Lisbon School of Economics & Management
Universidade de Lisboa

# MASTER

ACTUARIAL SCIENCES

# MASTER´S FINAL WORK

DISSERTATION

MINIMIZING RUIN PROBABILITY: AN OPTIMAL REINSURANCE PROBLEM USING A DYNAMICAL SETTING INCLUDING DEPENDENCES

ADRIALINA BOTNARIUC

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# Abstract

Reinsurance is one of the key risk management tools used by insurance companies to spread risk and receive financial protection against large losses. This comes at the price of the reinsurance premium which reduces the insurer's profits in exchange for safety.

This thesis focuses on analytically finding the optimal retention levels under three different excess of loss contracts, with the purpose of minimizing the ruin probability in infinite time and from the point of view of the insurance company. The expected value premium principle is used for the calculation of both the insurer's and reinsurer's premiums. The same analysis is developed when considering two dependent classes of risk.

A diffusion approximation of the classical Crámer-Lundberg risk process with reinsurance is considered. After building the model, the ruin probability function is characterized and conclusions regarding the optimal strategies are drawn. For the dependent case, the optimal strategy depends not only on the marginal distributions of the underlying risk, but also on the distribution of the sum of the claim severities. To better contextualize the analytical results, a numerical analysis is developed in each case, using the R software, considering different distributions and several values for its parameters.

The analytical results show that, for some particular cases of the excess of loss treaty, it is always optimum to transfer part of the risk to the reinsurer; for other cases, the optimal strategy is to retain all the risk and, for the remaining cases, it depends on the distribution of the underlying risk. The numerical results corroborate the analytical ones. In particular, the optimal reinsurance strategy under dependences is different if the two classes of risk are considered independently.

**Keywords:** Optimal reinsurance; Ruin probability; Excess of loss Treaty; Expected value premium principle; Dependent risks.

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# 1 Introduction

The main function of an insurance company is to take on risk in exchange for the payment of premiums, by selling policies to policyholders. According to Albrecher et al. (2017), this leveraging activity is a competitive advantage, although it is so risky that the insurer becomes highly vulnerable to insolvency. Reinsurance is one of the most important risk management tools used by insurers in order to deal with large risks. It is the practice whereby insurers transfers part of their risk portfolio to the reinsurance company, under some form of agreement which stipulates the part of the insurer's claims to be covered by the reinsurer in exchange for the payment of a premium. As Kopf (1929) stated, reinsurance spreads risks so widely and effectively that even the largest risk can be insured without unduly burdening the insurance company. Not only does it play a significant role in securing the financial stability of insurers but also for the whole insurance industry. Put in other words, reinsurance is the insurance for insurance companies and is considered to be the backbone of the insurance industry.

When entering a reinsurance contract, the insurer has to be mindful of the strategy he chooses, as it bears additional costs, namely because of the reinsurance premium. To create the optimal reinsurance strategy, the following must be kept in mind: (i) the form of reinsurance and (ii) the amount of risk to reinsure. This subject has been extensively studied, either analytically or by using numerical techniques, in continuous or discrete time, considering distinct optimization criterion, premium principles or including some other constraints. In this thesis, we study the optimal reinsurance problem by analytically finding the optimal retention levels which minimize the ruin probability for three different excess of loss contracts:(i) without superior limit, (ii) without inferior limit and (iii) with inferior and superior limits. Some numerical examples are developed in order to analyse the compliance with the analytical results. Furthermore, we include an analytical and numerical analysis of the optimal reinsurance with two dependent classes of risks, dependent through the number of claims.

Firstly, we consider the surplus process with reinsurance. From the premium paid by the insurance company, which includes a safety loading that buffers losses over the average, we discount the management expenses arising from reinsurance, for instance, the payment of the reinsurance premium. In this work, the expected value premium principle is used for the calculation of both insurer's and reinsurer's premium with corresponding safety loading $\eta > 0$ and $\theta > 0$, respectively. The reinsurers safety loading must be greater than the insurer's, $\eta < \theta$, as, if otherwise, the insurer would transfer all risk, making a profit without bearing any risk. This surplus process is modelled by the Cramér-Lundberg model, which follows a compound Poisson process, and can be approximated by a diffusion process, which results in a Brownian motion with drift.

As already mentioned, the optimization criteria considered in this study is the minimization of the ruin probability, with the moment of ruin analysed in continuous time and in an infinite time horizon. After defining the diffusion process parameters and introducing them into the expression of the ruin probability, we characterize our ruin probability function by finding its optimum domain and by studying its behaviour near the extremes of this optimum interval. The goal is to use this characterization to help draw conclusions on the existence and uniqueness of a minimum solution as function of the excess of loss reinsurance parameters. For the excess of loss contract without inferior limit, we were able to demonstrate analytically additional results for the specific cases of the Exponential and Pareto distributions.

With the purpose of verifying the analytical results, a numerical analysis is developed in R software. Also, the numerical results are used to study the influence that the distribution of the individual claims had on the ruin probability. The safety loadings are assumed fixed and three different distributions are considered: the Exponential distribution, the Pareto distribution and the Gamma distribution. We study our problem for distinct values of the distribution's parameters, and considering both the independent and dependent cases.

This thesis is organized as follows. In Chapter 2, the literature review is provided to give an understanding of the developments already made on this topic. Chapter 3 describes the surplus process with reinsurance and how it is approximated to a diffusion process. Still in this chapter, the expected value premium principle is introduced, the time of ruin and ruin probability, in infinite time, are defined, the formalization of the excess of loss treaty, for all three different contracts, is provided as well as the corresponding parameters of the diffusion process. In Chapter 4, the analytical analysis of the optimal reinsurance strategy is conducted for a single risk and, in Chapter 5, the optimal reinsurance problem with dependent risks is studied. Also, these chapters include the numerical results when considering the different distributions. Lastly, Chapter 6 presents the main conclusions of this study.

# 2 Literature review

Reinsurance is one of the key tools used by insurance companies to share risk and receive financial protection against large losses. By spreading its risk, the insurance company can insure clients with large coverage without being at risk of insolvency. However, this comes at the price of the reinsurance premium, reducing the insurer's profits in exchange for safety. Besides deciding on the form of reinsurance, the insurer must also decide on the amount of risk to reinsure.

The topic of optimal reinsurance has been studied for quite some time, with papers like Borch (1969) dating back to almost sixty years ago. Even now, this problem is yet being investigated in the actuarial community with new approaches and twists to the settings, as for example in Zanotto & Clemente (2022).

The problem has been tackled numerically, when looking for a simpler analysis, but many authors have developed a more analytical approach. Different reinsurance treaties have been considered under distinct optimization criteria and premium principles. Schmidli (2001), Hald & Schmidli (2004) and Liang & Guo (2007) considered the optimal proportional reinsurance problem while others, like Hipp & Vogt (2003) and Centeno (2002), considered the excess of loss treaty, the latter in a finite horizon. Combinations of these treaties have also been studied, for example by Zhang et al. (2007).

In this thesis, the excess of loss treaty is considered. The optimization criterion is the minimization of the ruin probability of the insurer and both the insurer's and reinsurer's premiums are computed according to the expected value premium principle. The same optimization criteria was used by Schmidli (2001), Browne (1995), Hipp & Vogt (2003), Liang & Guo (2007), Taksar & Markussen (2003). In Meng et al. (2016), the minimum ruin probability was pursued with the particularity that the insurance risk was partly transferred to two reinsurers, instead of one, where one of the reinsurer's premium was calculated using the expected value principle and the other using the variance principle. Liang et al. (2020) minimized the ruin probability with the premium computed according to the mean-variance premium principle, which is a combination of the expected value and variance premium principles. The same combination of premium principles was assumed by Kaluszka (2004), when deriving the optimal reinsurance rules provided the cedent traded off between the variance and the expected value of his gain. Also Han et al. (2020) used a combination of premium principles, when studying the optimal reinsurance strategy that minimized the probability of drawdown. In Meng et al. (2019), both the minimum ruin probability and the maximum expected utility of wealth were analysed for different premium principles: expected value, variance and exponential premium principles.

The maximization of the adjustment coefficient was studied by Hald & Schmidli (2004), Liang & Guo (2008) and Centeno (1986). Correia (2021) found, numerically, the optimal reinsurance for the quota-share and the excess of loss treaties, under three different simultaneous optimization criteria: (1) minimizing the ruin probability in infinite time; (2) maximizing the expected value; and (3) minimizing the variance of the process.

Many authors, like Liang & Yuen (2016), Bi & Chen (2019), de Moura (2017) and Guerra & de Moura (2021), have acknowledged the importance of considering dependence between classes of risk on the optimal strategies. Liang & Yuen (2016) provided closed-form expressions for the optimal strategies that maximized the expected exponential utility and gave some numerical examples. Cai & Wei (2012), focused their study on positively dependent risks with a particular dependence structure, the stochastic ordering. The authors show that the excess of loss treaty is the optimal form of reinsurance when considering the expected value premium principle. In Centeno (2005), the optimal excess of loss retention limits for two dependent classes of insurance risks were compared regarding the maximization of the expected utility of wealth and the maximization of the adjustment coefficient.

Liang & Guo (2011) studied the optimal combination of quota-share with excess of loss reinsurance to maximize the expected utility of the wealth and concluded that, under some conditions, the pure excess of loss reinsurance was better than any other combination. In Centeno (1985), the insurer could choose between all three treaties: pure quota-share, pure excess-of-loss and combinations of these two, with the purpose of minimizing the skewness coefficient of the insurer's retained risk, concluding that the optimal solution was a pure excess of loss treaty if the the premium principle is the expected value or the standard deviation. However this was not necessarily true when the variance principle was considered.

The surplus process of the insurance company can be modeled by the classical risk model, where the aggregate claims process is a compound Poisson. The distribution of the aggregate claims makes it hard to study the characteristics of the model, reason why many papers consider a diffusion approximation, the Brownian motion model. In the present study, the surplus process is approximated to the diffusion model in infinite time. Zhang et al. (2007), Højgaard & Taksar (1998), Liang et al. (2020) and Luo & Taksar (2011) consider the same approximation. Others, like Golubin (2008), Dickson & Waters (2006) and Hipp & Vogt (2003), work with the classical risk model. Some even work with both, for example Liang & Yuen (2016), Bi & Chen (2019) . Alternatively, Liang & Guo (2007), Hald & Schmidli (2004), Liang & Guo (2008) and Liang & Guo (2011) , have studied the surplus process by considering the diffusion approximation as well as the jump-diffusion model.

Stochastic control theory and the Hamilton-Jacobi-Bellman (HJB) equation have been heavily used to deal with the optimal reinsurance problem, either in a static or in a dynamic setting. While the first allows to derive closed-form expressions for the optimal strategy for both the compound Poisson risk model and the Brownian motion model, the second usually requires a numerical solving. In Hipp & Vogt (2003), the optimal dynamic excess of loss reinsurance that minimized ruin probability was examined, in infinite time, with the risk process modeled as a compound Poisson process. The existence of a smooth solution to the corresponding HJB equation was proved. In the analysis of Cani & Thonhauser (2017), the goal was to find a dynamic reinsurance policy that maximized the expected discounted surplus level by using analytical methods to identify the value function as a particular solution to the HJB equation. Taksar & Markussen (2003) approximated the surplus process by a diffusion and made use of stochastic control theory to determine the optimal proportional reinsurance policy that minimized ruin probability. Mao et al. (2016) determined the optimal solution for an insurer whose wealth followed a diffusion process by establishing the Hamilton-Jacobi-Bellman-Isaacs (HJBI) equations. This last research extended the work of Zhang & Siu (2009).

