MASTER IN

ACTUARIAL SCIENCE

FINAL MASTER WORK

INTERNSHIP REPORT

A MODEL FOR RISK ADJUSTMENT FOR LONGEVITY RISK IN LIFE INSURANCE UNDER IFRS 17

DIOGO FILIPE JERÓNIMO RODRIGUES

SUPERVISORS:

ALEXANDRA BUGALHO DE MOURA
CARLA SÁ PEREIRA

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Abstract

In this Master Final Work, a new Model for the Risk Adjustment for Longevity Risk in Life Insurance is proposed and it is compared with the classical techniques for calculating the Risk Adjustment, discussing its advantages and disadvantages.

In recent years, the insurance sector has evolved to standardize financial, actuarial and accounting reporting processes in order to be able to assess and compare the insurance companies' performance. New standards have emerged to achieve this shared vision, such as the IFRS 17 - Insurance Contracts.

The objective of this internship was to study the Risk Adjustment in Life Insurance. The Risk Adjustment is one of the components of liabilities measurement under IFRS 17 which reflects the compensation that an insurance company requires to support the uncertainty arising from non-financial risks. The standard does not prescribe any technique for its calculation, making it one of the major challenges for insurance companies in implementing the IFRS 17.

In this work, an original model for calculating the Risk Adjustment for Longevity Risk in Life Insurance is developed. It is based on statistical concepts, such as: Convex Ordering of Risks; Comonotonicity; and the Bounds of Sums of Random Variables. The new model creates an interval containing the Risk Adjustment to be applied to a given life insurance portfolio under longevity risk by applying an analytical formula. The analytical expression for the bounding interval is easy to implement and obtain results from. By using data inspired on a real life insurance portfolio of a leading insurance company operating in Portugal, it is shown that this model, which focuses on the volatility of the historical behavior of the portfolio in terms of exposure and deaths, has several good characteristics when compared to the classical techniques for calculating the Risk Adjustment: the Quantile Techniques; and the Cost-of-Capital Technique.

Keywords: Risk Adjustment; Longevity Risk; IFRS 17; Comonotonicity; Convex Ordering; Life Insurance.

Resumo

Neste Trabalho Final de Mestrado, um novo Modelo para o *Risk Adjustment* para o Risco de Longevidade em Seguros de Vida é proposto e comparado com as técnicas clássicas de cálculo do *Risk Adjustment*, discutindo as suas vantagens e desvantagens.

Nos últimos anos, o setor segurador tem evoluído no sentido de uniformizar os processos de relato financeiro, atuarial e contabilístico, de forma a poder avaliar e comparar o desempenho das seguradoras. Novas normas têm surgido para alcançar esta visão partilhada, como é o caso da IFRS 17 - Contratos de Seguro.

O objetivo deste estágio foi estudar o *Risk Adjustment* em Seguros de Vida. O *Risk Adjustment* é uma das componentes da mensuração de responsabilidades ao abrigo da IFRS 17 que reflete a compensação que uma seguradora necessita para suportar a incerteza decorrente de riscos não financeiros. A norma não prescreve qualquer técnica para o seu cálculo, tornando-o num dos maiores desafios para as seguradoras na implementação da IFRS 17.

Neste trabalho, desenvolve-se um modelo original para o cálculo do *Risk Adjustment* para o Risco de Longevidade em Seguros de Vida. Este é baseado em conceitos estatísticos, tais como: Ordenação Convexa de Riscos; Comonotonicidade; e os Limites de Somas de Variáveis Aleatórias. O novo modelo cria um intervalo que contém o *Risk Adjustment* a aplicar a um determinado portfólio de seguros de vida sob o risco de longevidade, através da aplicação de uma fórmula analítica. A expressão analítica para o intervalo é de fácil implementação e obtenção de resultados. Utilizando dados inspirados num portfólio real de seguros de vida de uma companhia de seguros líder a operar em Portugal, mostra-se que este modelo, que se foca na volatilidade do comportamento histórico do portfólio em termos de exposição e de mortes, tem várias características positivas quando comparado com as técnicas clássicas de cálculo do *Risk Adjustment*: as Técnicas de Quantis; e a Técnica *Cost-of-Capital.*

Palavras-Chave: *Risk Adjustment*; Risco de Longevidade; IFRS 17; Comonotonicidade; Ordenação Convexa; Seguros de Vida.

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Acronyms

ASF	Autoridade de Supervisão de Seguros e Fundos de Pensões
BB	Building Block
BBA	Building Blocks Approach
CoC	Cost-of-Capital
CSM	Contractual Service Margin
CLTE	Conditional Left Tail Expectation
CTE	Conditional Tail Expectation
EC	European Community
EIOPA	European Insurance and Occupational Pensions Authority
EU	European Union
EY	Ernst & Young, S.A.
FCF	Fulfilment Cash Flows
IAA	International Actuarial Association
IAS	International Accounting Standards
IASB	International Accounting Standards Board
IASC	International Accounting Standards Committee
IFRS	International Financial Reporting Standards
IFRS 17	IFRS 17 - Insurance Contracts
INE	Instituto Nacional de Estatística
LIC	Liabilities for Incurred Claims
LRC	Liabilities for Remaining Coverage
LTVaR	Left Tail Value at Risk
MLE	Maximum Likelihood Estimate
PAA	Premium Allocation Approach
PVFCF	Present Value of Future Cash Flows
SCR	Solvency Capital Requirement
SII	Solvency II
RA	Risk Adjustment
RM	Risk Margin
TVaR	Tail Value at Risk
UFR	Ultimate Forward Rate
VFA	Variable Fee Aproach
VaR	Value at Risk

Introduction

For decades, insurance companies have worked to create products to cover the most diverse risks, using a wide range of techniques to model and calculate premiums, expenses and liabilities inherent to these products. In recent years, the sector has evolved to standardize financial, accounting and actuarial reporting processes in order to be able to assess and compare insurance companies' performance more accurately. The rise of new standards, such as the Solvency II (SII) in the European Union (EU) and more recently the IFRS 17 - Insurance Contracts (IFRS 17), are examples of this shared vision. In part, this concern of the sector stems from the impacts of financial crises, which is why this new regulations also seeks to protect policyholders and prevent insolvency and bankruptcy situations through the continuous monitoring of the insurance companies. Nevertheless, these new standards and their implementation are a major challenge for insurance companies, consultancies and authorities responsible for the sector worldwide.

