$$cov(s,t) = \sigma^2 \min(s,t), \tag{3.4b}$$

where μ and σ are again the drift and volatility, respectively. The imposition of the training set $(T, \log \frac{S_T}{S_0})$ will particularize equations 3.3. Noting that $cov(t_i, T) = \sigma^2 t_i$, $\forall_{i=0,\dots,n}$, eq. 3.3a simplifies significantly to

$$m_{\mathcal{D}}(t) = \left(\mu - \frac{\sigma^2}{2}\right) t + \frac{\sigma^2 t}{\sigma^2 T} \left[\log \frac{S_T}{S_0} - \left(\mu - \frac{\sigma^2}{2}\right) T \right] = \frac{t}{T} \log \frac{S_T}{S_0}. \tag{3.5}$$

This result as the important meaning that $m_{\mathcal{D}}(t)$ does not depend on μ, which also means that the GBM tied to one point gives place to the natural reparametrization of μ by $\tilde{\mu} = \frac{1}{T} \log \frac{S_T}{S_0} = \log \left[\left(\frac{S_T}{S_0} \right)^{1/T} \right]$. We should in fact expect this result, because the risky asset is assumed log-normal which means that the annualized return will be

$e^{\tilde{\mu}} = \left(\frac{S_T}{S_0}\right)^{1/T}$. Additionally eq. 3.3b can also be reduced to

$$cov_{\mathcal{D}}(s,t) = \sigma^2(\min(s,t) - st) = \sigma^2 \begin{cases} s - st & , s \leq t \\ t - st & , s > t \end{cases} \tag{3.6}$$

Now the Brownian bridge path can be obtained by (see Glasserman (2003))

$$\mathbf{B} = \mathbf{m}_{\mathcal{D}} + \mathbf{CZ}, \tag{3.7}$$

where $\mathbf{C}$ is the Cholesky decomposition of $\mathbf{cov}_{\mathcal{D}}$, i.e. the covariance matrix whose elements are given by eq. (3.6), and $\mathbf{Z}$ is a vector of the standard normally distributed $\mathcal{N}(0, I)$ random numbers. Finally, the GBM paths are simulated by exponentiation of the Brownian bridge process, i.e $S_t = S_0 e^{B_t}$ so that $S_T = S_0 e^{\log S_T/S_0}$.

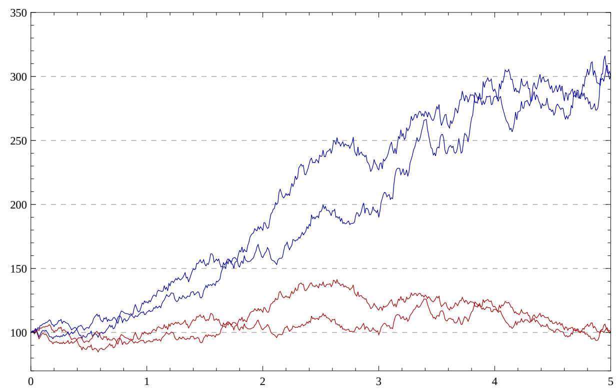


Figure 3.1: Two geometric Brownian motion paths conditioned to $S_T = 100$ (red) and $S_T = 300$ (blue), simulated with gaussian processes for machine learning regression. $T = 5$ and $\Delta t = 1/100$.

3.2 CPPI Implementation

We now proceed with an intuitive approach to CPPI in a discrete-time basis, making it more identifiable with the real world. For a more formal approach see Brandl (2009). Contrary to the continuous case, in the 'real world' traders are restricted to the official discrete trading days. Therefore, one must be prudent by choosing a proper multiplier, as the strategy can only insure that $V_t \geq F_t$ for a limited drop in the market between two consecutive trading days. To the risk of the stock dropping at a rate greater than the threshold we call it gap risk. A smaller period between the trading days reduces path-dependency and gap risk.

Let us then consider the simplest case of a partition of the time interval $[0, T]$ consisting of $n + 1$ equidistant t_k time steps, i.e $t_0 = 0 < t_1 < \ldots < t_n = T$ and

$t_{k+1} - t_k = T/n \equiv \Delta t, \forall_{k=0,\ldots,n}$. Now from the self financing condition in discrete form (see eq.(A.3)), eq.(2.12) can be rewritten in terms of t_k, i.e $\Delta V_{k+1} \equiv V_{k+1} - V_k$ is given by (to ease the notation let $x_{t_k} \equiv x_k$):

$$\Delta V_{k+1} = \frac{E_k}{S_k}\Delta S_{k+1} + \frac{V_k - E_k}{B_k}\Delta B_{k+1}. \tag{3.8}$$

As we consider the non-existence of short-selling, the CPPI exposure has to be defined with an inferior barrier of zero, i.e

$$E_k = \max[mC_k, 0] = \begin{cases} m(V_k^{cppi} - F_k) & \text{if } V_k^{cppi} \geq F_k \\ 0 & \text{if } V_k^{cppi} < F_k. \end{cases} \tag{3.9}$$

Note that in the continuous case, the null branch is not necessary because continuous rebalancing makes sure that $V_t \geq F_t, \forall t \in [0, T]$. Hence we see that the assets' numbers are $\nu_k^S = \frac{\max[mC_k,0]}{S_k}$ and $\nu_k^B = \frac{V_k - \max[mC_k,0]}{B_k}$ and so the portfolio value is given by

$$V_{k+1}^{cppi} = \begin{cases} m\left(V_k^{cppi} - F_k\right)\dfrac{S_{k+1}}{S_k} + \left(V_k^{cppi}(1-m) + mF_k\right)\dfrac{B_{k+1}}{B_k} & \text{if } V_k^{cppi} \geq F_k \\ V_k^{cppi}\dfrac{B_{k+1}}{B_k} & \text{if } V_k^{cppi} < F_k, \end{cases} \tag{3.10}$$

Thus, given the inputs V_0, η, r, T and m we obtain F_0 and E_0; with $\frac{B_k}{B_{k-1}} = \frac{e^{-r(T-t_k)}}{e^{-r(T-t_{k-1})}} = e^{r(t_k - t_{k-1})} = e^{rT/n}$ we are given $F_k = \frac{B_k}{B_{k-1}}F_{k-1}$; and at last given S_k we have all that is necessary to know the following V_k, $(k = 1, \ldots, n)$ by the recursion expression in eq.(3.10). But if we want to track the evolution of each the cushion, the exposure and the Z-C bond parts, then we can create the following loop:

Algorithm 1 CPPI algorithm pseudo-code.

for $k = 1 \rightarrow n$ **do**
  Evaluate V_k from eq.(3.10);
  Calculate $F_k = \eta V_0 e^{r(T-t_k)}$ and C_k with the condition:
  if $V_k \geq F_k$ **then**
    $C_k = V_k - F_k$;
  else
    $C_k = 0$;
  end if
  Calculate risky asset exposure $E_k = mC_k$ and the rest $V_k - E_k$ on risk-free asset;
end for

In other words, in every time step t_{k+1}, CPPI algorithm invests the previous V_k, allocates $E_k = m(V_k - F_k) \geq 0$ in S and $V_k - E_k$ in B, and obtains V_{k+1} by the stochastic variations of S and the known growth of B. Figure 3.2 shows an application of the CPPI strategies on a world stock index, where we can again observe cash-lock occurrences for CPPI 3 and 5.

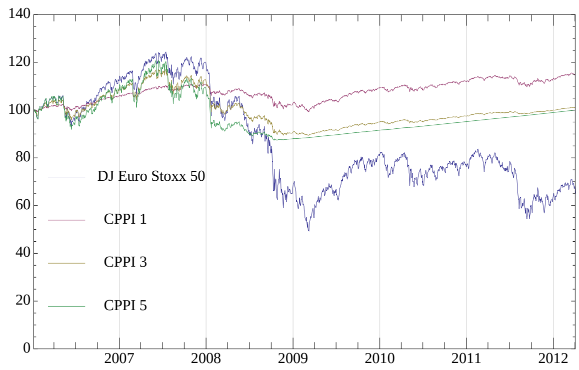


Figure 3.2: CPPI 1, 3 and 5 applied over DJ Euro Stoxx 50 index. $V_0 = 100$; $r = 4\%$; $\eta = 100\%$.

3.3 Parameters and Performance Measures

In this thesis all algorithms to generate the stock values and the corresponding portfolio insurance strategies were implemented in Mathematica. We compare tree of the most proliferated PI strategies: CPPI, SLPI and OBPI. We separate the CPPI strategy according to its multiplier values $m = 1, 3, 5$ and treat them as different strategies to be confronted. In fact, CPPI 1 is essentially different from the other two for its path-independent characteristic. It can be thought of as the simple strategy of buying the risky asset plus the risk-free asset separately, so that at each point in time, the portfolio value is always the sum of both assets till it matures, regardless of the path taken.

The simulations count on two types of parameters to implement: the procedural which will be fixed for every simulation and the scenario parameters which will assume different values that will recreate different scenarios. Among the first group are the initial portfolio investment $V_0 = 100$, the rebalancing frequency, i.e, constant time increments are $\Delta t = 1/100$, which can be thought as the trading days being separated periodically by $(1, 2, 2)$ days, the number of time steps is $n = T/\Delta t$, the number of paths / simulations $N = 10000$ (as in Annaert et al. (2009)) and the risk-free interest rate is always $r = 4\%$. The choice of the latter is also among the general usage (e.g. 5% in Costa and Gaspar (2011), 3% and 4% in Cont and Tankov (2009)).

The scenario parameters will vary so that their impact on PI performances and distributions are analyzed. Following the literature these parameters and the respective reference values are the volatility of the stock, σ:$\{15\%, 40\%\}$, the percentage of the initial portfolio to be insured η:$\{100\%, 80\%\}$, the maturity of the investment T:$\{5, 15\}$ and the stock value at maturity S_T:$\{100, 150, 200, 250, 300\}$. The combination of these parameters result in 40 different cases. The main difference on this scenario setup with respect to other literature, is the fixation of S_T and the consequent substitution of μ (see 3.1). The present work also extends the scenarios used in Costa and Gaspar (2011) by introducing $T = 15$ to the analysis, since long maturities can be found in some PI products. Another particularity is the choice of S_T values all above $S_0 = 100$. The reason is due to the fact that for negative rates of return of the underlying risky asset, PI strategies will return only the guarantee as they end invested almost entirely on the risk-free asset. We are concerned to find which PI perform better on large positive market trends, i.e. potentiate most upside performance. We also choose S_T instead of the annualized return $\frac{1}{T}\log\frac{S_T}{S_0}$ because the latter varies with T and as we study two different maturity cases, it seems more intuitive to a certain investor to know exactly which value he invested in the PI and its payoff value. All the probability density distributions can be found in Appendix B as an aiding tool for comparison.

In order to analyze and confront the aforementioned PI strategies we have chosen two of the most significant statistical methods for comparison in literature. One is the study of the first four moments. The moments are often used in literature because they can easily be interpreted and much information can be withdrawn about the behavior of the payoff distributions (see e.g. Prigent and Bertrand (2003); Pezier and Scheller (2011) and Khuman et al. (2008) uses log-moments). We calculated the central moments with the built-in functions delivered by Mathematica and the respective results are given in the next chapter. Additionally, moment analysis such as the mean-variance rule is suited only for investors with quadratic expected utility functions, i.e., investors whose risk-aversion increases with growing wealth. As this restricts most of the investors' risk-profiles, we must consider other measures that the sake of consistency of the present analysis.

Hence, we also use Stochastic Dominance (SD) criteria because it incorporates a wider class of expected utility functions in the analysis. Stochastic Dominance was introduced by Quirk and Saposnik (1962) as a more general decision rule than the moment analysis and based on assumptions about the utility function of the investor. Subsequent studies introduced higher order stochastic dominances (second and third order by Hadar and Russell (1969); Whitmore (1970) respectively) restricting the set of utility functions. The general framework assumes investors are von Neumann-Morgenstern-rational and maximize expected utility (Linton et al., 2003). In this thesis we consider the three orders: first-order SD (FSD) on which is assumed that investors choose only the portfolio with the highest payoff, i.e have utility functions with positive first derivative (Biswas, 2012); second-order SD (SSD) that implies a concave utility function, meaning that risk aversion increases; and the third-order SD (TSD) which requires that investors have convex utility functions, i.e., are risk-seekers when their wealth grows.

Some studies also take into account the analysis of a variety of performance ratios. In the present work however we do not make this approach because in our particular framework the PI payoff distributions of path-independent vs path-dependent are completely different: path-independent strategies result in degenerate distributions while path-dependent are flat. Comparing both types in terms of ratios that have less into account the specific shapes of the distributions can be misleading (Annaert et al., 2009). Furthermore, SD criteria are founded on expected utility theory, which is more desirable for making conclusions based on the risk profiles of investor. The formal definition is given in the respective section on the following chapter.

