Lisbon School of Economics & Management
Universidade de Lisboa

MASTER

MATHEMATICAL FINANCE

MASTER'S FINAL WORK

DISSERTATION

A REAL OPTIONS MODEL FOR HARVESTING

JOÃO AFONSO LAMPREIA DA COSTA BRAZÃO

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SUPERVISION:

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Abstract

Harvesting from natural resources, particularly fisheries, has played a central role in supporting human society, both as a source of food and economic activity. Decisions related to harvesting are influenced not only by biological and environmental conditions but also by economic incentives. Understanding how to manage these resources sustainably is essential for balancing short-term gains with long-term viability.

Despite its importance, the fishing industry faces significant challenges. When fishing efforts exceed ecological limits, fish stocks can collapse due to direct human impact. In contexts where access to fishing grounds is open or poorly regulated, excessive effort can be applied simultaneously by many players, placing unsustainable pressure on the resource.

To better manage this uncertainty and irreversibility of the investment, the decision to harvest can be framed as a real option. This approach treats the opportunity to fish as a right, not an obligation, allowing fishermen to delay harvesting until conditions are favourable.

In this work, we formulate the optimal harvesting policy as a stochastic control problem, leading to a Hamilton-Jacobi-Bellman (HJB) partial differential equation (PDE). We solve the HJB equation numerically, allowing us to simulate and analyse optimal policies under various scenarios. The results contribute to a better understanding of sustainable exploitation and highlight the economic value of flexibility in fisheries management.

Keywords: Stochastic Differential Equations, Partial Differential Equations, Real Options, Harvesting Option, Hamilton-Jacobi-Bellman Equation, Crank-Nicolson Scheme, Numerical Methods.

Resumo

A exploração de recursos naturais, particularmente a pesca, desempenha um papel central no apoio à sociedade humana, tanto como fonte de alimento como de atividade económica. As decisões relacionadas com a exploração da pesca são influenciadas não apenas por condições biológicas e ambientais, mas também por incentivos económicos. Compreender como gerir estes recursos de forma sustentável é essencial para equilibrar os lucros de curto prazo com a viabilidade a longo prazo.

Apesar da sua importância, a indústria da pesca enfrenta desafios significativos. Quando o esforço de pesca excede os limites ecológicos, os stocks de peixe podem colapsar devido ao impacto direto da atividade humana. Em contextos onde o acesso às zonas de pesca é livre ou mal regulamentado, pode verificar-se um esforço excessivo por parte de muitos indivíduos em simultâneo, criando uma pressão insustentável sobre o recurso.

Para melhor gerir esta incerteza e a irreversibilidade do investimento, a decisão de explorar pode ser enquadrada como uma opção real. Esta abordagem trata a oportunidade de pescar como um direito, e não uma obrigação, permitindo aos pescadores adiar a exploração até que as condições sejam favoráveis.

Nesta trabalho, formulamos a estratégia ótima de exploração como um problema de controlo estocástico, dando origem a uma equação diferencial parcial de Hamilton-Jacobi-Bellman (HJB). Resolvemos numericamente a equação HJB, permitindo-nos simular e analisar políticas ótimas sob diversos cenários. Os resultados contribuem para uma melhor compreensão da sustentabilidade na pesca e evidenciam o valor económico da flexibilidade na sua gestão.

Palavras-chave: Equações Diferenciais Estocásticas, Equações com Derivadas Parciais, Opções Reais, Opção de Exploração, Equação de Hamilton-Jacobi-Bellman, Esquema de Crank-Nicolson, Métodos Numéricos.

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Abbreviations and Notations List

ODE - Ordinary Differential Equation

SDE - Stochastic Differential Equation

PDE - Partial Differential Equation

HJB - Hamilton-Jacobi-Bellman

ENPV - Expected Net Present Value

SFU - Standardised Fishing Unit

DP - Dynamic Programming

SOCP - Stochastic Optimal Control Problem

CAPM - Capital Asset Pricing Model

r.v - Random Variable

$a.s.$ - almost surely

1 Introduction

For centuries, humans have relied on the ocean as a source of food, trade, and livelihood. It is often assumed that the seas possess an inexhaustible supply of resources. Although this belief is widespread, it carries significant risks. Overfishing, mismanagement, and lack of effective regulation can lead to the rapid depletion of stocks, threatening both ecosystems and economic sustainability.

In the complex world of fishing, many factors such as fish population dynamics, market price volatility, and harvesting costs affect the ability to predict outcomes and often lead to suboptimal decisions. For example, in regions where fishing rights are widely available or poorly regulated, a large number of players may choose to harvest simultaneously. This can result in an unsustainable race for resources, creating pressure on fish stocks and threatening the long-term viability of the ecosystem.

Initial academic approaches to optimal resource use in fisheries used the Expected Net Present Value (ENPV) rule to evaluate harvesting investments. This rule states that if the present value of expected future cash flows exceeds the investment cost, then the project is profitable and worth pursuing.

However, this approach has its limitations. It does not incorporate the ability to capture uncertainty or adapt to new information over time. Prices, costs, and biological variables are all subject to fluctuations and non-flexible policies may fail under changing conditions. Furthermore, fishing decisions often involve irreversible investments in vessels, gear, and fuel, which cannot be recovered once committed.

In [1], the harvesting opportunity was framed as a real option where an individual has the right, but not the obligation, to exploit the fishery resource. In this framework, the decision to fish is only exercised if it is economically viable. Their model considered a deterministic evolution of fish stocks and stochastic fish prices.

This work was extended by [2], who introduced a model where both the fish stock and the price evolved stochastically, with the population following a Gompertz model. This more realistic representation incorporated uncertainties in both the biological and economic dimensions.

Building on these foundational studies, this work proposes a different approach to the harvesting problem. Our main objective is to derive the optimal harvesting policy that maximizes the value of the opportunity to exploit the fishery, i.e., harvesting using a real options model.

The remainder of this dissertation is structured as follows. In Chapter 2, we present the theoretical and mathematical foundations required to conduct this work. In Chapter 3, we formulate the harvesting problem and derive the associated Hamilton-Jacobi-Bellman (HJB) partial differential equation (PDE) that governs the value of the harvesting option. Next, in Chapter 4, we provide a detailed numerical solution of the HJB equation using the Python programming language. In Chapter 5, we present and analyse the simulation results, offering insights into the behaviour of the optimal policy under uncertainty. Finally, in Chapter 6, we conclude the work, summarizing key findings and suggesting directions for future research.

2 Recap on SDEs, Options and Stochastic Optimal Control

The study of population dynamics across different species has long been a subject of scientific interest. These populations evolve in ways that are closely influenced by their specific environmental conditions. In particular, assessing aquatic populations is considerably more challenging due to the vastness and inaccessibility of oceanic environments. To address these challenges, this chapter provides a brief overview of the fundamental theoretical concepts required to formulate the optimal harvesting problem, including SDEs, real options theory, optimal stochastic control, and numerical methods.

2.1 Stochastic Differential Equations

In simple terms, stochastic refers to something that involves uncertainty overtime, meaning its outcome is not entirely predictable. Stochastic Processes are sequences of random variables used to describe the evolution of a system over time.

In a more theoretical way, a stochastic process is a family of random variables $\{X_t, t \in T\}$, defined within a probability space $(\Omega, \mathcal{F}, P)$, where T is the set on which the parameter t is defined. The process is said to be discrete if $T = \mathbb{N}_0$, and continuous if $T = [a, b] \subset \mathbb{R}$ or $T = \mathbb{R}$. In most cases where time is involved, T is taken to be $\mathbb{R}_+$. The state space is defined as the set of values that the process X_t can take.

Since each random variable $X_t = X_t(\omega)$ depends on the outcome $\omega \in \Omega$, a stochastic process can be viewed as a function of two variables: $t \in T$ and $\omega \in \Omega$. It is a mapping that associates each pair (t, ω) to the random variable $X_t(\omega)$. For each fixed $\omega \in \Omega$, the mapping $t \mapsto X_t(\omega)$ describes a trajectory of the process, as detailed in [3].

Consider a probability space $(\Omega, \mathcal{F}, P)$ and a stochastic process $\{X_t\}_{t\in T}$. Following [3], a family of sub-σ-algebras of $\mathcal{F}$, denoted by $\{\mathcal{F}_t\}_{t\in T}$, is called a filtration if it satisfies the condition:

$$\mathcal{F}_s \subseteq \mathcal{F}_t, \quad \text{for all } s \leq t.$$

This property expresses the idea that, at each time instant t, we do not have access to the full information contained in $\mathcal{F}$ but rather a progressively enlarging sequence of σ-algebras $\mathcal{F}_t$. These represent the accumulated information up to time t, incorporating both past and present data. A common choice of filtration is the natural filtration of a stochastic process, defined as:

$$\mathcal{F}_t = \sigma(X_s; 0 \leq s \leq t),$$

which is the smallest filtration to which X_t is adapted. The natural filtration contains all information generated by the process up to time t.

A stochastic process $\{X_t\}_{t\in T}$ is said to be adapted to the filtration $\{\mathcal{F}_t\}_{t\in T}$ if, for each $t \in T$, the random variable X_t is $\mathcal{F}_t$-measurable. This means that for every Borel set B, the preimage $X_t^{-1}(B)$ belongs to $\mathcal{F}_t$.

As discussed in [4], a stochastic process $\{X_t\}_{t\in T}$ on the probability space $(\Omega, \mathcal{F}, P)$ is a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\in T}$ if it satisfies the following conditions:

1. The process X_t is adapted to the filtration $\mathcal{F}_t$.
2. $\mathbb{E}[|X_t|] < \infty$ for all $t \in T$.
3. For any $s \leq t$, $\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s$ a.s.

A Markov Process is a type of stochastic process characterized by the "memoryless" property, which means that the future evolution of the process depends solely on its current state, and not on any previous states or the path taken to reach the current state.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $T = [0, N]$ with $0 \leq N \leq +\infty$. As shown in [4], a stochastic process $\{X_t\}_{t\in T}$ is a Markov process in continuous time if, for any $s, t \in T$ with $s \leq t$, and for any Borel set $B \in \mathcal{B}$, the following Markov property holds:

$$P(X_t \in B \mid X_u, 0 \leq u \leq s) = P(X_t \in B \mid X_s).$$

Equivalently, for any $n = 1, 2, \ldots; t_1 \leq t_2 \leq \ldots \leq t_n \leq t; x_1, \ldots, x_n \in \mathbb{R}$ and for any Borel set B, we have:

$$P(X_t \in B \mid X_{t_1} = x_1, \ldots, X_{t_n} = x_n) = P(X_t \in B \mid X_{t_n} = x_n).$$

The transition probability distributions, which are conditional distributions, can be defined through the transition probabilities:

$$P(t, B \mid s, x) := P(X_t \in B \mid X_s = x),$$

for $s \leq t$, as in [4]. This represents the probability that the process, starting at state x at time s, will transition to a state within the set B at time t.

A homogeneous Markov process is a Markov process whose transition probabilities are time-invariant, meaning that they depend only on the time difference rather than the absolute time. More precisely, the transition probabilities satisfy:

$$P(t + \tau, B \mid s + \tau, x) = P(t, B \mid s, x),$$

which implies that the probability of transitioning from state x to a set of states B over a time interval $t - s$ remains the same regardless of when the interval starts. Equivalently, if we denote by $\mathcal{F}_s$ the natural filtration, the Markov property for a homogeneous Markov process can be expressed as:

$$P(X_{s+\tau} \in B \mid \mathcal{F}_s) = P(X_\tau \in B \mid X_0),$$

for any $\tau \geq 0$ such that $s, s + \tau \in T$ and for any Borel set B, as presented in [4].

A Markov process with a discrete state space is called a Markov chain. When both the state space and the process are continuous, the Markov process is referred to as a diffusion process.

Let $(X_t)_{t\in[0,N]}$ be a stochastic process. It is said to be a diffusion process, according to [4], if it is a Markov process with almost surely (a.s.) continuous trajectories and satisfies the following properties for $s \in [0, N]$ and $x \in \mathbb{R}$:

(1) $\lim_{h\to 0^+} \frac{1}{h} P_{s,x} \left[|X_{s+h} - x| > \epsilon \right] = 0, \quad \forall \epsilon > 0$

(2) $\lim_{h\to 0^+} \mathbb{E}_{s,x} \left[\frac{X_{s+h} - x}{h} \right] = a(s, x)$

(3) $\lim_{h\to 0^+} \mathbb{E}_{s,x} \left[\frac{(X_{s+h} - x)^2}{h} \right] = b(s, x),$

where we write $\mathbb{E}\left[X_t \mid X_s = x\right]$ by $\mathbb{E}_{s,x}\left[X_t\right]$.

The function $a(s, x)$ is called the drift coefficient, while $b(s, x)$ is the diffusion coefficient.

- The drift coefficient $a(s, x)$ represents the rate of change of the expected value of the process at time s given that $X_s = x$.
- The diffusion coefficient $b(s, x)$ represents the rate of change of the variance of the process at time s given that $X_s = x$.

From this definition, we obtain the following approximations for small increments h:

(1) $\mathbb{E}_{s,x}\left[X_{s+h} - X_s\right] = a(s, x)h + o(h)$

(2) $\text{Var}_{s,x}\left[X_{s+h} - X_s\right] = b(s, x)h + o(h)$

This suggests the following approximation:

$$X_{s+h} - X_s \approx a(s, x)h + \sqrt{b(s, x)}Z,$$

where $Z \sim N(0, h)$ is a normally distributed random variable with mean zero and variance h.

Taking the limit as $h \to 0^+$, we can rewrite this expression in differential form, leading to the general formulation of a stochastic differential equation (SDE).

A Brownian Motion is a continuous stochastic process that models randomness, unpredictable movements, often used to describe phenomena such as the motion of particles suspended in a fluid, or to simulate randomness in diverse fields like finance, physics, and biology. Named after the botanist Robert Brown, who first observed this motion in 1827, it is also referred to as the Standard Wiener Process, inspired by the mathematician Norbert Wiener, who provided the first formal mathematical definition of the process.

As defined in [4], a stochastic process $B_t = \{B_t : t \geq 0\}$ is called a Brownian motion if it satisfies the following conditions:

1. $B_0 = 0$ a.s.
2. $\forall\, 0 \leq t_1 < \cdots < t_n$, the increments $B_{t_n} - B_{t_{n-1}}, \ldots, B_{t_2} - B_{t_1}$ are independent r.v.
3. For $s < t$, $B_t - B_s \sim N(0, t - s)$.
4. The process B_t has continuous trajectories.

Stochastic differential equations (SDEs) extend ordinary differential equations (ODEs) by incorporating a stochastic term. The deterministic part represents the system's average behaviour, while the stochastic component represents random fluctuations or noise affecting the system. When these random perturbations are absent, the SDE is simplified to an ODE.

Under the standard definition, the Riemann-Stieltjes integral is defined as the limit of Riemann-Stieltjes sums over all tagged partitions of $[0, t]$, as the mesh of the partition tends to zero. However, this approach fails when the integrator function is the Wiener process $B(t)$. The reason is that $B(t)$ almost surely has unbounded variation, which prevents the usual Riemann-Stieltjes integral from being well-defined.