This Master Final Work focuses on the Risk Adjustment (RA) quantification, namely the RA for surrender and longevity risks in life insurance products. The RA is one of the main challenges of the IFRS 17, since the standard does not prescribe any particular technique to compute it. This work is the result of a 6-month internship at Ernst & Young, S.A. (EY) in the Risk & Actuarial Services team. The internship involved several projects in a specialized team in the actuarial sector. As a result of team discussions, it was jointly decided to study the RA, given its current importance in the sector and considering that some insurance companies remain uncertain about which method to use. This is an important topic nowadays, also because the 1st IFRS 17 financial statements will be published in 2024, referring to 2023.

There are currently two main techniques used to calculate the RA in IFRS 17: the Quantile Techniques and the Cost-of-Capital (CoC) Technique. These techniques are well known in the sector, as they have been used in other actuarial calculations or standards for several years. Nevertheless, their extensive application has revealed some disadvantages. These disadvantages have led to the development of internal models in some insurance companies, after approval by the authorities.

In this Master Final Work, a new Model for RA for Longevity Risk will be developed as an original generalization of the work of Carlehed (2023) for the surrender risk in life insurance. With the new model, an analytically calculated interval containing the longevity RA of the portfolio is obtained. This interval provides the insurance company the possibility of choosing a more prudent value for the RA, depending on its risk aversion. Concepts such as Risk Measures, Risk Ordering and Comonotonicity will be applied to build this model. Focus in given to the volatility of the historical behavior of the portfolio in terms of exposure and deaths, in comparison to what the insurance company expects. To better understand the new model, a detailed analysis of the Model for RA for Surrender Risk by Carlehed (2023) is described, as well as its advantages, disadvantages and possible improvements.

The Master Final Work is composed by four Chapters. Chapter 1 provides a brief overview of the IFRS 17, the RA, its main characteristics and the classical techniques to calculate it. Subsequently, Chapter 2 serves as a theoretical framework to introduce the concepts required to build the models for the longevity RA. The new Model for RA for Longevity Risk is explained, after the Model for RA for Surrender Risk by Carlehed (2023) is presented, in Chapter 3. Finally, Chapter 4 illustrates a practical example of the application of the Model for RA for Longevity Risk to a life insurance portfolio inspired by a real portfolio of a leading insurance company operating in Portugal. This chapter concludes with a comparison of the Model for RA for Longevity Risk with the classical techniques, setting out the advantages and disadvantages of each approach.

Chapter 1
Risk Adjustment and IFRS 17

This Chapter provides a brief overview of the IFRS 17, as well as the definition, main characteristics and methods used to calculate the Risk Adjustment (RA). The focus is only on the fundamental aspects of the standard that will be applied throughout this work, leaving out many other key concepts and details of the general knowledge of the IFRS 17.

1.1 IFRS 17 - Insurance Contracts

The IFRS 17 - Insurance Contracts (IFRS 17) is the new international financial reporting standard for the insurance sector. It was published by the International Accounting Standards Board (IASB) in 2017, with the purpose of promoting the standardization of international accounting standards in the sector.

The IASB is an independent international organization that develops globally applicable accounting standards. As described in Flower and Ebbers (2002), it was founded in 2001 to replace its predecessor, the International Accounting Standards Committee (IASC), which, since 1973 and until 2001, had been the entity responsible for designing, developing, publishing and updating accounting standards, known as the International Accounting Standards (IAS). After its creation, the IASB became responsible for publishing what are known as the International Financial Reporting Standards (IFRS). However, some of the old IAS are still in force today as they have not been replaced by new standards.

According to IASB (2018a), the influence of the IFRS at the international level is notorious, with 166 jurisdictions worldwide applying these standards in line with the IASB's main objective. In Portugal, the application of these standards is required, since the European Union (EU) made it a mandatory rule in the Regulation (EC) No 1606/2002 (2002) for all EU listed companies in 2002.

1.1.1 History Overview, Objective and Scope of IFRS 17

The IFRS 17 enables the standardization of the international accounting standards for the insurance market. As described in IFRS 17 (2023a, § 1), the standard establishes "principles for the recognition, measurement, presentation and disclosure of insurance contracts within the scope of the standard", with the objective of ensuring that accurate and relevant information is provided on these contracts in order to assess their effect on the "entity's financial position, financial performance and cash flows".

The IFRS 17 is the result of an IASB project divided into two phases, which began in 1997 with the IASC, as described in IFRS 17 (2023b, §§ BC2-BC6). At the end of the first phase in 2004, IFRS 4 - Insurance Contracts emerged as a transitory standard which, over the years, revealed the weakness of allowing the use of local accounting standards, making it difficult to understand and compare the results of various insurance companies. In Portugal, the Autoridade de Supervisão de Seguros e Fundos de Pensões (ASF) has not fully implemented IFRS 4, adopting only the classifications of insurance contracts, as mentioned in ASF (2005).

In 2017, the IASB issued the IFRS 17 as the conclusion of the second phase of the project. The effective adoption of the standard to the portuguese insurance companies was in 2023, and the first financial statements will be published in 2024. In IFRS 17 (2023a, § 3), the scope of the standard is defined as being: the insurance contracts; the reinsurance contracts; and the investment contracts with discretionary participation features. The standard separates the components of an insurance contract in IFRS 17 (2023a, §§ 11-13), excluding from its scope the investment component which should be assessed under IFRS 9 - Financial Instruments, and other goods and non-insurance services under IFRS 15 - Revenue from Contracts with Customers, as shown in Figure 1 taken from EY (2021). The aggregate implementation of IFRS 9, IFRS 15 and IFRS 17 seeks to standardize the accounting system for the different types of contracts in order to increase transparency of information in the insurance sector.

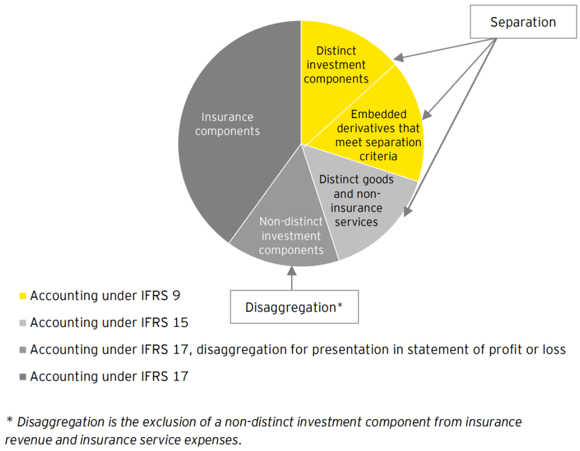


Figure 1: Scope of IFRS 9, IFRS 15 and IFRS 17 in Insurance Contracts. Taken from EY (2021)

1.1.2 Measurement Models and Building Blocks Approach

The IFRS 17 provides three different accounting measurement approaches, due to its objective of evaluating any type of insurance contract within its scope. As such, entities need to evaluate different groups of contracts using distinct measurement methods, depending on the specific conditions inherent in the contracts:

- The Building Blocks Approach (BBA) described in IFRS 17 (2023a, § 32), also known as the General Measurement Model, since it is the default model of the IFRS 17. It is based on an approach composed by four blocks that can be applied to all insurance contracts, except those with direct participation features, and therefore reported by the entities;
- The Premium Allocation Approach (PAA) introduced in IFRS 17 (2023a, § 53), which is a simplified model only for insurance contracts with a duration of one year or less, or if this simplification is a reasonable approximation that does not result in a materially different measurement of liabilities from that determined in the BBA;
- The Variable Fee Aproach (VFA) presented in IFRS 17 (2023a, § 45), which is applied to insurance contracts with direct participation features, thus requiring a different approach.