Some other extensions of the reinsurance problem have been analysed. For example, Li & Wang (2022) has derived closed-form expressions and gave numerical examples for the optimal dividend and proportional reinsurance problem. The work of Liang & Guo (2008), Browne (1995), Luo & Taksar (2011) and Mao et al. (2016) was focused on the optimal investment-reinsurance problem. Under the same problem, Schmidli (2002), allowed investment in a risky asset, modeled by a Black-Scholes model, but also allowed reinsurance. The optimal strategy was found considering the HJB approach and a numerical procedure was applied to solve its equation. David Promislow & Young (2005) and Liu & Yang (2004) included investment in a riskless asset under some borrowing constraints as, for example, that borrowing in order to invest in the risky asset was not allowed or that the insurer could only borrow at a higher rate than that earned on the riskless asset. The work of Cai & Tan (2007) and Cai & Chi (2020) is based on the analysis of the optimal reinsurance based on risk measures like VaR and CTE.

In this thesis, we develop an analytical approach for the optimal reinsurance problem. Under the optimization criterion of minimizing the ruin probability for the excess of loss treaty with premiums computed through the expected value principle and considering the approximation of the surplus classical process, the Cramér-Lundberg process, to a Brownian motion, we investigate the existence and uniqueness of the optimal strategies. Furthermore, this problem is studied for a process with two dependent classes of risk, dependent through the number of claims. Some numerical examples are provided in order to analyse the compliance with the obtained analytical results.

## 3 The surplus model and the probability of ruin

The classical risk model was introduced by Filip Lundberg and, afterwards, developed by Harald Cramér, reason why it is also known as the Cramér–Lundberg model. According to this model, the surplus process of a collective contract is given by

$$X_t = x + Pt - \sum_{i=0}^{N_t} Y_i$$

where $x$ is the initial surplus, $P > 0$ is the premium rate and $Y_i$, for $i = 0, 1, 2, ...$, is a sequence of non-negative independent and identically distributed random variables which are independent of $N_t$, representing the claims' size with $Y_0 \equiv 0$. Also, $N_t$ is a Poisson process modelling the incoming claims with intensity $\lambda$ and, for this reason, the aggregate claims process, $S_t = \sum_{i=0}^{N_t} Y_i$, is a compound Poisson process.

Under a reinsurance contract, the reinsurer has the responsibility to cover part of the insurer's claims in exchange for the payment of a premium. Hence, this value has to be deducted from the premium received from policyholders. Therefore, with reinsurance, the surplus process becomes

$$X_t = x + (P_T - P_R)t - (S_t - H_t),$$

where $P_T$ is the premium rate paid by the policyholder to the insurer, $P_R$ is the premium rate paid by the insurer to the reinsurer. We consider $S$ to represent the aggregate claims amount, $Z(S)$ represents the risk retained by the insurer and $H(S) = S - Z(S)$ the risk transferred to the reinsurer. Since we assume that the insurer's and reinsurer's premiums are both computed according to the expected value premium principle, with corresponding safety loadings $\eta$ and $\theta$, then

$$P_T(S) = (1 + \eta)E(S) \tag{1}$$

and

$$P_R(H(S)) = (1 + \theta)E(H(S)) = (1 + \theta)(E(S) - E(Z(S))). \tag{2}$$

The fact that the aggregate claim process is a compound Poisson process makes it hard to study the classical risk model, reason why a diffusion approximation is used. According to Glynn (1990), diffusion processes have a more convenient analytical structure, which makes them mathematically easier to handle than the original process with which one starts.

From Liao (2013) and Correia (2021), the diffusion approximation for the surplus process can be written as:

$$dX_t = \alpha dt + \sigma W_t.$$

where the drift parameter $\alpha$ represents the amount that the insurance company is gaining, by unit of time, with the premium income, the diffusion parameter $\sigma$ represents the volatility of the model and $W_t$ is the standard Brownian motion.

According to Correia (2021), these diffusion parameters can be written as follows:

$$\begin{aligned}\alpha &= P_T(S) - P_R(H(S)) - E(Z(S)) \\ &= \lambda\left[(\eta-\theta)E(S) + \theta E(Z(S))\right]\end{aligned} \tag{3}$$

and

$$\sigma^2 = Var(Z(S)) = \lambda E(Z(S)^2). \tag{4}$$

One of the objectives in studying the surplus model is to analyse the probability of ruin in infinite time, i.e. that the insurer's surplus level eventually falls below zero. We represent by $\tau_c$ the instant when the stochastic variable $X_t$ reaches a fixed barrier c, assuming $X_0 > c$, and it is given by:

$$\tau_c = \inf\{t \geq 0 : X_t \leq c\}.$$

Ruin occurs when the surplus process is negative in a given moment in time so, we consider $c = 0$. According to Liao (2013), in infinite time, the probability that the capital of an insurance company ever drops below the level $c$, can be written as

$$\Psi(c) = P(\tau_c < \infty) = P\left(\inf_{t\geq 0} X_t \leq c\right) = \begin{cases} exp\left[-2(X_0 - c)\frac{\alpha}{\sigma^2}\right] & ,\alpha > 0 \\ 1 & ,\alpha \leq 0 \end{cases}$$

with the drift parameter $\alpha$ representing the insurer's gains from premiums and the diffusion parameter $\sigma$ representing the volatility of the surplus model. But, by introducing the expressions (3) and (4) of the diffusion process parameters into the expression of the probability of ruin, when $\alpha > 0$, we obtain

$$\begin{aligned}\Psi(c) &= exp\left[-2(X_0 - c)\frac{\lambda\left[(\eta-\theta)E(S) + \theta E(Z(S))\right]}{\lambda E(S^2)}\right] \\ &= exp\left[-2(X_0 - c)\frac{(\eta-\theta)E(S) + \theta E(Z(S))}{E(S^2)}\right]\end{aligned}$$

which does not depend on the intensity $\lambda$ of the process. Therefore, without loss of generality, we assume that $\lambda = 1$.

In order to reduce risk, an insurance company makes an agreement, called the reinsurance treaty, to transfer part of its business portfolio to a reinsurer. The reinsurance company is responsible for all the risk falling within the terms of the treaty. There exist two classes of reinsurance treaties, namely, proportional and non-proportional. Quota-share and surplus are proportional treaties while excess of loss and stop loss are non-proportional.

The quota-share is the most popular form of proportional treaty. It is a reinsurance treaty in which the insurance company is responsible for a portion of the loss associated with a claim. Let $a \in (0, 1)$ be the retained percentage of the risk, and $(1 - a)$ be the transferred percentage to the reinsurer. Then, for each individual claim $Y$, the insurer retains $Z(Y) = aY$ and transfers

$H(Y) = (1-a)Y$.

The distribution of the individual claims retained is $F_{Z(Y)}(y) = F_Y(\frac{y}{b})$ and, from here, the first and second raw moments are computed and are given by $E(Z(Y)) = aE(Y)$ and $E(Z(Y)^2) = a^2E(Y^2)$, respectively. From equations (3) and (4), the diffusion process parameters are defined as $\alpha = (\eta - \theta(1-a))E(Y)$ and $\sigma^2 = a^2E(Y^2)$.

### Excess of loss treaty

The main focus of this study is on the excess of loss treaty. Before tackling the optimal reinsurance problem, a formalization of the parameters under this treaty is provided. The expressions for the excess of loss treaty are as follow (see for instance Correia (2021)).

Under an excess of loss treaty $(M, L)$, for each claim, the insurer retains losses below M while the amount in excess of M and up to a certain amount L is the reisurer's responsibility. Hence, for each individual claim $Y$, the insurance company retains

$$Z(Y) = \begin{cases} Y & , Y \leq M \\ M & , M < Y \leq M + L \\ Y - L & , Y > M + L \end{cases}$$

and transfers

$$H(Y) = \begin{cases} 0 & , Y \leq M \\ Y - M & , M < Y \leq M + L \\ L & , Y > M + L \end{cases}.$$

Thus, the total ceded risk in time interval $[0, t]$ is $\sum_{i=0}^{N_t} H(Y_i) := H(S)$ and the total retained risk in time interval $[0, t]$ is $\sum_{i=0}^{N_t}(Y_i - H(Y_i)) = \sum_{i=0}^{N_t} Z(Y_i) := Z(S)$. The distribution of the individual retained claims is

$$F_{Z(Y)}(y) = \begin{cases} F_Y(y) & , y < M \\ F_Y(y + L) & , y \geq M \end{cases}$$

from where the first and second raw moments are computed and given by

$$E(Z(Y)) = \int_0^M S_Y(y)dy + \int_{M+L}^{\infty} S_Y(y)dy \tag{5}$$

$$E(Z(Y)^2) = \int_0^M 2yS_Y(y)dy + \int_{M+L}^{\infty} 2(y-L)S_Y(y)dy. \tag{6}$$

| | $(M, L=\infty)$ | $(M=0, L)$ | $(M, L)$ |
|---|---|---|---|
| $P_R(H(Y))$ | $(1+\theta)\int_M^\infty S_Y(y)dy$ | $(1+\theta)\int_0^L S_Y(y)dy$ | $(1+\theta)\int_M^{M+L} S_Y(y)dy$ |
| $\alpha(M, L)$ | $\eta E(Y)-\theta\int_M^\infty S_Y(y)dy$ | $\eta E(Y)-\theta\int_0^L S_Y(y)dy$ | $\eta E(Y)-\theta\int_M^{M+L} S_Y(y)dy$ |
| $\sigma^2(M, L)$ | $\int_0^M 2yS_Y(y)dy$ | $\int_L^\infty 2(y-L)S_Y(y)dy$ | $\int_0^M 2yS_Y(y)dy+\int_{M+L}^\infty 2(y-L)S_Y(y)dy$ |

Table 1: Parameters of the diffusion process and reinsurer's premium principles for three different excess of loss contracts.

We consider three different contracts: (i) without superior limit $(M, L=\infty)$; (ii) without inferior limit $(M=0, L)$; (iii) with inferior limit $M$ and superior limit $L$, $(M, L)$. The diffusion process parameters and the reinsurer's premium are summarized in Table 1.

For the $(M, L)$ contract, the drift parameter $\alpha$ is computed from equations (1), (5) and (6), the diffusion parameter $\sigma^2$ is computed from equation (6) and the reinsurer's premium is calculated by using equation (5). For the excess of loss $(M, L=\infty)$ contract the same equations are used but replacing $L=\infty$. The calculation of the parameter for the excess of loss $(M=0, L)$ contract makes use of the same equations but replacing $M=0$.

## 4 The optimal excess of loss treaty

The main purpose of this chapter is to study the optimal reinsurance strategy which minimizes the ruin probability of the insurance company. Intuitively, when the insurer's safety loading exceeds the reinsurer's, the insurer can transfer all risk to the reinsurer making a profit without bearing any risk.

**Proposition 1:** If $\eta > \theta$ then the optimal strategy for the general excess of loss treaty $(M, L)$ is $\tilde{M} = 0$ and $\tilde{L} = \infty$ (transfer all risk to the reinsurer).

*Proof.* The goal is to minimize the ruin probability $\Psi(M, L) = exp(-2X_0 \frac{\alpha(M,L)}{\sigma^2(M,L)})$, but this is the same as maximizing $\frac{\alpha(M,L)}{\sigma^2(M,L)} = \frac{\eta E(Y) - \theta \int_M^{M+L} S_Y(y)dy}{\int_0^M 2yS_Y(y)dy + \int_{M+L}^{\infty} 2(y-L)S_Y(y)dy}$.

Since we assume $\eta > \theta$ and since

$$\lim_{M \to 0; L \to +\infty} \alpha(M, L) = (\eta - \theta)E(Y) > 0$$

and

$$\lim_{M \to 0; L \to +\infty} \sigma^2(M, L) = 0,$$

then $\lim_{M \to 0; L \to +\infty} \frac{\alpha(M,L)}{\sigma^2(M,L)} = +\infty$ and $\lim_{M \to 0; L \to +\infty} \Psi(M, L) = 0$. This means that, if $\eta > \theta$, $M = 0$ and $L = \infty$ minimize the ruin probability, i.e. the optimal strategy is to transfer all risk to the reinsurer leading to a zero probability of ruin for the insurance company. □

Clearly, this is not a realistic situation. So, from now on, we make the more interesting and reasonable assumption: $\eta < \theta$.