By [5], a continuous and adapted stochastic process $\{X_t, 0 \leq t \leq T\}$ is called an Itô process if it admits the representation:

$$X_t = X_0 + \int_0^t \mu_s \, ds + \int_0^t \sigma_s \, dB_s,$$

where $X_0 \in \mathbb{R}$, and $\mu(\cdot)$ and $\sigma(\cdot)$ are adapted processes. In differential notation, this is written as:

$$dX_t = \mu_t \, dt + \sigma_t \, dB_t.$$

Assuming that X is an Itô process and $f(t, x)$ is of class $C^{1,2}$, the process $Y_t = f(t, X_t)$ is given by Itô's formula:

$$Y_t = f(t, X_t) = f(0, X_0) + \int_0^t \frac{\partial f}{\partial t}(s, X_s) ds + \int_0^t \frac{\partial f}{\partial x}(s, X_s) \sigma_s dB_s + \int_0^t \frac{\partial f}{\partial x}(s, X_s) \mu_s ds + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_s) \sigma_s^2 ds.$$

In differential notation, Itô's formula can be expressed as:

$$\begin{aligned} df(t, X_t) &= \frac{\partial f}{\partial t}(t, X_t) dt + \frac{\partial f}{\partial x}(t, X_t) dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, X_t)(dX_t)^2 \\ &= \left(\frac{\partial f}{\partial t}(t, X_t) + \mu_t \frac{\partial f}{\partial x}(t, X_t) + \frac{1}{2} \sigma_t^2 \frac{\partial^2 f}{\partial x^2}(t, X_t) \right) dt + \sigma_t \frac{\partial f}{\partial x}(t, X_t) dB_t. \end{aligned}$$

Finally, as mentioned in [4], a stochastic process X_t is said to be a solution of the SDE:

$$\begin{cases} dX_t = \mu(t, X_t) dt + \sigma(t, X_t) dB_t, \\ X_0 = x_0. \end{cases} \tag{1}$$

if the following conditions hold:

1. The process X_t is adapted to the filtration of the Wiener process and has continuous sample paths.

2. The expectation of the integral of the squared diffusion term remains finite, i.e.,

$$\mathbb{E} \left[\int_0^T \sigma(s, X_s)^2 ds \right] < \infty$$

3. The process X_t satisfies (1).

To guarantee the existence and uniqueness of the solution, the drift and diffusion coefficients must meet the appropriate Lipschitz and growth conditions to avoid explosion (for more information please see [3]).

2.2 Options

An option is a financial contract that gives the holder the right, but not the obligation, to perform a specific transaction involving an underlying asset (as seen in [6], for instance).

Key features of an option include the maturity date, which is the final time the option can be exercised, and the strike price, which is the fixed price at which the underlying asset can be bought or sold according to the terms of the contract.

American options can be exercised at any time up to and including the expiration date, while European options can only be exercised on the expiration date.

There are two main types of options:

- A call option gives the holder the right, but not the obligation, to buy the underlying asset at the strike price on or before the expiration date.
- A put option gives the holder the right, but not the obligation, to sell the underlying asset at the strike price on or before the expiration date.

Each option contract involves two parties: the holder, who takes the long position and holds the right to exercise the option, and the writer, who takes the short position and has the obligation to fulfil the contract if the option is exercised. The holder pays a premium to acquire the option. The writer receives this premium but assumes the risk of potential losses. The profit or loss of the writer is the exact opposite of that of the holder.

Let K denote the strike price and S_T the price of the underlying asset at expiration. Then, the payoffs for a long position in a European option are given by:

- For a call option:

  $$\max(S_T - K, 0)$$

  The option is exercised if $S_T > K$. Otherwise, it expires worthless.

- For a put option:

  $$\max(K - S_T, 0)$$

  The option is exercised if $S_T < K$. Otherwise, it expires with no value.

Options can also be classified as in the money, at the money or out of the money based on the relationship between the underlying price S and the strike price K. A call option is in the money if $S > K$, at the money when $S = K$, and out of the money if $S < K$.

2.3 Real Options

Having discussed financial options, we now turn to their real-world counterparts: real options. In this work, we explore how real options theory can be applied to decision-making in the fisheries sector, where uncertainty and the ability to delay or adjust actions are crucial. As shown in [7] and [6], unlike traditional investment analysis, real options capture the value of flexibility, such as the choice to wait before harvesting.

Investment decisions in resource-based industries like fisheries are particularly challenging due to uncertainty in fluctuating market prices and variability of biological stocks. Conventional methods such as the ENPV, which typically assumes fixed plans and predictable returns, tend to fall short in such contexts.

Real options theory provides a more dynamic and realistic framework by treating investment decisions as opportunities with embedded choices, much like financial options. Rather than requiring full commitment upfront, it allows individuals to respond to new information over time. For instance, fishers frequently face decisions that involve postponing harvests for better prices, reducing effort during low-yield periods, or exiting the industry when operations become unprofitable. These adaptive behaviours represent forms of strategic flexibility that traditional evaluation tools overlook.

This perspective reframes harvesting as an option-like decision, similar to holding a call option in financial markets: the fisher has the right, but not the obligation, to harvest. The value of this option depends on both biological and economic factors. By incorporating real options, we can better model these complex decisions and evaluate policies that maximize long-term returns while acknowledging uncertainty, irreversibility, and flexibility.

2.4 Stochastic Optimal Control

Optimal control theory provides a system for achieving efficiency in complex problems by identifying optimal strategies. These problems typically consist of an objective function to be maximised or minimised, control variables that influence the evolution of the system and the constraints that the system must satisfy.

A variety of methods exist for solving such optimisation problems. One approach is Dynamic Programming (DP). Instead of directly analysing the optimal control u^* and trajectory x^*, DP focuses on maximising the objective function itself. This approach decomposes a large problem into a series of smaller sub-problems.

Following [8], a process $\{u_t\}_{t \in T}$ is progressively measurable if for any $t \in T$, the mapping $(s, \omega) \mapsto u_s(\omega)$ is measurable on $[0, t] \times \Omega$ equipped with the product σ-field $\mathcal{B}([0, t]) \otimes \mathcal{F}_t$.

Let $U \subset \mathbb{R}^k$ be a given subset. We denote by $\mathcal{U}$ the set of all progressively measurable processes $u = \{u_t,\ t < N\}$ valued in U, as defined in [9]. The elements of $\mathcal{U}$ are called control processes.

We begin by considering the following ODE:

$$\begin{cases} \dot{x}(t) = f(x(t), \alpha(t)) \quad (t \geq 0) \\ x(0) = x_0 \end{cases} \tag{2}$$

As in [10], the payoff functional is defined as:

$$P[\alpha(\cdot)] = \int_0^T L(x(t), \alpha(t))dt + \Phi(x(T)) \tag{3}$$

where $T > 0$ is the terminal time, $L : \mathbb{R}^n \times U \to \mathbb{R}$ is the running payoff and $\Phi : \mathbb{R}^n \to \mathbb{R}$ is the terminal payoff.

The goal is to find an optimal control $\alpha^*(\cdot)$ such that $P[\alpha^*(\cdot)] = \max_{\alpha(\cdot)\in U} P[\alpha(\cdot)]$, i.e. we aim to maximise the objective function starting from time 0 and initial state x_0.

As in [10], we define the Hamiltonian function:

$$H(x, p, a) := f(x, a) \cdot p + L(x, a), \quad (x, p \in \mathbb{R}^n, a \in U)$$

According to the Pontryagin Maximum Principle (see [10]), let $\alpha^*(\cdot)$ be the optimal control for Eq. 2 and 3, and $x^*(\cdot)$ the corresponding trajectory. Then, there exists a function $p^* : [0, T] \to \mathbb{R}^n$ such that:

- $\dot{x}^*(t) = \nabla_p H(x^*(t), p^*(t), \alpha^*(t))$,
- $\dot{p}^*(t) = -\nabla_x H(x^*(t), p^*(t), \alpha^*(t))$,
- $H(x^*(t), p^*(t), \alpha^*(t)) = \max_{a\in U} H(x^*(t), p^*(t), a) \quad (0 \le t \le T)$,
- $p^*(T) = \nabla\Phi(x^*(T))$.

Let the value function v be a C^1 function. Then v satisfies the HJB equation

$$v_t(x, t) + \max_{a\in U}\{f(x, a) \cdot \nabla_x v(x, t) + L(x, a)\} = 0, \quad (x \in \mathbb{R}^n, 0 \le t \le T),$$

with terminal condition $v(x, T) = \Phi(x)$, as in [10].

In a random setting, the system's evolution is described by a SDE of the form:

$$\begin{cases} dX(t) = f(X(t), A(t))dt + \sigma dB(t), \ (X \in \mathbb{R}^n, A \in \mathcal{U}, t \in [t_0, T]) \\ X(t_0) = x_0 \end{cases}$$

By [10], the expected payoff functional is given by $P_{x,t}[A(\cdot)] = \mathbb{E}\left[\int_t^T L(X(s), A(s))ds + \Phi(X(T))\right]$.

Similarly to the deterministic case, the objective is to find the optimal control $A^*(\cdot)$ such that:

$$P_{x,t}[A^*(\cdot)] = \max_{A(\cdot)\in\mathcal{U}} P_{x,t}[A(\cdot)]$$

Let v be the value function $v(x, t) := \sup_{A(\cdot)\in\mathcal{U}} P_{x,t}[A(\cdot)]$. Then, v satisfies the stochastic HJB PDE:

$$\begin{cases} v_t(x, t) + \frac{\sigma^2}{2}\Delta v(x, t) + \max_{a\in U}\{f(x, a) \cdot \nabla_x v(x, t) + L(x, a)\} = 0 \\ v(x, T) = \Phi(x) \end{cases}$$

The sketch proofs of the theorems and results discussed above can be found in [10].

2.5 Numerical Topics

PDEs often cannot be solved analytically due to their complexity. In such cases, numerical methods become essential tools for approximating solutions.

We can define the Taylor expansion as a way to approximate a function by a polynomial, given by:

$$f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(x) + \frac{1}{6}h^3 f'''(x) + O(h^4)$$

To assess how accurately a numerical method approximates the true solution, we rely on the ”Big-O” and ”little-o” theory, which describe the rate at which error terms decay.

As discussed in [11], let $g(h)$ be a function with $g \to 0$ as $h \to 0$. Then:

- $f \in O(g)$ if and only if

$$\limsup_{h \to 0} \left| \frac{f(h)}{g(h)} \right| < \infty$$

  This means that $f(h)$ decays to zero at least as fast as $g(h)$.

- $f \in o(g)$ if and only if

$$\lim_{h \to 0} \left| \frac{f(h)}{g(h)} \right| = 0$$

  This means that $f(h)$ decays to zero faster than $g(h)$.

It is common to write $f = O(g)$ and $f = o(g)$ instead of the set notation $f \in O(g)$ and $f \in o(g)$, respectively. Note that if $f(h) = o(g(h))$, then necessarily $f(h) = O(g(h))$, although the converse may not be true (as can be seen in [11]).

When a PDE or a SDE cannot be solved explicitly, we resort, for instance, to finite difference methods for approximating derivatives.

If the function f is sufficiently smooth, we can approximate the first order, as outlined in [12], using the following quotients:

- Forward Difference Quotient: $f'(x) \approx \dfrac{f(x+h) - f(x)}{h} + O(h)$
- Backward Difference Quotient: $f'(x) \approx \dfrac{f(x) - f(x-h)}{h} + O(h)$
- Centred Difference Quotient: $f'(x) \approx \dfrac{f(x+h) - f(x-h)}{2h} + O(h^2)$

The second order derivative can be approximated by:

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2)$$

Higher order derivatives can also be approximated using extended finite difference formulas, as discussed in [12].

To solve PDEs numerically, we discretize both time and space. Let the temporal grid be defined as

$$t_0 < t_1 < \cdots < t_N = T, \quad \Delta t_n = t_n - t_{n-1}, \quad 1 \le n \le N,$$

and the spatial grid by:

$$x_0 < x_1 < \cdots < x_M, \quad \Delta x_m = x_m - x_{m-1}, \quad 1 \le m \le M.$$

A common reformulation of a PDE isolates the time derivative of a function U, yielding a semi-discrete form

$$\frac{\partial U}{\partial t} = (1-\theta) f^n + \theta f^{n+1}.$$

Here, f represents the spatially discretized operator, and $\theta \in [0,1]$ is a parameter that determines the numerical scheme. As presented in [11], for different values of θ, we have the following schemes:

- Forward Euler Scheme ($\theta = 0$): this scheme is a fully explicit method, where solution at the next time step depends only on the current time step

$$U^{n+1} = U^n + \Delta t f^n$$

- Crank-Nicolson Scheme $\left(\theta = \frac{1}{2}\right)$: in this scheme, we take the average of the spatial operator between the current and next time steps

$$U^{n+1} = U^n + \frac{\Delta t}{2}\left(f^n + f^{n+1}\right)$$

  This is widely used for parabolic PDEs (see [12]).

- Backward Euler Scheme ($\theta = 1$): this scheme is a fully implicit method, where the future iterate is computed using the spatial operator at that same future time step

$$U^{n+1} = U^n + \Delta t f^{n+1}$$

A graphical representation of the Crank-Nicolson scheme can be used to better understand how the values of $U^*_{n,m}$ are obtained at each grid point:

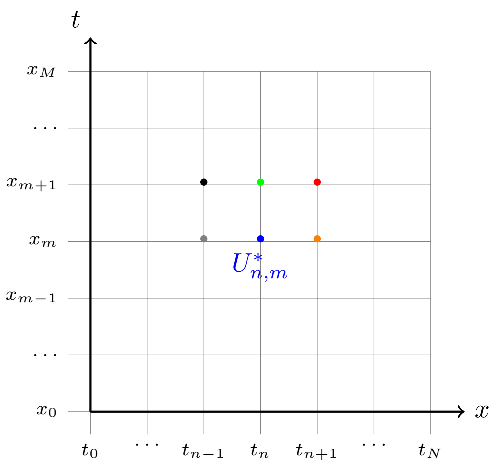

Figure 1: Crank-Nicolson Discretization Mesh Points

3 The Harvesting Option

3.1 Stochastic Model

To study the behaviour of a harvested population over time, we can model its dynamics using the following SDE:

$$dX(t) = f(X(t))dt - H(t)dt + \sigma_x X(t)dB_1(t),\ X(0) > 0$$

where:

- $X(t)$ denotes the size of the fish population at time t, expressed as biomass or number of individuals,
- $f(X(t))$ represents the natural growth rate of the population,
- $H(t)$ is the harvesting rate at time t,
- σ_x is the volatility of the population,
- $B_1(t)$ is a Standard Brownian Motion,
- and $X(0) > 0$ is the initial population size.

In practice, the harvesting function $H(t)$ is influenced by the fishing capacity of the fleet, which may depend on various factors such as the type and efficiency of the fishing equipment, the mesh size of the nets and the number of vessels in operation. To reflect this, we define $H(t)$ as:

$$H(t) = qE(t)X(t)$$

where $q > 0$ is the catchability coefficient, representing the proportion of the fish population harvested per unit of effort per unit time, and $E(t)$ denotes the fishing effort applied at time t (see [13] and [14]).

In this work, we will consider a stochastic optimal control problem to identify the optimal effort $E(t)$ that maximizes the expected profit from the fishery. The effort is subject to the constraint:

$$0 \leq E_{\text{min}} \leq E(t) \leq E_{\text{max}} < \infty,\ \forall t$$

where E_{min} and E_{max} are, respectively, the minimum and maximum levels of effort. These boundaries reflect the practical limits on the fishing activity: E_{max} account for the finite availability of resources such as vessels, labour, and fuel, while E_{min} corresponds to a minimal level of fishing activity, typically zero.

We consider the price of fish subject to uncertainty, according to the SDE:

$$dp(t) = \mu_p p(t)dt + \sigma_p p(t)dB_2(t),\ p(0) > 0,$$

where:

- $p(t)$ is the fish price at time t
- μ_p is the constant expected growth rate of the fish price,
- σ_p captures the volatility of the price,

- $B_2(t)$ is a Standard Brownian Motion
- and $p(0)$ is the initial price of fish at time 0.