In this study, only the general model of the IFRS 17, the BBA, will be applied. As described in IFRS 17 (2023a, § 32), and illustrated in Figure 2, taken from EY (2018), the BBA consists of four Building Blocks:

1. the 1st Building Block (BB) is the Estimate of Future Cash Flows;
2. the 2nd BB includes the discount effect, mentioned in the standard as the adjustment that “reflects the time value of money and the financial risks related to the future cash flows”;

3. the 3rd BB is Risk Adjustment (RA) for non-financial risks;
4. the 4th BB is the Contractual Service Margin (CSM).

The combination of the 1st and 2nd BB's, which corresponds to discounting the cash flow estimates, is known as the Present Value of Future Cash Flows (PVFCF). The sum of the 1st, 2nd and 3rd BB's is known as the Fulfilment Cash Flows (FCF), which refers to the present value already accounting for the adjustment for the risks associated with future cash flows. It is a result of the entity's information on the cash flows arising from the fulfillment of the insurance contracts.

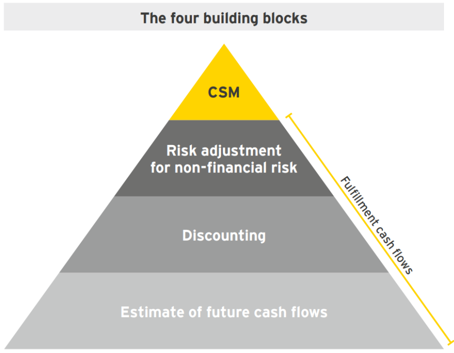


Figure 2: The 4 Building Blocks. Taken from EY (2018)

On the other hand, the 4th BB, i.e., the CSM, is "a component of the asset or liability for the group of insurance contracts that represents the unearned profit the entity will recognise as it provides insurance contract services in the future", as described in IFRS 17 (2023a, § 38).

Regarding the measurement of liabilities, as mentioned in IFRS 17 (2023a, § 40), the standard divides total liabilities into two types: the Liabilities for Remaining Coverage (LRC), which refer to the FCF related to future services to be provided by the insurance company, together with the CSM still to be recognized throughout the insurance contract; and the Liabilities for Incurred Claims (LIC), which refer to the FCF of the insurance company related to provided past service;

According to IFRS 17 (2023a, § 32 & § 38), on initial recognition, the insurance company measures a group of insurance contracts as the total of the FCF and the CSM. This sum will generally be equal to zero due to the fact that, at the inception, the CSM is considered to be symmetrical to the FCF.

At subsequent measurement, the total amount of liabilities is given by the sum of the LRC and the LIC as defined above and illustrated in Figure 3, taken from IFRS 17 (2018b). The LRC will be the FCF arising from future services to be provided and the CSM yet to be recognized, unless the insurance contract costs the company more to cover than it will receive in return. In this case, the CSM will be zero and a loss component will be recognized, as described in IFRS 17 (2023a, §§ 47-52). In the case of the LIC, these are the FCF resulting from past services provided. There is no CSM on the LIC since it has already been recognized as a profit or a loss.

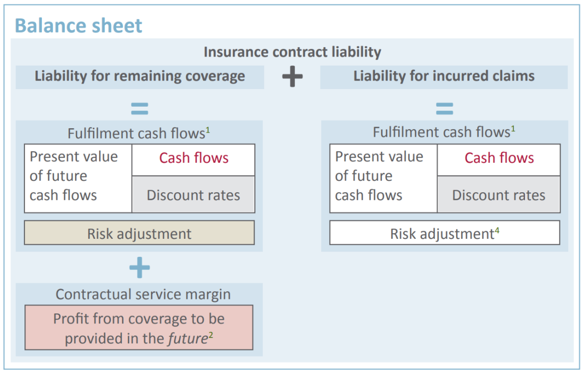


Figure 3: The BBA Balance Sheet. Taken from IFRS 17 (2018b)

1.2 Risk Adjustment

The Risk Adjustment (RA) for non-financial risks is the main subject of this study. In the following, a deeper understanding of its definition, characteristics, classical techniques and their advantages and disadvantages will be presented, which will be useful for the analysis and development of an original Model for the RA for Longevity Risk, as described in Chapter 3.

As mentioned above, the RA is the 3rd BB of the BBA. According to IFRS 17 (2023a, § 37), the RA is defined as "the compensation that the entity requires for bearing the uncertainty about the amount and timing of the cash flows that arises from non-financial risk". In IFRS 17 (2023a, § B86), the standard clarifies the concept of non-financial risk as being the set formed by insurance risks and other non-financial risks, which include the surrender risk. Therefore, the RA for specific insurance contracts measures the compensation that the insurance company would require to become indifferent between covering a liability with a wide range of possible outcomes caused by a non-financial risk or fulfilling a liability that will produce fixed cash flows with the same expected present value as those contracts, as described in IFRS 17 (2023a, § B87). Hence, it is the additional amount that must be added to the PVFCF, the 1st and 2nd BB's, to calculate the FCF.

It can be concluded that the RA is the BB that reflects the uncertainty in cash flows arising from non-financial risks. It also reveals the degree of diversification benefit and of risk aversion of the insurance company, as mentioned in IFRS 17 (2023a, §§ B88-B89).

1.2.1 Key Characteristics of Risk Adjustment

The IFRS 17 does not prescribe any technique that should be used to determine the RA, so the insurance company must assess which is the most appropriate technique to calculate it. The standard defines, in IFRS 17 (2023a, § B91), the following five criteria, which correspond to the main characteristics that should be considered when calculating the RA:

(a) "risks with low frequency and high severity will result in higher risk adjustments for non-financial risk than risks with high frequency and low severity;"

(b) "for similar risks, contracts with a longer duration will result in higher risk adjustments for non-financial risk than contracts with a shorter duration;"

(c) "risks with a wider probability distribution will result in higher risk adjustments for non-financial risk than risks with a narrower distribution;"

(d) "the less that is known about the current estimate and its trend, the higher will be the risk adjustment for non-financial risk; and"

(e) "to the extent that emerging experience reduces uncertainty about the amount and timing of cash flows, risk adjustments for non-financial risk will decrease and vice versa."