For the quota-share treaty, Correia (2021) has provided an explicit expression for the optimal strategy.

**Proposition 2:** Let Y be a non-negative random variable with infinite support. Consider the quota-share treaty with $a \in [0, 1]$ the retained percentage and $\eta < \theta$. Then, the optimal solution minimizing the ruin probability is $\tilde{a} = 2 - \frac{2\eta}{\theta}$.

In this thesis, we intended to provide similar results when considering three different versions of the excess of loss treaty: (1) without upper limit; (2) without lower limit and (3) with lower and upper limits. To do so, we study the existence and uniqueness of the optimal reinsurance strategy for each case.

## 4.1 The Excess of loss Treaty with no upper limit

**Proposition 3:** Let Y be a non-negative random variable with infinite support. Consider the excess of loss treaty $(M, L = \infty)$, with $M \geq 0$ and $\eta < \theta$.

Then, the optimal solution minimizing the ruin probability $\tilde{M}$ is the smallest solution to the equation $\alpha(M) = \frac{\theta}{2M}\sigma^2(M)$ that verifies $\tilde{M} > \hat{M}$, where $\hat{M}$ is the solution to $\alpha(M) = 0$.

*Proof.* The probability of ruin is given by

$$\Psi(M) = \begin{cases} exp\left[\frac{-2X_0\alpha(M)}{\sigma^2(M)}\right] & ,\alpha(M) > 0 \\ 1 & ,\alpha(M) \leq 0 \end{cases}$$

with the expressions of $\alpha(M)$ and $\sigma^2(M)$ given in Table 1.

Since, $\alpha(M = 0) = E(Y)(\eta - \theta) < 0$, $\lim_{M\to\infty} \alpha(M) = \eta E(Y) > 0$ and since $\alpha(M)$ is increasing with $M$ then $\exists! \hat{M} : \alpha(\hat{M}) = 0$. So, we can restrict the optimum domain of $M$ to the interval $(\hat{M}, \infty)$ .

Let

$$\Psi_1(M) = \frac{-2X_0\alpha(M)}{\sigma^2(M)}.$$

The derivative of $\Psi(M)$ is given by $\Psi'(M) = \Psi(M)\Psi_1'(M)$ with

$$\Psi_1'(M) = -2X_0 S_Y(M) \frac{\theta \int_0^M 2yS_Y(y)dy - 2M(\eta E(Y) - \theta \int_M^\infty S_Y(y)dy)}{(\int_0^M 2yS_Y(y)dy)^2}. \tag{7}$$

To minimize the ruin probability, we have to find the roots of its derivatives, i.e. $\Psi'(M) = 0$. Since $\Psi(M) > 0$, $X_0 > 0$, $S_Y(y) > 0$ and $\int_0^M 2yS_Y(y)dy > 0$ then, from equation (7), we get

$$\begin{aligned} \Psi'(M) = 0 &\Leftrightarrow \Psi_1'(M) = 0 \\ &\Leftrightarrow \eta E(Y) - \theta \int_M^\infty S_Y(y)dy = \frac{\theta}{2M}\int_0^M 2yS_Y(y)dy. \end{aligned} \tag{8}$$

We can rewrite equation (8) as

$$\alpha(M) = \frac{\theta}{2M}\sigma^2(M). \tag{9}$$

So, any critical point must verify equation (9).

To study the existence and uniqueness of critical points, the function $\Psi(M)$ and the sign of its derivative are analysed in the extremes of the optimal interval $(\hat{M}, \infty)$ and compared to the ruin probability in the critical points.

On the extremes, we have $\lim_{M\to\hat{M}}\Psi(M)=1$ and

$$\lim_{M\to\infty}\Psi(M)=exp\left[\frac{-2X_0\eta E(Y)}{E(Y^2)}\right]. \tag{10}$$

Also, from equation (7), we have $\Psi'(\hat{M}) = -2X_0 S_Y(\hat{M})\Psi(\hat{M})\frac{\theta\int_0^{\hat{M}}2yS_Y(y)dy}{(\int_0^{\hat{M}}2yS_Y(y)dy)^2} < 0$, since $\alpha(\hat{M}) = 0$. Moreover, $\lim_{M\to+\infty}\Psi'(M) = 0$, since $\lim_{M\to\infty}S_Y(M) = 0$ and, since we assume the existence of moments, the survival function $S_Y$ goes faster to zero than the numerator of equation (7) goes to infinity.

Now, the existence of solution to equation (9) depends on the sign of $\Psi'(M)$ changing in the interval $(\hat{M},\infty)$. Since $X_0>0$, $S_Y(M)>0$, $\Psi(M)>0$ and $(\int_0^M 2yS_Y(y)dy)^2>0$ then, the sign of $\Psi'(M)$ only depends on the sign of its numerator, i.e.
$f(M) := \theta\int_0^M 2yS_Y(y)dy - 2M(\eta E(Y)-\theta\int_M^\infty S_Y(y)dy)$. But, $f(\hat{M}) = \theta\int_0^{\hat{M}}2yS_Y(y)dy > 0$ and $\lim_{M\to\infty}f(M) = -\infty < 0$. This means that, in fact, the sign of $\Psi'(M)$ changes and, since $\Psi'(\hat{M})<0$, we can conclude that $\exists\tilde{M}>\hat{M}:\alpha(\tilde{M})=\frac{\theta}{2\tilde{M}}\sigma^2(\tilde{M})$, i.e. the ruin probability has at least one minimum critical point $\tilde{M}>\hat{M}$.

Assume $\tilde{M}$ is a critical point, i.e. verifies equation (9). Then,

$$\Psi(\tilde{M}) = exp\left[-\frac{\theta X_0}{\tilde{M}}\right].$$

$\Psi(\tilde{M})$ is increasing with $\tilde{M}$, i.e. if $\tilde{M}_1$ and $\tilde{M}_2$ are two critical points such that $\tilde{M}_1<\tilde{M}_2$, then $\Psi(\tilde{M}_1)<\Psi(\tilde{M}_2)$. If $\tilde{M}_1$ is not a minimum then $\tilde{M}_2$ must be. This means that $\Psi(\tilde{M}_1)>\Psi(\tilde{M}_2)$, which is absurd. Therefore, $\tilde{M}_1$ is the minimum critical point.

Finally, we can conclude that the optimum solution is $\tilde{M}$ the smallest solution to equation (9) that verifies $\tilde{M}>\hat{M}$ where $\hat{M}$ is the solution to $\alpha(M)=0$. □

### 4.1.1 Numerical analysis

To provide a better contextualization of the analytical results, we conducted a numerical analysis using R software, from where we have also studied the influence of the distribution of the individual claims on the ruin probability. We start by setting the insurer's safety loading to $\eta = 0.5$, the reinsurer's safety loading to $\theta = 0.8$ and the initial surplus to $x = 1$. To find the optimal solution that minimizes the ruin probability, the `optimize` function of R is used.

The family of Gamma distributions is a very common choice for modelling claims' severity but, in the cases of heavy-tailed data, the Pareto distribution is known to be a better choice. For this reason, firstly we consider that individual claims are modeled by the Exponential distribution with parameter $b$ and, secondly, by the Pareto distribution with scale parameter $d$ and shape parameter $k$. We considered different values for the parameters of each distribution and obtained the optimal retention levels and corresponding minimum ruin probability.

In this section, we conduct the numerical analysis of the excess of loss treaty without superior limit. Firstly, the Exponential distribution is considered and, secondly, the Pareto distribution. Table 2 presents the optimal levels of risk retention and the respective probability of ruin for four different values of parameter $b$ of the Exponential distribution. It also includes information on the optimal interval $(\hat{M}, \infty)$. Table 3 presents the same information for the Pareto distribution with six different scenarios for the values of parameters $k$ and $d$ of the distribution. In Figure 1 we show the plot of the ruin probability function for the Exponential distribution and, in Figures 2 and 3, for the Pareto distribution.

| $b$ | $E(Y)$ | $Var(Y)$ | $\hat{M}$ | $\tilde{M}$ | $\Psi(\tilde{M})$ |
|---|---|---|---|---|---|
| 0.5 | 2 | 4 | 0.94 | 2.05 | 0.68 |
| 1.5 | 0.67 | 0.44 | 0.31 | 0.68 | 0.31 |
| 3 | 0.33 | 0.11 | 0.16 | 0.34 | 0.10 |
| 5 | 0.20 | 0.04 | 0.09 | 0.20 | 0.02 |

Table 2: Optimal strategies for the excess of loss $(M, L = \infty)$ with claims' size modelled by the Exponential distribution with parameter $b$

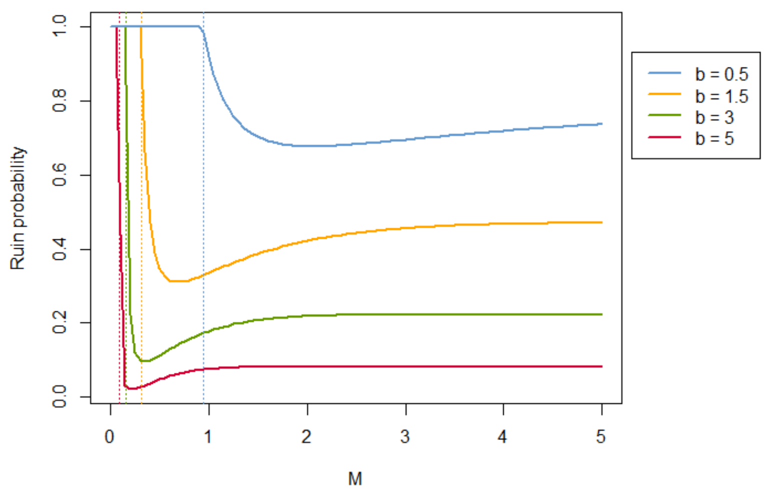


Figure 1: Ruin probability for the excess of loss treaty $(M, L = \infty)$ with claims' size modelled by the Exponential distribution with parameter $b$

When analysing the results on Table 2 and Figure 1, one can see that the higher the value of the distribution's parameter $b$, the wider is the optimal interval $(\hat{M}, +\infty)$, the smaller is the optimum retention level $\tilde{M}$ and the smaller is the minimum ruin probability, which becomes particularly close to zero when $b = 5$, meaning that almost all risk is transferred.

For the Pareto distribution, we fix the scale parameter $d$ and vary the shape parameter $k$ for the first three scenarios and vice-versa for the last three. By looking at the numerical results of Figure 2 and the first three scenarios of Table 3, we find a similar behaviour to the Exponential case. The larger the shape parameter $k$, the wider the optimal interval $(\hat{M}, +\infty)$ is, the smaller is the optimal retention level $M$ and the smaller is the minimum ruin probability. But, when analysing Figure 3 and the last three lines of Table 3, increasing the scale parameter $d$ leads to a symmetrical behaviour, i.e. the optimal interval becomes narrower, the optimal retention level increases and so does the corresponding ruin probability. Therefore, for fixed scale parameter $d$, the higher the shape parameter $k$ is the greater is the amount of risk to reinsurer and, for fixed shape parameter, the higher the scale parameter is the less risk is reinsured.