The profit at time t is defined as the difference between revenue from sales and the cost of harvesting:

$$\Pi(t) = P(t) - C(t) = p(t)qX(t)E(t) - (c_1 + c_2E(t))E(t), \tag{4}$$

where $P(t)$ is the revenue from selling the harvested fish, $C(t)$ is the harvesting cost, assumed to be a quadratic function in effort, and $c_1 \geq 0$ and $c_2 > 0$ are constants.

Accordingly, the total expected profit at time t is given by:

$$\mathbb{E}\left[\Pi(t)\right] = \mathbb{E}\left[(p(t)qX(t) - c_1 - c_2E(t))E(t)\right].$$

For the purpose of this study, an in line with previous works such as [15], [2] and [16], we model the fishing population dynamics using the Gompertz growth model, described by the following SDE:

$$dX(t) = rX(t)\ln\left(\frac{K}{X(t)}\right)dt - qE(t)X(t)dt + \sigma_x X(t)dB_1(t),$$

where $r > 0$ is the growth rate of the population, $K > 0$ is the environmental carrying capacity and the other terms are previously defined.

3.2 The harvesting opportunity PDE

Now, we define $J := J(t, p(t), X(t))$ as the value of the opportunity to exploit the fishery, which depends on the current time t, the price of fish $p(t)$ and the fish population $X(t)$. To determine how the value J changes over a small interval of time, we apply the Itô Lemma to $J(t, p(t), X(t))$:

$$\begin{aligned}
dJ &= \frac{\partial J}{\partial t}dt + \frac{\partial J}{\partial X}dX + \frac{\partial J}{\partial p}dp + \frac{1}{2}\frac{\partial^2 J}{\partial X^2}(dX)^2 + \frac{1}{2}\frac{\partial^2 J}{\partial p^2}(dp)^2 + \frac{\partial^2 J}{\partial X \partial p}dXdp \\
&= \frac{\partial J}{\partial t}dt + \frac{\partial J}{\partial X}\left(\left(rX(t)\ln\left(\frac{K}{X(t)}\right) - qE(t)X(t)\right)dt + \sigma_x X(t)dB_1(t)\right) \\
&+ \frac{\partial J}{\partial p}(\mu_p p(t)dt + \sigma_p p(t)dB_2(t)) + \frac{1}{2}\frac{\partial^2 J}{\partial X^2}\sigma_x^2 X^2(t)dt + \frac{1}{2}\frac{\partial^2 J}{\partial p^2}\sigma_p^2 p^2(t)dt \\
&+ \frac{\partial^2 J}{\partial X \partial p}\sigma_x X(t)\sigma_p p(t)dB_1(t)dB_2(t)
\end{aligned}$$

Considering that the two Brownian increments are correlated, with a correlation coefficient $-1 \leq \rho \leq 1$, where $\rho \neq 0$, we can express $dB_1(t)$ as a linear combination of two independent Brownian increments, $dB_2(t)$ and $dB_3(t)$, as follows:

$$dB_1(t) = udB_2(t) + vdB_3(t)$$

where u and v are constants.

The following properties of Brownian increments are assumed to hold:

- $\mathbb{E}\left[dB_i(t)\right] = 0,\ i = 1, 2, 3,$
- $\mathbb{E}\left[dB_i(t)^2\right] = dt,\ i = 1, 2, 3,$
- $\mathbb{E}\left[dB_2(t)dB_3(t)\right] = 0$, as they are uncorrelated.

We now compute the correlation between $dB_1(t)$ and $dB_2(t)$. By definition:

$$\rho = \text{Corr}\left[dB_1(t), dB_2(t)\right] = \frac{\mathbb{E}\left[dB_1(t)dB_2(t)\right] - \mathbb{E}\left[dB_1(t)\right]\mathbb{E}\left[dB_2(t)\right]}{\sqrt{\text{Var}\left[dB_1(t)\right]}\sqrt{\text{Var}\left[dB_2(t)\right]}} = \frac{\mathbb{E}\left[dB_1(t)dB_2(t)\right]}{dt}$$

Using the linear combination, we get:

$$\begin{aligned}
\mathbb{E}\left[dB_1(t)dB_2(t)\right] &= \mathbb{E}\left[(udB_2(t) + vdB_3(t))dB_2(t)\right] = u\mathbb{E}[dB_2(t)^2] + v\mathbb{E}[dB_3(t)dB_2(t)] = udt \\
&\implies \rho dt = udt \\
&\implies \rho = u
\end{aligned}$$

Next, using the variance of $dB_1(t)$, we have:

$$\begin{aligned}
\mathbb{E}[dB_1(t)^2] &= \mathbb{E}[(udB_2(t) + vdB_3(t))^2] = (u^2 + v^2)dt \\
&\implies dt = (u^2 + v^2)dt \\
&\implies 1 = u^2 + v^2 \\
&\implies v = \sqrt{1 - \rho^2}.
\end{aligned}$$

We then conclude that the Brownian increment $dB_1(t)$ can be expressed as:

$$dB_1(t) = \rho dB_2(t) + \sqrt{1 - \rho^2} dB_3(t)$$

Thus, we obtain the following expression:

$$\begin{aligned}
dJ = \frac{\partial J}{\partial t}dt &+ \frac{\partial J}{\partial X}\left(rX(t)\ln\left(\frac{K}{X(t)}\right) - qE(t)X(t)\right)dt \\
&+ \frac{\partial J}{\partial X}\sigma_x X(t)\left(\rho dB_2(t) + \sqrt{1 - \rho^2}dB_3(t)\right) \\
&+ \frac{\partial J}{\partial p}(\mu_p p(t)dt + \sigma_p p(t)dB_2(t)) + \frac{1}{2}\frac{\partial^2 J}{\partial X^2}\sigma_x^2 X^2(t)dt \\
&+ \frac{1}{2}\frac{\partial^2 J}{\partial p^2}\sigma_p^2 p^2(t)dt + \frac{\partial^2 J}{\partial X \partial p}\rho\sigma_x X(t)\sigma_p p(t)dt. \qquad (5)
\end{aligned}$$

In the case where the two Brownian increments, $dB_1(t)$ and $dB_2(t)$, are uncorrelated ($\rho = 0$), the expression simplifies to:

$$\begin{aligned}
dJ = &\frac{\partial J}{\partial t}dt + \frac{\partial J}{\partial X}\left(rX(t)\ln\left(\frac{K}{X(t)}\right) - qE(t)X(t)\right)dt + \frac{\partial J}{\partial X}\sigma_x X(t)dB_1(t) \\
&+ \frac{\partial J}{\partial p}(\mu_p p(t)dt + \sigma_p p(t)dB_2(t)) + \frac{1}{2}\frac{\partial^2 J}{\partial X^2}\sigma_x^2 X^2(t)dt + \frac{1}{2}\frac{\partial^2 J}{\partial p^2}\sigma_p^2 p^2(t)dt, \qquad (6)
\end{aligned}$$

3.3 Convenience Yield

Let $Y(t, p(t))$ be the convenience yield by holding one unit of fish. This concept, as discussed in [1] and [7], represents the net benefit of physically possessing the commodity.

From a financial perspective, when an investor owns a stock, they receive dividends. However, holding a derivative of that stock does not provide the same benefit. In a similar way, owning fish provides significant advantages, such as the ability to sell quickly, respond to unexpected demand, or avoid shortages. These advantages are not available if we hold the option to fish in the future.

In the case of the fishery, the decision is between harvesting and selling the fish now, or leaving them in the sea expecting future price increases. Choosing to wait means giving up the immediate benefit of having the fish available for sale. This benefit can be interpreted as an opportunity cost, which we represent through the convenience yield.

Still, this interpretation needs to be adjusted to reflect a more realistic view of fish as a commodity. Since fish are perishable and cannot be stored for long periods, the convenience yield here should not be seen as a benefit from physical storage. Instead, we interpret it as the flexibility to harvest and sell the fish instantly in response to market conditions.

Following [1], we assume that the convenience yield is proportional to the price of the fish:

$$Y(t, p(t)) = yp(t), \text{ where } y \text{ is a constant.}$$

3.4 The Spanning Asset

The fish industry faces several challenges due to the nature of the environment. Once harvested, fish are highly perishable and cannot be stored for long periods, making inventory management difficult and requiring careful balancing of stock levels. The market is also very fragmented, consists mainly of small-scale and regional fisherman. Additionally, fish vary in species, size and form, each with its own pricing dynamics. Together, these factors contribute to the absence of a futures market for fish.

Following the approach in [1], we assume that markets are sufficiently complete and there are no arbitrage opportunities. This implies that the stochastic component of the fish price can be replicated by a combination of existing traded assets. In line with [17], we also assume that trading in assets occurs continuously over time, there are no transaction costs or taxes, and all assets are perfectly divisible.

Under these assumptions, it is possible to find a traded asset or portfolio whose price dynamics are perfectly correlated with those of the fish price.

Let $S(t)$ denote the price of the spanning asset. Its evolution is described by the SDE:

$$dS(t) = \mu_s S(t)dt + \sigma_s S(t)dB_2(t), \tag{7}$$

where μ_s is the expected return and σ_s is the volatility, both assumed to be known. Since this asset replicates the same source of risk as the fish price, its stochastic component is driven by the same Brownian motion $B_2(t)$.

In our model, we assume that $\mu_s \geq \mu_p$. To understand the implications of this condition, consider first the alternative case where $\mu_s < \mu_p$. In this scenario, the fish price would be expected to grow at a faster rate than the spanning asset. As a result, the value of waiting increases indefinitely, since delaying the harvest offers higher future returns. Then, the optimal decision would always be to delay harvesting, and the option would never be exercised. Therefore, such scenario is unrealistic.

Now consider the case where $\mu_s \geq \mu_p$. While waiting still reflects the expectation that the price of fish will increase, it also means giving up the returns that could be earned by investing in the spanning asset. In this context, the harvesting option is exercised when the opportunity cost of waiting outweighs the expected benefit of a higher future fish price. Therefore, this condition leads to a more realistic model in which the option to harvest may be exercised at some point depending on various factors.

Following the CAPM, as discussed in [1], we denote by r_p and r_s the expected returns from the fish price $p(t)$ and the spanning asset $S(t)$, respectively, and by λ the risk-free rate of return, assumed to be known. The expected return from $p(t)$ includes both its drift and the benefit from the convenience yield, so we write:

$$r_p = \mu_p + y = \lambda + \theta \rho_{pm} \sigma_p, \tag{8}$$

where $\theta = \frac{\mu_m - \lambda}{\sigma_m}$ is the market price of risk, and ρ_{pm} is the correlation between the return on the fish price and the market portfolio.

The expected return from the spanning asset is given by:

$$r_s = \mu_s = \lambda + \theta \rho_{sm} \sigma_s, \tag{9}$$

with ρ_{sm} being the correlation between the return on the spanning asset and the market portfolio.

Since $S(t)$ replicates the stochastic behaviour of the fish price $p(t)$, the two prices are perfectly correlated. As in [1], this implies that their correlations with the market portfolio must be equal, that is, $\rho_{pm} = \rho_{sm}$. Moreover, as noted in [7], we can assume that the volatility of the spanning asset is equal to that of the fish price, i.e., $\sigma_s = \sigma_p$. If this assumption does not hold, it is still possible to construct a portfolio combining the risk-free asset with the spanning asset to match the volatility σ_p.

Using the equations 8 and 9, we obtain:

$$\mu_p + y = \mu_s. \tag{10}$$

Since we previously assumed $\mu_s \geq \mu_p$, it follows that $y \geq 0$.

We also assume, following [7], that the price of fish is uncorrelated with the market portfolio. This is justified by the reasons discussed at the beginning of this chapter. The local nature of fish markets leads to supply and demand conditions that are specific to each region. So, under the CAPM, this implies that the expected return r_p is equal to the risk-free rate λ, which gives:

$$\mu_p + y = \lambda. \tag{11}$$

Finally, as in [1], we assume that the option to exploit the fishery is perpetual. That is, once developed, the fishery operates indefinitely without costs associated with opening or closing.

3.5 Portfolio Construction

To derive the PDE that describes the value of the harvesting option, we construct a portfolio by taking a long position in the exploitation option and a short position of ν units in the spanning asset. We focus on the case where $\rho \neq 0$, the uncorrelated case follows similarly. The stochastic component driven by $dB_3(t)$ is driven by biological uncertainty and cannot be hedged. On the other hand, the uncertainty driven by $dB_2(t)$ can be hedged by using the portfolio. To eliminate this source of risk, we determine the number of units ν by equating the coefficients of $dB_2(t)$ in the dynamics of the option value dJ and the spanning asset dS. Thus,

$$\sigma_x X(t)\rho \frac{\partial J}{\partial X} + \sigma_p p(t) \frac{\partial J}{\partial p} = \nu \sigma_s S(t).$$

Since we have assumed that $\sigma_s = \sigma_p$, this simplifies to

$$\frac{\sigma_x}{\sigma_p} X(t)\rho \frac{\partial J}{\partial X} + p(t) \frac{\partial J}{\partial p} = \nu S(t). \tag{12}$$

Based on the approach in [7], we compute the expected return of the portfolio over a small time interval dt. The expected return has three components: the expected change in the value of the harvesting option J, $\mathbb{E}[dJ]$, the expected change from shorting ν units of the spanning asset S, $\mathbb{E}[-\nu dS]$, and the actual cash flow generated from harvesting, given by 4.