1.2.2 Classical Risk Adjustment Techniques

As mentioned above, IFRS 17 does not force the use of a particular technique for determining the RA. In IAA (2018), two classical techniques are described: the Quantile Techniques; and the Cost-of-Capital (CoC) Technique. Both have become very attractive for calculating the RA, since they are well-known methodologies in the insurance sector, as they are used in actuarial analyses or to comply with other regulations. Indeed, the Quantile Techniques are risk measures commonly applied in insurance companies, as they are key risk indicators for understanding the level of exposure of insurance contracts to certain risks. They are applied in the actuarial calculations to carry out various analyses, including the premium principles and the pricing of insurance contracts. The CoC Technique is especially well-known in the EU, since it is the applied methodology to calculate the Risk Margin (RM) under Solvency II (SII).

1.2.2.1 Quantile Techniques

As described in the IAA (2018), there are two types of quantile techniques that are commonly used by insurance companies to determine maximum risk limits: the Value at Risk (VaR), also known as the confidence level or percentile; and the Tail Value at Risk (TVaR), also referred to as the Conditional Tail Expectation (CTE).

The VaR is very popular given its simplicity, making its application and understanding easy. With this technique, the RA is determined as the amount added to the PVFCF such that the probability of the actual liability being less than the FCF is equal to a confidence level, fixed by the insurance company. On the other hand, the TVaR is more complex to compute, but it is more sensitive to skewness in the tails of the distributions. With this technique, the RA is calculated as the conditional expected value of the liabilities that exceed a confidence level, fixed by the insurance company.

To apply the quantile techniques, the company use its portfolio to create a distribution and then simulate a large number of outcomes in order to calculate the RA. Typically, simulation methods are applied. The most common simulation methods are the Monte Carlo Method or the Bootstrap Method. In many cases, fitting the PVFCF to a known distribution is considered a simplifying assumption.

The mathematical details of these techniques will be presented in Chapter 2.

1.2.2.2 Cost-of-Capital Technique - Solvency II Approach

The CoC Technique is based on the idea that the insurance company determines the appropriate amount of capital for the RA in order to reflect its risk aversion. As described in IAA (2018), this amount of capital is calculated using “one or more benchmarks or types of analyses, such as stress tests, stochastic or probability models”. Most companies prefer this technique when calculating the RA, especially in the EU, since it is the approach behind the calculation of the RM under the SII. Therefore, and in parallel to the calculation of the RM in SII (2015, § 37(2)), the RA would be defined as:

$$\mathrm{RA}_{\mathrm{SII}} = \mathrm{CoC} \cdot \sum_{t=1}^{T} \frac{\mathrm{SCR}_t}{(1 + i_{t+1})^{t+1}}, \tag{1.1}$$

where CoC is the selected cost-of-capital rate for year t, SCR_t is the Solvency Capital Requirement in year t, i_t is the selected interest rate for year t and T is the maximum contract boundary, established according to IFRS 17 (2023a, § 34 & § B61).

Despite the similarities between the calculation of the RA under IFRS 17 and the RM under SII using the CoC Technique, there are significant differences between the two concepts. Under IFRS 17, the choice of the CoC rate is selected by the insurance company, while under SII it is prescribed at 6%. The choice of confidence level is chosen in accordance with the risk aversion of the insurance company under IFRS 17, while under SII it is prescribed at 99.5%. Also, the RA under IFRS 17 should be calculated on an ultimate view basis, while the RM under SII is calculated on a one-year view basis, leading some entities to apply a time factor to capture the effect of the duration of liabilities when applying the CoC Technique to calculate the RA.

In this work, the CoC Technique will not be applied, since the focus will be on the analysis of the Longevity Risk. Therefore, in order to determine the RA according to the SII for Longevity Risk for a confidence level of $\alpha = 99.5\%$, an instantaneous permanent shock of a decrease of 20% in the mortality rates used to calculate the PVFCF will be applied, based on what is stated in SII (2015, § 138(1)). This is the methodology that will be applied for the practical example in Chapter 4.

1.2.2.3 Advantages and Disadvantages of each Technique

In order to better understand the classical techniques and to be able to develop the new methods for calculating the RA, presented in Chapter 3, it is necessary to analyze the advantages and disadvantages of these techniques.

The Quantile Techniques, combined with the use of simulation methods, have the advantage of enabling the generation of a wide range of different outcomes, helping to better understand the behavior of the portfolio and its distribution under certain risks. Typically, these techniques provide more information about the losses of the insurance company, since the greater the number of simulations performed, the greater the knowledge of extreme events. On the other hand, the high number of simulations needed is also seen as a disadvantage, representing a high computational cost. When comparing the two Quantile Techniques, the VaR has the advantage of being the simplest, as it is simply the difference between the estimated liability for a given confidence level and the expected liability. However, it ignores liabilities for situations in which the quantile is higher than the one chosen by the insurance company, having a huge impact on portfolios with more skewed distributions. On the other hand, the TVaR overcomes this disadvantage, since by calculating the conditional expected value of all results above a confidence level it has greater sensitivity to possible outcomes with high severity and low frequency. It is, thus, a more suitable technique for assessing the RA in these portfolios. The TVaR will be the quantile technique used in Chapter 4 for the practical example, due to its advantages over VaR and its similarity to the original Model for RA for Longevity Risk presented in Chapter 3.

Finally, regarding the CoC Techique, it is faster to calculate and requires less computational power, because it usually applies fixed shocks as in SII. Also, it does not require the calculation of a large number of simulations like the Quantile Techniques. It also has the advantage that, as it is the technique prescribed by the SII in the EU, it enables the standardization of the technique used to calculate the RA for a considerable number of insurance companies, making it easier to compare and benchmark the RA considered by each company. However, it has serious disadvantages, including the use of shocks predefined by the insurance company or by a standard, such as the SII, leading to a lack of sensitivity about the historical characteristics and volatility of a given portfolio. Moreover, if the shocks as defined in the SII are applied to calculate the RA, then the simplifying assumption of normality of the portfolio is implied, even if that is not the case in reality.