For both distributions, the numerical results corroborate the conclusions taken analytically: (1) the ruin probability is decreasing right after $\hat{M}$; (2) there is always a minimum critical point, i.e. optimum retention level $\tilde{M} > \hat{M}$; and (3) the function is increasing after $\tilde{M}$.

| $k$ | $d$ | $E(Y)$ | $Var(Y)$ | $\hat{M}$ | $\tilde{M}$ | $\Psi(\tilde{M})$ |
|---|---|---|---|---|---|---|
| 3 | 2 | 1 | 3 | 0.53 | 1.20 | 0.51 |
| 4 | 2 | 0.67 | 0.89 | 0.35 | 0.76 | 0.35 |
| 6 | 2 | 0.40 | 0.24 | 0.20 | 0.44 | 0.16 |
| 3 | 3 | 1.50 | 6.75 | 0.79 | 1.80 | 0.64 |
| 3 | 4 | 2 | 12 | 1.06 | 2.40 | 0.71 |
| 3 | 6 | 3 | 27 | 1.59 | 3.60 | 0.80 |

Table 3: Optimal strategies for the excess of loss $(M, L = \infty)$ with claims' size modelled by the Pareto distribution with scale parameter $d$ and shape parameter $k$

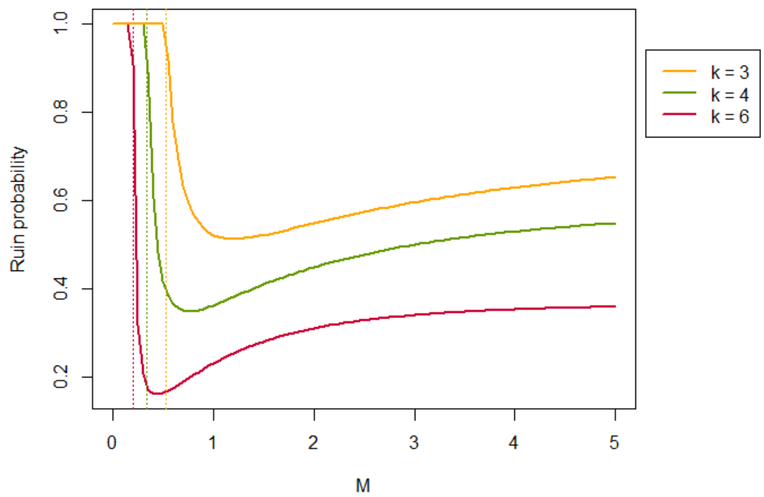


Figure 2: Ruin probability for the excess of loss treaty $(M, L = \infty)$ with claims' size modelled by the Pareto distribution with scale parameter $d = 2$ and shape parameter $k$

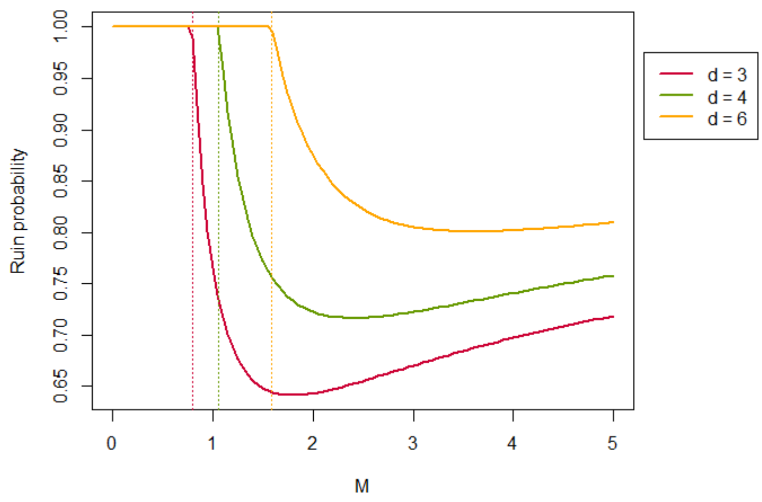


Figure 3: Ruin probability for the excess of loss treaty $(M, L = \infty)$ with claims' size modelled by the Pareto distribution with shape parameter $k = 3$ and scale parameter $d$

### 4.2 The Excess of Loss Treaty with no lower limit

**Proposition 4:** Let Y be a non-negative random variable with infinite support. Consider the excess of loss treaty $(M = 0, L)$, with $L > 0$ and $\eta < \theta$.

Assume that $\tilde{L}_1$ is such that

$$\tilde{L}_1 = \arg\min_{L \in (0,\hat{L})} \Psi(L)$$

with $\hat{L}$ the solution to $\alpha(L) = 0$. Then, the optimal treaty minimizing the ruin probability is given by $(M = 0, \tilde{L})$ such that:

(i) if $-\theta E(Y^2) + 2\eta(E(Y))^2 > 0$ then $\tilde{L} = \tilde{L}_1$;

(ii) if $-\theta E(Y^2) + 2\eta(E(Y))^2 \leq 0$ then

(a) $\tilde{L} = 0$ when $\eta > \frac{\theta S_Y(\tilde{L}_1) E(Y^2)}{2E(Y) \int_{\tilde{L}_1}^{\infty} S_Y(y)dy}$ or when there are no solutions to equation $\alpha(L) = \frac{\theta S_Y(L) \sigma^2(L)}{2 \int_L^{\infty} S_Y(y)dy}$;

(b) $\tilde{L} = \tilde{L}_1$ when $\eta < \frac{\theta S_Y(\tilde{L}_1) E(Y^2)}{2E(Y) \int_{\tilde{L}_1}^{\infty} S_Y(y)dy}$.

*Proof.* The probability of ruin is given by

$$\Psi(L) = \begin{cases} exp\left[\frac{-2X_0\alpha(L)}{\sigma^2(L)}\right] & , \alpha(L) > 0 \\ 1 & , \alpha(L) \leq 0 \end{cases}$$

with $\alpha(L)$ and $\sigma^2(L)$ given in Table 1.

Since, $\alpha(L=0)=\eta E(Y)>0$, $\lim_{L\to\infty}\alpha(L)=E(Y)(\eta-\theta)<0$ and since $\alpha(L)$ is decreasing with $L$ then $\exists!\hat{L}:\alpha(\hat{L})=0$. Hence, we restrict $L$'s optimal domain to $(0,\hat{L})$.

Let

$$\Psi_2(L)=\frac{-2X_0\alpha(L)}{\sigma^2(L)}.$$

The derivative of $\Psi(L)$ is given by $\Psi'(L)=\Psi(L)\Psi_2'(L)$ with

$$\Psi_2'(L)=-2X_0\frac{-\theta S_Y(L)\int_L^\infty 2(y-L)S_Y(y)dy+2(\eta E(Y)-\theta\int_0^L S_Y(y)dy)\int_L^\infty S_Y(y)dy}{(\int_L^\infty 2(y-L)S_Y(y)dy)^2}. \tag{11}$$

To minimize the ruin probability, we have to find the roots of its derivative, i.e. $\Psi'(L)=0$. Since $\Psi(L)>0$, $X_0>0$, $\int_L^\infty 2(y-L)S_Y(y)dy>0$ and, from equation (11), we get

$$\begin{aligned}\Psi'(L)=0 &\Leftrightarrow \Psi_2'(L)=0\\ &\Leftrightarrow \eta E(Y)-\theta\int_0^L S_Y(y)dy=\frac{\theta S_Y(L)\int_L^\infty 2(y-L)S_Y(y)dy}{2\int_L^\infty S_Y(y)dy}.\end{aligned} \tag{12}$$

We can rewrite equation (12) as

$$\alpha(L)=\frac{\theta S_Y(L)\sigma^2(L)}{2\int_L^\infty S_Y(y)dy}. \tag{13}$$

So, any critical point must verify equation (13).

To study the existence and uniqueness of critical points, the function $\Psi(L)$ and its derivative are analysed in the extremes of the optimal interval $(0,\hat{L})$ and compared with the ruin probability in the critical point. We have

$$\lim_{L\to\hat{L}}\Psi(L)=1,$$

since $\alpha(\hat{L})=0$. This means that we must have $\lim_{L\to\hat{L}}\Psi'(L)>0$, i.e. the ruin probability function must be increasing just before $\hat{L}$. Also,

$$\begin{aligned}\lim_{L\to 0}\Psi(L) &= exp\left[-2X_0\frac{\eta E(Y)-\theta\int_0^0 S_Y(y)dy}{\int_0^\infty 2yS_Y(y)dy}\right]\\ &= exp\left[\frac{-2X_0\eta E(Y)}{E(Y^2)}\right]\end{aligned}$$

and

$$\lim_{L\to 0}\Psi'(L)=-2X_0\frac{-\theta E(Y^2)+2\eta E(Y)^2}{(E(Y^2))}. \tag{14}$$

Assume $\tilde{L}$ verifies equation (13), i.e. it is a critical point. Then,

$$\Psi(\tilde{L}) = exp\left[\frac{-2X_0\theta S_Y(\tilde{L})}{2\int_{\tilde{L}}^{\infty} S_Y(y)dy}\right] \tag{15}$$

whose monotony depends on the distribution of the underlying risk, namely, on its survival function $S_Y$.

From equation (14), if $-\theta E(Y^2) + 2\eta(E(Y))^2 > 0$, then $\lim_{L\to 0}\Psi'(L) < 0$. This means that $\exists\tilde{L} > \hat{L} : \alpha(\tilde{L}) = \frac{\theta S_Y(\tilde{L})\sigma^2(\tilde{L})}{2\int_{\tilde{L}}^{\infty} S_Y(y)dy}$ and the optimal solution is $\tilde{L} = \tilde{L}_1$, with $\tilde{L}_1$ the solution to equation (13) that minimizes the ruin probability. But, if $-\theta E(Y^2) + 2\eta(E(Y))^2 \leq 0$, then $\lim_{L\to 0}\Psi'(L) > 0$ and, in this case, the existence of critical points is not guaranteed:

(i) if we don't have critical points, i.e. if equation (13) has no solution, then $\tilde{L} = 0$;

(ii) if otherwise, then we have to compare $\Psi(L = \tilde{L}_1)$ with $\Psi(L = 0)$ in order to find which of $L = \tilde{L}$ and $L = 0$ minimize the ruin probability, assuming $\tilde{L}_1$ is the solution to equation (13) with the smallest ruin probability.

(a) If

$$\begin{aligned}
\Psi(\tilde{L}_1) > \Psi(0) &\Leftrightarrow exp\left[-2X_0\frac{\theta S_Y(\tilde{L}_1)}{2\int_{\tilde{L}_1}^{\infty} S_Y(y)dy)})\right] > exp\left[-2X_0\frac{2\eta E(Y)}{E(Y^2)}\right] \\
&\Leftrightarrow \frac{\theta S_Y(\tilde{L}_1)}{\int_{\tilde{L}_1}^{\infty} S_Y(y)dy)} < \frac{2\eta E(Y)}{E(Y^2)} \\
&\Leftrightarrow \eta > \frac{\theta S_Y(\tilde{L}_1)E(Y^2)}{2E(Y)\int_{\tilde{L}_1}^{\infty} S_Y(y)dy}
\end{aligned}$$

then the optimum solution is $\tilde{L} = 0$;

(a) If

$$\eta < \frac{\theta S_Y(\tilde{L}_1)E(Y^2)}{2E(Y)\int_{\tilde{L}_1}^{\infty} S_Y(y)dy}$$

then the optimum solution is $\tilde{L} = \tilde{L}_1$.

□

Just like for the excess of loss $(M, L = \infty)$, we developed a numerical analysis for the excess of loss $(M = 0, L)$ considering both the Exponential distribution with parameter $b$ and the Pareto distribution with shape parameter $k$ and scale parameter $d$. On the analytical analysis we have proved that the optimum strategy depends on the sign of $-\theta E(Y^2)+2\eta E(Y)^2$ and on the existence of solution to equation (13). We prove below that, for these two distributions, the optimal strategy is to retain all the risk to the insurance company.

**Corollary 1:** Let $Y \sim Exponential(b)$ be a non-negative random variable with infinite support and $b > 0$. Consider the excess of loss treaty $(M = 0, L)$, with $L > 0$ and $\eta < \theta$. Then, $\forall b > 0$, the optimum solution is $\tilde{L} = 0$.