Chapter 4
Results

4.1 Moment Analysis

This section is dedicated to the presentation of the first four moments, i.e, the mean, variance, skewness and kurtosis. Actually the second and fourth moments are adjusted to their most used and more easily interpretable forms: respectively the standard deviation which is the square root of the variance, and the excess kurtosis which is simply equal to kurtosis - 3. The latter adjustment takes advantage of the fact that normal distribution has 0 kurtosis, hence makes comparison more intuitive. All moments are obtained from the density distribution functions of the portfolio payoff at maturity, V_T, for each PI strategy in all scenarios. In the present section we only catalog the key results of each strategy for the different parameter cases and leave its discussion to the respective chapter.

Before we move on to the analysis of results, recall that CPPI 1 and OBPI are the only path-independent strategies chosen for the comparisons and that all simulated paths of the risky asset end at value S_T with probability 1. Therefore, CPPI 1 and OBPI distributions are degenerate with a single 100% weighted bar. The OBPI values were computed according to the BS model (recall in section 2.2 we use the Synthetic OBPI) while the CPPI 1 payoff at maturity is simply the sum of the insured amount ηV_0 plus the amount invested in the stock, i.e. $V_0 - \eta V_0 e^{-rT}$. One must also note that the different S_T values displayed on the left column can be associated with a Buy and Hold (B&H) strategy with initial investment of $S_0 = 100$. Hence we can compare directly the path-independent strategies with the simplest B&H strategy, but the path-dependent must be framed carefully with the other moments. In this regard we have $B_T = \{122.14, 182.21\}$ for $T = \{5, 15\}$ respectively. We begin the analysis with the mean values in Table 4.1.

CPPI 1 strategy mean values do not vary with volatility (path-independence). In general, CPPI 1 exhibits a slight improvement from $\eta = 100\%$ to $\eta = 80\%$ and longer maturity. It outperforms the B&H strategy in the ($S_T = 100$, $T = 5$) and ($S_T = \{100, 150\}$, $T = 15$). Moreover, CPPI 1 has a better performance than the OBPI strategy for high volatility and long maturity, but also for $S_T \leq 150$ when ($\sigma = 15\%$, $T = 15$) and ($\sigma = 45\%$, $T = 5$). **CPPI 3** is highly dependent on σ which is due to its path-dependency. The low mean values for $\sigma = 40\%$ suggest high cash-lock occurrences. While in both volatility cases those occurrences may obviously decrease under higher S_T realizations, only for $\sigma = 15\%$ we can see possible cases of CPPI 3 performing better than B&H for ($S_T = 300$, $\eta = 100\%$) and ($S_T \geq 200$, $\eta = 80\%$). For the **CPPI 5** strategy the means also show an extreme dependence on the volatility and maturity as cash-lock events may happen for almost every simulation for $\sigma = 40\%$ and $T = 15$. However, for $\sigma = 15\%$ this strategy can outperform B&H not only for ($S_T = 300$, $\eta = 100\%$) but also in ($S_T \geq 200$, $\eta = 80\%$) cases. As for the **OBPI** we can see that even though it is a path-independent strategy the mean values decrease with volatility because the synthetic OBPI is model dependent on which σ is a required parameter. Therefore we verify that there in any case OBPI outperforms the B&H, but in low volatile markets it is close to B&H and does perform better than CPPI 1 in the majority of S_T values. Finally, the **SLPI** mean values are very similar to the OBPI, with exception of some cases of $\sigma = 40\%$, but the two strategies are different in respect to path-dependency.

Table 4.1: Mean of the PI payoff distributions. $V_0 = 100$

η = 100%											
		σ = 15%					σ = 40%				
	S_T	CPPI1	CPPI3	CPPI5	OBPI	SLPI	CPPI1	CPPI3	CPPI5	OBPI	SLPI
T=5	100	118.1	108.7	102.6	100.0	100.0	118.1	101.1	100.0	100.0	100.0
	150	127.2	129.3	120.1	139.7	139.1	127.2	103.7	100.0	100.0	110.6
	200	136.3	169.4	184.8	186.3	192.4	136.3	108.7	100.1	131.5	132.8
	250	145.3	235.4	358.4	232.8	245.4	145.3	117.1	100.2	164.3	161.4
	300	154.4	333.7	741.6	279.4	296.9	154.4	129.6	100.6	197.2	193.6
T=15	100	145.1	104.9	100.1	100.0	100.0	145.1	100.0	100.0	100.0	100.0
	150	167.7	116.6	101.0	144.5	138.7	167.7	100.0	100.0	108.9	109.9
	200	190.2	139.5	104.4	192.7	192.1	190.2	100.1	100.0	145.2	130.6
	250	212.8	177.2	113.6	240.9	244.5	212.8	100.1	100.0	181.5	157.2
	300	235.4	233.4	133.9	289.1	296.5	235.4	100.3	100.0	217.8	187.1
η = 80%											
		σ = 15%					σ = 40%				
	S_T	CPPI1	CPPI3	CPPI5	OBPI	SLPI	CPPI1	CPPI3	CPPI5	OBPI	SLPI
T=5	100	114.5	96.49	84.99	98.56	96.64	114.5	82.04	80.00	82.06	84.83
	150	131.8	135.7	118.2	147.8	149.5	131.8	86.97	80.03	123.1	116.2
	200	149.0	212.0	241.3	197.1	199.9	149.0	96.61	80.14	164.1	158.2
	250	166.3	337.7	571.9	246.4	250.0	166.3	112.6	80.43	205.1	203.5
	300	183.5	524.8	1301.	295.7	300.0	183.5	136.4	81.10	246.2	250.3
T=15	100	136.1	86.12	80.17	98.62	93.58	136.1	80.01	80.00	81.41	83.11
	150	164.1	100.7	81.30	147.9	147.0	164.1	80.04	80.00	122.1	105.6
	200	192.2	129.1	85.51	197.2	198.8	192.2	80.09	80.00	162.8	137.9
	250	220.2	175.9	96.88	246.5	249.6	220.2	80.19	80.00	203.5	174.6
	300	248.3	245.8	122.1	295.9	299.8	248.3	80.32	80.00	244.2	213.7

In the following three moments, the analysis is reduced to the path-dependent strategies, CPPI 3 and 5. CPPI 1 and OBPI are obviously left aside because of their degeneracy. SLPI is also path-dependent, but in a different manner, because it has only two possible outcomes: B_T or S_T, whichever is the highest at maturity. This means that we do not need the higher order moments to interpret the characteristics of this strategy. All the information is on the probabilities of the two outcomes which are depicted in Table 4.2. Yet, we still deliver some observations about the skewness and kurtosis of SLPI.

Table 4.2: SLPI probabilities.

		η=100%				η=80%			
		σ=15%		σ=40%		σ=15%		σ=40%	
	ST	S_T	F_T	S_T	F_T	S_T	F_T	S_T	F_T
T=5	100	1.	1.	1.	1.	0.8319	0.1681	0.2414	0.7586
	150	0.7819	0.2181	0.211	0.789	0.9924	0.0076	0.5178	0.4822
	200	0.9243	0.0757	0.3277	0.6723	0.9993	0.0007	0.6518	0.3482
	250	0.969	0.031	0.4096	0.5904	0.9998	0.0002	0.7267	0.2733
	300	0.9843	0.0157	0.468	0.532	0.9999	0.0001	0.7739	0.2261
T=15	100	1.	1.	1.	1.	0.679	0.321	0.1553	0.8447
	150	0.7744	0.2256	0.1971	0.8029	0.9566	0.0434	0.3653	0.6347
	200	0.9214	0.0786	0.3063	0.6937	0.9903	0.0097	0.4823	0.5177
	250	0.9633	0.0367	0.3815	0.6185	0.9977	0.0023	0.5565	0.4435
	300	0.9823	0.0177	0.4354	0.5646	0.9989	0.0011	0.6077	0.3923

In Table 4.3 it is very clear that with higher volatility the standard deviations decrease. We see that this translates in more situations where the CPPI 3 and 5 strategies end up cash-locked, according to the means previously analyzed. Equivalently, in all cases the higher the floor (higher η) the same observation happens because there is more probability of cash-lock events. We note that both strategies suffer a decrease in the standard deviation for longer maturities which is enhanced by higher volatilities corroborating the idea that those conditions imply more cash-lock occurrences. We must also emphasize the fact that higher multipliers amplify the negative effect of longer maturities. For $S_T = 100$ we see that the SLPI distribution is obviously degenerate with only one possible outcome 100 because the final floor value coincides with S_T.

The skewness of a distribution measures its asymmetry with respect to the mean. Specifically, a negative or left-skewed distribution has a longer left tail whereas a distribution with a broader right tail has positive or right skewness. Hence zero-skewed strategies are symmetric. Investors tend to favor positively skewed payoffs, so an analysis merely based on mean and variance measures would overrate the strategies which reduce skewness. In Table 4.4 we can see that for CPPI 3 and 5, η does not influence skewness (not even kurtosis as can be seen further) but the increasing volatility makes distributions more positive-skewed. In addition higher S_T values give place to very small decreases in skewness while longer T gives more positive skewness. For the SLPI skewness along with the mean values show the bimodal aspect of the distribution.

Table 4.3: Standard Deviation of the PI payoff distributions. $V_0 = 100$

		η = 100%									
		σ = 15%					σ = 40%				
	S_T	CPPI1	CPPI3	CPPI5	OBPI	SLPI	CPPI1	CPPI3	CPPI5	OBPI	SLPI
T=5	100	0	0.1854	0.1892	0	0	0	0.1684	0.001230	0	0
	150	0	0.6242	1.438	0	20.65	0	0.5715	0.009846	0	20.40
	200	0	1.475	6.039	0	26.45	0	1.358	0.04289	0	46.94
	250	0	2.872	18.34	0	26.00	0	2.657	0.1340	0	73.77
	300	0	4.948	45.38	0	24.86	0	4.595	0.3393	0	99.80
T=15	100	0	0.1816	0.01701	0	0	0	0.002558	1.388×10^{-6}	0	0
	150	0	0.6135	0.1302	0	20.90	0	0.008712	6.225×10^{-10}	0	19.89
	200	0	1.454	0.5510	0	26.91	0	0.02078	2.735×10^{-9}	0	46.10
	250	0	2.840	1.686	0	28.21	0	0.04076	8.613×10^{-9}	0	72.87
	300	0	4.906	4.201	0	26.37	0	0.07068	2.198×10^{-8}	0	99.17
		η = 80%									
		σ = 15%					σ = 40%				
	S_T	CPPI1	CPPI3	CPPI5	OBPI	SLPI	CPPI1	CPPI3	CPPI5	OBPI	SLPI
T=5	100	0	0.3530	0.3601	0	7.479	0	0.3205	0.002341	0	8.559
	150	0	1.188	2.737	0	6.080	0	1.088	0.01874	0	34.98
	200	0	2.807	11.49	0	3.174	0	2.586	0.08164	0	57.17
	250	0	5.466	34.91	0	2.404	0	5.058	0.2550	0	75.76
	300	0	9.417	86.38	0	2.200	0	8.746	0.6457	0	92.03
T=15	100	0	0.2258	0.02114	0	9.338	0	0.003180	1.725×10^{-6}	0	7.244
	150	0	0.7627	0.1619	0	14.26	0	0.01083	7.739×10^{-10}	0	33.71
	200	0	1.808	0.6850	0	11.76	0	0.02583	3.400×10^{-9}	0	59.97
	250	0	3.531	2.096	0	8.144	0	0.05068	1.071×10^{-8}	0	84.46
	300	0	6.099	5.223	0	7.293	0	0.08788	2.732×10^{-8}	0	107.4

For $\sigma = 15\%$ it is always left skewed (with exception of $S_T = 100$) because there were more $V_T = S_T$ realizations than $V_T = B_T$ conferring an effective left tail to the distribution. For higher S_T values the left-skewness intensifies because there are less chances of triggering the stop-loss rule and therefore more weight is given on the right bar.

The exact interpretation of 'tailedness' and 'peakedness' of a distribution function provided by the kurtosis has been subject to wide discussion (and often confusion) over the past century (DeCarlo, 1997). Yet presently there is still room for presumptions that can give alternative measurements of a distributions peak sharpness and tail fatness, because different shaped distributions with equal kurtosis have been already found. However it is consensual that shape has to incorporate those two aspects (peak and tails). Therefore the kurtosis measurement basically assumes that the shoulders of a distribution are located at the mean plus (and minus) a standard deviation and scales the fourth moment to its variance. Another common meaning used for kurtosis is the 'departure from normality'. Hence, normal/mesokurtic distributions have excess kurtosis $\gamma_2 = 0$ (or 3 for kurtosis), $\gamma_2 > 0$ correspond to leptokurtic curves, i.e., with

Table 4.4: Skewness (third moment).