Under the no-arbitrage assumption, a risk-free portfolio must grow at the risk-free rate λ. Therefore, the expected return from the portfolio must equal $\lambda(J - \nu S(t))dt$. Combining these two expressions, we get:

$$\lambda(J - \nu S(t))dt = \mathbb{E}[dJ - \nu dS] + (p(t)qX(t)E(t) - (c_1 + c_2 E(t))E(t))dt. \tag{13}$$

Simplifying the expression, replacing dJ and dS from equations 5 and 7, and dividing both sides by dt, we obtain:

$$\begin{aligned}
\lambda(J - \nu S(t)) = {} & \frac{\partial J}{\partial t} + \frac{\partial J}{\partial X}\left(rX(t)\ln\left(\frac{K}{X(t)}\right) - qE(t)X(t)\right) \\
& + \frac{\partial J}{\partial p}\mu_p p(t) + \frac{1}{2}\frac{\partial^2 J}{\partial X^2}\sigma_x^2 X^2(t) + \frac{1}{2}\frac{\partial^2 J}{\partial p^2}\sigma_p^2 p^2(t) \\
& + \frac{\partial^2 J}{\partial X \partial p}\rho\sigma_x X(t)\sigma_p p(t) - \nu\mu_s S(t) \\
& + (p(t)qX(t) - c_1 - c_2 E(t))E(t).
\end{aligned}$$

Rearranging terms and bringing all to one side, we have:

$$\begin{aligned}
0 = {} & \frac{\partial J}{\partial t} - \lambda J + \frac{\partial J}{\partial X}\left(rX(t)\ln\left(\frac{K}{X(t)}\right) - qE(t)X(t)\right) + \frac{\partial J}{\partial p}\mu_p p(t) \\
& + \frac{1}{2}\frac{\partial^2 J}{\partial X^2}\sigma_x^2 X^2(t) + \frac{1}{2}\frac{\partial^2 J}{\partial p^2}\sigma_p^2 p^2(t) + \frac{\partial^2 J}{\partial X \partial p}\rho\sigma_x X(t)\sigma_p p(t) \\
& + (\lambda - \mu_s)\nu S(t) + (p(t)qX(t) - c_1 - c_2 E(t))E(t).
\end{aligned}$$

Using equations 10 and 11, we rewrite the equation as:

$$0 = \frac{\partial J}{\partial t} - \lambda J + \frac{\partial J}{\partial X}\left(rX(t)\ln\left(\frac{K}{X(t)}\right) - qE(t)X(t)\right) + (\lambda - y)p(t)\frac{\partial J}{\partial p}$$
$$+ \frac{1}{2}\frac{\partial^2 J}{\partial X^2}\sigma_x^2 X^2(t) + \frac{1}{2}\frac{\partial^2 J}{\partial p^2}\sigma_p^2 p^2(t) + \frac{\partial^2 J}{\partial X \partial p}\rho\sigma_x X(t)\sigma_p p(t)$$
$$+ (p(t)qX(t) - c_1 - c_2 E(t))E(t).$$

To determine the optimal harvesting strategy, we maximize the right-hand side of this equation with respect to the control variable $E(t)$. Hence, we need to solve:

$$0 = \max_{E(t)}\Bigg[\frac{\partial J}{\partial t} - \lambda J + \frac{\partial J}{\partial X}\left(rX(t)\ln\left(\frac{K}{X(t)}\right) - qE(t)X(t)\right) + (\lambda - y)p(t)\frac{\partial J}{\partial p}$$
$$+ \frac{1}{2}\frac{\partial^2 J}{\partial X^2}\sigma_x^2 X^2(t) + \frac{1}{2}\frac{\partial^2 J}{\partial p^2}\sigma_p^2 p^2(t) + \frac{\partial^2 J}{\partial X \partial p}\rho\sigma_x X(t)\sigma_p p(t)$$
$$+ (p(t)qX(t) - c_1 - c_2 E(t))E(t)\Bigg].$$

Equivalently, this can be written in the form:

$$-\frac{\partial J}{\partial t} = \max_{E(t)}\Bigg[-\lambda J + \frac{\partial J}{\partial X}\left(rX(t)\ln\left(\frac{K}{X(t)}\right) - qE(t)X(t)\right) + (\lambda - y)p(t)\frac{\partial J}{\partial p}$$
$$+ \frac{1}{2}\frac{\partial^2 J}{\partial X^2}\sigma_x^2 X^2(t) + \frac{1}{2}\frac{\partial^2 J}{\partial p^2}\sigma_p^2 p^2(t) + \frac{\partial^2 J}{\partial X \partial p}\rho\sigma_x X(t)\sigma_p p(t)$$
$$+ (p(t)qX(t) - c_1 - c_2 E(t))E(t)\Bigg], \tag{14}$$

where the optimal effort level $E(t)$ is constrained to belong within the range $[E_{\text{min}}, E_{\text{max}}]$.

3.6 Optimal Effort

The next step is to determine the optimal harvesting effort that maximizes the value of the harvesting option. To do this, we isolate the terms in 14 that depend explicitly on the control variable $E(t)$, and define the control problem:

$$\mathcal{C} = \max_{E(t)}\left[(p(t)qX(t) - c_1 - c_2 E(t))E(t) - qE(t)X(t)\frac{\partial J^*(X(t), p(t), t)}{\partial X(t)}\right]$$

where $J^*(X(t), p(t), t)$ denotes the value function under the optimal control policy.

Let $E^*_{\mathcal{C}}(t)$ be the optimal harvesting strategy in $\mathcal{C}$. Then to calculate this $E^*_{\mathcal{C}}(t)$ we need to solve the equation $\frac{\partial \mathcal{C}}{\partial E} = 0$:

$$\frac{\partial \mathcal{C}}{\partial E} = 0 \iff p(t)qX(t) - c_1 - 2c_2 E^*_{\mathcal{C}}(t) - qX(t)\frac{\partial J^*(X(t), p(t), t)}{\partial X(t)} = 0.$$

Solving for $E^*_{\mathcal{C}}(t)$, we obtain:

$$E^*_{\mathcal{C}}(t) = \frac{qX(t)}{2c_2}\left(p(t) - \frac{\partial J^*(X(t), p(t), t)}{\partial X(t)}\right) - \frac{c_1}{2c_2}. \tag{15}$$

Replacing the optimal effort in 14 yields:

$$
\begin{aligned}
-\frac{\partial J^*(X(t),p(t),t)}{\partial t} &= -\lambda J^*(X(t),p(t),t) + \frac{\partial J^*(X(t),p(t),t)}{\partial X(t)}\left(rX(t)\ln\left(\frac{K}{X(t)}\right) - qE^*(t)X(t)\right) \\
&+ (\lambda - y)p(t)\frac{\partial J^*(X(t),p(t),t)}{\partial p(t)} + \frac{1}{2}\frac{\partial^2 J^*(X(t),p(t),t)}{\partial X(t)^2}\sigma_x^2 X^2(t) \\
&+ \frac{1}{2}\frac{\partial^2 J^*(X(t),p(t),t)}{\partial p(t)^2}\sigma_p^2 p^2(t) + \frac{\partial^2 J^*(X(t),p(t),t)}{\partial X(t)\partial p(t)}\rho\sigma_x X(t)\sigma_p p(t) \\
&+ (p(t)qX(t) - c_1 - c_2E^*(t))E^*(t), \qquad (16)
\end{aligned}
$$

where

$$
E^*(t) = \begin{cases} 0, & E_C^*(t) < E_{\text{min}} \\ \dfrac{qX(t)}{2c_2}\left(p(t) - \dfrac{\partial J^*(X(t),p(t),t)}{\partial X(t)}\right) - \dfrac{c_1}{2c_2}, & E_{\text{min}} \le E_C^*(t) \le E_{\text{max}} \\ E_{\text{max}}, & E_C^*(t) > E_{\text{max}} \end{cases}
$$

Eq. 16 is the HJB equation associated with the stochastic optimal control problem for harvesting. To solve it, we impose the following boundary conditions:

1. $E^*(X_{\text{min}}, p(t), t) = 0$, since the fishery is closed when the population reaches the minimum viable level X_{min}.
2. $J^*(X_{\text{min}}, p(t), t) = 0$, given that the population is at its minimum viable point, eventually close to extinction, the value of the harvesting option will be worthless.
3. $J^*(X(T), p(T), T) = 0$, since at the final time T, the opportunity to delay harvesting no longer exists.

4 Computational Solution

The HJB equation derived in the previous section is a non-linear PDE involving a time variable, a population variable and a price variable. Due to its complexity and the lack of an analytical solution, we resort to numerical methods. In particular, we implement the Crank-Nicolson scheme combined with finite differences approximations.

The problem is defined over the domain $[0, T] \times \mathbb{R} \times \mathbb{R}$. However, to obtain a numerical solution, we must discretize the domain by specifying minimum and maximum bounds for each variable. This results in a domain of the form $[t_0, T] \times [x_0, x_m] \times [p_0, p_l]$. The discretized domain is shown in Figure 2.

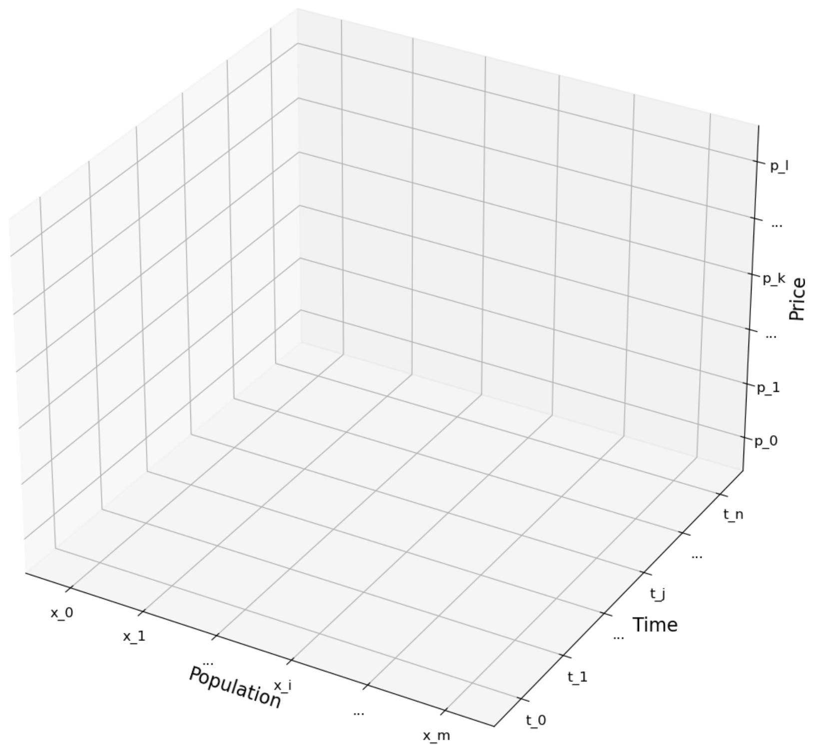

Figure 2: Discretization of the Domain

Since the maximum sustainable population is denoted by K, there is a non-zero probability that the population may exceed this level. To mitigate this, we choose $x_{\text{max}} > K$ such that the probability of the population surpassing x_{max} is negligible.

Now, we do the discretization of each variable as follows:

- Population grid: $x_i = x_0 + i\Delta x; \quad i = 0, 1, \ldots, m; \quad \Delta x = \dfrac{x_{\text{max}} - x_0}{m};$

- Price grid: $p_k = p_0 + k\Delta p; \quad k = 0, 1, \ldots, l; \quad \Delta p = \dfrac{p_{\text{max}} - p_0}{l};$

- Time grid: $t_j = t_0 + j\Delta t; \quad j = 0, 1, \ldots, n; \quad \Delta t = \dfrac{T - t_0}{n};$

Each point in the grid is represented by a triplet (x_i, p_k, t_j), which forms a three-dimensional mesh that resembles a cube. At each grid point, the value of the harvesting option and the optimal control are represented by:

$$J^*_{i,k,j} = J^*(x_i, p_k, t_j), \quad E^*_{i,k,j} = E^*(x_i, p_k, t_j),$$

for $0 \leq i \leq m,\ 0 \leq k \leq l$ and $0 \leq j \leq n$.

Since the boundary condition is defined at the final time by the terminal condition $J^*(X(T), p(T), T) = 0$, the numerical scheme must be solved backwards in time, starting from $t = T$ and proceeding step by step until $t = 0$. We approximate the first-order time derivative using a forward difference scheme:

$$\frac{\partial J^*_{i,k,j}}{\partial t} \approx \frac{J^*_{i,k,j+1} - J^*_{i,k,j}}{\Delta t}, 0 \leq i \leq m,\ 0 \leq k \leq l,\ 0 \leq j \leq n-1.$$

To approximate the partial derivatives with respect to the population variable in the HJB equation 16, we proceed as follows. For $1 \leq i \leq m-1$, we use a centred difference scheme for the first-order derivative and a three-point difference quotient for the second-order derivative:

$$\frac{\partial J^*_{i,k,j}}{\partial x} \approx \frac{J^*_{i+1,k,j} - J^*_{i-1,k,j}}{2\Delta x},$$

$$\frac{\partial^2 J^*_{i,k,j}}{\partial x^2} \approx \frac{J^*_{i+1,k,j} - 2J^*_{i,k,j} + J^*_{i-1,k,j}}{\Delta x^2}.$$

At the boundary $i = m$, the centred differences are no longer applicable due to the absence of forward values. Therefore, we use a backward difference scheme to approximate the first-order derivative and a backward difference quotient applied twice to approximate the second-order derivative:

$$\frac{\partial J^*_{m,k,j}}{\partial x} \approx \frac{J^*_{m,k,j} - J^*_{m-1,k,j}}{\Delta x},$$

$$\frac{\partial^2 J^*_{m,k,j}}{\partial x^2} \approx \frac{J^*_{m,k,j} - 2J^*_{m-1,k,j} + J^*_{m-2,k,j}}{\Delta x^2}.$$

The partial derivatives with respect to the price variable are approximated analogously.

For $1 \leq k \leq l-1$ and $k = l$, we use:

$$\frac{\partial J^*_{i,k,j}}{\partial p} \approx \frac{J^*_{i,k+1,j} - J^*_{i,k-1,j}}{2\Delta p},$$

$$\frac{\partial^2 J^*_{i,k,j}}{\partial p^2} \approx \frac{J^*_{i,k+1,j} - 2J^*_{i,k,j} + J^*_{i,k-1,j}}{\Delta p^2},$$

$$\frac{\partial J^*_{i,l,j}}{\partial p} \approx \frac{J^*_{i,l,j} - J^*_{i,l-1,j}}{\Delta p},$$

$$\frac{\partial^2 J^*_{i,l,j}}{\partial p^2} \approx \frac{J^*_{i,l,j} - 2J^*_{i,l-1,j} + J^*_{i,l-2,j}}{\Delta p^2}.$$

At the lower price boundary, $k = 0$, we approximate the first-order derivative with a forward difference quotient and the second-order derivative applying a forward difference scheme twice:

$$\frac{\partial J^*_{i,0,j}}{\partial p} \approx \frac{J^*_{i,1,j} - J^*_{i,0,j}}{\Delta p},$$

$$\frac{\partial^2 J^*_{i,0,j}}{\partial p^2} \approx \frac{J^*_{i,2,j} - 2J^*_{i,1,j} + J^*_{i,0,j}}{\Delta p^2}.$$

The cross-partial derivatives with respect to the population and price variables are computed using backward differences at the upper boundaries, forward differences at the lower boundaries and centred difference schemes for the remaining points:

$$\frac{\partial J^*_{i,k,j}}{\partial x \partial p} \approx \frac{J^*_{i+1,k+1,j} - J^*_{i-1,k+1,j} - J^*_{i+1,k-1,j} + J^*_{i-1,k-1,j}}{4\Delta x \Delta p}, \ 1 \le i \le m-1, \ 1 \le k \le l-1,$$

$$\frac{\partial J^*_{m,k,j}}{\partial x \partial p} \approx \frac{J^*_{m,k+1,j} - J^*_{m-1,k+1,j} - J^*_{m,k-1,j} + J^*_{m-1,k-1,j}}{2\Delta x \Delta p}, \ i = m, \ 1 \le k \le l-1,$$

$$\frac{\partial J^*_{i,l,j}}{\partial x \partial p} \approx \frac{J^*_{i+1,l,j} - J^*_{i-1,l,j} - J^*_{i+1,l-1,j} + J^*_{i-1,l-1,j}}{2\Delta x \Delta p}, \ 1 \le i \le m-1, \ k = l,$$

$$\frac{\partial J^*_{m,l,j}}{\partial x \partial p} \approx \frac{J^*_{m,l,j} - J^*_{m-1,l,j} - J^*_{m,l-1,j} + J^*_{m-1,l-1,j}}{\Delta x \Delta p}, \ i = m, \ k = l,$$

$$\frac{\partial J^*_{i,0,j}}{\partial x \partial p} \approx \frac{J^*_{i+1,1,j} - J^*_{i-1,1,j} - J^*_{i+1,0,j} + J^*_{i-1,0,j}}{2\Delta x \Delta p}, \ 1 \le i \le m-1, \ k = 0,$$

$$\frac{\partial J^*_{m,0,j}}{\partial x \partial p} \approx \frac{J^*_{m,1,j} - J^*_{m-1,1,j} - J^*_{m,0,j} + J^*_{m-1,0,j}}{\Delta x \Delta p}, \ i = m, \ k = 0.$$