Chapter 2

Risk Measures, Comonotonicity and Ordering of Risks

This Chapter will serve as the Theoretical Framework for the Model for Risk Adjustment for Longevity Risk that will be developed in Chapter 3. The concepts of Risk Measures, Ordering of Risks and Comonotonicity will be presented, as well as a deeper look at the Bounds of Sums of Random Variables. The Chapter is mainly based on Kaas et al. (2000), Dhaene et al. (2006), Vanduffel et al. (2005) and Vanduffel et al. (2008).

2.1 Risk Measures

Risk Measures are important key risk indicators for actuaries to assess the level of exposure to certain risks to which the insurance company is subject. There is a wide variety of risk measures analyzed and used nowadays, and there are some well known properties that are considered important to be verified by a risk measure. Artzner et al. (1999) introduced the concept of coherent risk measure as a function of the risk X, $\rho(X)$, where X is a random variable, with the properties of subadditivity, monotonicity, positive homogeneity and translation invariance (see, for instance, Klugman et al., 2008). These four properties are typically used to assess the quality of risk measures. Examples of well-known risk measures (see, for instance, Dhaene et al., 2006) are the Value at Risk (VaR) and the Tail Value at Risk (TVaR).

The **Value at Risk (VaR)** of X at a confidence level α (also know as the α-quantile risk measure), denoted by $\mathrm{VaR}_\alpha(X)$, is defined by:

$$\mathrm{VaR}_\alpha(X) = \inf\{x \in \mathbb{R} \mid F_X(x) \geq \alpha\}, \quad \alpha \in]0,1[. \tag{2.1}$$

The VaR is a non-decreasing and left-continuous function of α, so it verifies the following equivalence relation:

$$\mathrm{VaR}_\alpha(X) \leq x \iff \alpha \leq F_X(x), \quad \forall x \in \mathbb{R} \wedge \alpha \in]0,1[. \tag{2.2}$$

When $F_X(x)$ is continuous, the equivalence relation (2.2) can be rewritten as:

$$\mathrm{VaR}_\alpha(X) = x \iff \alpha = F_X(x), \quad \forall x \in \mathbb{R} \wedge \alpha \in]0,1[. \tag{2.3}$$

Thus, for $F_X(x)$ continuous, the VaR can be defined by $\mathrm{VaR}_\alpha(X) = F_X^{-1}(\alpha)$. Although VaR is a well-established and popular risk measure, due to its simplicity, it is not coherent since it does not verify the subadditivity property.

The **Tail Value at Risk (TVaR)** of X at a confidence level α, denoted by $\mathrm{TVaR}_\alpha(X)$, is defined by:

$$\mathrm{TVaR}_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_\alpha(X)\, dq, \quad \alpha \in]0,1[. \tag{2.4}$$

.

The **Conditional Tail Expectation (CTE)** of X at a confidence level α, denoted by $\mathrm{CTE}_\alpha(X)$, is another risk measure, defined by:

$$\mathrm{CTE}_\alpha(X) = \mathbb{E}[X \mid X > \mathrm{VaR}_\alpha(X)], \quad \alpha \in]0,1[. \tag{2.5}$$

When $F_X(x)$ is continuous, the TVaR and the CTE are equivalent.

Another important property of the TVaR is its relation with the expectation of X:

$$\lim_{\alpha \to 0} \text{TVaR}_\alpha(X) = \mathbb{E}[X]. \tag{2.6}$$

Unlike the VaR, the TVaR has the advantage of being a coherent risk measure.

The Left Tail Value at Risk (LTVaR) is also introduced, which is going to be important to understand the results of Carlehed (2023) that are presented in detail in Chapter 3 and that will be the inspiration for the original results of this thesis.

Let X be a random variable such that $F_X(x)$ is continuous. The **Left Tail Value at Risk (LTVaR)** of X at a confidence level α (also know as Conditional Left Tail Expectation (CLTE)) and denoted by $\text{LTVaR}_\alpha(X)$, is defined by:

$$\text{LTVaR}_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha \text{VaR}_\alpha(X)\, dq = \mathbb{E}[X \mid X < \text{VaR}_\alpha(X)], \quad \alpha \in]0,1[. \tag{2.7}$$

The LTVaR is, in fact, a combination of two definitions (LTVaR and CLTE). However, it has been decided to present it this way since it will only be used for the case where $F_X(x)$ is continuous. The relations behind it are in everything analogous to the definitions of TVaR and CTE.

2.2 Ordering of Risks

In the actuarial world, being able to compare random variables is essential for analyzing and measuring risk, which supports decision making. In the following, the definitions of three Orders of Risks with important actuarial applications will be presented, according to Dhaene et al. (2006).

Definition 2.2.1 Consider two random variables X and Y. X is said to precede Y in the **stochastic dominance sense**, denoted by $X \leq_{st} Y$, if the cumulative distribution function of X always exceeds that of Y:

$$F_X(x) \geq F_Y(x), \quad -\infty < x < +\infty. \tag{2.8}$$

Definition 2.2.2 Consider two random variables X and Y. X is said to precede Y in the **stop-loss order sense**, denoted by $X \leq_{sl} Y$, if and only if:

$$\mathbb{E}[(X-d)_+] \leq \mathbb{E}[(Y-d)_+], \quad -\infty < d < +\infty. \tag{2.9}$$

Definition 2.2.3 Consider two random variables X and Y. X is said to precede Y in the **convex order sense**, denoted by $X \leq_{cx} Y$, if and only if, $X \leq_{sl} Y$ and in addition $\mathbb{E}[X] = \mathbb{E}[Y]$.

In the next two theorems (see Dhaene et al., 2006), the relation between ordering of risks and risk measures is highlighted.

Theorem 2.2.4 Consider two random variables X and Y. X precedes Y in the stochastic dominance sense if and only if the corresponding VaR are ordered:

$$X \leq_{st} Y \iff \text{VaR}_\alpha(X) \leq \text{VaR}_\alpha(Y), \quad \forall\alpha \in]0,1[. \tag{2.10}$$

Theorem 2.2.5 Consider two random variables X and Y. X precedes Y in the stop-loss order sense if and only if the corresponding TVaR are ordered:

$$X \leq_{sl} Y \iff \mathrm{TVaR}_\alpha(X) \leq \mathrm{TVaR}_\alpha(Y), \quad \forall \alpha \in]0,1[. \tag{2.11}$$

From Theorem 2.2.5, it is possible to show the following corollary, that will be essential in Chapter 3.

Corollary 2.2.6 Consider two random variables X and Y. X precedes Y in the convex order sense if and only if their expected values are equal and their TVaR are ordered:

$$X \leq_{cx} Y \iff \mathbb{E}[X] = \mathbb{E}[Y] \wedge \mathrm{TVaR}_\alpha(X) \leq \mathrm{TVaR}_\alpha(Y), \quad \forall \alpha \in]0,1[. \tag{2.12}$$

It is possible to develop similar results to Theorem 2.2.5 and Corollary 2.2.6 for the risk measure LTVaR, as presented next.