						η = 100%					
		σ = 15%					σ = 40%				
	S_T	CPPI1	CPPI3	CPPI5	OBPI	SLPI	CPPI1	CPPI3	CPPI5	OBPI	SLPI
T=5	100	Ind	−0.06993	0.08492	Ind	Ind	Ind	0.3413	1.797	Ind	Ind
	150	Ind	−0.07020	0.08331	Ind	−1.365	Ind	0.3393	1.777	Ind	1.417
	200	Ind	−0.07039	0.08217	Ind	−3.208	Ind	0.3379	1.763	Ind	0.7342
	250	Ind	−0.07053	0.08129	Ind	−5.412	Ind	0.3368	1.752	Ind	0.3677
	300	Ind	−0.07065	0.08058	Ind	−7.792	Ind	0.3359	1.743	Ind	0.1283
T=15	100	Ind	0.04592	0.3100	Ind	Ind	Ind	0.7670	−99.98	Ind	Ind
	150	Ind	0.04577	0.3091	Ind	−1.313	Ind	0.7657	4.696	Ind	1.523
	200	Ind	0.04567	0.3084	Ind	−3.132	Ind	0.7648	4.679	Ind	0.8404
	250	Ind	0.04559	0.3079	Ind	−4.928	Ind	0.7642	4.666	Ind	0.4879
	300	Ind	0.04552	0.3075	Ind	−7.315	Ind	0.7636	4.654	Ind	0.2606
						η = 80%					
		σ = 15%					σ = 40%				
	S_T	CPPI1	CPPI3	CPPI5	OBPI	SLPI	CPPI1	CPPI3	CPPI5	OBPI	SLPI
T=5	100	Ind	−0.06993	0.08492	Ind	−1.775	Ind	0.3413	1.797	Ind	1.209
	150	Ind	−0.07020	0.08331	Ind	−11.34	Ind	0.3393	1.777	Ind	−0.07125
	200	Ind	−0.07039	0.08217	Ind	−37.76	Ind	0.3379	1.763	Ind	−0.6373
	250	Ind	−0.07053	0.08129	Ind	−70.69	Ind	0.3368	1.752	Ind	−1.017
	300	Ind	−0.07065	0.08058	Ind	−99.98	Ind	0.3359	1.743	Ind	−1.310
T=15	100	Ind	0.04592	0.3100	Ind	−0.7668	Ind	0.7670	−99.98	Ind	1.903
	150	Ind	0.04577	0.3091	Ind	−4.482	Ind	0.7657	4.696	Ind	0.5595
	200	Ind	0.04567	0.3084	Ind	−10.01	Ind	0.7648	4.679	Ind	0.07084
	250	Ind	0.04559	0.3079	Ind	−20.78	Ind	0.7642	4.665	Ind	−0.2275
	300	Ind	0.04552	0.3075	Ind	−30.10	Ind	0.7636	4.654	Ind	−0.4412

sharp peak and fat tails, while platykurtic shapes measure $\gamma_2 < 0$, are flat at the peak and have short tails. This being said it can be observed in Table 4.5 the same independence on η as in the skewness values. CPPI 3 and 5 are always leptokurtic but almost normal for $\sigma = 15\%$ and CPPI 3 has still low positive kurtosis for $\sigma = 45\%$. However CPPI 5 bypasses positively the normal range for high $\sigma = 45\%$, but even more heavily when adding long maturities large γ_2. For the SLPI strategy again kurtosis shows a different behavior. In general, for low S_T the two possible outcome bars are more close and equitably distributed hence decreasing the absolute value of skewness and kurtosis. Has S_T rises, the left bar stays fixed and the right bard increasingly detaches from the other as it gains more weight simultaneously.

4.2 Stochastic Dominance

Consider two random variables V_1 and V_2, and their respective cumulative distribution functions (CDF), $F_1(x)$ and $F_2(x)$. Then, we say that V_1 ith order stochastically dominates V_2 if and only if $D_1^{(i)}(x) \leq D_2^{(i)}(x)$ $\forall x$ (with strict inequality for at least one x), where $D_k^{(i)} = \int_{-\infty}^{x} D_k^{(i-1)} dx$ and $D_k^{(1)} = F_k(x)$ (Davidson and Duclos, 2000; Annaert

Table 4.5: Excess Kurtosis (fourth moment −3).

η = 100%

		σ = 15%					σ = 40%				
	S_T	CPPI1	CPPI3	CPPI5	OBPI	SLPI	CPPI1	CPPI3	CPPI5	OBPI	SLPI
T=5	100	Ind	0.06056	0.06018	Ind	Ind	Ind	0.2492	6.003	Ind	Ind
	150	Ind	0.06063	0.05974	Ind	−0.1360	Ind	0.2468	5.866	Ind	0.006764
	200	Ind	0.06069	0.05943	Ind	8.292	Ind	0.2451	5.771	Ind	−1.461
	250	Ind	0.06073	0.05920	Ind	27.29	Ind	0.2438	5.699	Ind	−1.865
	300	Ind	0.06076	0.05902	Ind	58.71	Ind	0.2428	5.641	Ind	−1.984
T=15	100	Ind	0.02965	0.1870	Ind	Ind	Ind	1.073	9995.	Ind	Ind
	150	Ind	0.02963	0.1859	Ind	−0.2761	Ind	1.069	42.37	Ind	0.3191
	200	Ind	0.02962	0.1852	Ind	7.808	Ind	1.067	42.06	Ind	−1.294
	250	Ind	0.02961	0.1847	Ind	22.29	Ind	1.065	41.82	Ind	−1.762
	300	Ind	0.02960	0.1842	Ind	51.52	Ind	1.063	41.63	Ind	−1.932

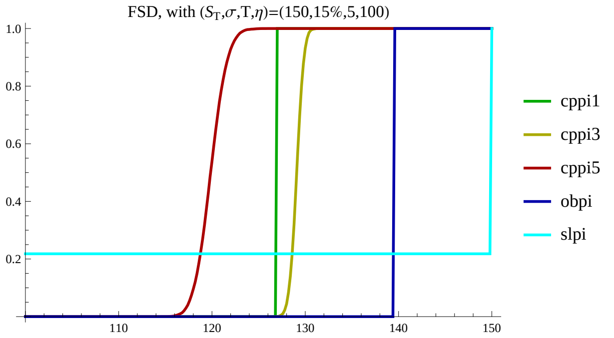
η = 80%

		σ = 15%					σ = 40%				
	S_T	CPPI1	CPPI3	CPPI5	OBPI	SLPI	CPPI1	CPPI3	CPPI5	OBPI	SLPI
T=5	100	Ind	0.06056	0.06018	Ind	1.151	Ind	0.2492	6.003	Ind	−0.5393
	150	Ind	0.06063	0.05974	Ind	126.6	Ind	0.2468	5.866	Ind	−1.995
	200	Ind	0.06069	0.05943	Ind	1424.	Ind	0.2451	5.771	Ind	−1.594
	250	Ind	0.06073	0.05920	Ind	4995.	Ind	0.2438	5.699	Ind	−0.9649
	300	Ind	0.06076	0.05902	Ind	9995.	Ind	0.2428	5.641	Ind	−0.2850
T=15	100	Ind	0.02965	0.1870	Ind	−1.412	Ind	1.073	9995.	Ind	1.623
	150	Ind	0.02963	0.1859	Ind	18.09	Ind	1.069	42.37	Ind	−1.687
	200	Ind	0.02962	0.1852	Ind	98.10	Ind	1.067	42.06	Ind	−1.995
	250	Ind	0.02961	0.1847	Ind	429.8	Ind	1.065	41.82	Ind	−1.948
	300	Ind	0.02960	0.1842	Ind	904.1	Ind	1.063	41.63	Ind	−1.805

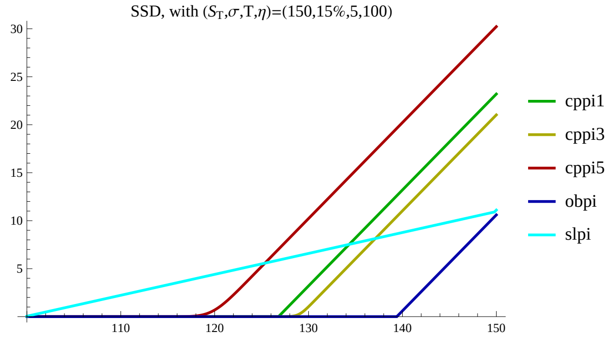
et al., 2009). We denote V_1 stochastically dominates V_2 on first (second and third) order by V_1 FSD (respectively SSD, TSD) V_2 (as in e.g. Levy and Wiener (1998)). Therefore, if the CDF of the two strategies intersect or are equal, there is no SD between them. Fig. 4.1 illustrates an example of three orders of stochastic dominance in a scenario described in the caption bellow.

Hence, the test is made in both directions because if V_1 does not SD V_2, it does not mean that V_2 SD V_1. Contrarily, it is obvious that if V_1 SD V_2 we know the reverse does not. Therefore this study organizes the stochastic dominance results so that no duplications arise. In addition, successive narrowing of the class of utility functions contemplated on higher order SD suggests that lower degree SD imply necessarily the SD on the subsequent orders, i.e., FSD⇒SSD⇒TSD. We now present the tests of stochastic dominance between the five proposed strategies for each one of the forty cases. Each column represents the stochastic dominance of the corresponding PI strategy over the remaining four strategies. The observations on (F, S and T)SD (see Tables (4.6, 4.7 and 4.8) correspondingly) are made separately but the higher the order, the less observations since the rest are resumed in the lower order SD.

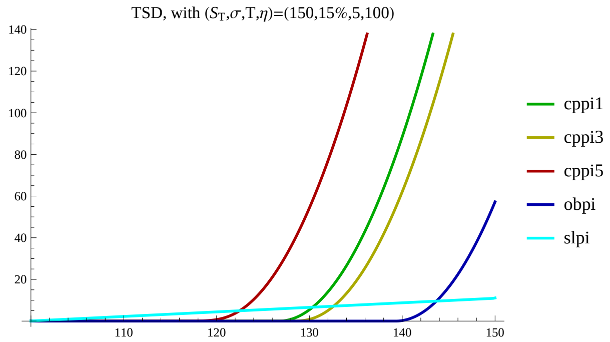
For first order SD, investors who are concerned simply with higher payoff,



(a) CDF



(b) $\int CDF$



(c) $\int\int CDF$

Figure 4.1: Cumulative distribution functions and its sums, depicting the first three orders of stochastic dominance. Scenario: $\{\sigma, T, \eta\} = \{15\%, 5, 100\%\}$.

prefer always **CPPI 1** to all other strategies for $S_T = 100$ in every scenario and for $(ST = 150,\ T = 15)$, confirming the mean analysis. It also FSD CPPI 3 and 5 in all scenarios except for $\sigma = 15\%$, $(S_T \geq 150,\ T = 5)$ and $(S_T = 300,\ T = 15)$. The choices of insurance percentage generally do not influence CPPI 1's dominance, existing only one exception, where the dominance over OBPI in $(S_T = 200,\ \sigma = 40\%,\ T = 5)$ and $\eta = 100\%$ is lost for $\eta = 80\%$. **CPPI 3** FSD all strategies except CPPI 1 for the lowest S_T and $\eta = 100\%$ in all volatility and maturity cases. It also FSD CPPI 5 in every scenario except for $(S_T \geq 250,\ \sigma = 15\%,\ T = 5)$ which are the only cases it dominates CPPI 1. **CPPI 5** FSD all strategies for $(S_T \geq 250,\ \sigma = 15\%,\ T = 5)$ for both floor choices. Also dominates on first order OBPI and SLPI for $S_T = 100$. **OBPI** dominates all strategies except SLPI for most cases where $S_T \geq 200$ except when CPPI 3 and 5 dominate. For $S_T \geq 150$ it also presents some dominance on low volatile markets. **SLPI** first order SD CPPI 3 and 5 for high volatility markets and long maturity.

In respect to second order stochastic dominance, the investors who are risk averse would choose **CPPI 1** over SLPI in some cases of high volatile markets, such as for $(S_T = 150,\ T = 5)$ and for $(S_T \geq 200,\ T = 15)$ for both η. CPPI 1 also dominates CPPI 3 on second order for $(S_T = 300,\ \sigma = 15\%,\ T = 15)$ for both insurance percentages as well. **OBPI** dominates CPPI 3 only in two very different cases: $(\sigma =$

40%, $T = 5$) and ($\sigma = 15\%$, $T = 15$) in both cases for $S_T = 100$ and $\eta = 80\%$.

Concerning third order stochastic dominance, the investor whose risk aversion decreases with growing wealth, chooses CPPI 1 over SLPI and OBPI in few cases of high volatility with long maturity, or low volatility with short maturity, but both cases for $S_T = \{200, 250\}$. CPPI 3 and 5 are also preferable to this investor than SLPI for some cases of low volatility and $T = 5$: the first strategy for ($S_T = 150$, $\eta = 80\%$) and ($S_T = 100$, $\eta = 80\%$), and the second for ($S_T = 200$, $\eta = 100\%$). Finally, OBPI also stochastically dominates on third order the SLPI strategy for $S_T \geq 150$ and ($\sigma = 15\%$, $T = 15$).

Table 4.6: First order stochastic dominance.