Using these approximations, we derive the discretized form of the equation 16. For $1 \le i \le m-1$ and $1 \le k \le l-1$, the scheme is given by:

$$\begin{aligned}
-\frac{J^*_{i,k,j+1} - J^*_{i,k,j}}{\Delta t} = \frac{1}{2}\Bigg[& -\lambda J^*_{i,k,j} + \frac{J^*_{i+1,k,j} - J^*_{i-1,k,j}}{2\Delta x}\left(rX_i \ln\left(\frac{K}{X_i}\right) - qE^*_{i,k,j}X_i\right) \\
& + \frac{1}{2}\frac{J^*_{i+1,k,j} - 2J^*_{i,k,j} + J^*_{i-1,k,j}}{\Delta x^2}\sigma_x^2 X_i^2 \\
& + \frac{1}{2}\frac{J^*_{i,k+1,j} - 2J^*_{i,k,j} + J^*_{i,k-1,j}}{\Delta p^2}\sigma_p^2 p_k^2 \\
& + \frac{J^*_{i+1,k+1,j} - J^*_{i-1,k+1,j} - J^*_{i+1,k-1,j} + J^*_{i-1,k-1,j}}{4\Delta x \Delta p}\rho\sigma_x\sigma_p X_i p_k \\
& + (\lambda - y)p_k \frac{J^*_{i,k+1,j} - J^*_{i,k-1,j}}{2\Delta p} \\
& + \left(p_k q X_i - c_1 - c_2 E^*_{i,k,j}\right) E^*_{i,k,j}\Bigg] \\
+ \frac{1}{2}\Bigg[& -\lambda J^*_{i,k,j+1} + \frac{J^*_{i+1,k,j+1} - J^*_{i-1,k,j+1}}{2\Delta x}\left(rX_i \ln\left(\frac{K}{X_i}\right) - qE^*_{i,k,j+1}X_i\right) \\
& + \frac{1}{2}\frac{J^*_{i+1,k,j+1} - 2J^*_{i,k,j+1} + J^*_{i-1,k,j+1}}{\Delta x^2}\sigma_x^2 X_i^2 \\
& + \frac{1}{2}\frac{J^*_{i,k+1,j+1} - 2J^*_{i,k,j+1} + J^*_{i,k-1,j+1}}{\Delta p^2}\sigma_p^2 p_k^2 \\
& + \frac{J^*_{i+1,k+1,j+1} - J^*_{i-1,k+1,j+1} - J^*_{i+1,k-1,j+1} + J^*_{i-1,k-1,j+1}}{4\Delta x \Delta p}\rho\sigma_x\sigma_p X_i p_k \\
& + (\lambda - y)p_k \frac{J^*_{i,k+1,j+1} - J^*_{i,k-1,j+1}}{2\Delta p} \\
& + \left(p_k q X_i - c_1 - c_2 E^*_{i,k,j+1}\right) E^*_{i,k,j+1}\Bigg].
\end{aligned}$$

When $i = m$ and $1 \leq k \leq l-1$:

$$
\begin{aligned}
-\frac{J^*_{m,k,j+1} - J^*_{m,k,j}}{\Delta t} = \frac{1}{2}\Bigg[& -\lambda J^*_{m,k,j} + \frac{J^*_{m,k,j} - J^*_{m-1,k,j}}{\Delta x}\left(rX_m \ln\left(\frac{K}{X_m}\right) - qE^*_{m,k,j}X_m \right) \\
& + \frac{1}{2}\frac{J^*_{m,k,j} - 2J^*_{m-1,k,j} + J^*_{m-2,k,j}}{\Delta x^2}\sigma_x^2 X_m^2 \\
& + \frac{1}{2}\frac{J^*_{m,k+1,j} - 2J^*_{m,k,j} + J^*_{m,k-1,j}}{\Delta p^2}\sigma_p^2 p_k^2 \\
& + \frac{J^*_{m,k+1,j} - J^*_{m-1,k+1,j} - J^*_{m,k-1,j} + J^*_{m-1,k-1,j}}{2\Delta x \Delta p}\rho\sigma_x\sigma_p X_m p_k \\
& + (\lambda - y)p_k\frac{J^*_{m,k+1,j} - J^*_{m,k-1,j}}{2\Delta p} + \left(p_k q X_m - c_1 - c_2 E^*_{m,k,j}\right)E^*_{m,k,j}\Bigg] \\
+ \frac{1}{2}\Bigg[& -\lambda J^*_{m,k,j+1} + \frac{J^*_{m,k,j+1} - J^*_{m-1,k,j+1}}{\Delta x}\left(rX_m \ln\left(\frac{K}{X_m}\right) - qE^*_{m,k,j+1}X_m \right) \\
& + \frac{1}{2}\frac{J^*_{m,k,j+1} - 2J^*_{m-1,k,j+1} + J^*_{m-2,k,j+1}}{\Delta x^2}\sigma_x^2 X_m^2 \\
& + \frac{1}{2}\frac{J^*_{m,k+1,j+1} - 2J^*_{m,k,j+1} + J^*_{m,k-1,j+1}}{\Delta p^2}\sigma_p^2 p_k^2 \\
& + \frac{J^*_{m,k+1,j+1} - J^*_{m-1,k+1,j+1} - J^*_{m,k-1,j+1} + J^*_{m-1,k-1,j+1}}{2\Delta x \Delta p}\rho\sigma_x\sigma_p X_m p_k \\
& + (\lambda - y)p_k\frac{J^*_{m,k+1,j+1} - J^*_{m,k-1,j+1}}{2\Delta p} + \left(p_k q X_m - c_1 - c_2 E^*_{m,k,j+1}\right)E^*_{m,k,j+1}\Bigg].
\end{aligned}
$$

When $1 \leq i \leq m-1$ and $k = l$:

$$
\begin{aligned}
-\frac{J^*_{i,l,j+1} - J^*_{i,l,j}}{\Delta t} = \frac{1}{2}\Bigg[& -\lambda J^*_{i,l,j} + \frac{J^*_{i+1,l,j} - J^*_{i-1,l,j}}{2\Delta x}\left(rX_i \ln\left(\frac{K}{X_i}\right) - qE^*_{i,l,j}X_i \right) \\
& + \frac{1}{2}\frac{J^*_{i+1,l,j} - 2J^*_{i,l,j} + J^*_{i-1,l,j}}{\Delta x^2}\sigma_x^2 X_i^2 \\
& + \frac{1}{2}\frac{J^*_{i,l,j} - 2J^*_{i,l-1,j} + J^*_{i,l-2,j}}{\Delta p^2}\sigma_p^2 p_l^2 \\
& + \frac{J^*_{i+1,l,j} - J^*_{i-1,l,j} - J^*_{i+1,l-1,j} + J^*_{i-1,l-1,j}}{2\Delta x \Delta p}\rho\sigma_x\sigma_p X_i p_l \\
& + (\lambda - y)p_l\frac{J^*_{i,l,j} - J^*_{i,l-1,j}}{\Delta p} + \left(p_l q X_i - c_1 - c_2 E^*_{i,l,j}\right)E^*_{i,l,j}\Bigg] \\
+ \frac{1}{2}\Bigg[& -\lambda J^*_{i,l,j+1} + \frac{J^*_{i+1,l,j+1} - J^*_{i-1,l,j+1}}{2\Delta x}\left(rX_i \ln\left(\frac{K}{X_i}\right) - qE^*_{i,l,j+1}X_i \right) \\
& + \frac{1}{2}\frac{J^*_{i+1,l,j+1} - 2J^*_{i,l,j+1} + J^*_{i-1,l,j+1}}{\Delta x^2}\sigma_x^2 X_i^2 \\
& + \frac{1}{2}\frac{J^*_{i,l,j+1} - 2J^*_{i,l-1,j+1} + J^*_{i,l-2,j+1}}{\Delta p^2}\sigma_p^2 p_l^2 \\
& + \frac{J^*_{i+1,l,j+1} - J^*_{i-1,l,j+1} - J^*_{i+1,l-1,j+1} + J^*_{i-1,l-1,j+1}}{2\Delta x \Delta p}\rho\sigma_x\sigma_p X_i p_l \\
& + (\lambda - y)p_l\frac{J^*_{i,l,j+1} - J^*_{i,l-1,j+1}}{\Delta p} + \left(p_l q X_i - c_1 - c_2 E^*_{i,l,j+1}\right)E^*_{i,l,j+1}\Bigg].
\end{aligned}
$$

When $i = m$ and $k = l$:

$$
\begin{aligned}
-\frac{J^*_{m,l,j+1} - J^*_{m,l,j}}{\Delta t} = \frac{1}{2}\Bigg[& -\lambda J^*_{m,l,j} + \frac{J^*_{m,l,j} - J^*_{m-1,l,j}}{\Delta x}\left(rX_m \ln\left(\frac{K}{X_m}\right) - qE^*_{m,l,j}X_m\right) \\
& + \frac{1}{2}\frac{J^*_{m,l,j} - 2J^*_{m-1,l,j} + J^*_{m-2,l,j}}{\Delta x^2}\sigma_x^2 X_m^2 \\
& + \frac{1}{2}\frac{J^*_{m,l,j} - 2J^*_{m,l-1,j} + J^*_{m,l-2,j}}{\Delta p^2}\sigma_p^2 p_l^2 \\
& + \frac{J^*_{m,l,j} - J^*_{m-1,l,j} - J^*_{m,l-1,j} + J^*_{m-1,l-1,j}}{\Delta x \Delta p}\rho\sigma_x\sigma_p X_m p_l \\
& + (\lambda - y)p_l \frac{J^*_{m,l,j} - J^*_{m,l-1,j}}{\Delta p} + \left(p_l q X_m - c_1 - c_2 E^*_{m,l,j}\right)E^*_{m,l,j}\Bigg] \\
+ \frac{1}{2}\Bigg[& -\lambda J^*_{m,l,j+1} + \frac{J^*_{m,l,j+1} - J^*_{m-1,l,j+1}}{\Delta x}\left(rX_m \ln\left(\frac{K}{X_m}\right) - qE^*_{m,l,j+1}X_m\right) \\
& + \frac{1}{2}\frac{J^*_{m,l,j+1} - 2J^*_{m-1,l,j+1} + J^*_{m-2,l,j+1}}{\Delta x^2}\sigma_x^2 X_m^2 \\
& + \frac{1}{2}\frac{J^*_{m,l,j+1} - 2J^*_{m,l-1,j+1} + J^*_{m,l-2,j+1}}{\Delta p^2}\sigma_p^2 p_l^2 \\
& + \frac{J^*_{m,l,j+1} - J^*_{m-1,l,j+1} - J^*_{m,l-1,j+1} + J^*_{m-1,l-1,j+1}}{\Delta x \Delta p}\rho\sigma_x\sigma_p X_m p_l \\
& + (\lambda - y)p_l \frac{J^*_{m,l,j+1} - J^*_{m,l-1,j+1}}{\Delta p} + \left(p_l q X_m - c_1 - c_2 E^*_{m,l,j+1}\right)E^*_{m,l,j+1}\Bigg].
\end{aligned}
$$

When $1 \leq i \leq m-1$ and $k = 0$:

$$
\begin{aligned}
-\frac{J^*_{i,0,j+1} - J^*_{i,0,j}}{\Delta t} = \frac{1}{2}\Bigg[& -\lambda J^*_{i,0,j} + \frac{J^*_{i+1,0,j} - J^*_{i-1,0,j}}{2\Delta x}\left(rX_i \ln\left(\frac{K}{X_i}\right) - qE^*_{i,0,j}X_i\right) \\
& + \frac{1}{2}\frac{J^*_{i+1,0,j} - 2J^*_{i,0,j} + J^*_{i-1,0,j}}{\Delta x^2}\sigma_x^2 X_i^2 \\
& + \frac{1}{2}\frac{J^*_{i,2,j} - 2J^*_{i,1,j} + J^*_{i,0,j}}{\Delta p^2}\sigma_p^2 p_0^2 \\
& + \frac{J^*_{i+1,1,j} - J^*_{i-1,1,j} - J^*_{i+1,0,j} + J^*_{i-1,0,j}}{2\Delta x \Delta p}\rho\sigma_x\sigma_p X_i p_0 \\
& + (\lambda - y)p_0 \frac{J^*_{i,1,j} - J^*_{i,0,j}}{\Delta p} + \left(p_0 q X_i - c_1 - c_2 E^*_{i,0,j}\right)E^*_{i,0,j}\Bigg] \\
+ \frac{1}{2}\Bigg[& -\lambda J^*_{i,0,j+1} + \frac{J^*_{i+1,0,j+1} - J^*_{i-1,0,j+1}}{2\Delta x}\left(rX_i \ln\left(\frac{K}{X_i}\right) - qE^*_{i,0,j+1}X_i\right) \\
& + \frac{1}{2}\frac{J^*_{i+1,0,j+1} - 2J^*_{i,0,j+1} + J^*_{i-1,0,j+1}}{\Delta x^2}\sigma_x^2 X_i^2 \\
& + \frac{1}{2}\frac{J^*_{i,2,j+1} - 2J^*_{i,1,j+1} + J^*_{i,0,j+1}}{\Delta p^2}\sigma_p^2 p_0^2 \\
& + \frac{J^*_{i+1,1,j+1} - J^*_{i-1,1,j+1} - J^*_{i+1,0,j+1} + J^*_{i-1,0,j+1}}{2\Delta x \Delta p}\rho\sigma_x\sigma_p X_i p_0 \\
& + (\lambda - y)p_0 \frac{J^*_{i,1,j+1} - J^*_{i,0,j+1}}{\Delta p} + \left(p_0 q X_i - c_1 - c_2 E^*_{i,0,j+1}\right)E^*_{i,0,j+1}\Bigg].
\end{aligned}
$$

When $i = m$ and $k = 0$:

$$
\begin{aligned}
-\frac{J^*_{m,0,j+1} - J^*_{m,0,j}}{\Delta t} &= \frac{1}{2}\Bigg[-\lambda J^*_{m,0,j} + \frac{J^*_{m,0,j} - J^*_{m-1,0,j}}{\Delta x}\left(rX_m \ln\left(\frac{K}{X_m}\right) - qE^*_{m,0,j}X_m \right) \\
&\quad + \frac{1}{2}\frac{J^*_{m,0,j} - 2J^*_{m-1,0,j} + J^*_{m-2,0,j}}{\Delta x^2}\sigma_x^2 X_m^2 \\
&\quad + \frac{1}{2}\frac{J^*_{m,2,j} - 2J^*_{m,1,j} + J^*_{m,0,j}}{\Delta p^2}\sigma_p^2 p_0^2 \\
&\quad + \frac{J^*_{m,1,j} - J^*_{m-1,1,j} - J^*_{m,0,j} + J^*_{m-1,0,j}}{\Delta x \Delta p}\rho\sigma_x\sigma_p X_m p_0 \\
&\quad + (\lambda - y)p_0 \frac{J^*_{m,1,j} - J^*_{m,0,j}}{\Delta p} + \left(p_0 q X_m - c_1 - c_2 E^*_{m,0,j}\right)E^*_{m,0,j} \Bigg] \\
&\quad + \frac{1}{2}\Bigg[-\lambda J^*_{m,0,j+1} + \frac{J^*_{m,0,j+1} - J^*_{m-1,0,j+1}}{\Delta x}\left(rX_m \ln\left(\frac{K}{X_m}\right) - qE^*_{m,0,j+1}X_m \right) \\
&\quad + \frac{1}{2}\frac{J^*_{m,0,j+1} - 2J^*_{m-1,0,j+1} + J^*_{m-2,0,j+1}}{\Delta x^2}\sigma_x^2 X_m^2 \\
&\quad + \frac{1}{2}\frac{J^*_{m,2,j+1} - 2J^*_{m,1,j+1} + J^*_{m,0,j+1}}{\Delta p^2}\sigma_p^2 p_0^2 \\
&\quad + \frac{J^*_{m,1,j+1} - J^*_{m-1,1,j+1} - J^*_{m,0,j+1} + J^*_{m-1,0,j+1}}{\Delta x \Delta p}\rho\sigma_x\sigma_p X_m p_0 \\
&\quad + (\lambda - y)p_l \frac{J^*_{m,1,j+1} - J^*_{m,0,j+1}}{\Delta p} + \left(p_0 q X_m - c_1 - c_2 E^*_{m,0,j+1}\right)E^*_{m,0,j+1} \Bigg].
\end{aligned}
$$