Theorem 2.2.7 Consider two random variables X and Y. X precedes Y in the stop-loss order sense if and only if the corresponding LTVaR are contrary ordered:

$$X \leq_{sl} Y \iff \mathrm{LTVaR}_\alpha(X) \geq \mathrm{LTVaR}_\alpha(Y), \quad \forall \alpha \in]0,1[. \tag{2.13}$$

Corollary 2.2.8 Consider two random variables X and Y. X precedes Y in the convex order sense if and only if their expected values are equal and their LTVaR are contrary ordered:

$$X \leq_{cx} Y \iff \mathbb{E}[X] = \mathbb{E}[Y] \wedge \mathrm{LTVaR}_\alpha(X) \geq \mathrm{LTVaR}_\alpha(Y), \quad \forall \alpha \in]0,1[. \tag{2.14}$$

2.3 Comonotonicity

The concept of Comonotonicity is highly important when studying dependent random variables. It has applications in copula's construction and in bounding multi-dimensional random variables. Below, its definition and some important results linked with risk measures will be introduced.

Definition 2.3.1 The n-dimensional random vector $\underline{X} = (X_1, X_2, \ldots, X_n)$ is said to be **comonotonic** if and only if:

$$F_{\underline{X}}(x_1, x_2, \ldots, x_n) = \min(F_{X_1}(x_1), F_{X_2}(x_2), \ldots, F_{X_n}(x_n)), \quad \forall x_1, x_2, \ldots, x_n \in \mathbb{R}, \tag{2.15}$$

that is

$$\underline{X} \stackrel{d}{=} \left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \ldots, F_{X_n}^{-1}(U)\right), \tag{2.16}$$

where U is a uniformly distributed random variable on the unit interval $]0,1[$ and the notation $\stackrel{d}{=}$ is used to indicate "equality in distribution".

The following theorem, based on Dhaene et al. (2006), shows why the use of comonotonic bounds has interesting applications in the calculation of risk measures of sums of dependent random variables.

Theorem 2.3.2 Consider a comonotonic n-dimensional random vector $\underline{X} = (X_1, X_2, \ldots, X_n)$. Then, the VaR, TVaR and LTVaR are additive for all $\alpha \in]0,1[$:

$$\mathrm{VaR}_\alpha\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathrm{VaR}_\alpha[X_i], \quad \forall \alpha \in]0,1[, \tag{2.17}$$

$$\mathrm{TVaR}_\alpha\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathrm{TVaR}_\alpha[X_i], \quad \forall \alpha \in]0,1[, \tag{2.18}$$

$$\mathrm{LTVaR}_\alpha\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathrm{LTVaR}_\alpha[X_i], \quad \forall \alpha \in]0,1[. \tag{2.19}$$

The concept of Comonotonicity has been exhaustively studied and developed over the last decades, together with the appearance of a wide variety of applications in actuarial science and finance. For a better understanding of the theory and its applications, refer to Dhaene et al. (2002a) and Dhaene et al. (2002b), and to Deelstra et al. (2011) for an extensive bibliographic overview and for awareness of further developments of the theory and its applications.

2.4 Bounds for Sums of Random Variables

The sum of random variables has always been a challenging problem. Finding bounds to the sums of random variables, using different sums with better properties, has been the subject of active research. Next, results joining the concepts of comonotonicity and convex order to bound the sum of random variables are presented.

The main result here presented joins two theorems where convex comonotonic (upper and lower) bounds for the sum of random variables are derived. According to Deelstra et al. (2011), the first references regarding the upper bound are attributed to Meilijson and Nádas (1979), Tchen (1980) and Rüschendorf (1983), while Kaas et al. (2000) were the ones who introduced the lower bound. The theorem here presented is based on the versions in Deelstra et al. (2011) and Carlehed (2023).

Theorem 2.4.1 Let $\underline{X} = (X_1, X_2 \ldots, X_n)$ be any n-dimensional random vector, U a uniformly distributed random variable on the unit interval $]0,1[$ and Λ any random variable. Then:

$$\sum_{i=1}^{n} \mathbb{E}[X_i \,|\, \Lambda] \leq_{cx} \sum_{i=1}^{n} X_i \leq_{cx} \sum_{i=1}^{n} F_{X_i}^{-1}(U). \tag{2.20}$$

By denoting $S = \sum_{i=1}^{n} X_i$, $S_l = \sum_{i=1}^{n} \mathbb{E}[X_i \,|\, \Lambda]$ and $S_u = \sum_{i=1}^{n} F_{X_i}^{-1}(U)$, inequality (2.20) can be rewritten as:

$$S_l \leq_{cx} S \leq_{cx} S_u. \tag{2.21}$$

Note that the upper bound S_u is always a comonotonic sum. Regarding the lower bound, Kaas et al. (2000) state that if all terms $\mathbb{E}[X_i \,|\, \Lambda]$ are increasing (or all are decreasing) functions of Λ then S_l is a comonotonic sum. The importance of this remark is that it may be possible to apply Theorem 2.3.2 to the comonotonic bounds.

The ability to bound the sum of random variables by a convex order has notable advantages for calculating risk measures or intervals that bound them. The next theorem clarifies this and it is based on Corollaries 2.2.6 and 2.2.8. Due to the convex order, it is also possible to relate the variance of the sum of random variables to that of their bounds. For further details refer to Vanduffel et al. (2008) and Denuit et al. (2005).

Theorem 2.4.2 Let S, S_l and S_u be random variables such that the following convex order relations hold:

$$S_l \leq_{cx} S \leq_{cx} S_u. \tag{2.22}$$

Then, the relations below are verified:

$$\mathbb{E}[S_l] = \mathbb{E}[S] = \mathbb{E}[S_u], \tag{2.23}$$

$$\mathbb{V}[S_l] \leq \mathbb{V}[S] \leq \mathbb{V}[S_u], \tag{2.24}$$

$$\mathrm{TVaR}_\alpha[S_l] \leq \mathrm{TVaR}_\alpha[S] \leq \mathrm{TVaR}_\alpha[S_u], \quad \forall \alpha \in]0,1[, \tag{2.25}$$

$$\mathrm{LTVaR}_\alpha[S_l] \geq \mathrm{LTVaR}_\alpha[S] \geq \mathrm{LTVaR}_\alpha[S_u], \quad \forall \alpha \in]0,1[. \tag{2.26}$$

2.4.1 Application to the Sum of Lognormals

Theorem 2.4.1 allows to bound the sum of random variables, in particular the Sum of Lognormals. As stated in Vanduffel et al. (2008), many problems in actuarial science, finance and even physics and engineering, involve evaluating the cumulative distribution function of a random variable S of the form:

$$S = \sum_{i=1}^{n} \alpha_i \, e^{Z_i}, \tag{2.27}$$

where $\alpha_i \in \mathbb{R}_0^+, \forall i \in \{1, 2, \ldots, n\}$ and $\underline{Z} = (Z_1, Z_2 \ldots, Z_n)$ is a n-dimensional multivariate normal random vector. From Theorem 2.4.1, Kaas et al. (2000) introduced the following theorem that defines comonotonic bounds for the sum of lognormals random variables (see also Vanduffel et al., 2005).