			η = 100% , σ = 15% , T = 5		
ST	**CPPI1**	**CPPI3**	**CPPI5**	**OBPI**	**SLPI**
100	All	cppi5,obpi,slpi	obpi,slpi	slpi	None
150	cppi5	cppi5	None	cppi1,cppi3,cppi5	None
200	None	cppi1	cppi1	cppi1,cppi3	None
250	None	cppi1	All	cppi1	None
300	None	cppi1,obpi,slpi	All	cppi1	None
			η = 100% , σ = 40% , T = 5		
ST	**CPPI1**	**CPPI3**	**CPPI5**	**OBPI**	**SLPI**
100	All	cppi5,obpi,slpi	obpi,slpi	slpi	None
150	cppi3,cppi5,obpi	cppi5,obpi	obpi	None	cppi5,obpi
200	cppi3,cppi5,obpi	cppi5	None	cppi3,cppi5	cppi5
250	cppi3,cppi5	cppi5	None	cppi1,cppi3,cppi5	cppi5
300	cppi3,cppi5	cppi5	None	cppi1,cppi3,cppi5	None
			η = 100% , σ = 15% , T = 15		
ST	**CPPI1**	**CPPI3**	**CPPI5**	**OBPI**	**SLPI**
100	All	cppi5,obpi,slpi	obpi,slpi	slpi	None
150	All	cppi5	None	cppi3,cppi5	None
200	cppi3,cppi5	cppi5	None	cppi1,cppi3,cppi5	None
250	cppi3,cppi5	cppi5	None	cppi1,cppi3,cppi5	None
300	cppi5	cppi5	None	cppi1,cppi3,cppi5	None
			η = 100% , σ = 40% , T = 15		
ST	**CPPI1**	**CPPI3**	**CPPI5**	**OBPI**	**SLPI**
100	All	cppi5,obpi,slpi	obpi,slpi	slpi	None
150	All	cppi5	None	cppi3,cppi5	cppi3,cppi5
200	cppi3,cppi5,obpi	cppi5	None	cppi3,cppi5	cppi3,cppi5
250	cppi3,cppi5,obpi	cppi5	None	cppi3,cppi5	cppi3,cppi5
300	cppi3,cppi5,obpi	cppi5	None	cppi3,cppi5	cppi3,cppi5
			η = 80% , σ = 15% , T = 5		
ST	**CPPI1**	**CPPI3**	**CPPI5**	**OBPI**	**SLPI**
100	All	cppi5	None	cppi3,cppi5	None
150	cppi5	cppi5	None	cppi1,cppi3,cppi5	None
200	None	cppi1,obpi,slpi	cppi1,obpi,slpi	cppi1	None
250	None	cppi1,obpi,slpi	All	cppi1	None
300	None	cppi1,obpi,slpi	All	cppi1	None
			η = 80% , σ = 40% , T = 5		
ST	**CPPI1**	**CPPI3**	**CPPI5**	**OBPI**	**SLPI**
100	All	cppi5	None	cppi5	cppi5
150	cppi3,cppi5,obpi	cppi5	None	cppi3,cppi5	cppi5
200	cppi3,cppi5	cppi5	None	cppi1,cppi3,cppi5	cppi5
250	cppi3,cppi5	cppi5	None	cppi1,cppi3,cppi5	cppi5
300	cppi3,cppi5	cppi5	None	cppi1,cppi3,cppi5	None
			η = 80% , σ = 15% , T = 15		
ST	**CPPI1**	**CPPI3**	**CPPI5**	**OBPI**	**SLPI**
100	All	cppi5	None	cppi3,cppi5	cppi5
150	All	cppi5	None	cppi3,cppi5	None
200	cppi3,cppi5	cppi5	None	cppi1,cppi3,cppi5	None
250	cppi3,cppi5	cppi5	None	cppi1,cppi3,cppi5	None
300	cppi5	cppi5	None	cppi1,cppi3,cppi5	None
			η = 80% , σ = 40% , T = 15		
ST	**CPPI1**	**CPPI3**	**CPPI5**	**OBPI**	**SLPI**
100	All	cppi5	None	cppi3,cppi5	cppi3,cppi5
150	All	cppi5	None	cppi3,cppi5	cppi3,cppi5
200	cppi3,cppi5,obpi	cppi5	None	cppi3,cppi5	cppi3,cppi5
250	cppi3,cppi5,obpi	cppi5	None	cppi3,cppi5	cppi3,cppi5
300	cppi3,cppi5,obpi	cppi5	None	cppi3,cppi5	cppi3,cppi5

Table 4.7: Second order stochastic dominance.

		η = 100% , σ = 15% , T = 5			
ST	**CPPI1**	**CPPI3**	**CPPI5**	**OBPI**	**SLPI**
100	All	cppi5,obpi,slpi	obpi,slpi	slpi	None
150	cppi5	cppi5	None	All	None
200	None	cppi1	cppi1	cppi1,cppi3,cppi5	None
250	None	cppi1	All	cppi1	None
300	None	cppi1,obpi,slpi	All	cppi1	None
		η = 100% , σ = 40% , T = 5			
ST	**CPPI1**	**CPPI3**	**CPPI5**	**OBPI**	**SLPI**
100	All	cppi5,obpi,slpi	obpi,slpi	slpi	None
150	All	cppi5,obpi	obpi	None	cppi5,obpi
200	All	cppi5	None	cppi3,cppi5	cppi5
250	cppi3,cppi5	cppi5	None	All	cppi5
300	cppi3,cppi5	cppi5	None	All	None
		η = 100% , σ = 15% , T = 15			
ST	**CPPI1**	**CPPI3**	**CPPI5**	**OBPI**	**SLPI**
100	All	cppi5,obpi,slpi	obpi,slpi	slpi	None
150	All	cppi5	None	cppi3,cppi5,slpi	None
200	cppi3,cppi5	cppi5	None	All	None
250	cppi3,cppi5	cppi5	None	cppi1,cppi3,cppi5	None
300	cppi3,cppi5	cppi5	None	cppi1,cppi3,cppi5	None
		η = 100% , σ = 40% , T = 15			
ST	**CPPI1**	**CPPI3**	**CPPI5**	**OBPI**	**SLPI**
100	All	cppi5,obpi,slpi	obpi,slpi	slpi	None
150	All	cppi5	None	cppi3,cppi5	cppi3,cppi5
200	All	cppi5	None	cppi3,cppi5,slpi	cppi3,cppi5
250	All	cppi5	None	cppi3,cppi5,slpi	cppi3,cppi5
300	All	cppi5	None	cppi3,cppi5,slpi	cppi3,cppi5
		η = 80% , σ = 15% , T = 5			
ST	**CPPI1**	**CPPI3**	**CPPI5**	**OBPI**	**SLPI**
100	All	cppi5	None	cppi3,cppi5,slpi	None
150	cppi5	cppi5	None	cppi1,cppi3,cppi5	None
200	None	cppi1,obpi,slpi	cppi1,obpi,slpi	cppi1	None
250	None	cppi1,obpi,slpi	All	cppi1	None
300	None	cppi1,obpi,slpi	All	cppi1	None
		η = 80% , σ = 40% , T = 5			
ST	**CPPI1**	**CPPI3**	**CPPI5**	**OBPI**	**SLPI**
100	All	cppi5	None	cppi3,cppi5	cppi5
150	All	cppi5	None	cppi3,cppi5,slpi	cppi5
200	cppi3,cppi5	cppi5	None	All	cppi5
250	cppi3,cppi5	cppi5	None	All	cppi5
300	cppi3,cppi5	cppi5	None	cppi1,cppi3,cppi5	None
		η = 80% , σ = 15% , T = 15			
ST	**CPPI1**	**CPPI3**	**CPPI5**	**OBPI**	**SLPI**
100	All	cppi5	None	cppi3,cppi5,slpi	cppi5
150	All	cppi5	None	cppi3,cppi5,slpi	None
200	cppi3,cppi5	cppi5	None	cppi1,cppi3,cppi5	None
250	cppi3,cppi5	cppi5	None	cppi1,cppi3,cppi5	None
300	cppi3,cppi5	cppi5	None	cppi1,cppi3,cppi5	None
		η = 80% , σ = 40% , T = 15			
ST	**CPPI1**	**CPPI3**	**CPPI5**	**OBPI**	**SLPI**
100	All	cppi5	None	cppi3,cppi5	cppi3,cppi5
150	All	cppi5	None	cppi3,cppi5,slpi	cppi3,cppi5
200	All	cppi5	None	cppi3,cppi5,slpi	cppi3,cppi5
250	All	cppi5	None	cppi3,cppi5,slpi	cppi3,cppi5
300	All	cppi5	None	cppi3,cppi5,slpi	cppi3,cppi5

Table 4.8: Third order stochastic dominance.

		η = 100% , σ = 15% , T = 5			
ST	**CPPI1**	**CPPI3**	**CPPI5**	**OBPI**	**SLPI**
100	All	cppi5,obpi,slpi	obpi,slpi	slpi	None
150	cppi5	cppi5,slpi	None	All	None
200	None	cppi1	cppi1,slpi	All	None
250	None	cppi1	All	cppi1	None
300	None	cppi1,obpi,slpi	All	cppi1	None
		η = 100% , σ = 40% , T = 5			
ST	**CPPI1**	**CPPI3**	**CPPI5**	**OBPI**	**SLPI**
100	All	cppi5,obpi,slpi	obpi,slpi	slpi	None
150	All	cppi5,obpi	obpi	None	cppi5,obpi
200	All	cppi5	None	cppi3,cppi5,slpi	cppi5
250	cppi3,cppi5,slpi	cppi5	None	All	cppi5
300	cppi3,cppi5	cppi5	None	All	None
		η = 100% , σ = 15% , T = 15			
ST	**CPPI1**	**CPPI3**	**CPPI5**	**OBPI**	**SLPI**
100	All	cppi5,obpi,slpi	obpi,slpi	slpi	None
150	All	cppi5	None	cppi3,cppi5,slpi	None
200	cppi3,cppi5,slpi	cppi5	None	All	None
250	cppi3,cppi5	cppi5	None	All	None
300	cppi3,cppi5	cppi5	None	All	None
		η = 100% , σ = 40% , T = 15			
ST	**CPPI1**	**CPPI3**	**CPPI5**	**OBPI**	**SLPI**
100	All	cppi5,obpi,slpi	obpi,slpi	slpi	None
150	All	cppi5	None	cppi3,cppi5,slpi	cppi3,cppi5
200	All	cppi5	None	cppi3,cppi5,slpi	cppi3,cppi5
250	All	cppi5	None	cppi3,cppi5,slpi	cppi3,cppi5
300	All	cppi5	None	cppi3,cppi5,slpi	cppi3,cppi5
		η = 80% , σ = 15% , T = 5			
ST	**CPPI1**	**CPPI3**	**CPPI5**	**OBPI**	**SLPI**
100	All	cppi5,slpi	None	cppi3,cppi5,slpi	None
150	cppi5	cppi5	None	All	None
200	None	cppi1,obpi,slpi	cppi1,obpi,slpi	cppi1	None
250	None	cppi1,obpi,slpi	All	cppi1	None
300	None	cppi1,obpi,slpi	All	cppi1	None
		η = 80% , σ = 40% , T = 5			
ST	**CPPI1**	**CPPI3**	**CPPI5**	**OBPI**	**SLPI**
100	All	cppi5	None	cppi3,cppi5	cppi5
150	All	cppi5	None	cppi3,cppi5,slpi	cppi5
200	cppi3,cppi5,slpi	cppi5	None	All	cppi5
250	cppi3,cppi5,slpi	cppi5	None	All	cppi5
300	cppi3,cppi5	cppi5	None	All	None
		η = 80% , σ = 15% , T = 15			
ST	**CPPI1**	**CPPI3**	**CPPI5**	**OBPI**	**SLPI**
100	All	cppi5	None	cppi3,cppi5,slpi	cppi5
150	All	cppi5	None	cppi3,cppi5,slpi	None
200	cppi3,cppi5,slpi	cppi5	None	All	None
250	cppi3,cppi5	cppi5	None	All	None
300	cppi3,cppi5	cppi5	None	All	None
		η = 80% , σ = 40% , T = 15			
ST	**CPPI1**	**CPPI3**	**CPPI5**	**OBPI**	**SLPI**
100	All	cppi5	None	cppi3,cppi5	cppi3,cppi5
150	All	cppi5	None	cppi3,cppi5,slpi	cppi3,cppi5
200	All	cppi5	None	cppi3,cppi5,slpi	cppi3,cppi5
250	All	cppi5	None	cppi3,cppi5,slpi	cppi3,cppi5
300	All	cppi5	None	cppi3,cppi5,slpi	cppi3,cppi5

Chapter 5
Discussion

We now come to the selection and discussion of the most important results presented in the previous chapter. Our results allow us to make some important conclusions about the path-(in)dependent behavior of each studied PI. Taking into consideration the setup for simulations which was carried out in this study, we must always bear in mind that the simulations highlight the path-dependent behavior of CPPI 3, 5 and SLPI in contrast with the certainty of the CPPI 1 and OBPI outcomes. For this reason, we separate the analysis making the comparison between the path-dependent strategies - CPPI 3 and 5 - and the path-independent strategies - CPPI 1 and OBPI. We must note that despite SLPI also being a path-dependent strategy, it is only so because of the two possible outcomes it can assume. Therefore, its distribution is very different than the distributions of the other path-dependent strategies. In this regard, we treat SLPI separately, because in some cases it can almost be path-independent, i.e., have one only possible outcome. Another consequence of simulating conditioned S_T, is that this study focuses only on high trend markets, because for negative returns, PI strategies return a value equal or insignificantly greater than the guarantee. In other words, taking the investors perspective, if we know that a stock will fall, we invest in a bond, a saving account, or simply do nothing. We are concerned to find in which cases cash-lock events occur for the CPPI 3 and 5, and which PI perform better under large positive market trends, i.e., assess how these strategies really potentiate upside performance.