*Proof.* Let $Y \sim Exponential(b)$. The first and second raw moments of the individual claim $Y$ are given by $E(Y) = \frac{1}{b}$ and $E(Y^2) = \frac{2}{b^2}$, respectively. Then,

$$\begin{aligned} -\theta E(Y^2) + 2\eta E(Y)^2 &= -\frac{2\theta}{b^2} + \frac{2\eta}{b^2} \\ &= \frac{2}{b^2}(\eta - \theta) < 0 \end{aligned}$$

since $\eta < \theta$. From the equations on Table 1, the parameters of the diffusion process are computed and given by

$$\alpha(L) = \frac{\eta}{b} - \theta \int_0^L e^{-by} dy = \frac{\eta}{b} + \frac{\theta}{b}(e^{-bL} - 1))$$

and

$$\begin{aligned} \sigma^2(L) &= \int_L^\infty 2ye^{-by} dy - \int_L^\infty 2Le^{-by} dy \\ &= -\frac{2}{b}(0 - Le^{-bL}) + \frac{2}{b}\int_L^\infty e^{-by} dy + \frac{2L}{b}(0 - e^{-bL}) \\ &= \frac{2e^{-bL}}{b^2}. \end{aligned}$$

Let us assume $\tilde{L}$ verifies equation (13). Then,

$$\begin{aligned} \alpha(\tilde{L}) &= \frac{\theta e^{-b\tilde{L}} \sigma^2(\tilde{L})}{\frac{2e^{-b\tilde{L}}}{b}} \\ \Leftrightarrow \frac{\eta}{b} + \frac{\theta}{b}(e^{-b\tilde{L}} - 1)) &= \frac{\theta e^{-b\tilde{L}}}{b} \\ \Leftrightarrow \frac{\eta}{b} - \frac{\theta}{b} &= 0 \end{aligned}$$

which is absurd as $\eta < \theta$. Hence, equation (13) has no solution and, therefore, the optimal strategy is $\tilde{L} = 0$.

□

A similar result follows when considering that individual claims are modelled by the Pareto distribution instead of the Exponential.

**Corollary 2:** Let $Y \sim Pareto(k, d)$ be a non-negative random variable with infinite support, with $k > 2$ the shape parameter and $d > 0$ the scale parameter of the distribution. Consider the excess of loss treaty $(M = 0, L)$, with $L > 0$ and $\eta < \theta$ . Then, $\forall k > 2, d > 0$, the optimum solution is $\tilde{L} = 0$.

*Proof.* Let $Y \sim Pareto(k, d)$. We consider $k > 2$ so that the first and second raw moments of the individual claim $Y$ exist and are given, respectively, by $E(Y) = \frac{d}{k-1}$ and $E(Y^2) = \frac{2d^2}{(k-1)(k-2)}$. Then,

$$\begin{aligned} -\theta E(Y^2) + 2\eta E(Y)^2 &= \frac{-2\theta d^2}{(k-1)(k-2)} + \frac{2\eta d^2}{(k-1)^2} \\ &= 2d^2 \frac{\eta(k-2) - \theta(k-1)}{(k-1)^2(k-2)} \end{aligned}$$

but, since $\eta < \theta$ and $0 < k-2 < k-1$, then $\eta(k-1) < \theta(k-2)$. Therefore, we can conclude that $-\theta E(Y^2) + 2\eta E(Y)^2 < 0, \forall k > 2, d > 0$. From the equations on Table 1, the parameters of the diffusion process are computed and given by

$$\begin{aligned} \alpha(L) &= \frac{\eta d}{k-1} - \theta \int_0^L (\frac{d}{d+y})^k dy \\ &= \frac{\eta d}{k-1} + \frac{\theta d^k}{k-1}((d+L)^{1-k} - d^{1-k}) \\ &= \frac{(\eta - \theta)d}{k-1} + \frac{\theta d^k}{(d+L)^{k-1}(k-1)} \end{aligned}$$

and

$$\begin{aligned} \sigma^2(L) &= \int_L^\infty 2y(\frac{d}{d+y})^k dy - \int_L^\infty 2L(\frac{d}{d+y})^k dy \\ &= \frac{d^k}{1-k}(0 - 2L(d+L)^{1-k}) - \frac{2d^k}{(1-k)(2-k)}(0 - (d+L)^{2-k}) - \\ &- \frac{d^k}{1-k}(0 - 2L(d+L)^{1-k}) \\ &= \frac{2d^k(d+L)^{2-k}}{(k-1)(k-2)}. \end{aligned}$$

Let's assume $\tilde{L}$ verifies equation (13). Then,

$$\begin{aligned} \alpha(\tilde{L}) &= \frac{\theta(\frac{d}{d+\tilde{L}})^k \sigma^2(\tilde{L})}{2\int_{\tilde{L}}^\infty (\frac{d}{d+\tilde{L}})^k dy} \\ \Leftrightarrow \alpha(\tilde{L}) &= \frac{\theta(\frac{d}{d+\tilde{L}})^k (d+\tilde{L})^{k-1}(k-1)\sigma^2(\tilde{L})}{2d^k} \\ \Leftrightarrow \alpha(\tilde{L}) &= \frac{\theta(k-1)\sigma^2(\tilde{L})}{2(d+\tilde{L})} \\ \Leftrightarrow \alpha(\tilde{L}) &= \frac{\theta(k-1)\frac{2d^k(d+\tilde{L})^{2-k}}{(k-1)(k-2)}}{2(d+\tilde{L})} \\ \Leftrightarrow \frac{(\eta-\theta)d}{k-1} + \frac{\theta d^k}{(d+\tilde{L})^{k-1}(k-1)} &= \frac{\theta d^k}{(d+\tilde{L})^{k-1}(k-2)} \\ \Leftrightarrow \frac{(\eta-\theta)d}{k-1} &= \frac{\theta d^k}{(d+\tilde{L})^{k-1}(k-1)(k-2)} \end{aligned}$$

which is absurd since $\frac{\theta d^k}{(d+L)^{k-1}(k-1)(k-2)} > 0$ and $\frac{(\eta-\theta)d}{k-1} < 0$. Hence, equation (13) has no solution and, therefore, the optimal strategy is $\tilde{L} = 0$. $\square$

Our goal, when developing the numerical analysis for the excess of loss $(M = 0, L)$ is to study the ruin probability, both when $-\theta E(Y^2) + 2\eta E(Y)^2 < 0$ and $-\theta E(Y^2) + 2\eta E(Y)^2 > 0$, so that better conclusions can be drawn. For this reason, we consider the Exponential, the Pareto but also the Gamma distribution with shape parameter $e$ and scale parameter $f$ as it can be proven that, for this distribution, the sign of $-\theta E(Y^2) + 2\eta E(Y)^2$ depends on its shape parameter.

Let $Y \sim Gamma(e, f)$. The first and second raw moments of the individual claim $Y$ are given, respectively, by $E(Y) = ef$ and $E(Y^2) = e(e-1)f^2$. Then,

$$\begin{aligned} -\theta E(Y^2) + 2\eta E(Y)^2 &= -\theta e(e+1)f^2 + 2\eta e^2 f^2 \\ &= ef^2(-\theta(e+1) + 2\eta e). \end{aligned}$$

If $-\theta(e+1)+2\eta e < 0$ then $-\theta E(Y^2)+2\eta E(Y)^2 < 0$. On the other hand, if $-\theta(e+1)+2\eta e > 0$ then $-\theta E(Y^2) + 2\eta E(Y)^2 > 0$. Clearly, the sign of $-\theta E(Y^2) + 2\eta E(Y)^2$ only depends on the shape parameter $e$ of the distribution.

Under the excess of loss treaty without inferior limit, small losses are transferred to the reinsurer. It makes sense to believe that the optimal retention level depends on the distribution chosen to model the claim sizes and on its tail behavior. The Pareto distribution is heavy-tailed and, for this case, it is not worth paying for a reinsurance treaty which covers small losses. Hence, nothing is transferred to the reinsurer. The Gamma distribution is more dense on the left (small losses, close to zero) and on the center and, therefore, it may be worth transferring part of the risk under this contract. It seems that this depends on the parameters of the distribution as, when considering the shape parameter $e = 1$, i.e the Exponential distribution, the optimal under this type of reinsurance is to retain all risk.

### 4.2.1 Numerical analysis

We proceed by giving some numerical results for the excess of loss $(M = 0, L)$. We consider that individual claims are modeled by (1) the Exponential distribution with parameter $b$, (2) the Pareto distribution with shape parameter $k$ and scale parameter $d$ and (3) the Gamma distribution with shape parameter $e$ and scale parameter $f$. For each distribution, we consider different values for the parameters and obtain the optimum retention level $\tilde{L}$ and corresponding ruin probability $\Psi(\tilde{L})$.

Table 4, Table 5 and Table 6 present the optimal levels of risk retention and the respective probability of ruin for different values of the distribution's parameters of the Exponential, of the Pareto and of the Gamma distribution, respectively. It also includes information on the optimal interval $(\hat{M}, \infty)$. We show the plot of the ruin probability function for the Exponential distribution in Figure 4. In Figures 5 and 6, we present the plots of the ruin probability for the Pareto distribution and, in Figures 7 and 8, for the Gamma distribution.

By analysing the numerical results on Figure 4 and Table 4, one can see that the higher the value of parameter $b$, the narrower is the optimum interval $(0, \hat{L})$. For all cases we have $-\theta E(Y^2) + 2\eta E(Y)^2 < 0$ and the insurer's optimal strategy is to retain all risk, i.e. $\tilde{L} = 0$. Also, the higher the value of $b$ the smaller is the probability of ruin, which gets very close to zero when $b = 5$.

| $b$ | $E(Y)$ | $Var(Y)$ | $-\theta E(Y^2) + 2\eta E(Y)^2$ | $\hat{L}$ | $\tilde{L}$ | $\Psi(\tilde{L})$ |
|---|---|---|---|---|---|---|
| 0.5 | 2 | 4 | -2.40 | 1.96 | 0.00 | 0.78 |
| 1.5 | 0.67 | 0.44 | -0.27 | 0.65 | 0.00 | 0.47 |
| 3 | 0.33 | 0.11 | -0.07 | 0.33 | 0.00 | 0.22 |
| 5 | 0.20 | 0.04 | -0.02 | 0.20 | 0.00 | 0.08 |

Table 4: Optimal strategies for the excess of loss $(M = 0, L)$ with claims' size modelled by the Exponential distribution with parameter $b$

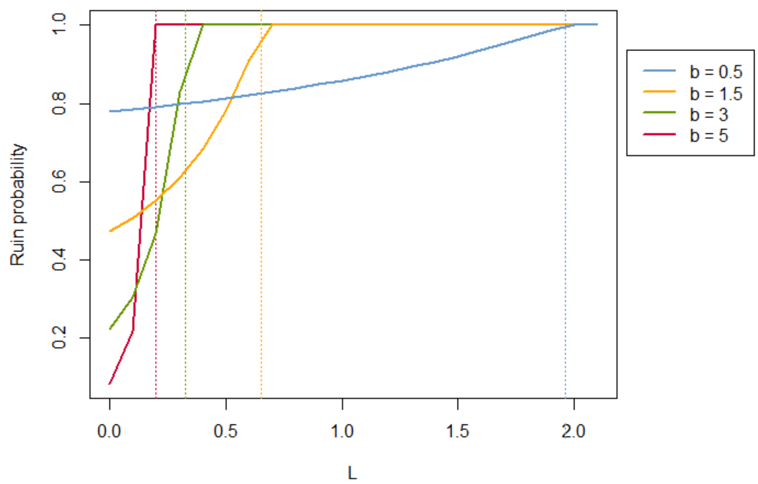


Figure 4: Ruin probability for the excess of loss treaty $(M = 0, L)$ with claims' size modelled by the Exponential distribution with parameter $b$

For the Pareto distribution, we've fixed the scale parameter $d$ and varied the shape parameter $k$, for the first three scenarios, and vice-versa, for the last three. By analysing Figures 5 and 6 and Table 5, similar conclusions to the Exponential case can be taken: the sign of $-\theta E(Y^2)+2\eta E(Y)^2$ is always negative, regardless of the parameters' values, and the optimal strategy is to retain all risk ($\hat{L} = 0$). But, while for the first three scenarios of Table 5 we see that the optimal interval $(0, \hat{L})$ is becoming narrower and the ruin probability is increasing, for the last three we see the opposite behaviour.