Theorem 2.4.3 Let S be given by (2.27) and, for some given choice of the γ_i, consider the conditioning random variable Λ, given by:

$$\Lambda = \sum_{i=1}^{n} \gamma_i Z_i. \tag{2.28}$$

Also, consider the random variables S_l and S_u defined by:

$$S_l = \sum_{i=1}^{n} \alpha_i \, e^{\mathbb{E}[Z_i] + \frac{1}{2}\left(1-\rho_i^2\right)\sigma_{Z_i}^2 + \rho_i \sigma_{Z_i} \Phi^{-1}(U)}, \tag{2.29}$$

and

$$S_u = \sum_{i=1}^{n} \alpha_i \, e^{\mathbb{E}[Z_i] + \sigma_{Z_i} \Phi^{-1}(U)}, \tag{2.30}$$

where U is a uniformly distributed random variable on the unit interval $]0,1[$, Φ is the cumulative distribution function of the standard normal distribution and ρ_i is the correlation between Z_i and Λ defined by:

$$\rho_i = Corr[Z_i, \Lambda] = \frac{Cov[Z_i, \Lambda]}{\sigma_{Z_i}\sigma_\Lambda} = \frac{1}{\sigma_{Z_i}\sigma_\Lambda}\sum_{j=1}^{n}\gamma_j Cov[Z_i, Z_j]. \tag{2.31}$$

Then, for the random variables S, S_l and S_u, the following convex order relations hold:

$$S_l \leq_{cx} S \leq_{cx} S_u. \tag{2.32}$$

Similar to Theorem 2.4.1, S_u is always a comonotonic sum. As for S_l, it is a comonotonic sum if all ρ_i are positive.

Since both S_l and S_u are comonotonic sums (in the case where all ρ_i are positive), it is possible to apply Theorem 2.3.2, obtaining the following theorem (see Dhaene et al., 2006).

Theorem 2.4.4 Let S_l and S_u be random variables defined as in Theorem 2.4.3 and $\Lambda = \sum_{i=1}^{n}\gamma_i Z_i$, for some given choice of the γ_i such that all ρ_i are positive.
Then, the risk measures VaR, TVaR and LTVaR for S_l and S_u are given by:

$$\mathrm{VaR}_\alpha(S_l) = \sum_{i=1}^{n}\alpha_i\, e^{\mathbb{E}[Z_i]+\frac{1}{2}\left(1-\rho_i^2\right)\sigma_{Z_i}^2+\rho_i\sigma_{Z_i}z_\alpha}, \tag{2.33}$$

$$\mathrm{TVaR}_\alpha(S_l) = \sum_{i=1}^{n}\alpha_i\, e^{\mathbb{E}[Z_i]+\frac{1}{2}\sigma_{Z_i}^2} \cdot \frac{\Phi(\rho_i\sigma_{Z_i} - z_\alpha)}{1-\alpha}, \tag{2.34}$$

$$\mathrm{LTVaR}_\alpha(S_l) = \sum_{i=1}^{n}\alpha_i\, e^{\mathbb{E}[Z_i]+\frac{1}{2}\sigma_{Z_i}^2} \cdot \frac{1-\Phi(\rho_i\sigma_{Z_i} - z_\alpha)}{\alpha}, \tag{2.35}$$

and

$$\mathrm{VaR}_\alpha(S_u) = \sum_{i=1}^{n}\alpha_i\, e^{\mathbb{E}[Z_i]+\sigma_{Z_i}z_\alpha}, \tag{2.36}$$

$$\mathrm{TVaR}_\alpha(S_u) = \sum_{i=1}^{n}\alpha_i\, e^{\mathbb{E}[Z_i]+\frac{1}{2}\sigma_{Z_i}^2} \cdot \frac{\Phi(\sigma_{Z_i} - z_\alpha)}{1-\alpha}, \tag{2.37}$$

$$\mathrm{LTVaR}_\alpha(S_u) = \sum_{i=1}^{n}\alpha_i\, e^{\mathbb{E}[Z_i]+\frac{1}{2}\sigma_{Z_i}^2} \cdot \frac{1-\Phi(\sigma_{Z_i} - z_\alpha)}{\alpha}. \tag{2.38}$$

for all $\alpha \in]0,1[$, where z_α is the α-quantile of the standard normal distribution.

2.4.2 Improvements in the Lower Bound of the Sum of Lognormals

The previous section introduced comonotonic bounds for the sum of lognormals. By using them, it is possible to create intervals for the risk measures TVaR and LTVaR, according to Theorem 2.4.2. However, the size of the bounding interval for the risk measure can vary widely in this case, leading to poor quality results and compromising their applicability.

Looking in detail at the bounds introduced in Theorem 2.4.3, it can be seen that the upper bound S_u is fixed. As for the lower bound S_l, it varies depending on the definition of Λ and, hence, of the γ_i. The variability of S_l was seen by several authors as an opportunity to improve the efficiency of the interval. As such, there is a vast research in the optimal choices for Λ. In this section, three different choices for Λ are introduced, based on the works of Kaas et al. (2000), Vanduffel et al. (2005) and Vanduffel et al. (2008).

The way optimal choices for Λ are constructed is intrinsically related to Theorem 2.4.2, and also to the possible approximations that can be made to the variance or the TVaR of the lower bound S_l. The following theorem introduces the expected value and variance of the random variables S, S_l and S_u, as can be seen in Vanduffel et al. (2005).