5.1 Path-Dependent Strategies and Cash-Lock

The first issue we address is that path-dependent strategies exhibit high cash-lock occurrences. For example, on a 40% volatility market and maturity of 15 years, we can see that for every S_T value, the payoffs of the path-dependent strategies end up cash-locked almost 100% of the simulations. This can be observed by the mean almost coinciding with the floor value, at the same time that the standard deviation ranges from values of the order of 10^{-3} to 10^{-2} for the CPPI 3, and from 10^{-6} to 10^{-10} for the CPPI 5. In these cases the low values of skewness and kurtosis for the CPPI 3 indicate us the non-existence of significant outliers and thus, almost no exceptions. Despite the high leptokurtic shape of CPPI 5, cash-lock events are even more frequent given the extremely low standard deviations. The reason for such frequency of cash-lock events is because a longer maturity is equivalent to a longer path which *ceteris paribus* amplifies path-dependency. But mostly, it is due to the high volatility, which increases the probability of larger drops in the underlying risky asset. Still looking at $\sigma = 40\%$, we see that even for a 5year-maturity investment, the path-dependent strategies do not escape a large set of cash-lock events. This can also be observed by the mean values - also near the floor - and standard deviations ranging from orders of 10^{-1} to 10 for the CPPI 3 and 10^{-3} to 10^{-1} for the CPPI 5. The only scenario where the path-dependent strategies perform better than the others, is for the combination of low volatility, short maturity and high returns of the risky asset: $S_T > 200$ with a guarantee floor of 100%, where the inequality loses its strictness for $\eta = 80\%$.

The SLPI is a rather peculiar strategy under the present framework's perspective. This strategy resumes to a two outcome lottery: either one receives the insured amount, or wins the risky asset as if it has been fully invested on it. The obtained probabilities of each outcome and a comparison with the other PI mean values, tell us that SLPI is probably the best choice in 6 cases, all of which with low volatility: for long maturity - 80% guarantee and $S_T \geq 200$; 100% guarantee and $S_T \geq 250$ - and for short maturity - 100% guarantee and $S_T = 200$. We carefully use the word probably because it is not clear for instance that an investor will prefer a SLPI which has 98.23% probability of returning 300 with the remaining 1.77% chance of returning 100, as opposed to the OBPI strategy whose only possible outcome is 289.1%. This situation refers to the scenario of $T = 15$, $\sigma = 15\%$, $S_T = 300$ and $\eta = 100\%$.

5.2 Path-Independent Strategies

The study of the path-independent strategies is more direct in the present context. In general, the obtained moments show that the path-independent strategies are better suited for high volatile markets and longer maturities, regardless of the risky asset's payoffs. This is because they have less probability of being exaggeratedly invested on the risk-free asset. In particular, the CPPI 1 is better for moderate market increases and outperforms OBPI for a few cases of high volatility and long maturity. Conversely, the OBPI is a better choice than CPPI in some low volatile market scenarios.

5.3 Stochastic Dominance

So far we have identified in which scenarios path-dependent strategies are preferable than path-independent, and vice-versa on the perspective of the analysis of moments. However, in many cases, it is unclear only by the descriptive statistical analysis to grasp such conclusions. Therefore, we used stochastic dominance tests which take into account the whole cumulative distribution of the payoffs at maturity of two different strategies and provide an answer to whether an investor choses between those two strategies. Nevertheless, we see that the results of the stochastic dominant test confirm all the conclusions made with the analysis of moments. These results show in fact that investors who are simply interested in the higher payoffs, choose both CPPI 1 and OBPI over CPPI 3 and 5 strategies in almost all scenarios of high volatility. The same conclusions were also obtained for the dominance of the path-dependent strategies, which occurs only in low volatile markets, and short maturities. The SLPI exhibits dominance over CPPI 3 and 5 only on the combination of high volatility and long maturity. Between equally path-independent strategies, it becomes more clear with SD that in general CPPI 1 is chosen over OBPI for high volatile markets and longer maturities, while the opposite is observed for short maturity investments and low volatility. In addition, both dominate each other in different situations, CPPI 1 mainly for choices of $\eta = 100\%$ and OBPI for $\eta = 80\%$. As there have been many cases found of first order SD, few exceptions emerged for investors who can be both risk averse and decreasingly risk averse (second order SD), or for investors who have only the latter risk profile (third order). However, almost every second and third order of SD happen over SLPI.

Chapter 6
Conclusion

The present thesis addresses an important issue concerning path-dependent CPPI strategies which is extremely undesirable for investors and has not yet received an empirical study. This issue is usually called cash-lock and refers to the event of a PI becoming excessively invested in the risk-free asset, which transgresses a fundamental purpose of PI: allow participation in upside performance of the risky asset. Hence the question that arises is: *When and how often do these cash-lock events happen?*, which leads necessarily to an even more important question: *Taking into consideration the cash-lock issue, which PI should an investor choose?* In this work we provide an answer to both questions and emphasize the negative impact of this path-dependent behavior on PI performance.

To answer the aforementioned questions, we begin by acknowledging that if we simulate risky asset paths all conditioned to the same final value, we obtain a single outcome for a path-independent strategy, while a path-dependent gives rise to a distribution. Hence, the difference between both types of strategies is highlighted with this approach, which is not encountered in previous studies on this subject. To achieve this, we assumed the risky asset follows a geometric Brownian motion which is a Gaussian process and can thus be simulated and conditioned to a fixed final value using Gaussian Processes for Machine Learning regression.

The main finding of this thesis is that, in fact, cash-lock occurrences on the path-dependent CPPI 3 and 5 strategies happen very often and prohibit upside participation, even in cases where the risky asset triples at maturity. This is particularly patent on high volatile markets and for long maturities which is where the path-dependencies have more presence. Hence, under such market scenarios this undesirable risk makes the path-dependent strategies less attractive than the path-independent CPPI 1 and OBPI strategies. This conclusion is in consonance with previous studies and is corroborated with our analysis of the moments and stochastic dominance. However, the Buy and Hold strategy still remains a better choice for higher returns of the risky asset. Furthermore, in cases where volatility is low, the SLPI is almost identical to the Buy and Hold strategy. However, SLPI is more dependent on the risk profile of an investor and the stochastic dominance tests were not conclusive.

We conclude this thesis with our goal achieved: to answer the questions posed above, presenting a different approach for the analysis of PI strategies. We also find it contributes as a warning for investors who think of purchasing those products, which still need much improvement in the design process so that cash-lock risk is reduced.

We believe this topic alone has much more to be studied and discussed. In particular, there are other sources of path-dependency that can be introduced, e.g., borrowing constraints or different trading schedules. Such aspects can increase the cash-lock risk and thus we suggest further investigations on this matter, introducing more frictions on both market and PI levels. Additionally, a more realistic model for the risky asset such as with stochastic volatility should be introduced as we consider the GBM choice a limitation to this study. In this case however, conditioning paths with Gaussian processes for machine learning could prove to be a quite challenging task. We also consider that an analytic approach to the cash-lock behavior under discrete-time trading is an interesting matter of research.

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Appendix A
Auxiliary Calculations

Self-Financing Portfolios

Consider a portfolio p consisting of $(S^1, \dots, S^a)$ assets and the respective asset numbers $(\nu^1, \dots, \nu^a)$. To ease the notation let $x_{t_k} \equiv x_k$. Hence, the portfolio value at time t_k is given by (see e.g. Bjork (2009)):

$$V_k = \sum_{i=1}^{a} \nu_k^i S_k^i. \tag{A.1}$$

Now let us define in discrete-time the increment $\Delta V_k = V_k - V_{k-1}$. Then

$$\begin{aligned}
\Delta V_k =& \Delta\Big(\sum_{i=1}^{a} \nu^i S^i\Big)_k = \sum_{i=1}^{a} \Delta(\nu^i S^i)_k = \sum_{i=1}^{a} \big[\nu_k^i S_k^i - \nu_{k-1}^i S_{k-1}^i\big] = \\
=& \sum_{i=1}^{a} \big[(\nu_k^i - \nu_{k-1}^i) S_k^i + \nu_{k-1}^i (S_k^i - S_{k-1}^i)\big] = \\
=& \sum_{i=1}^{a} \big[\Delta\nu_k^i S_k^i + \nu_{k-1}^i \Delta S_k^i\big].
\end{aligned} \tag{A.2}$$

where the passage from the first line to the second line is done by adding and subtracting $\sum_{i=1}^{a} \nu_{k-1}^i S_k^i$. The self-financing condition means that all changes in the portfolio value are due to the assets changes exclusively, i.e. there is no external money inflow to or outflow from the portfolio. Mathematically this is the same as stating that $\nu_{k-1}^i S_k^i = \nu_k^i S_k^i$, i.e $\Delta\nu_k^i S_k^i = 0, \forall i = 0, \dots, a$ and thus (A.2) becomes:

$$\Delta V_k = \sum_{i=1}^{a} \nu_{k-1}^i \Delta S_k^i. \tag{A.3}$$

CPPI Cushion SDE

We obtain the cushion SDE simply by putting V_t and E_t in terms of C_t in eq.(2.12) and using the B-S choice for S_t and B_t in (2.8):

$$\begin{aligned}
dV_t^{cppi} &= \frac{E_t}{S_t}dS_t + \frac{V_t - E_t}{B_t}dB_t \\
&= mC_t\frac{dS_t}{S_t} + (F_t + C_t - mC_t)\frac{dB_t}{B_t} \\
&= mC_t(\mu dt + \sigma dW_t) + r(1-m)C_t dt + F_t r dt \\
&= C_t\Big([m\mu + (1-m)r]dt + m\sigma dW_t\Big) + dF_t \\
d(V_t^{cppi} - F_t) &= dC_t = C_t(a dt + b dW_t),
\end{aligned} \tag{A.4}$$

where $a = m\mu + (1-m)r$ and $b = m\sigma$. Hence, the cushion is a GBM with drift a and volatility b, and which solution is given by eq.(2.8b):

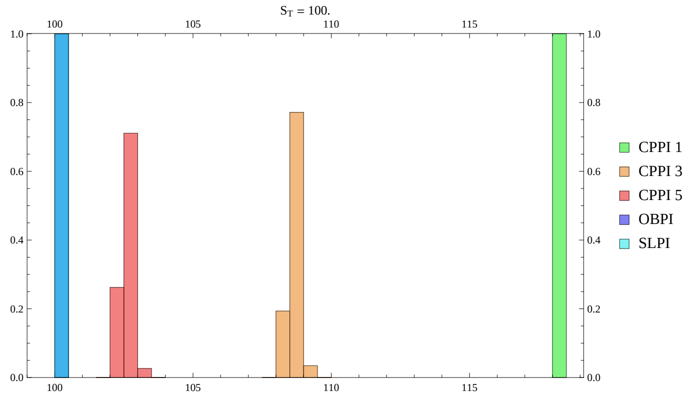
$$\begin{aligned}
C_t &= C_0 e^{[m\mu + (1-m)r - \frac{(m\sigma)^2}{2}]t + m\sigma W_t} \\
&= (V_0^{cppi} - F_0)e^{[(1-m)r - \frac{(m\sigma)^2}{2} + \frac{m\sigma^2}{2}]t + (m\mu - m\frac{\sigma^2}{2})t + m\sigma W_t} \\
&= V_0^{cppi}(1 - \eta e^{-rT})\left(\frac{S_t}{S_0}\right)^m e^{\lambda t}
\end{aligned} \tag{A.5}$$

where $\lambda = (1-m)r - (m - m^2)\frac{\sigma^2}{2} = (1-m)(r - m\frac{\sigma^2}{2})$. Now V_t comes immediately by $F_t + C_t$:

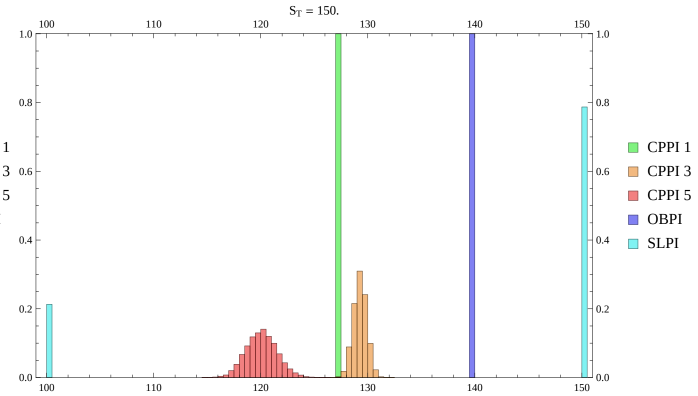
$$V_t^{cppi} = V_0^{cppi}\left[\eta e^{-r(T-t)} + (1 - \eta e^{-rT})\left(\frac{S_t}{S_0}\right)^m e^{\lambda t}\right]. \tag{A.6}$$