The numerical results obtained for the Exponential distribution and for the Pareto distribution support the analytical results: (1) the sign of $-\theta E(Y^2) + 2\eta E(Y)^2$ is always negative; (2) there aren't minimum critical points and (3) the ruin probability is always increasing, reason why the optimum retention level is $\tilde{L} = 0$ for all cases.

| $k$ | $E(Y)$ | $Var(Y)$ | $d$ | $-\theta E(Y^2) + 2\eta E(Y)^2$ | $\hat{L}$ | $\tilde{L}$ | $\Psi(\tilde{L})$ |
|---|---|---|---|---|---|---|---|
| 3 | 2 | 1 | 3 | -2.20 | 1.27 | 0.00 | 0.78 |
| 4 | 2 | 0.67 | 0.89 | -0.62 | 0.77 | 0.00 | 0.61 |
| 8 | 2 | 0.29 | 0.11 | -0.07 | 0.30 | 0.00 | 0.22 |
| 3 | 3 | 1.50 | 6.75 | -4.95 | 1.90 | 0.00 | 0.85 |
| 3 | 4 | 2 | 12 | -8.80 | 2.53 | 0.00 | 0.88 |
| 3 | 8 | 4 | 48 | -35.2 | 5.06 | 0.00 | 0.94 |

Table 5: Optimal strategies for the excess of loss $(M = 0, L)$ with claims' size modelled by the Pareto distribution with scale parameter $d$ and shape parameter $k$

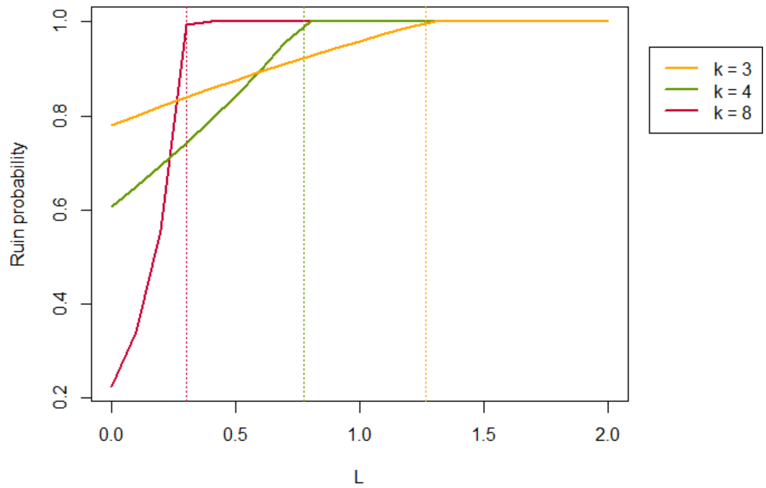


Figure 5: Ruin probability for the excess of loss treaty $(M = 0, L)$ with claims' size modelled by the Pareto distribution with scale parameter $d = 2$ and shape parameter $k$

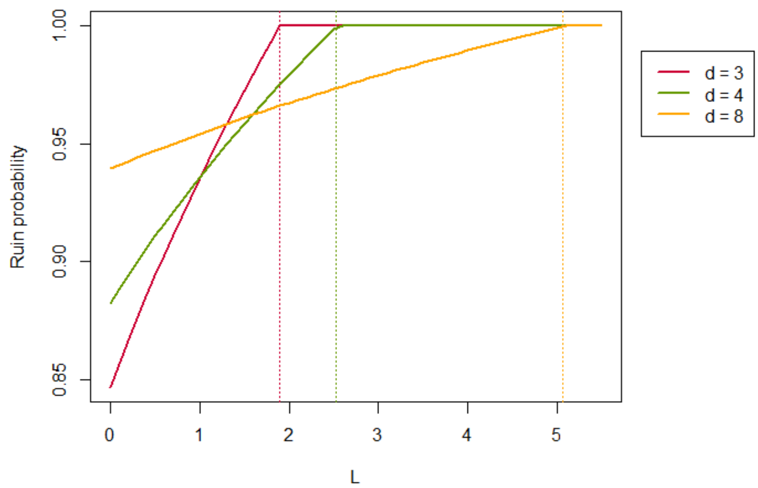


Figure 6: Ruin probability for the excess of loss treaty $(M = 0, L)$ with claims' size modelled by the Pareto distribution with shape parameter $k = 3$ and scale parameter $d$

When considering the Gamma distribution, we fix the scale parameter $f$ and vary the shape parameter $e$, for the first four scenarios, and vice-versa, for the last three. When looking at Figure 7 and the first four scenarios of Table 6, we can conclude that, the higher the value of $e$, the wider is the optimal interval $(0, \tilde{L})$. Also, for small values of $e$, the sign of $-\theta E(Y^2)+2\eta E(Y)^2$ is negative and the optimal retention level $\tilde{L}$ is zero. For the last two cases, $-\theta E(Y^2) + 2\eta E(Y)^2$ becomes positive and the optimal retention level is a positive number, which increases as $e$ increases. The minimum ruin probability is also increasing with the shape parameter $e$. If we look at the numerical results from Figure 8 and the last three scenarios of Table 6, we see that $-\theta E(Y^2) + 2\eta E(Y)^2$ is always negative and the optimal retention level is $\tilde{L} = 0$ for all scenarios.

For the Gamma distribution, the sign of $-\theta E(Y^2) + 2\eta E(Y)^2$ changes, depending on the value of $e$. When it's sign is negative there aren't any minimum critical points but, when positive, there is one minimum critical point. Also, the ruin probability is always increasing close to $\hat{L}$. Therefore, the numerical results once again comply with the analytical ones.

| $e$ | $f$ | $E(Y)$ | $Var(Y)$ | $-\theta E(Y^2) + 2\eta E(Y)^2$ | $\hat{L}$ | $\tilde{L}$ | $\Psi(\tilde{L})$ |
|---|---|---|---|---|---|---|---|
| 1.5 | 2 | 3 | 6 | -3.00 | 2.53 | 0.00 | 0.82 |
| 3 | 2 | 6 | 12 | -2.40 | 4.28 | 0.00 | 0.88 |
| 8 | 2 | 16 | 32 | 25.60 | 10.28 | 1.75 | 0.95 |
| 15 | 2 | 30 | 60 | 132.00 | 18.88 | 5.09 | 0.97 |
| 3 | 3 | 9 | 27 | -5.40 | 6.43 | 0.00 | 0.92 |
| 3 | 8 | 24 | 192 | -38.40 | 17.13 | 0.00 | 0.97 |
| 3 | 15 | 45 | 675 | -135.00 | 32.12 | 0.00 | 0.98 |

Table 6: Optimal strategies for the excess of loss $(M = 0, L)$ with claims' size modelled by the Gamma distribution with shape parameter $e$ and scale parameter $f$

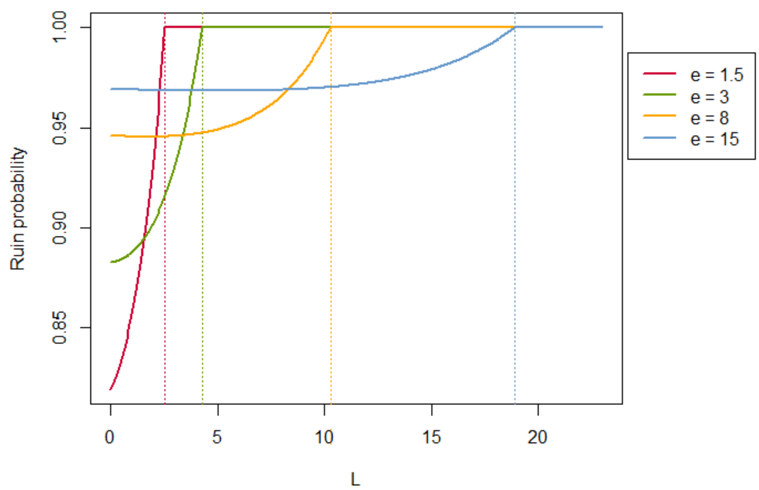


Figure 7: Ruin probability for the excess of loss treaty $(M = 0, L)$ with claims' size modelled by the Gamma distribution with scale parameter $f = 2$ and shape parameter $e$

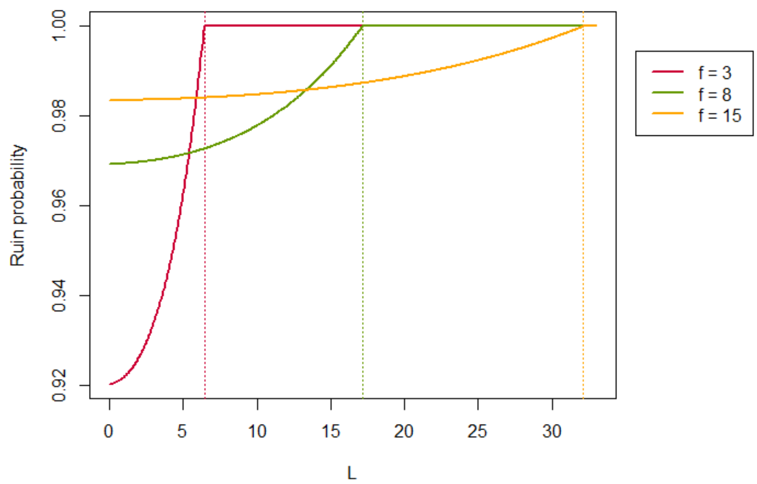


Figure 8: Ruin probability for the excess of loss treaty $(M = 0, L)$ with claims' size modelled by the Gamma distribution with shape parameter $e = 3$ and scale parameter $f$

### 4.3 The Excess of Loss Treaty with lower and upper limits

**Proposition 5:** Let Y be a non-negative random variable with infinite support. Consider the excess of loss treaty $(M, L)$, with $0 \leq M < L$ and $\eta < \theta$.

Then, the optimal solution minimizing the ruin probability is $\tilde{L} = \infty$ and $\tilde{M}$ the smallest solution to the equation $\alpha(M, \infty) = \frac{\theta}{2M}\sigma^2(M, \infty)$ that verifies $\tilde{M} > \hat{M}$, where $\hat{M}$ is the solution to $\alpha(M, \infty) = 0$.

*Proof.* The probability of ruin is given by

$$\Psi(M, L) = \begin{cases} exp\left[\frac{-2X_0\alpha(M,L)}{\sigma^2(M,L)}\right] & , \alpha(M, L) > 0 \\ 1 & , \alpha(M, L) \leq 0 \end{cases}$$

with $\alpha(M, L)$ and $\sigma^2(M, L)$ given by Table 1.