Now, by rearranging the terms in the equations, we isolate the option value at each grid point corresponding to time step j on the left-hand side and time step $j+1$ on the right-hand side. When $1 \leq i \leq m-1$ and $1 \leq k \leq l-1$, the discretized equation becomes:

$$
\begin{aligned}
&\left(1 + \frac{\Delta t}{2}\lambda + \frac{\Delta t}{2\Delta x^2}\sigma_x^2 X_i^2 + \frac{\Delta t}{2\Delta p^2}\sigma_p^2 p_k^2\right) J^*_{i,k,j} \\
&- \left(\frac{\Delta t}{4\Delta x}\left(rX_i \ln\left(\frac{K}{X_i}\right) - qE^*_{i,k,j}X_i\right) + \frac{\Delta t}{4\Delta x^2}\sigma_x^2 X_i^2\right) J^*_{i+1,k,j} \\
&+ \left(\frac{\Delta t}{4\Delta x}\left(rX_i \ln\left(\frac{K}{X_i}\right) - qE^*_{i,k,j}X_i\right) - \frac{\Delta t}{4\Delta x^2}\sigma_x^2 X_i^2\right) J^*_{i-1,k,j} \\
&- \left(\frac{\Delta t}{4\Delta p}(\lambda - y)p_k + \frac{\Delta t}{4\Delta p^2}\sigma_p^2 p_k^2\right) J^*_{i,k+1,j} + \left(\frac{\Delta t}{4\Delta p}(\lambda - y)p_k - \frac{\Delta t}{4\Delta p^2}\sigma_p^2 p_k^2\right) J^*_{i,k-1,j} \\
&- \left(\frac{\Delta t}{8\Delta x \Delta p}\rho\sigma_x X_i \sigma_p p_k\right) J^*_{i+1,k+1,j} + \left(\frac{\Delta t}{8\Delta x \Delta p}\rho\sigma_x X_i \sigma_p p_k\right) J^*_{i-1,k+1,j} \\
&+ \left(\frac{\Delta t}{8\Delta x \Delta p}\rho\sigma_x X_i \sigma_p p_k\right) J^*_{i+1,k-1,j} - \left(\frac{\Delta t}{8\Delta x \Delta p}\rho\sigma_x X_i \sigma_p p_k\right) J^*_{i-1,k-1,j} \\
&- \left(\frac{\Delta t}{2}\left(p_k q X_i - c_1 - c_2 E^*_{i,k,j}\right) E^*_{i,k,j}\right) \\
&= \left(1 - \frac{\Delta t}{2}\lambda - \frac{\Delta t}{2\Delta x^2}\sigma_x^2 X_i^2 - \frac{\Delta t}{2\Delta p^2}\sigma_p^2 p_k^2\right) J^*_{i,k,j+1} \\
&+ \left(\frac{\Delta t}{4\Delta x}\left(rX_i \ln\left(\frac{K}{X_i}\right) - qE^*_{i,k,j+1}X_i\right) + \frac{\Delta t}{4\Delta x^2}\sigma_x^2 X_i^2\right) J^*_{i+1,k,j+1} \\
&- \left(\frac{\Delta t}{4\Delta x}\left(rX_i \ln\left(\frac{K}{X_i}\right) - qE^*_{i,k,j+1}X_i\right) - \frac{\Delta t}{4\Delta x^2}\sigma_x^2 X_i^2\right) J^*_{i-1,k,j+1}
\end{aligned}
$$

$$+\left(\frac{\Delta t}{4\Delta p}(\lambda-y)p_k+\frac{\Delta t}{4\Delta p^2}\sigma_p^2p_k^2\right)J^*_{i,k+1,j+1}-\left(\frac{\Delta t}{4\Delta p}(\lambda-y)p_k-\frac{\Delta t}{4\Delta p^2}\sigma_p^2p_k^2\right)J^*_{i,k-1,j+1}$$
$$+\left(\frac{\Delta t}{8\Delta x\Delta p}\rho\sigma_xX_i\sigma_pp_k\right)J^*_{i+1,k+1,j+1}-\left(\frac{\Delta t}{8\Delta x\Delta p}\rho\sigma_xX_i\sigma_pp_k\right)J^*_{i-1,k+1,j+1}$$
$$-\left(\frac{\Delta t}{8\Delta x\Delta p}\rho\sigma_xX_i\sigma_pp_k\right)J^*_{i+1,k-1,j+1}+\left(\frac{\Delta t}{8\Delta x\Delta p}\rho\sigma_xX_i\sigma_pp_k\right)J^*_{i-1,k-1,j+1}$$
$$+\left(\frac{\Delta t}{2}\left(p_kqX_i-c_1-c_2E^*_{i,k,j+1}\right)E^*_{i,k,j+1}\right).$$

When $i=m$ and $1\le k\le l-1$:

$$\left(1+\frac{\Delta t}{2}\lambda-\frac{\Delta t}{2\Delta x}\left(rX_m\ln\left(\frac{K}{X_m}\right)-qE^*_{m,k,j}X_m\right)-\frac{\Delta t}{4\Delta x^2}\sigma_x^2X_m^2+\frac{\Delta t}{2\Delta p^2}\sigma_p^2p_k^2\right)J^*_{m,k,j}$$
$$+\left(\frac{\Delta t}{2\Delta x}\left(rX_m\ln\left(\frac{K}{X_m}\right)-qE^*_{m,k,j}X_m\right)+\frac{\Delta t}{2\Delta x^2}\sigma_x^2X_m^2\right)J^*_{m-1,k,j}$$
$$-\left(\frac{\Delta t}{4\Delta x^2}\sigma_x^2X_m^2\right)J^*_{m-2,k,j}$$
$$-\left(\frac{\Delta t}{4\Delta p}(\lambda-y)p_k+\frac{\Delta t}{4\Delta p^2}\sigma_p^2p_k^2+\frac{\Delta t}{4\Delta x\Delta p}\rho\sigma_xX_m\sigma_pp_k\right)J^*_{m,k+1,j}$$
$$+\left(\frac{\Delta t}{4\Delta p}(\lambda-y)p_k-\frac{\Delta t}{4\Delta p^2}\sigma_p^2p_k^2+\frac{\Delta t}{4\Delta x\Delta p}\rho\sigma_xX_m\sigma_pp_k\right)J^*_{m,k-1,j}$$
$$+\left(\frac{\Delta t}{4\Delta x\Delta p}\rho\sigma_xX_m\sigma_pp_k\right)J^*_{m-1,k+1,j}-\left(\frac{\Delta t}{4\Delta x\Delta p}\rho\sigma_xX_m\sigma_pp_k\right)J^*_{m-1,k-1,j}$$
$$-\left(\frac{\Delta t}{2}\left(p_kqX_m-c_1-c_2E^*_{m,k,j}\right)E^*_{m,k,j}\right)$$
$$=\left(1-\frac{\Delta t}{2}\lambda+\frac{\Delta t}{2\Delta x}\left(rX_m\ln\left(\frac{K}{X_m}\right)-qE^*_{m,k,j+1}X_m\right)+\frac{\Delta t}{4\Delta x^2}\sigma_x^2X_m^2-\frac{\Delta t}{2\Delta p^2}\sigma_p^2p_k^2\right)J^*_{m,k,j+1}$$
$$-\left(\frac{\Delta t}{2\Delta x}\left(rX_m\ln\left(\frac{K}{X_m}\right)-qE^*_{m,k,j+1}X_m\right)+\frac{\Delta t}{2\Delta x^2}\sigma_x^2X_m^2\right)J^*_{m-1,k,j+1}$$
$$+\left(\frac{\Delta t}{4\Delta x^2}\sigma_x^2X_m^2\right)J^*_{m-2,k,j+1}$$
$$+\left(\frac{\Delta t}{4\Delta p}(\lambda-y)p_k+\frac{\Delta t}{4\Delta p^2}\sigma_p^2p_k^2+\frac{\Delta t}{4\Delta x\Delta p}\rho\sigma_xX_m\sigma_pp_k\right)J^*_{m,k+1,j+1}$$
$$-\left(\frac{\Delta t}{4\Delta p}(\lambda-y)p_k-\frac{\Delta t}{4\Delta p^2}\sigma_p^2p_k^2+\frac{\Delta t}{4\Delta x\Delta p}\rho\sigma_xX_m\sigma_pp_k\right)J^*_{m,k-1,j+1}$$
$$-\left(\frac{\Delta t}{4\Delta x\Delta p}\rho\sigma_xX_m\sigma_pp_k\right)J^*_{m-1,k+1,j+1}+\left(\frac{\Delta t}{4\Delta x\Delta p}\rho\sigma_xX_m\sigma_pp_k\right)J^*_{m-1,k-1,j+1}$$
$$+\left(\frac{\Delta t}{2}\left(p_kqX_m-c_1-c_2E^*_{m,k,j+1}\right)E^*_{m,k,j+1}\right).$$

When $1\le i\le m-1$ and $k=l$:

$$\left(1+\frac{\Delta t}{2}\lambda-\frac{\Delta t}{2\Delta p}\left(\lambda-y\right)p_l+\frac{\Delta t}{2\Delta x^2}\sigma_x^2X_i^2-\frac{\Delta t}{4\Delta p^2}\sigma_p^2p_l^2\right)J^*_{i,l,j}$$
$$-\left(\frac{\Delta t}{4\Delta x}\left(rX_i\ln\left(\frac{K}{X_i}\right)-qE^*_{i,l,j}X_i\right)+\frac{\Delta t}{4\Delta x^2}\sigma_x^2X_i^2+\frac{\Delta t}{4\Delta x\Delta p}\rho\sigma_xX_i\sigma_pp_l\right)J^*_{i+1,l,j}$$
$$+\left(\frac{\Delta t}{4\Delta x}\left(rX_i\ln\left(\frac{K}{X_i}\right)-qE^*_{i,l,j}X_i\right)-\frac{\Delta t}{4\Delta x^2}\sigma_x^2X_i^2+\frac{\Delta t}{4\Delta x\Delta p}\rho\sigma_xX_i\sigma_pp_l\right)J^*_{i-1,l,j}$$
$$+\left(\frac{\Delta t}{2\Delta p}(\lambda-y)p_l+\frac{\Delta t}{2\Delta p^2}\sigma_p^2p_l^2\right)J^*_{i,l-1,j}-\left(\frac{\Delta t}{4\Delta p^2}\sigma_p^2p_l^2\right)J^*_{i,l-2,j}$$

$$
\begin{aligned}
&+\left(\frac{\Delta t}{4\Delta x\Delta p}\rho\sigma_x X_i\sigma_p p_l\right)J^*_{i+1,l-1,j}-\left(\frac{\Delta t}{4\Delta x\Delta p}\rho\sigma_x X_i\sigma_p p_l\right)J^*_{i-1,l-1,j}\\
&-\left(\frac{\Delta t}{2}\left(p_l q X_i-c_1-c_2E^*_{i,l,j}\right)E^*_{i,l,j}\right)\\
&=\left(1-\frac{\Delta t}{2}\lambda+\frac{\Delta t}{2\Delta p}\left(\lambda-y\right)p_l-\frac{\Delta t}{2\Delta x^2}\sigma_x^2X_i^2+\frac{\Delta t}{4\Delta p^2}\sigma_p^2p_l^2\right)J^*_{i,l,j+1}\\
&+\left(\frac{\Delta t}{4\Delta x}\left(rX_i\ln\left(\frac{K}{X_i}\right)-qE^*_{i,l,j+1}X_i\right)+\frac{\Delta t}{4\Delta x^2}\sigma_x^2X_i^2+\frac{\Delta t}{4\Delta x\Delta p}\rho\sigma_x X_i\sigma_p p_l\right)J^*_{i+1,l,j+1}\\
&-\left(\frac{\Delta t}{4\Delta x}\left(rX_i\ln\left(\frac{K}{X_i}\right)-qE^*_{i,l,j+1}X_i\right)-\frac{\Delta t}{4\Delta x^2}\sigma_x^2X_i^2+\frac{\Delta t}{4\Delta x\Delta p}\rho\sigma_x X_i\sigma_p p_l\right)J^*_{i-1,l,j+1}\\
&-\left(\frac{\Delta t}{2\Delta p}(\lambda-y)p_l+\frac{\Delta t}{2\Delta p^2}\sigma_p^2p_l^2\right)J^*_{i,l-1,j+1}+\left(\frac{\Delta t}{4\Delta p^2}\sigma_p^2p_l^2\right)J^*_{i,l-2,j+1}\\
&-\left(\frac{\Delta t}{4\Delta x\Delta p}\rho\sigma_x X_i\sigma_p p_l\right)J^*_{i+1,l-1,j+1}+\left(\frac{\Delta t}{4\Delta x\Delta p}\rho\sigma_x X_i\sigma_p p_l\right)J^*_{i-1,l-1,j+1}\\
&+\left(\frac{\Delta t}{2}\left(p_l q X_i-c_1-c_2E^*_{i,l,j+1}\right)E^*_{i,l,j+1}\right).
\end{aligned}
$$

When $i=m$ and $k=l$:

$$
\begin{aligned}
&\left(1+\frac{\Delta t}{2}\lambda-\frac{\Delta t}{2\Delta x}\left(rX_m\ln\left(\frac{K}{X_m}\right)-qE^*_{m,l,j}X_m\right)-\frac{\Delta t}{2\Delta p}\left(\lambda-y\right)p_l-\frac{\Delta t}{4\Delta x^2}\sigma_x^2X_m^2\right.\\
&\left.-\frac{\Delta t}{4\Delta p^2}\sigma_p^2p_l^2-\frac{\Delta t}{2\Delta x\Delta p}\rho\sigma_x X_m\sigma_p p_l\right)J^*_{m,l,j}\\
&+\left(\frac{\Delta t}{2\Delta x}\left(rX_m\ln\left(\frac{K}{X_m}\right)-qE^*_{m,l,j}X_m\right)+\frac{\Delta t}{2\Delta x^2}\sigma_x^2X_m^2+\frac{\Delta t}{2\Delta x\Delta p}\rho\sigma_x X_m\sigma_p p_l\right)J^*_{m-1,l,j}\\
&-\left(\frac{\Delta t}{4\Delta x^2}\sigma_x^2X_m^2\right)J^*_{m-2,l,j}+\left(\frac{\Delta t}{2\Delta p}(\lambda-y)p_l+\frac{\Delta t}{2\Delta p^2}\sigma_p^2p_l^2+\frac{\Delta t}{2\Delta x\Delta p}\rho\sigma_x X_m\sigma_p p_l\right)J^*_{m,l-1,j}\\
&-\left(\frac{\Delta t}{4\Delta p^2}\sigma_p^2p_l^2\right)J^*_{m,l-2,j}-\left(\frac{\Delta t}{2\Delta x\Delta p}\rho\sigma_x X_m\sigma_p p_l\right)J^*_{m-1,l-1,j}\\
&-\left(\frac{\Delta t}{2}\left(p_l q X_m-c_1-c_2E^*_{m,l,j}\right)E^*_{m,l,j}\right)\\
&=\left(1-\frac{\Delta t}{2}\lambda+\frac{\Delta t}{2\Delta x}\left(rX_m\ln\left(\frac{K}{X_m}\right)-qE^*_{m,l,j+1}X_m\right)+\frac{\Delta t}{2\Delta p}\left(\lambda-y\right)p_l+\frac{\Delta t}{4\Delta x^2}\sigma_x^2X_m^2\right.\\
&\left.+\frac{\Delta t}{4\Delta p^2}\sigma_p^2p_l^2+\frac{\Delta t}{2\Delta x\Delta p}\rho\sigma_x X_m\sigma_p p_l\right)J^*_{m,l,j+1}\\
&-\left(\frac{\Delta t}{2\Delta x}\left(rX_m\ln\left(\frac{K}{X_m}\right)-qE^*_{m,l,j+1}X_m\right)+\frac{\Delta t}{2\Delta x^2}\sigma_x^2X_m^2+\frac{\Delta t}{2\Delta x\Delta p}\rho\sigma_x X_m\sigma_p p_l\right)J^*_{m-1,l,j+1}\\
&+\left(\frac{\Delta t}{4\Delta x^2}\sigma_x^2X_m^2\right)J^*_{m-2,l,j+1}-\left(\frac{\Delta t}{2\Delta p}(\lambda-y)p_l+\frac{\Delta t}{2\Delta p^2}\sigma_p^2p_l^2+\frac{\Delta t}{2\Delta x\Delta p}\rho\sigma_x X_m\sigma_p p_l\right)J^*_{m,l-1,j+1}\\
&+\left(\frac{\Delta t}{4\Delta p^2}\sigma_p^2p_l^2\right)J^*_{m,l-2,j+1}+\left(\frac{\Delta t}{2\Delta x\Delta p}\rho\sigma_x X_m\sigma_p p_l\right)J^*_{m-1,l-1,j+1}\\
&+\left(\frac{\Delta t}{2}\left(p_l q X_m-c_1-c_2E^*_{m,l,j+1}\right)E^*_{m,l,j+1}\right).
\end{aligned}
$$