Appendix B

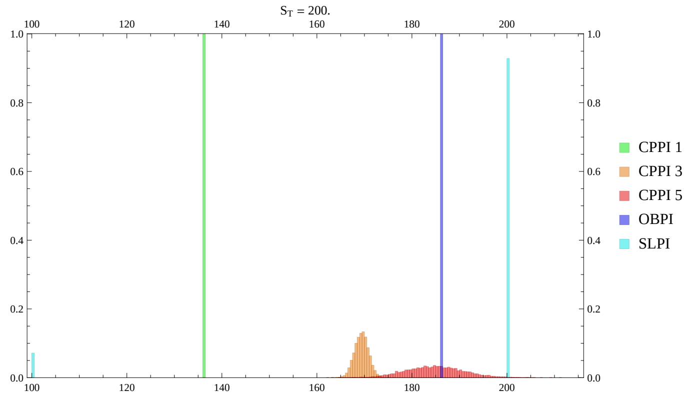
Probability Density Functions



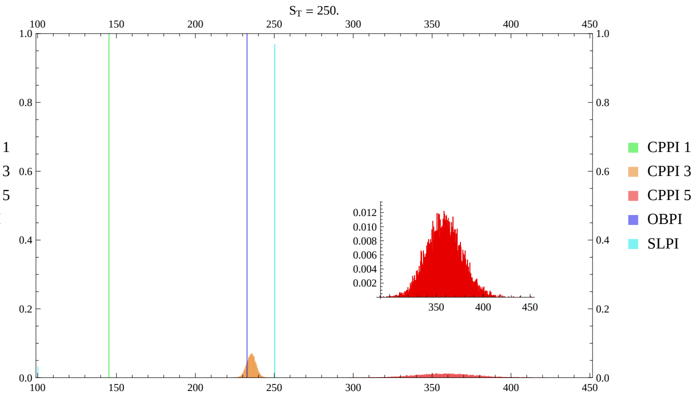
(a) $S_T = 100$



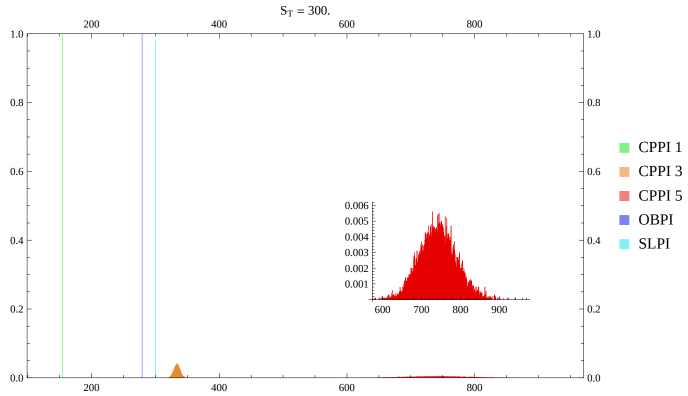
(b) $S_T = 150$



(c) $S_T = 200$

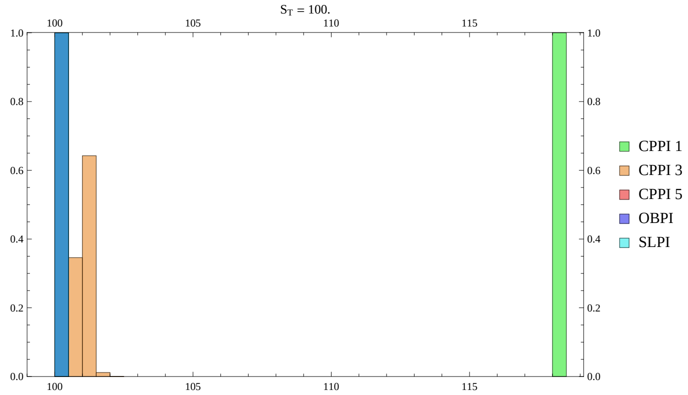


(d) $S_T = 250$

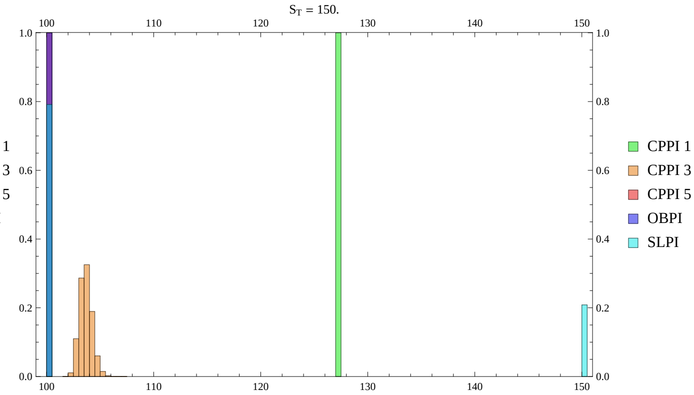


(e) $S_T = 300$

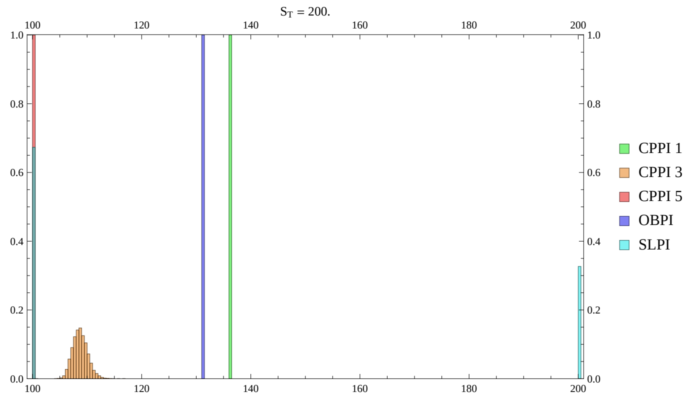
Figure B.1: Probability density functions of the PI's payoff at maturity.
Scenario: $\{\sigma, T, \eta\} = \{15\%, 5, 100\%\}$. Procedural parameters: $\{N, \Delta t, r, V_0\} = \{10000, 0.01, 4\%, 100\}$



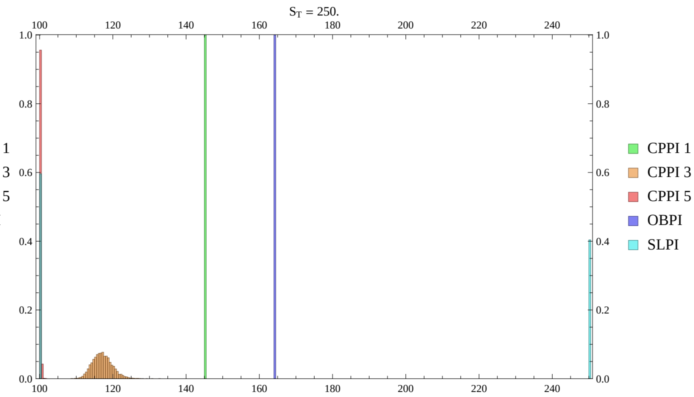
(a) $S_T = 100$



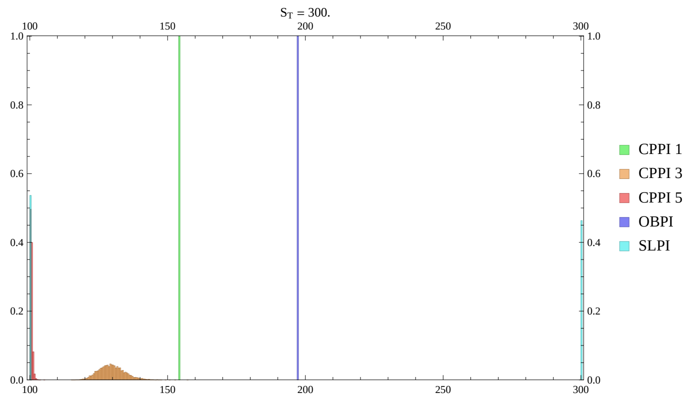
(b) $S_T = 150$



(c) $S_T = 200$

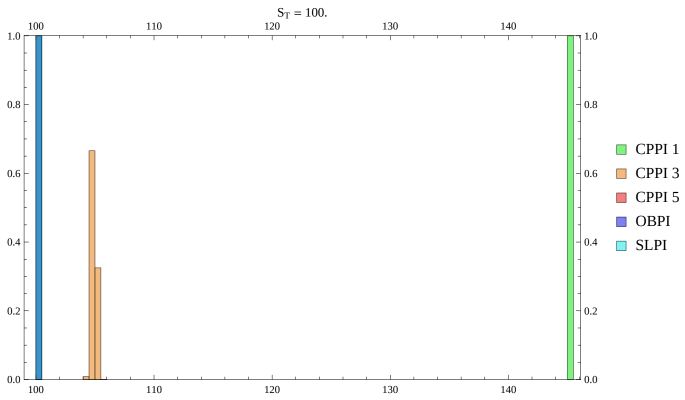


(d) $S_T = 250$

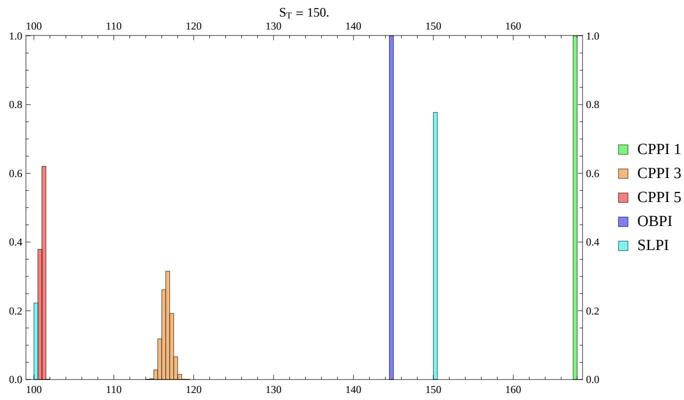


(e) $S_T = 300$

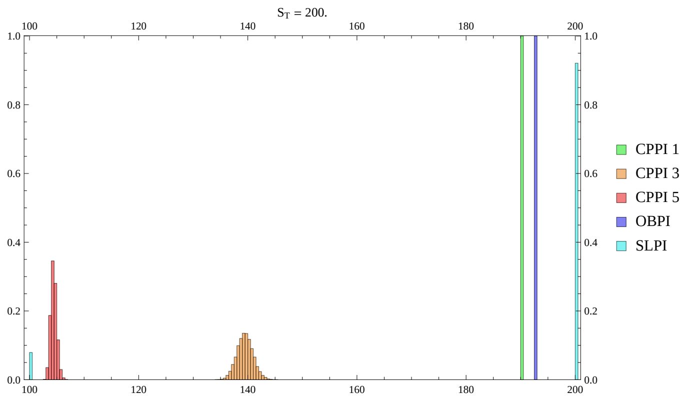
Figure B.2: Probability density functions of the PI's payoff at maturity.
Scenario: $\{\sigma, T, \eta\} = \{40\%, 5, 100\%\}$. Procedural parameters: $\{N, \Delta t, r, V_0\} = \{10000, 0.01, 4\%, 100\}$