Let

$$\Psi_3(M, L) = \frac{-2X_0\alpha(M, L)}{\sigma^2(M, L)}.$$

Its partial derivatives are given by

$$\begin{cases} \frac{\partial\Psi_3(M,L)}{\partial M} = -2X_0(S_Y(M + L) - S_Y(M))\frac{-\theta\sigma^2(M,L)+2M\alpha(M,L)}{\sigma^4(M,L)} \\ \frac{\partial\Psi_3(M,L)}{\partial L} = -2X_0\frac{-\theta S_Y(M+L)\sigma^2(M,L)-2\alpha(M,L)(MS_Y(M+L)+\int_{M+L}^{\infty}S_Y(y)dy)}{\sigma^4(M,L)}. \end{cases} \quad (16)$$

To minimize the ruin probability we have to find the roots of its partial derivatives. Since $\Psi(M, L) > 0$, $X_0 > 0$, $S_Y(M + L) > 0$, $S_Y(M + L) - S_Y(M) < 0$ and, from equations (16), we get

$$\begin{cases} \frac{\partial \Psi(M,L)}{\partial M} = 0 \\ \frac{\partial \Psi(M,L)}{\partial L} = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{\partial \Psi_3(M,L)}{\partial M} = 0 \\ \frac{\partial \Psi_3(M,L)}{\partial L} = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha(M,L) = \frac{\theta}{2M}\sigma^2(M,L) \\ \alpha(M,L) = \frac{\theta S_Y(M+L)}{2MS_Y(M+L)+2\int_{M+L}^{\infty} S_Y(y)dy}\sigma^2(M,L) \end{cases} . \tag{17}$$

Therefore,

$$\begin{aligned} \frac{\theta}{2M} &= \frac{\theta S_Y(M+L)}{2MS_Y(M+L) + 2\int_{M+L}^{\infty} S_Y(y)dy} \\ \Leftrightarrow M &= \frac{MS_Y(M+L) + \int_{M+L}^{\infty} S_Y(y)dy}{S_Y(M+L)} \\ \Leftrightarrow \int_{M+L}^{\infty} & S_Y(y)dy = 0. \end{aligned} \tag{18}$$

From equation (18), we conclude that $L = \infty$. This means that our the problem is reduced to the first excess of loss case studied $(M, L = \infty)$. Therefore, under the excess of loss treaty with inferior and superior limits, the conclusions follow from the excess of loss treaty without superior limit, i.e., the optimal solution is $\tilde{L} = \infty$ and $\tilde{M}$ the smallest solution to the equation $\alpha(M,\infty) = \frac{\theta}{2M}\sigma^2(M,\infty)$ that verifies $\tilde{M} > \hat{M}$, where $\hat{M}$ is the solution to $\alpha(M,\infty) = 0$. □

## 5 Including dependences

Similarly to what was done by Liang & Yuen (2016), in this section we consider the optimal reinsurance strategy in a risk model with two dependent classes of risk, where the two claim number processes are correlated through a common shock component, under the optimization criteria of minimizing the ruin probability. Let $X_i$ be the size of claim $i$ for one of the classes of risk, with $X_i$ iid to $X$, and $Y_j$ the size of claim $j$ for the other risk class, with $Y_j$ iid to $Y$. The corresponding distributions are given by $F_X(x)$ and $F_Y(y)$. We assume that $X_i$ and $Y_j$ are independent, for all $i$ and $j$, and that the moment generating functions, given by $M_X(u)$ and $M_Y(u)$, exist. The aggregate claims processes for the two classes are given by

$$S_1(t) = \sum_{i=1}^{N_1(t)+N(t)} X_i \text{ and } S_2(t) = \sum_{j=1}^{N_2(t)+N(t)} Y_j \tag{19}$$

where $N_1(t)$, $N_2(t)$ and $N(t)$ are three independent Poisson processes with corresponding intensities $\lambda_1$, $\lambda_2$ and $\lambda$. Then, $S_1(t)$ is a compound Poisson process with intensity $\lambda_1 + \lambda$ and $S_2(t)$ is a compound Poisson process with intensity $\lambda_2 + \lambda$. Moreover, the aggregate claims process generated from these two classes of risk is given by

$$S_t = S_1(t) + S_2(t) = \sum_{i=1}^{N_1(t)+N(t)} X_i + \sum_{j=1}^{N_2(t)+N(t)} Y_j.$$

**Theorem 1:** $S_t$ is a compound Poisson process with intensity $\xi = \lambda_1 + \lambda_2 + \lambda$ and secondary distribution $F_T(x) = \frac{\lambda_1}{\xi} F_X(x) + \frac{\lambda_2}{\xi} F_Y(x) + \frac{\lambda}{\xi} F_{X+Y}(x)$, where $F_X(x)$, $F_Y(x)$ and $F_{X+Y}(x)$ are the distribution functions of $X$, $Y$ and $X+Y$, respectively.

*Proof.* We start by computing the expressions for the moment generating functions of $S_1$ and $S_2$. Since $X_i$ are independent random variables then

$$\begin{aligned} M_{S_1}(u) &= E\left[e^{u\sum_{i=1}^{n+N_1(t)} X_i}\right] \\ &= E\left[e^{u\sum_{i=1}^{n} X_i} e^{u\sum_{i=1}^{N_1(t)} X_{i+n}}\right] \\ &= E[e^{uX}]^n E\left[e^{u\sum_{i=1}^{N_1(t)} X_{i+n}}\right] \\ &= (M_X(u))^n E\left[M_X(u)^{N_1(t)}\right] \\ &= (M_X(u))^n e^{\lambda_1 t(M_X(u)-1)}. \end{aligned}$$

By following the same line of thought, and since $Y_i$ are independent random variables, we have

$$M_{S_2}(u) = (M_Y(u))^n e^{\lambda_2 t(M_Y(u)-1)}.$$

Now, we can compute the moment generating function for the aggregated claims process $S_t$:

$$\begin{aligned} E\left[e^{u(S_1(t)+S_2(t))}\right] &= e^{\lambda_1 t(M_X(u)-1)+\lambda_2 t(M_Y(u)-1)} E\left[M_X(u)^{N_t} M_Y(u)^{N_t}\right] \\ &= e^{\lambda_1 t(M_X(u)-1)+\lambda_2 t(M_Y(u)-1)+\lambda t(M_X(u)M_Y(u)-1)} \\ &= e^{\xi t(\frac{\lambda_1}{\xi}M_X(u)+\frac{\lambda_1}{\xi}M_Y(u)+\frac{\lambda}{\xi}M_X(u)M_Y(u)-1)}. \end{aligned} \tag{20}$$

Since $X$ and $Y$ are independent, then $M_{X+Y}(u) = E\left[e^{uX+uY}\right] = M_X(u)M_Y(u)$. Hence, equation (19) is the expression of the moment generating function of a compound Poisson process with intensity $\xi = \lambda_1 + \lambda_2 + \lambda$ and secondary distribution $\frac{\lambda_1}{\xi}F_X(x) + \frac{\lambda_2}{\xi}F_Y(x) + \frac{\lambda}{\xi}F_{X+Y}(x)$. Therefore, $S_t$ is a compound Poisson process. □

This result allows us to take some direct conclusions from the ones made in Proposition 2, Proposition 3, Proposition 4 and Proposition 5 on the optimal reinsurance strategy when considering two dependent risks, assuming that both risk classes are reinsured under the same reinsurance contract.

**Corollary 3:**Let $S_1(t)$ and $S_2(t)$ be two compound Poisson processes dependent through a common shock component as in (19). Consider $S_t = S_1(t) + S_2(t)$ and the quota-share treaty with $a \in [0, 1]$ the retained percentage of the risk and $\eta < \theta$. The optimal reinsurance strategy minimizing the ruin probability is $\tilde{a} = 2 - \frac{2\eta}{\theta}$.

*Proof.* From Theorem 1, $S_t = S_1 + S_2$ is a compound Poisson process with intensity $\xi = \lambda_1 + \lambda_2 + \lambda$ and secondary distribution $F_T(x) = \frac{\lambda_1}{\xi}F_X(x) + \frac{\lambda_2}{\xi}F_Y(x) + \frac{\lambda}{\xi}F_{X+Y}(x)$. Therefore, from Proposition 2, the optimal strategy is $\tilde{a} = 2 - \frac{2\eta}{\theta}$. □

**Corollary 4:** Let $S_1(t)$ and $S_2(t)$ be two compound Poisson processes dependent through a common shock component as in (19). Consider $S_t = S_1(t) + S_2(t)$ and the excess of loss treaty $(M, L = \infty)$, with $M > 0$ and $\eta < \theta$. The optimal reinsurance strategy minimizing the ruin probability is $\tilde{M}$ the optimum solution to the equation $\alpha(M) = \frac{\theta}{2M}\sigma^2(M)$ which verifies $\tilde{M} > \hat{M}$, where $\hat{M}$ is the solution to $\alpha(M) = 0$.

*Proof.* From Theorem 1, $S_t = S_1 + S_2$ is a compound Poisson process with intensity $\xi = \lambda_1 + \lambda_2 + \lambda$ and secondary distribution $F_T(x) = \frac{\lambda_1}{\xi}F_X(x) + \frac{\lambda_2}{\xi}F_Y(x) + \frac{\lambda}{\xi}F_{X+Y}(x)$. Therefore, from Proposition 3, the optimal strategy is $\tilde{M}$ the optimum solution to the equation $\alpha(M) = \frac{\theta}{2M}\sigma^2(M)$ which verifies $\tilde{M} > \hat{M}$, where $\hat{M}$ is the solution to $\alpha(M) = 0$. □

**Corollary 5:** Let $S_1(t)$ and $S_2(t)$ be two compound Poisson processes dependent through a common shock component as in (19). Consider $S_t = S_1(t) + S_2(t)$ and the excess of loss treaty $(M = 0, L)$, with $L > 0$ and $\eta < \theta$.

Assume that $\tilde{L}_1$ is such that

$$\tilde{L}_1 = \arg\min_{L \in (0, \hat{L})} \Psi(L),$$

with $\hat{L}$ the solution to $\alpha(L) = 0$. Then, the optimal reinsurance treaty minimizing the ruin probability is given by $\tilde{L}$ such that:

(i) if $-\theta E(T^2) + 2\eta(E(T))^2 > 0$ then $\tilde{L} = \tilde{L}_1$;

(ii) if $-\theta E(T^2) + 2\eta(E(T))^2 \leq 0$ then

   (a) $\tilde{L} = 0$ when $\eta > \frac{\theta S_T(\tilde{L}_1) E(T^2)}{2E(T) \int_{\tilde{L}_1}^{\infty} S_T(y)dy}$ or when there are no solutions to equation $\alpha(L) = \frac{\theta S_T(L)\sigma^2(L)}{2\int_L^{\infty} S_T(y)dy}$;

   (b) $\tilde{L} = \tilde{L}_1$ when $\eta < \frac{\theta S_T(\tilde{L}_1) E(T^2)}{2E(T) \int_{\tilde{L}_1}^{\infty} S_T(y)dy}$.

*Proof.* From Theorem 1, $S_t = S_1 + S_2$ is a compound Poisson process with intensity $\xi = \lambda_1 + \lambda_2 + \lambda$ and secondary distribution $F_T(x) = \frac{\lambda_1}{\xi} F_X(x) + \frac{\lambda_2}{\xi} F_Y(x) + \frac{\lambda}{\xi} F_{X+Y}(x)$. Assume that $\tilde{L}_1$ is the solution to the critical point's equation $\alpha(L) = \frac{\theta S_T(L)\sigma^2(L)}{2\int_L^{\infty} S_T(y)dy}$ that minimizes the ruin probability, i.e. $\tilde{L}_1 = \arg\min_{L \in (0, \hat{L})} \Psi(L)$. Therefore, from Proposition 4,

(i) If $-\theta E(T^2) + 2\eta(E(T))^2 > 0$ then $\tilde{L} = \tilde{L}_1$;

(ii) If $-\theta E(T^2) + 2\eta(E(T))^2 \leq 0$ then

   (a) $\tilde{L} = 0$ when $\eta > \frac{\theta S_T(\tilde{L}_1) E(T^2)}{2E(T) \int_{\tilde{L}_1}^{\infty} S_T(y)dy}$ or when there are no solutions to equation $\alpha(L) = \frac{\theta S_T(L)\sigma^2(L)}{2\int_L^{\infty} S_T(y)dy}$;

   (b) $\tilde{L} = \tilde{L}_1$ when $\eta < \frac{\theta S_T(\tilde{L}_1) E(T^2)}{2E(T) \int_{\tilde{L}_1}^{\infty} S_T(y)dy}$.

□

**Corollary 6:** Let $S_1(t)$ and $S_2(t)$ be two compound Poisson processes dependent through a common shock component as in (19). Consider $S_t = S_1(t) + S_2(t)$ and the excess of loss treaty $(M, L)$, with $0 < M < L$ and $\eta < \theta$.

Then, the optimal strategy minimizing the ruin probability is $\tilde{L} = \infty$ and $\tilde{M}$ the smallest solution to the equation $\alpha(M, \infty) = \frac{\theta}{2M}\sigma^2(M, \infty)$ that verifies $\tilde{M} > \hat{M}$, where $\hat{M}$ is the solution to $\alpha(M, \infty) = 0$.