When $1 \leq i \leq m-1$ and $k=0$:

$$
\begin{aligned}
&\left(1+\frac{\Delta t}{2}\lambda+\frac{\Delta t}{2\Delta p}\left(\lambda-y\right)p_0+\frac{\Delta t}{2\Delta x^2}\sigma_x^2X_i^2-\frac{\Delta t}{4\Delta p^2}\sigma_p^2p_0^2\right)J^*_{i,0,j}\\
&-\left(\frac{\Delta t}{4\Delta x}\left(rX_i\ln\left(\frac{K}{X_i}\right)-qE^*_{i,0,j}X_i\right)+\frac{\Delta t}{4\Delta x^2}\sigma_x^2X_i^2-\frac{\Delta t}{4\Delta x\Delta p}\rho\sigma_xX_i\sigma_pp_0\right)J^*_{i+1,0,j}\\
&+\left(\frac{\Delta t}{4\Delta x}\left(rX_i\ln\left(\frac{K}{X_i}\right)-qE^*_{i,0,j}X_i\right)-\frac{\Delta t}{4\Delta x^2}\sigma_x^2X_i^2-\frac{\Delta t}{4\Delta x\Delta p}\rho\sigma_xX_i\sigma_pp_0\right)J^*_{i-1,0,j}\\
&-\left(\frac{\Delta t}{2\Delta p}(\lambda-y)p_0-\frac{\Delta t}{2\Delta p^2}\sigma_p^2p_0^2\right)J^*_{i,1,j}-\left(\frac{\Delta t}{4\Delta p^2}\sigma_p^2p_0^2\right)J^*_{i,2,j}\\
&-\left(\frac{\Delta t}{4\Delta x\Delta p}\rho\sigma_xX_i\sigma_pp_0\right)J^*_{i+1,1,j}+\left(\frac{\Delta t}{4\Delta x\Delta p}\rho\sigma_xX_i\sigma_pp_0\right)J^*_{i-1,1,j}\\
&-\left(\frac{\Delta t}{2}\left(p_0qX_i-c_1-c_2E^*_{i,0,j}\right)E^*_{i,0,j}\right)\\
&=\left(1-\frac{\Delta t}{2}\lambda-\frac{\Delta t}{2\Delta p}\left(\lambda-y\right)p_0-\frac{\Delta t}{2\Delta x^2}\sigma_x^2X_i^2+\frac{\Delta t}{4\Delta p^2}\sigma_p^2p_0^2\right)J^*_{i,0,j+1}\\
&+\left(\frac{\Delta t}{4\Delta x}\left(rX_i\ln\left(\frac{K}{X_i}\right)-qE^*_{i,0,j+1}X_i\right)+\frac{\Delta t}{4\Delta x^2}\sigma_x^2X_i^2-\frac{\Delta t}{4\Delta x\Delta p}\rho\sigma_xX_i\sigma_pp_0\right)J^*_{i+1,0,j+1}\\
&-\left(\frac{\Delta t}{4\Delta x}\left(rX_i\ln\left(\frac{K}{X_i}\right)-qE^*_{i,0,j+1}X_i\right)-\frac{\Delta t}{4\Delta x^2}\sigma_x^2X_i^2-\frac{\Delta t}{4\Delta x\Delta p}\rho\sigma_xX_i\sigma_pp_0\right)J^*_{i-1,0,j+1}\\
&+\left(\frac{\Delta t}{2\Delta p}(\lambda-y)p_0-\frac{\Delta t}{2\Delta p^2}\sigma_p^2p_0^2\right)J^*_{i,1,j+1}+\left(\frac{\Delta t}{4\Delta p^2}\sigma_p^2p_0^2\right)J^*_{i,2,j+1}\\
&+\left(\frac{\Delta t}{4\Delta x\Delta p}\rho\sigma_xX_i\sigma_pp_0\right)J^*_{i+1,1,j+1}-\left(\frac{\Delta t}{4\Delta x\Delta p}\rho\sigma_xX_i\sigma_pp_0\right)J^*_{i-1,1,j+1}\\
&+\left(\frac{\Delta t}{2}\left(p_0qX_i-c_1-c_2E^*_{i,0,j+1}\right)E^*_{i,0,j+1}\right).
\end{aligned}
$$

When $i=m$ and $k=0$:

$$
\begin{aligned}
&\left(1+\frac{\Delta t}{2}\lambda-\frac{\Delta t}{2\Delta x}\left(rX_m\ln\left(\frac{K}{X_m}\right)-qE^*_{m,0,j}X_m\right)+\frac{\Delta t}{2\Delta p}\left(\lambda-y\right)p_0-\frac{\Delta t}{4\Delta x^2}\sigma_x^2X_m^2\right.\\
&\left.-\frac{\Delta t}{4\Delta p^2}\sigma_p^2p_0^2+\frac{\Delta t}{2\Delta x\Delta p}\rho\sigma_xX_m\sigma_pp_0\right)J^*_{m,0,j}\\
&+\left(\frac{\Delta t}{2\Delta x}\left(rX_m\ln\left(\frac{K}{X_m}\right)-qE^*_{m,0,j}X_m\right)+\frac{\Delta t}{2\Delta x^2}\sigma_x^2X_m^2-\frac{\Delta t}{2\Delta x\Delta p}\rho\sigma_xX_m\sigma_pp_0\right)J^*_{m-1,0,j}\\
&-\left(\frac{\Delta t}{4\Delta x^2}\sigma_x^2X_m^2\right)J^*_{m-2,0,j}-\left(\frac{\Delta t}{2\Delta p}(\lambda-y)p_0-\frac{\Delta t}{2\Delta p^2}\sigma_p^2p_0^2+\frac{\Delta t}{2\Delta x\Delta p}\rho\sigma_xX_m\sigma_pp_0\right)J^*_{m,1,j}\\
&-\left(\frac{\Delta t}{4\Delta p^2}\sigma_p^2p_0^2\right)J^*_{m,2,j}+\left(\frac{\Delta t}{2\Delta x\Delta p}\rho\sigma_xX_m\sigma_pp_0\right)J^*_{m-1,1,j}-\left(\frac{\Delta t}{2}\left(p_0qX_m-c_1-c_2E^*_{m,0,j}\right)E^*_{m,0,j}\right)\\
&=\left(1-\frac{\Delta t}{2}\lambda+\frac{\Delta t}{2\Delta x}\left(rX_m\ln\left(\frac{K}{X_m}\right)-qE^*_{m,0,j+1}X_m\right)-\frac{\Delta t}{2\Delta p}\left(\lambda-y\right)p_0+\frac{\Delta t}{4\Delta x^2}\sigma_x^2X_m^2\right.\\
&\left.+\frac{\Delta t}{4\Delta p^2}\sigma_p^2p_0^2-\frac{\Delta t}{2\Delta x\Delta p}\rho\sigma_xX_m\sigma_pp_0\right)J^*_{m,0,j+1}\\
&-\left(\frac{\Delta t}{2\Delta x}\left(rX_m\ln\left(\frac{K}{X_m}\right)-qE^*_{m,0,j+1}X_m\right)+\frac{\Delta t}{2\Delta x^2}\sigma_x^2X_m^2-\frac{\Delta t}{2\Delta x\Delta p}\rho\sigma_xX_m\sigma_pp_0\right)J^*_{m-1,0,j+1}\\
&+\left(\frac{\Delta t}{4\Delta x^2}\sigma_x^2X_m^2\right)J^*_{m-2,0,j+1}+\left(\frac{\Delta t}{2\Delta p}(\lambda-y)p_0-\frac{\Delta t}{2\Delta p^2}\sigma_p^2p_0^2+\frac{\Delta t}{2\Delta x\Delta p}\rho\sigma_xX_m\sigma_pp_0\right)J^*_{m,1,j+1}\\
&+\left(\frac{\Delta t}{4\Delta p^2}\sigma_p^2p_0^2\right)J^*_{m,2,j+1}-\left(\frac{\Delta t}{2\Delta x\Delta p}\rho\sigma_xX_m\sigma_pp_0\right)J^*_{m-1,1,j+1}+\left(\frac{\Delta t}{2}\left(p_0qX_m-c_1-c_2E^*_{m,0,j+1}\right)E^*_{m,0,j+1}\right).
\end{aligned}
$$

The unconstrained harvesting effort is discretized as:

$$E_{i,k,j} = \frac{qX_i}{2c2}\left(p_k - \frac{J^*_{i+1,k,j} - J^*_{i-1,k,j}}{2\Delta x}\right) - \frac{c_1}{2c_2},\ 1 \le i \le m-1,\ 0 \le k \le l,\ 0 \le j \le n,$$

$$E_{m,k,j} = \frac{qX_m}{2c2}\left(p_k - \frac{J^*_{m,k,j} - J^*_{m-1,k,j}}{\Delta x}\right) - \frac{c_1}{2c_2},\ i = m,\ 0 \le k \le l,\ 0 \le j \le n.$$

This problem can be reformulated as a system of $(m+1) \times (l+1)$ equations of the form:

$$AJ^*_- = BJ^*_+ + C$$

where:

- $J^*_- = \left[J^*_0 \mid J^*_1 \mid \cdots \mid J^*_{n-1}\right],$
- $J^*_+ = \left[J^*_1 \mid J^*_2 \mid \cdots \mid J^*_n\right],$
- $J^*_j = \left[J^*_{0,0,j} \mid \cdots \mid J^*_{i,k,j} \mid \cdots \mid J^*_{m,l,j}\right], \quad 0 \le i \le m, \quad 0 \le k \le l, \quad 0 \le j \le n.$

By solving this system recursively backwards in time, we get the optimal option values at each point on the grid. Finally, to determine the optimal solution at a specific population level X and time t_d, we perform a polynomial interpolation by approximating the value of X using its neighbouring points in the partition $\{x_0, x_1, \ldots, x_m\}$.

5 Numerical Results and Interpretation

In this section, we will implement the discretization scheme described previously and conduct simulations to analyse the behaviour of the fish population, the optimal harvesting effort and, consequently, the value of the option to exploit the fishery.

As before, the value of the harvesting opportunity is denoted by J^*. When the option to fish reaches time t, it is interpreted as the time that has passed since the option became exercisable, which means that the fishing agent has $T - t$ time remaining to exercise the harvesting option.

Obtaining reliable data on marine species is very challenging due to the complex nature of the aquatic environment. To improve the approximation of our model, we gathered empirical data for the shrimp population in Bangladesh, as reported in [18] and [15]. The parameters estimated in the article are implemented in our simulations. All simulations are performed in the Python programming language. The baseline parameter configuration is summarised in the following table:

Table 1: Values of the parameters used in the simulation

Parameter	Value	Units
Time horizon: T	10	year
Population sub-intervals: m	50	
Price sub-intervals: l	50	
Time sub-intervals: n	120	
Number of simulations	1000	
Wiener processes correlation: ρ	0	
Population growth rate: r	1.331	year^{-1}
Population carrying capacity: K	11400	tonnes
Initial population size: x	$0.5K$	tonnes
Minimum population size: x_{min}	$0.1K$	tonnes
Maximum population size: x_{max}	$2K$	tonnes
Population volatility: σ_x	0.2	year$^{-1/2}$
Catchability: q	$9.77 \cdot 10^{-5}$	SFU^{-1}year^{-1}
Maximum allowed effort: E_{max}	$0.5r/q$	SFU
Minimum allowed effort: E_{min}	0	SFU
Discount rate: λ	0.05	year^{-1}
Convenience yield: y	0.0475	year^{-1}
Price volatility: σ_p	0.01	year$^{-1/2}$
Initial price value: p_0	8362.3	BDT $\cdot$ tonnes^{-1}
Linear cost coefficient: c_1	1156.8	BDT $\cdot$ SFU^{-1}year^{-1}
Quadratic cost coefficient: c_2	0.01	BDT $\cdot$ SFU^{-2}year^{-1}

Here, BDT is an abbreviation for Bangladesh Taka, the national currency. The unit SFU (Standardised Fishing Unit) represents a standardised measure of fishing effort, accounting for different types of vessel, gear and fishing practices.

Figure 3 shows 1000 simulated shrimp price trajectories.

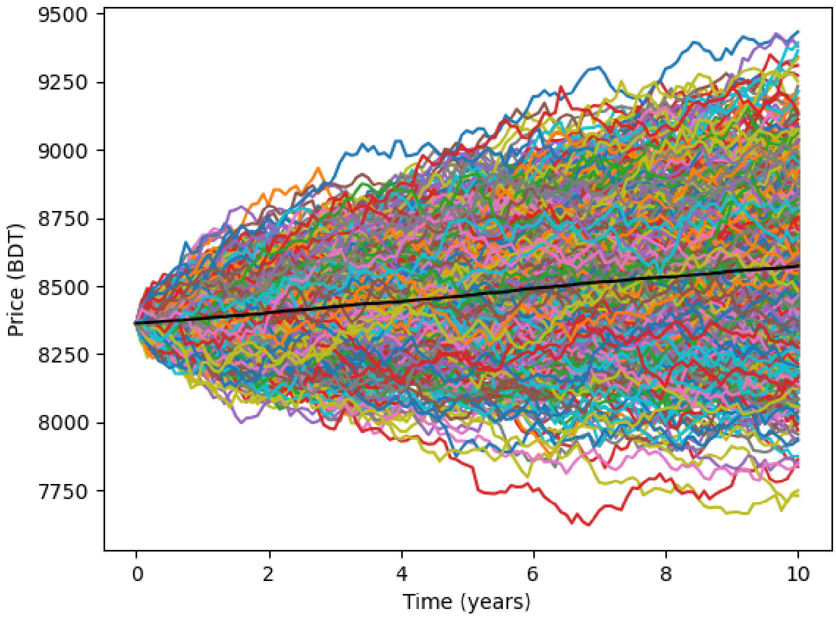

Figure 3: Simulated Price Trajectories and Mean

Effort is a key component in this problem, since it represents the intensity of fishing activity, including the number of vessels, workers, and the time spent fishing, among others. This variable also plays a central role in determining how the shrimp population evolves over time. When the effort is too high, the population may grow more slowly or even decline. On the other hand, when the effort is low, the population tends to recover and increase faster. Therefore, it is essential to maintain levels of effort that ensure sustainability and avoid the risk of population collapse.