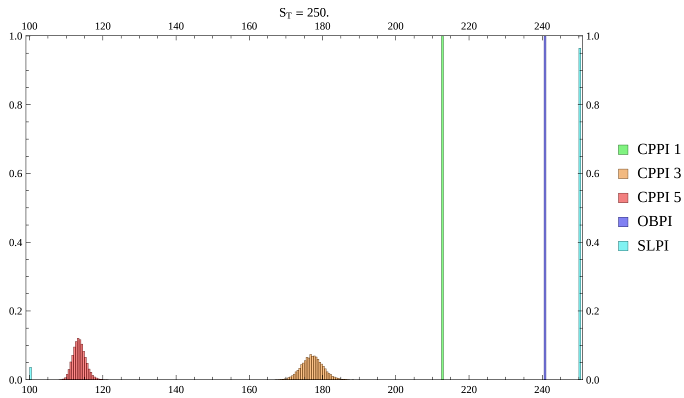
(a) $S_T = 100$



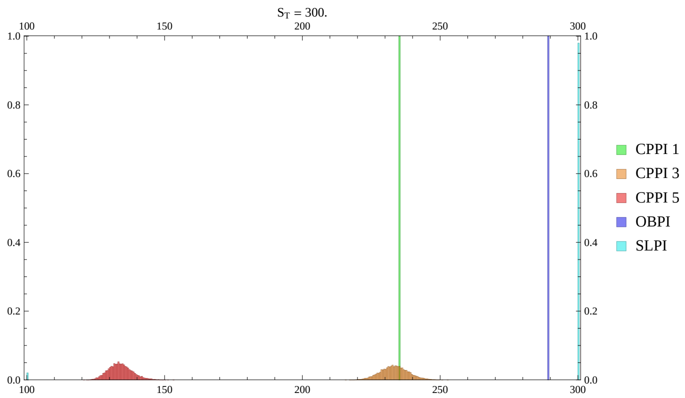
(b) $S_T = 150$



(c) $S_T = 200$

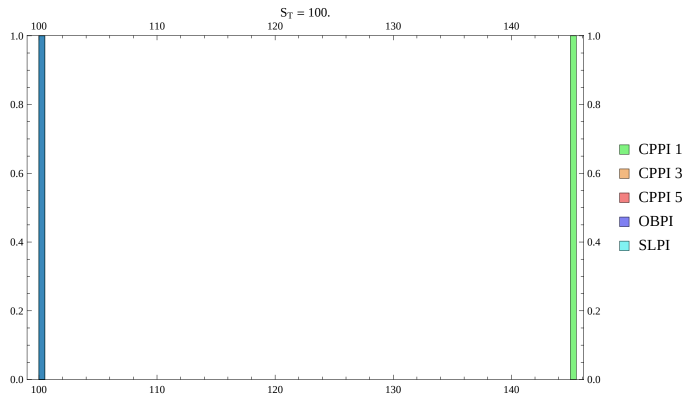


(d) $S_T = 250$

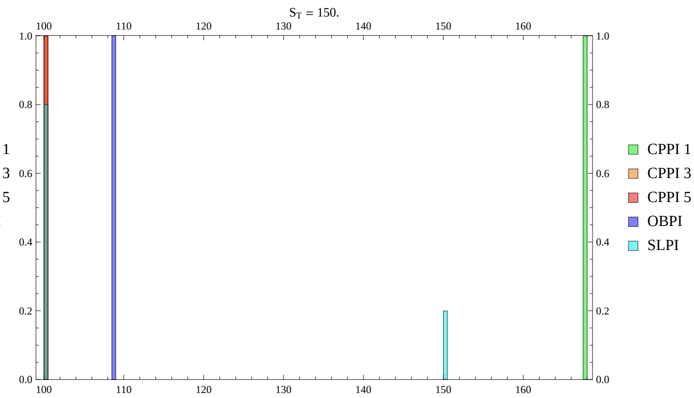


(e) $S_T = 300$

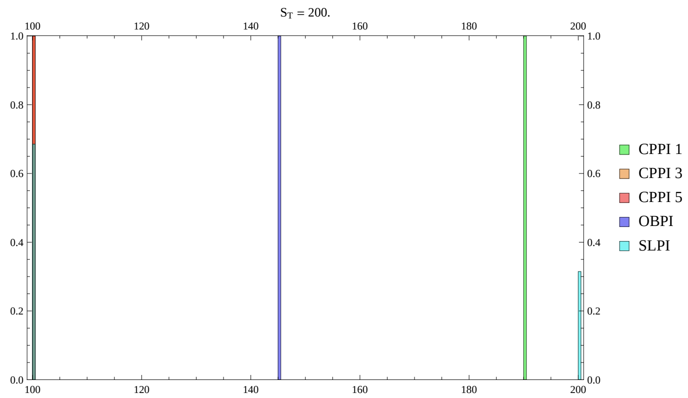
Figure B.3: Probability density functions of the PI's payoff at maturity.
Scenario: $\{\sigma, T, \eta\} = \{15\%, 15, 100\%\}$. Procedural parameters: $\{N, \Delta t, r, V_0\} = \{10000, 0.01, 4\%, 100\}$



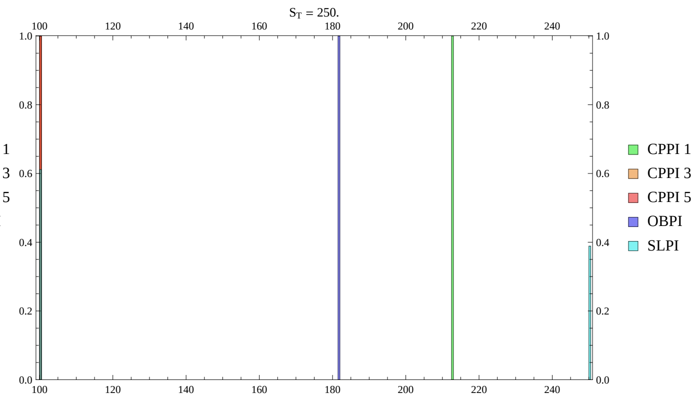
(a) $S_T = 100$



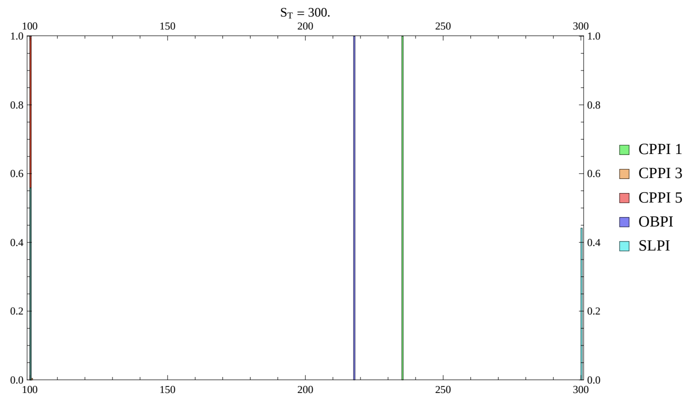
(b) $S_T = 150$



(c) $S_T = 200$

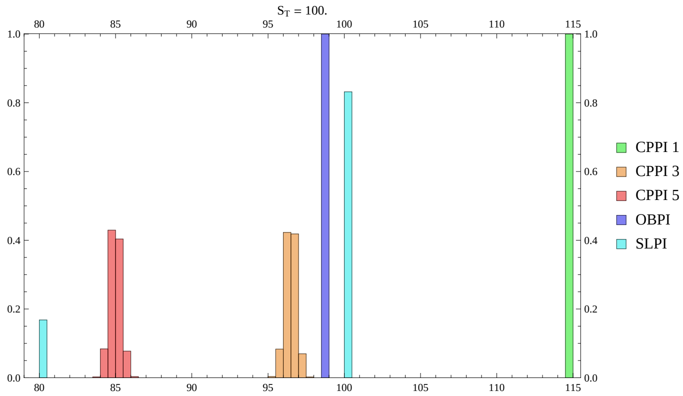


(d) $S_T = 250$

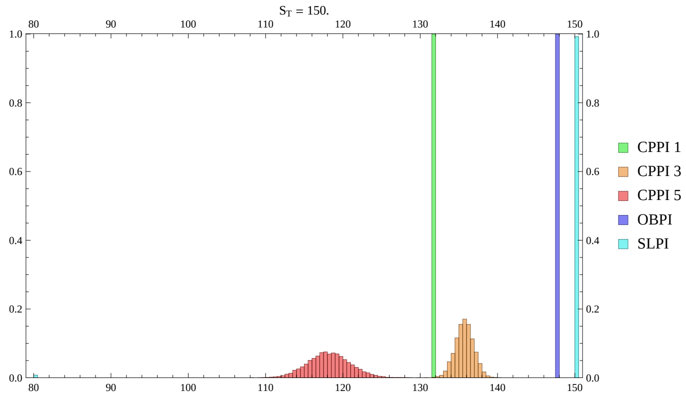


(e) $S_T = 300$

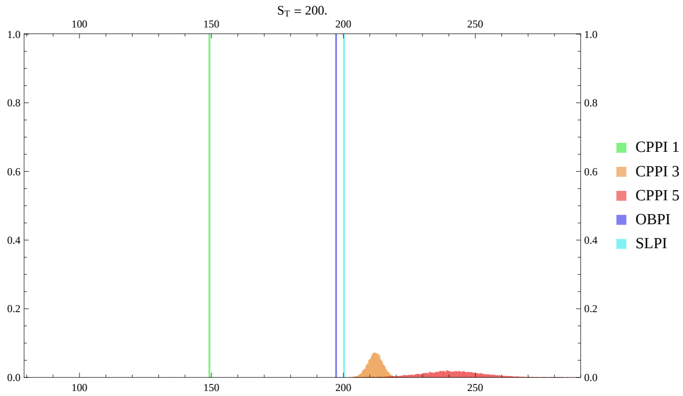
Figure B.4: Probability density functions of the PI's payoff at maturity.
Scenario: $\{\sigma, T, \eta\} = \{40\%, 15, 100\%\}$. Procedural parameters: $\{N, \Delta t, r, V_0\} = \{10000, 0.01, 4\%, 100\}$



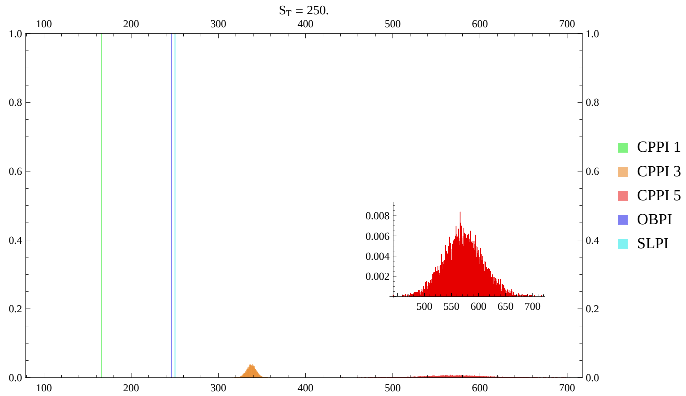
(a) $S_T = 100$



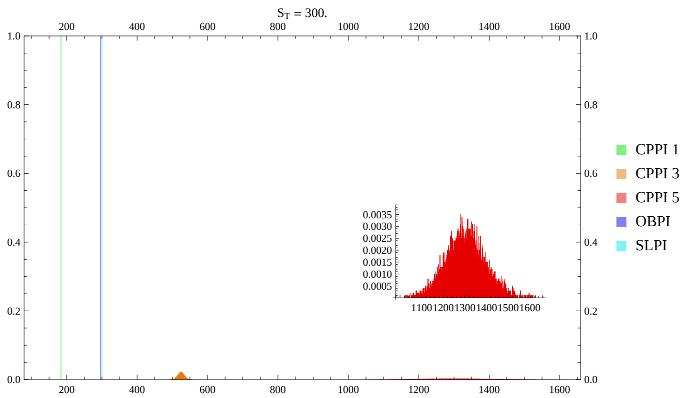
(b) $S_T = 150$



(c) $S_T = 200$

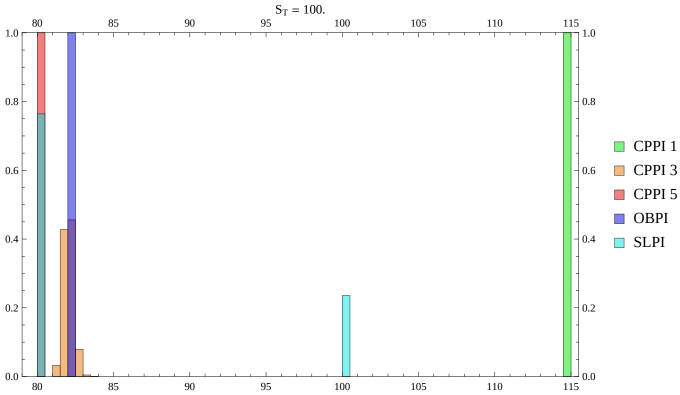


(d) $S_T = 250$

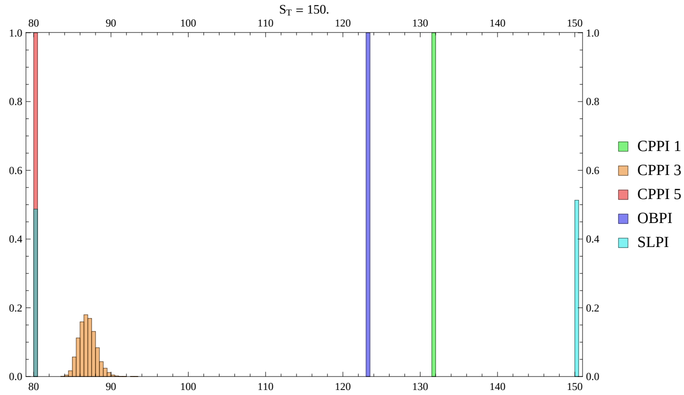


(e) $S_T = 300$

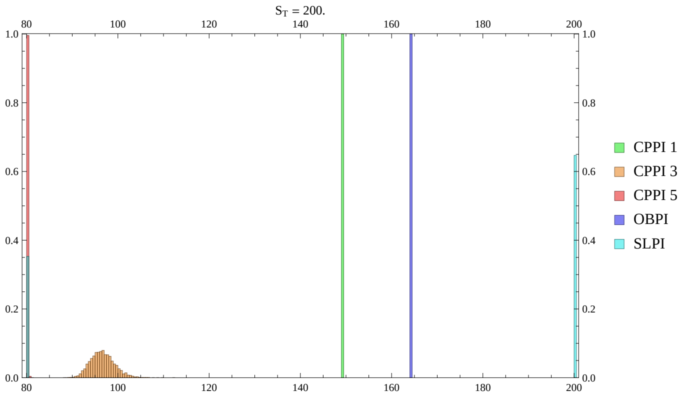
Figure B.5: Probability density functions of the PI's payoff at maturity.
Scenario: $\{\sigma, T, \eta\} = \{15\%, 5, 80\%\}$. Procedural parameters: $\{N, \Delta t, r, V_0\} = \{10000, 0.01, 4\%, 100\}$