*Proof.* From Theorem 1, $S_t = S_1 + S_2$ is a compound Poisson process with intensity $\xi = \lambda_1 + \lambda_2 + \lambda$ and secondary distribution $F_T(x) = \frac{\lambda_1}{\xi}F_X(x) + \frac{\lambda_2}{\xi}F_Y(x) + \frac{\lambda}{\xi}F_{X+Y}(x)$. Therefore, from Proposition 5, the optimal strategy is $\tilde{L} = \infty$ and $\tilde{M}$ the smallest solution to the equation $\alpha(M,\infty) = \frac{\theta}{2M}\sigma^2(M,\infty)$ that verifies $\tilde{M} > \hat{M}$, where $\hat{M}$ is the solution to $\alpha(M,\infty) = 0$. □

## 5.1 Numerical analysis

Now, we conduct a numerical analysis to analyse the influence of considering dependent risks on the optimal reinsurance strategy. To do so, the optimal retention levels for the individual claims processes, $S_1$ and $S_2$, are compared to the optimal retention levels for the process with two dependent risks, $S_t$.

Let $X \sim Gamma(e_1, f)$ and $Y \sim Gamma(e_2, f)$. Then, $X + Y \sim Gamma(e_1 + e_2, f)$. From Theorem 1, the secondary distribution of $S_t$ depends on the distribution of $X$, $Y$ and $X+Y$ as well as on the intensities $\lambda_1$, $\lambda_2$ and $\lambda$. We fix the parameters $\lambda_1 = \lambda_2 = 1$ and $\lambda = 2$, the safety loadings for both risk classes, $\eta = 0.5$ and $\theta = 0.8$, and the initial surplus $x = 1$ .

| | Parameters | | | Expected Value | | | Variance | | |
|---|---|---|---|---|---|---|---|---|---|
| Scenario | $e_1$ | $e_2$ | $f$ | $E(X)$ | $E(Y)$ | $E(T)$ | $Var(X)$ | $Var(Y)$ | $Var(T)$ |
| I | 2 | 3 | 3 | 6 | 9 | 11.25 | 18 | 27 | 48.94 |
| II | 2 | 5 | 3 | 6 | 15 | 15.75 | 18 | 25 | 84.94 |
| III | 2 | 8 | 3 | 6 | 24 | 22.50 | 18 | 72 | 164.25 |
| IV | 5 | 8 | 5 | 25 | 40 | 48.75 | 125 | 200 | 535.94 |
| V | 5 | 9 | 5 | 25 | 45 | 52.50 | 125 | 225 | 618.75 |
| VI | 5 | 12 | 5 | 25 | 60 | 63.75 | 125 | 300 | 923.44 |

Table 7: Different scenarios for the parameters $e_1$, $e_2$ and $f$ of the Gamma distributions

For the excess of loss treaty $(M, L = \infty)$, we present a plot of the ruin probability function, when dependent risks are considered, in Figure 9. A similar plot for the excess of loss treaty $(M = 0, L)$ is represented in Figure 10. The optimal retention level, and corresponding minimum ruin probability, for each one of the individual claims process, $S_1$ and $S_2$, and for the process $S_t$, are presented in Table 8, for the excess of loss without superior limit $L$, and in Table 9, for the excess of loss without inferior limit $M$.

From Figure 9, we can conclude that the analytical results for the excess of loss treaty $(M, L = \infty)$ are confirmed: (1) the ruin probability function is increasing right after $\hat{M}$; (2) there is always a minimum critical point $\tilde{M} > \hat{M}$ and (3) the ruin probability function is decreasing right after this minimum. From Table 8, we can compare the optimal independent retention levels for $S_1$ and $S_2$ to the optimal retention level for the process with dependent risks $S_t$ and see that they are distinct. Therefore, considering dependences influences the optimal reinsurance strategy.

By analysing Figure 10 and Table 9, we can see that, for the excess of loss $(M = 0, L)$, the numerical results for $S_t$ verify the analytical ones: (1) the ruin probability function is increasing right before $\hat{L}$; (2) when $-\theta E_2+2\eta E_1^2 < 0$, the optimal strategy is to retain all risk to the insurance company, i.e. $\tilde{L} = 0$ and the ruin probability is always increasing and (3) when $-\theta E_2+2\eta E_1^2 > 0$, there is a minimum critical point $0 < \tilde{L} < \hat{L}$ and the ruin probability is decreasing right before this minimum. Also from Table 9, we can see that the optimal retention level for the individual claims processes $S_1$ and $S_2$ are different from the optimal retention level when considering dependent risks.

| | $\tilde{M}$ | | | $\Psi(\tilde{M})$ | | |
|---|---|---|---|---|---|---|
| Scenario | $S_1$ | $S_2$ | $S_t$ | $S_1$ | $S_2$ | $S_t$ |
| I | 5.18 | 7.33 | 7.73 | 0.86 | 0.90 | 0.90 |
| II | 5.18 | 11.70 | 11.08 | 0.86 | 0.93 | 0.93 |
| III | 5.18 | 18.33 | 16.77 | 0.86 | 0.96 | 0.95 |
| IV | 19.50 | 30.54 | 39.02 | 0.95 | 0.97 | 0.98 |
| V | 19.50 | 34.24 | 42.22 | 0.96 | 0.98 | 0.98 |
| VI | 19.50 | 45.38 | 52.20 | 0.96 | 0.98 | 0.98 |

Table 8: Optimal strategies for the excess of loss $(M, L = \infty)$ for the process with dependent risks and for the individual claims process $S_1$ and $S_2$

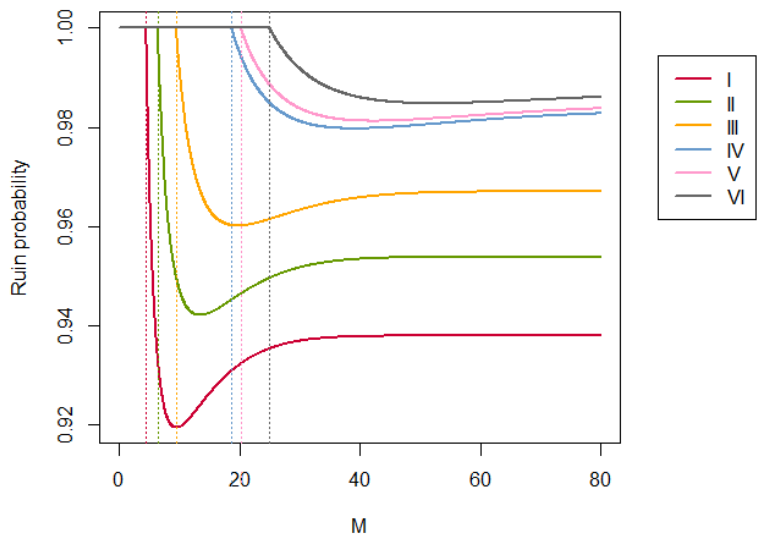


Figure 9: Ruin probability for the excess of loss treaty $(M, L = \infty)$ for the process with dependent risks

| | $-\theta E_2 + 2\eta E_1^2$ | | | $\tilde{L}$ | | | $\Psi(\tilde{L})$ | | |
|---|---|---|---|---|---|---|---|---|---|
| Scenario | $S_1$ | $S_2$ | $S_t$ | $S_1$ | $S_2$ | $S_t$ | $S_1$ | $S_2$ | $S_t$ |
| I | -7.20 | -5.40 | -13.00 | 0.00 | 0.00 | 0.00 | 0.89 | 0.92 | 0.94 |
| II | -7.20 | 9.00 | -18.34 | 0.00 | 0.62 | 0.00 | 0.89 | 0.95 | 0.95 |
| III | -7.20 | 57.60 | -30.15 | 0.00 | 2.63 | 0.00 | 0.89 | 0.96 | 0.97 |
| IV | 25.00 | 160.00 | 112.86 | 1.03 | 4.38 | 3.42 | 0.97 | 0.98 | 0.98 |
| V | 25.00 | 225.00 | 126.00 | 1.03 | 5.55 | 3.55 | 0.97 | 0.98 | 0.98 |
| VI | 25.00 | 480.00 | 153.96 | 1.03 | 9.11 | 3.51 | 0.97 | 0.98 | 0.98 |

Table 9: Optimal strategies for the excess of loss treaty $(M = 0, L)$ for the process with dependencies $S_t$ and for the individual claims processes $S_1$ and $S_2$, with $E_1$ and $E_2$ the first and second raw moments

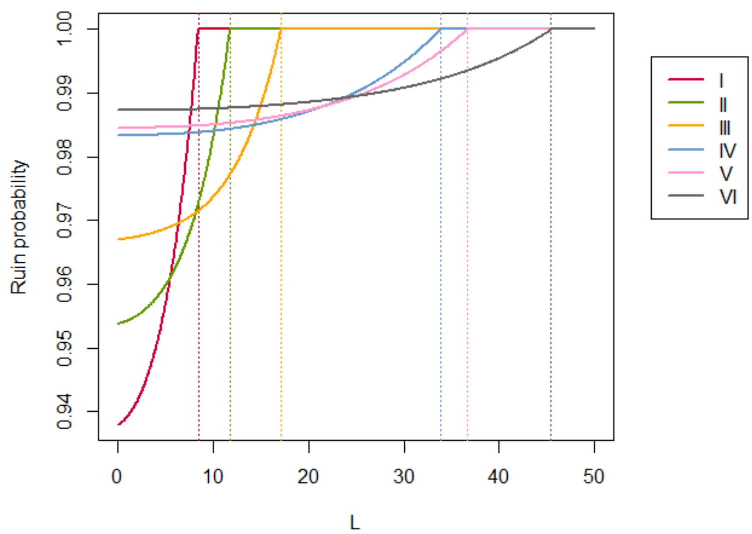


Figure 10: Ruin probability for the excess of loss treaty $(M = 0, L)$ for the process with dependent risks

# 6 Conclusions

This thesis purpose was to study the optimal reinsurance strategy in continuous time by finding the optimal retention levels when considering the excess of loss treaty and by including dependences. The optimization criterion used is the minimization of the ruin probability in an infinite time horizon, from the point of view of the insurance company, and the expected value premium principle was considered for the insurer's premium and also for the reinsurer's.

Firstly, the optimal reinsurance problem was analytically addressed. The ruin probability function was characterized by analysing the optimal domain and by studying the sign of its derivative in the extremes of this interval. For the excess of loss treaty without superior limit, $(M, L = \infty)$, the analytical results show that it is always optimal to transfer part of the insurer's risk to the reinsurer. On the other hand, the results for the excess of loss treaty without inferior limit, $(M = 0, L)$, show that the decision to transfer, or not, part of the risk is heavily dependent on the individual claims' size distribution. For this contract, it is shown that, when assuming individual claims are modelled by an Exponential or by a Pareto distribution, the optimal strategy is to retain all the risk to the insurer, independently of the parameters considered. Lastly, when looking for the optimal reinsurance strategy which minimizes the ruin probability of the insurance company under the excess of loss contract with both the inferior and superior limits, $(M, L)$, the analytical results show that the optimal strategy for this contract degenerates into the excess of loss contract without superior limit. Hence, for this last contract, the conclusions on the optimum solution follow from those taken on the first reinsurance contract studied. We have proved that, when considering dependence between two classes of risk, where the dependence is introduced on the number of claims as a common Poisson shock, then we still have a compound Poisson process and, hence, the surplus process can be approximated to a Brownian motion with drift process through a diffusion approximation, as for the single risk case. Therefore, conclusions on the optimal strategies for the excess of loss contract with dependent risks follow directly from the first three cases previously studied.

By using R software, a numerical analysis was conducted. It helped to determine the influence of the distribution of the underlying risk on the optimal retention levels. The excess of loss contract $(M, L = \infty)$ was numerically analysed in two ways, firstly considering the Exponential distribution and, secondly, the Pareto, for different values of the distribution's parameters. Besides these two distributions, an analysis with the Gamma distribution is included in the numerical study of the excess of loss contract $(M = 0, L)$. For the dependent case, we considered two different Gamma as secondary distributions for the individual claims processes and studied the optimal retention levels for the excess of loss contracts under dependences and for each risk separately, comparing the two cases. The numerical results show that, in fact, the optimal retention levels, with and without dependences, are different.

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