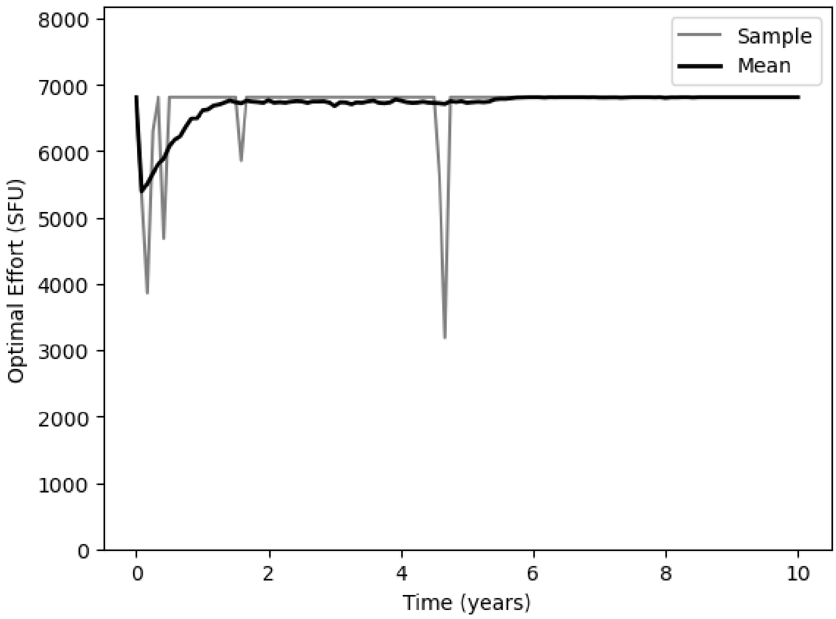

Figure 4: Simulated Effort Trajectory and Mean

In Figure 4, two different trajectories are shown. The grey line represents a single trajectory from the 1000 simulations, while the black line shows the mean of all simulated trajectories.

We observe that the mean effort trajectory starts at its maximum level and decreases rapidly shortly after. This behaviour is a consequence of setting a low initial population size, making it necessary to reduce the effort to avoid a possible extinction. If a higher initial population had been chosen, then the mean effort trajectory would have evolved differently.

Following the initial drop, we can see that the mean effort gradually increases and eventually stabilizes, with minor variations, at the maximum sustainable level. Near the final time, the average effort shows almost no variation and remains fixed at the maximum value. This is likely due to the time-limited nature of the optimization problem, which imposes artificial constraints on the optimal solution. These constraints include an early sharp decline in fishing activity to preserve the stock, followed by intensive exploitation in the final years to maximise the return before the time runs out.

Focusing on the sample trajectory, we view that it generally follows the pattern of the mean trajectory. However, since each sample path has its own unique random fluctuations, individual trajectories can differ significantly.

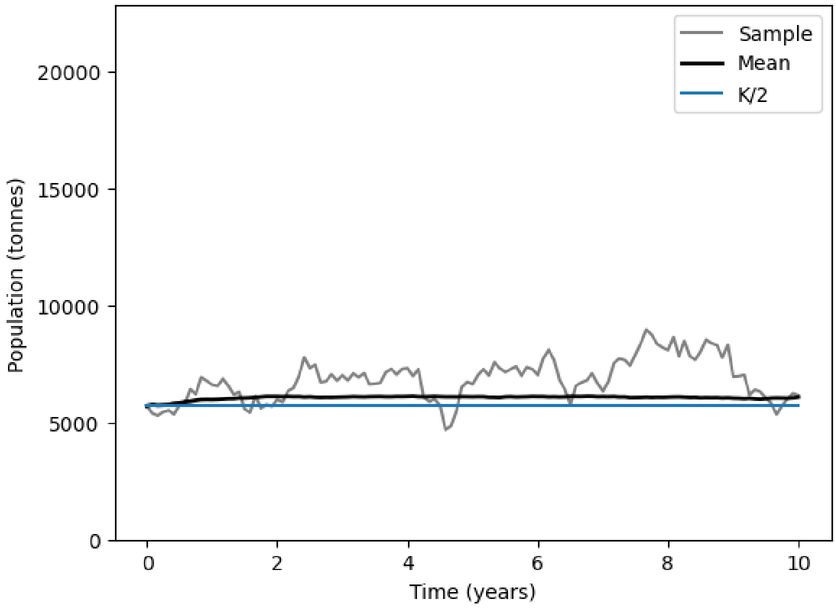

Figure 5: Simulated Population Trajectory and Mean

Now, looking at Figure 5, we observe three plotted lines. The blue line indicates the initial population level, set at $K/2$, the grey path represents a single sample trajectory from the 1000 simulations, and the black trajectory shows the average of all simulated trajectories.

In this figure, we visualize how the shrimp population relates to the effort graph. Focusing first on the mean population curve, we see that the population size begins its trajectory at the predefined value and remains stable while the effort is initially high. As the mean effort approaches its minimum, the mean population rises, reaching approximately 6000 tonnes. Finally, both mean effort and population stabilize as the effort returns to its maximum. This pattern suggests that the adopted optimal effort strategy allows the population to regenerate without being overexploited, ensuring the long-term sustainability of the fishery.

It is worth noticing that a decrease in effort leads to an increase in population, which is consistent with theoretical expectations, and appears to support the goal of achieving population stability. This relationship is clearly reflected in both Figures.

Additionally, to reinforce these conclusions, we can examine the sample population trajectory. Notably, periods of larger population growth tend to coincide with sudden drops in effort along the sample trajectory.

By observing Figure 6, we can draw conclusions about the behaviour of the harvesting option value and how it changes in response to shifts in optimal effort and population size.

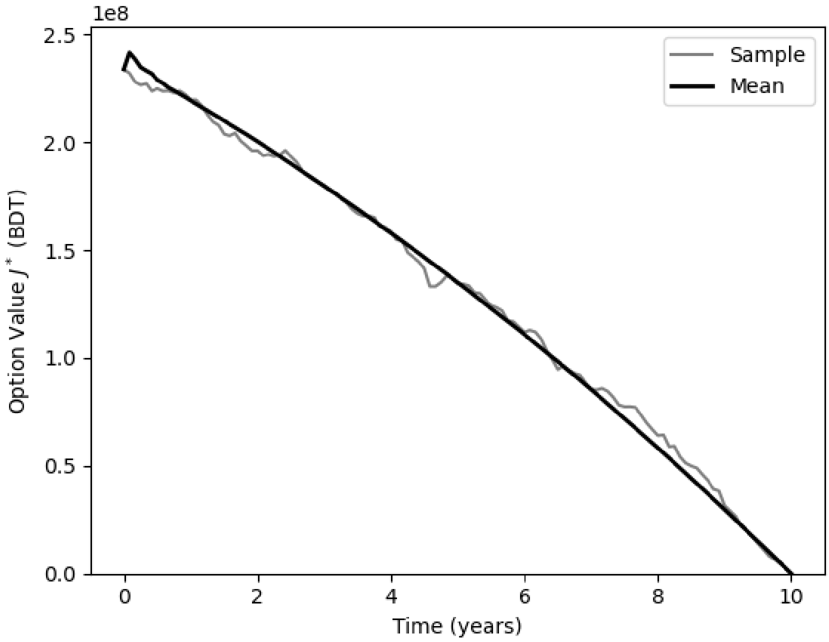

Figure 6: Simulated Harvesting Option Value Trajectory and Mean

Upon closer examination, we identify that the mean option value curve starts with an upward spike. This initial increase is likely caused by the sudden drop in mean effort, which, as illustrated in Figures 4 and 5, leads to a subsequent rise in the average shrimp population. Consequently, a larger available stock translates into higher potential revenues, resulting in an increased value of the harvesting option.

After this initial spike, the exploitation option enters a declining trajectory as both the mean population size and harvesting effort stabilize, and as time progresses towards the expiration date. By the terminal time of 10 years, the fishing option reaches the value of zero, which is consistent with the boundary condition imposed. While this assumption simplifies the context of the model, it reduces the realism of the scenario, as real-world fishing operations often extend over longer and more flexible time horizons.

We cannot fully understand the results of this experiment without testing how sensitive the harvesting option value is to certain parameters. The parameters we will vary are: the convenience yield y and the initial population of shrimp x_0.

Firstly, we will analyse the effect of changes in the convenience yield. The selected values are: 4.75%, 4%, and 3%.

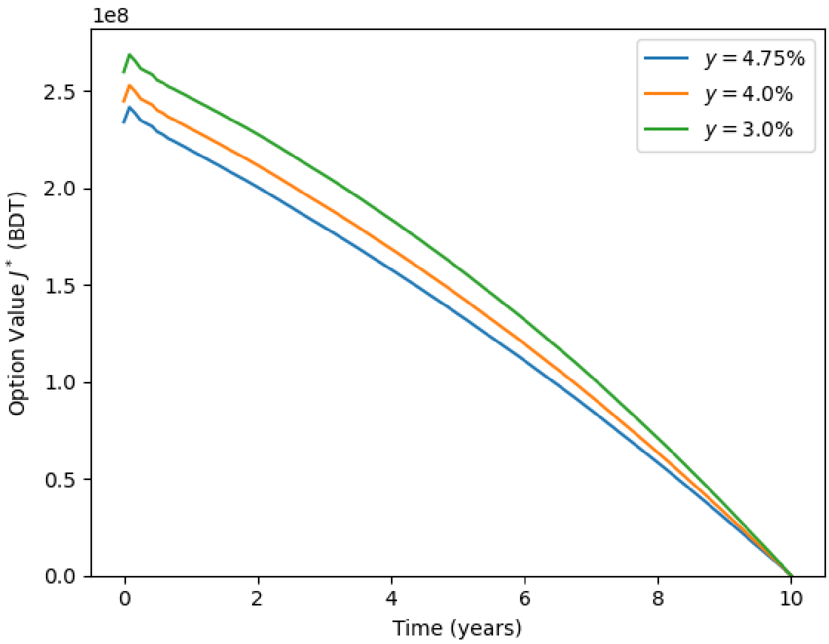

Figure 7: Simulated Option Value Sensitivity to the Convenience Yield

In Figure 7, it is clearly illustrated that the harvesting option value increases as the convenience yield decreases. This trend can be explained by the role of the convenience yield, which operates similarly to a dividend. In financial markets, a dividend is a monetary reward paid to the owner of an asset, and it typically leads to a drop in the asset's price. Consequently, the higher the dividend, the larger the reduction in the asset's value. Since the holder of an option on that underlying asset does not receive the dividend itself, the value of the option is negatively impacted by this price depreciation.

As discussed in Section 3.3, the convenience yield is not interpreted as the benefit of physical storage, since fish, unlike grain or oil, cannot be practically stored for long periods. Instead, we view it as the benefit of being able to harvest and sell fish immediately in response to market pressures. Thus, the value lies in having the ability to react swiftly to favourable conditions, without delay.

Just like a dividend, a higher convenience yield increases the incentive to harvest immediately, as it raises the return for acting quickly under attractive market circumstances. This diminishes the value of waiting, making the harvesting option less advantageous. As a result, harvesting becomes more aggressive, causing a faster decline in the shrimp population, further reducing the value of the option to harvest in the future. This effect is clearly visible in Figure 7. The trajectories with higher convenience yields consistently result in lower harvesting option curves compared to those with lower yields, converging to zero at the end.

We now assess how changes in the initial shrimp stock, x_0, affect the value of the harvesting option, J^*. For this purpose, we consider the values $0.5K$, $0.7K$ and $0.9K$. As shown in Figure 8, the option value at time $t = 0$ increases with the initial population size. This observation aligns with the conclusions drawn from the previous analysis, since the option becomes more valuable when more shrimp are harvestable.

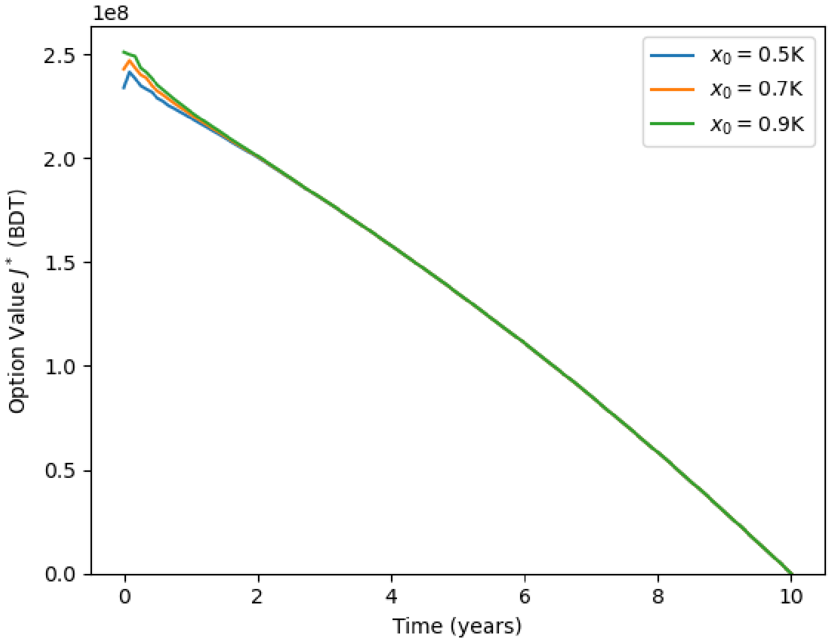

Figure 8: Simulated Option Value Sensitivity to the Initial Population Size

Interestingly, although the initial option values differ between the three cases, these differences are relatively small. Moreover, the option values quickly converge to a common trajectory. This indicates that, regardless of the initial stock quantity, the mean populations stabilize over time at the same level, leading to equal option values in the long run.

This behaviour highlights a key feature of the model: the optimal harvesting strategy guides the average population size towards a stable path, moving to approximately 6000 tonnes, independent of the initial stock. Therefore, while the initial conditions influence the early option value, the long-term dynamics are driven by the convergence of the population to this optimal equilibrium.

6 Conclusions

In this work, we tackle the optimal harvesting problem by integrating stochastic fluctuations in both fish population growth and price dynamics, each modelled by two distinct Brownian motions. To solve this problem, we adopt a real options approach, treating the fishing activity as an option that grants the fisher the right, but not the obligation, to harvest.

We constructed a portfolio consisting of a long position in the harvesting option and a short position of ν units in a spanning asset. This formulation resulted in a non-linear PDE, which cannot be solved analytically. Consequently, we applied numerical methods to approximate the solution, using the Crank-Nicolson scheme.

Using the Gompertz model to describe the population dynamics, along with realistic data from a harvested shrimp population, we computed the optimal value of the harvesting option. Our results confirmed the expected behaviour: the population size tends to increase as harvesting effort decreases and the option value rises when shrimp stock is abundant.

We conducted sensitivity tests with respect to the convenience yield and the initial population. As anticipated, the harvesting option value declines with an increase in the convenience yield, reflecting its role similar to a dividend. Moreover, we found that the option value increases with the initial population size, and that, regardless of the starting population, trajectories tend to converge to a common long-run equilibrium.

The algorithm was implemented in the Python programming language. Readers interested in the code may request it directly.

For future research, it would be highly valuable to extend the model by incorporating realistic, time-varying interest rates and seasonal patterns in fisheries, as most are not accessible year-round. These enhancements would bring the model closer to real-world conditions and improve its practical applicability.

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