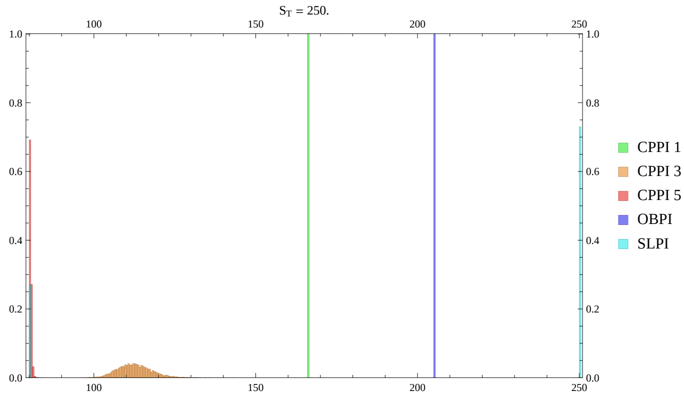
(a) $S_T = 100$



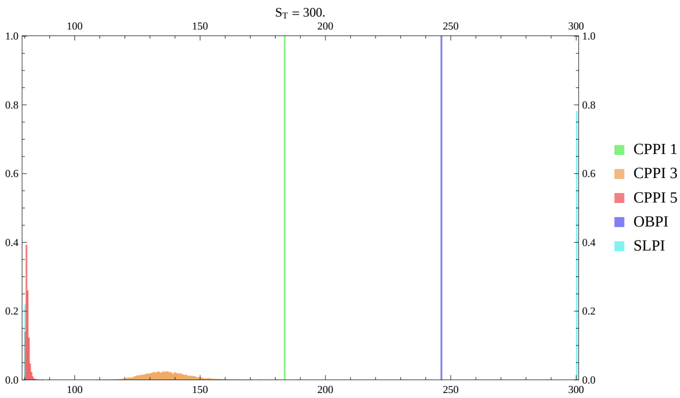
(b) $S_T = 150$



(c) $S_T = 200$

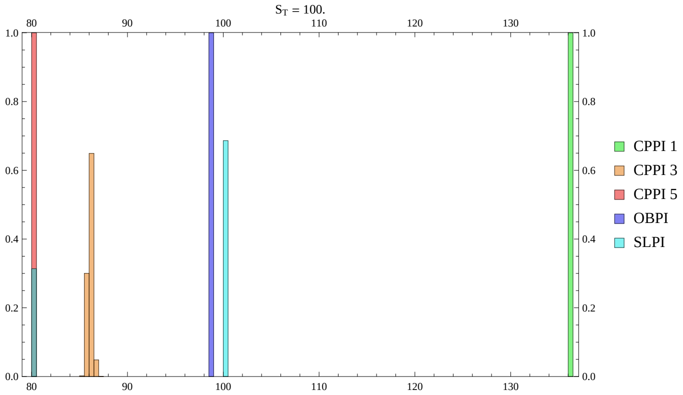


(d) $S_T = 250$

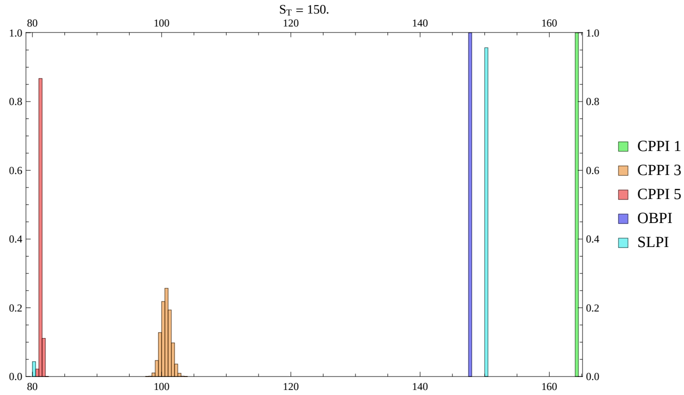


(e) $S_T = 300$

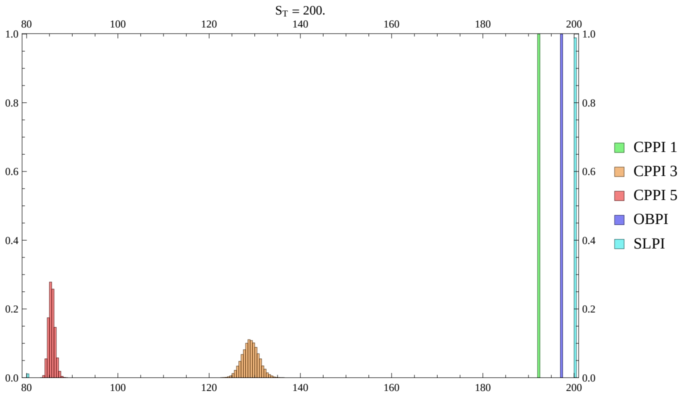
Figure B.6: Probability density functions of the PI's payoff at maturity.
Scenario: $\{\sigma, T, \eta\} = \{40\%, 5, 80\%\}$. Procedural parameters: $\{N, \Delta t, r, V_0\} = \{10000, 0.01, 4\%, 100\}$



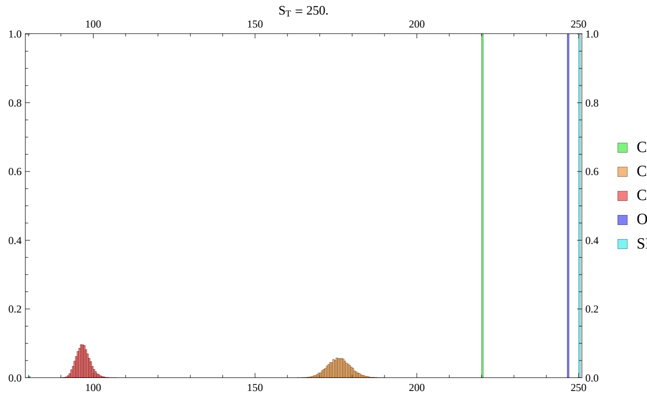
(a) $S_T = 100$



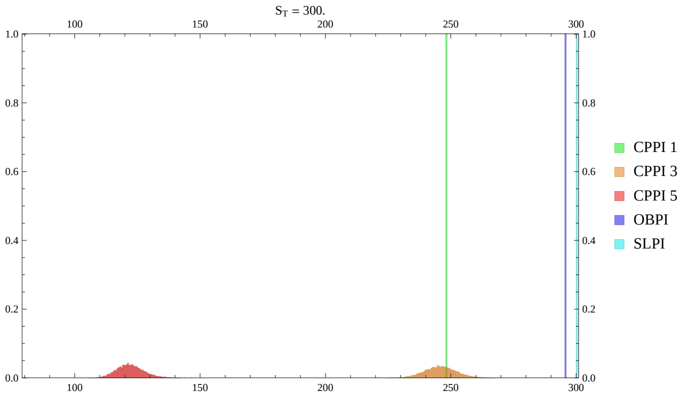
(b) $S_T = 150$



(c) $S_T = 200$

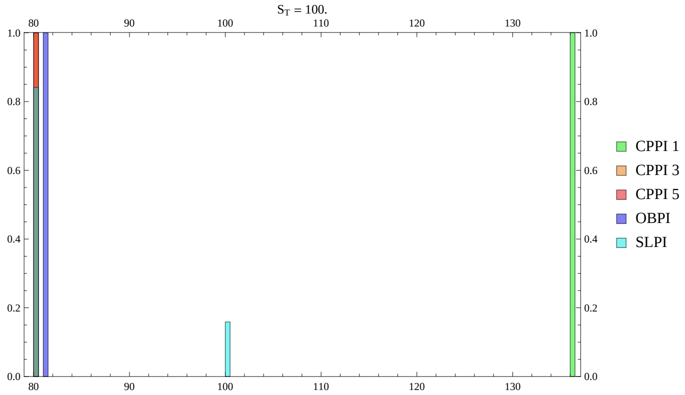


(d) $S_T = 250$

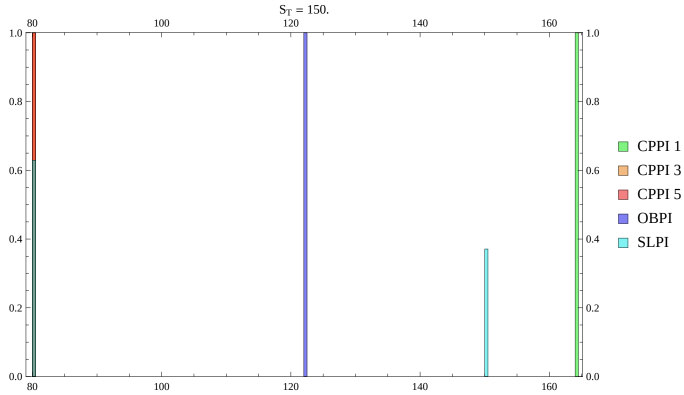


(e) $S_T = 300$

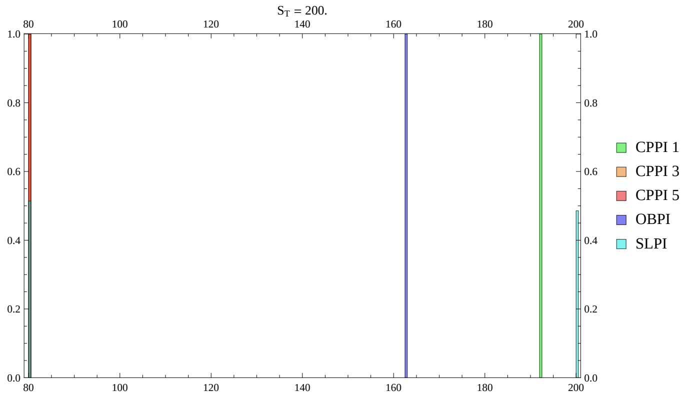
Figure B.7: Probability density functions of the PI's payoff at maturity.
Scenario: $\{\sigma, T, \eta\} = \{15\%, 15, 80\%\}$. Procedural parameters: $\{N, \Delta t, r, V_0\} = \{10000, 0.01, 4\%, 100\}$



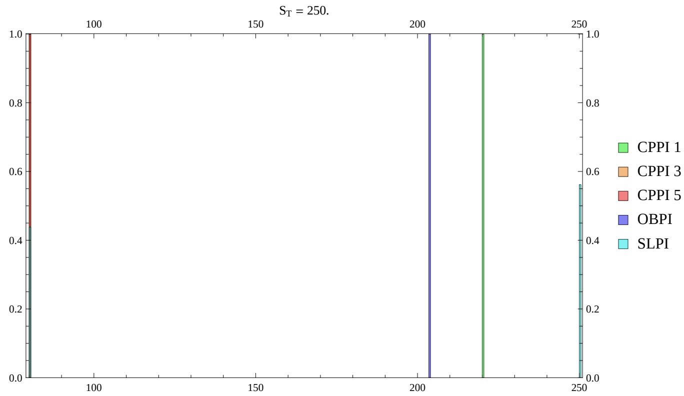
(a) $S_T = 100$



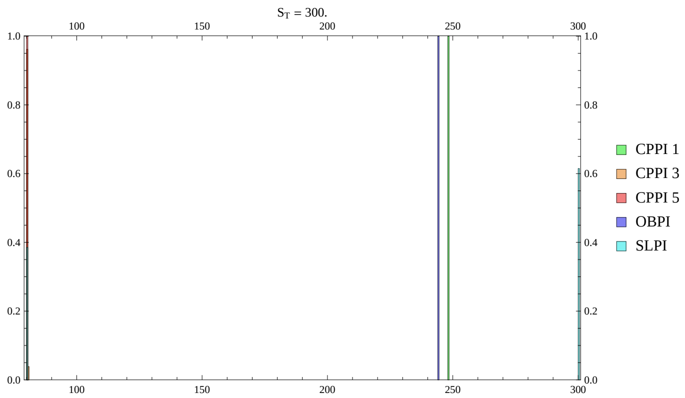
(b) $S_T = 150$



(c) $S_T = 200$



(d) $S_T = 250$



(e) $S_T = 300$

Figure B.8: Probability density functions of the PI's payoff at maturity.
Scenario: $\{\sigma, T, \eta\} = \{40\%, 15, 80\%\}$. Procedural parameters: $\{N, \Delta t, r, V_0\} = \{10000, 0.01, 4\%, 100\}$