Theorem 2.4.5 Let S, S_l and S_u be random variables defined as in Theorem 2.4.3. Then, the expected value and variance of S, S_l and S_u are given by:

$$\mathbb{E}[S] = \mathbb{E}[S_l] = \mathbb{E}[S_u] = \sum_{i=1}^{n} \alpha_i \, e^{\mathbb{E}[Z_i]+\frac{1}{2}\sigma_{Z_i}^2}, \tag{2.39}$$

$$\mathbb{V}[S] = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j \, e^{\mathbb{E}[Z_i]+\mathbb{E}[Z_j]+\frac{1}{2}\left(\sigma_{Z_i}^2+\sigma_{Z_j}^2\right)} \left(e^{Cov(Z_i,Z_j)} - 1\right), \tag{2.40}$$

$$\mathbb{V}[S_l] = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j \, e^{\mathbb{E}[Z_i]+\mathbb{E}[Z_j]+\frac{1}{2}\left(\sigma_{Z_i}^2+\sigma_{Z_j}^2\right)} \left(e^{\rho_i\rho_j\sigma_{Z_i}\sigma_{Z_j}} - 1\right), \tag{2.41}$$

$$\mathbb{V}[S_u] = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j \, e^{\mathbb{E}[Z_i]+\mathbb{E}[Z_j]+\frac{1}{2}\left(\sigma_{Z_i}^2+\sigma_{Z_j}^2\right)} \left(e^{\sigma_{Z_i}\sigma_{Z_j}} - 1\right). \tag{2.42}$$

2.4.2.1 Optimal choice of Λ according to the "Taylor-based" Approximation

As previously mentioned, Kaas et al. (2000) introduced the lower bound S_l. In their numerical illustration, γ_i are defined so that Λ is a linear transformation of the first-order approximation of S. Indeed, the authors considered that Λ and S should be as similar as possible. Through this choice, the lower bound S_l is closer to S, resulting in a reduction of the interval bounding S, making the approximation of the risk measures TVaR and LTVaR for the random variable S more accurate. Below is the definition of γ_i and Λ according to this "Taylor-based" Approximation.

Definition 2.4.6 Let S and S_l be random variables defined as in Theorem 2.4.3. The **"Taylor-based" Approximation** is the lower bound S_l which approximates S, based on the coefficients γ_i^{TB} defined by:

$$\gamma_i^{TB} = \alpha_i \, e^{\mathbb{E}[Z_i]}, \quad \forall i \in \{1, 2, \ldots, n\}. \tag{2.43}$$

Therefore, the conditioning random variable Λ^{TB}, the correlation between Z_i and Λ^{TB}, ρ_i^{TB}, and the standard deviation of Λ^{TB}, σ_Λ^{TB}, are, respectively:

$$\Lambda^{TB} = \sum_{i=1}^{n} \alpha_i \, e^{\mathbb{E}[Z_i]} Z_i, \tag{2.44}$$

$$\rho_i^{TB} = \frac{1}{\sigma_{Z_i}\sigma_\Lambda^{TB}} \sum_{j=1}^{n} \alpha_i\, e^{\mathbb{E}[Z_i]}\, Cov[Z_i, Z_j], \quad \forall i \in \{1, 2, \ldots, n\}, \tag{2.45}$$

$$\sigma_\Lambda^{TB} = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j\, e^{\mathbb{E}[Z_i]+\mathbb{E}[Z_j]}\, Cov[Z_i, Z_j] \right)^{\frac{1}{2}}. \tag{2.46}$$

2.4.2.2 Optimal choice of Λ according to the "Maximal Variance" Approximation

The "Taylor-based" Approximation consisted on an intuitive approach. Vanduffel et al. (2005) developed instead an explicit approach for calculating Λ based on the maximization of the variance of S_l.

As can be seen from Theorem 2.4.2, one has $S_l \leq_{cx} S$, $\mathbb{E}[S_l] = \mathbb{E}[S]$ and $\mathbb{V}[S_l] \leq \mathbb{V}[S]$. Therefore, the larger the variance of S_l, the closer S_l will be to S and the better the bounding interval for the random variable S. Thus, Vanduffel et al. (2005) obtained the following approximation for the variance of S_l based on Theorem 2.4.5 and the fact that $e^x - 1 \approx x$ by the first-order Taylor expansion:

$$\begin{aligned}
\mathbb{V}[S_l] &= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j\, e^{\mathbb{E}[Z_i]+\mathbb{E}[Z_j]+\frac{1}{2}\left(\sigma_{Z_i}^2+\sigma_{Z_j}^2\right)} \left(e^{\rho_i \rho_j \sigma_{Z_i}\sigma_{Z_j}} - 1\right) \approx \\
&\approx \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j\, e^{\mathbb{E}[Z_i]+\mathbb{E}[Z_j]+\frac{1}{2}\left(\sigma_{Z_i}^2+\sigma_{Z_j}^2\right)} \left(\rho_i \rho_j \sigma_{Z_i}\sigma_{Z_j}\right) = \\
&= \left(Corr\left[\sum_{i=1}^{n} \alpha_i\, e^{\mathbb{E}[Z_i]+\frac{1}{2}\sigma_{Z_i}^2} Z_i, \Lambda \right] \right)^2 \mathbb{V}\left[\sum_{i=1}^{n} \alpha_i\, e^{\mathbb{E}[Z_i]+\frac{1}{2}\sigma_{Z_i}^2} Z_i \right].
\end{aligned} \tag{2.47}$$

Using this approximation of the variance of S_l, Vanduffel et al. (2005) defined the "Maximum Variance" Approximation as the lower bound S_l based on the coefficients γ_i, i.e. Λ, that maximize the first-order approximation of $\mathbb{V}[S_l]$.

Definition 2.4.7 Let S and S_l be random variables defined as in Theorem 2.4.3. The **"Maximum Variance" Approximation** is the lower bound S_l which approximates S, based on the coefficients γ_i^{MV} defined by:

$$\gamma_i^{MV} = \alpha_i\, e^{\mathbb{E}[Z_i]+\frac{1}{2}\sigma_{Z_i}^2}, \quad \forall i \in \{1, 2, \ldots, n\}. \tag{2.48}$$

Therefore, the conditioning random variable Λ^{MV}, the correlation between Z_i and Λ^{MV}, ρ_i^{MV}, and the standard deviation of Λ^{MV}, σ_Λ^{MV}, are, respectively:

$$\Lambda^{MV} = \sum_{i=1}^{n} \alpha_i\, e^{\mathbb{E}[Z_i]+\frac{1}{2}\sigma_{Z_i}^2} Z_i, \tag{2.49}$$

$$\rho_i^{MV} = \frac{1}{\sigma_{Z_i}\sigma_\Lambda^{MV}} \sum_{j=1}^{n} \alpha_i\, e^{\mathbb{E}[Z_i]+\frac{1}{2}\sigma_{Z_i}^2}\, Cov[Z_i, Z_j], \quad \forall i \in \{1, 2, \ldots, n\}, \tag{2.50}$$

$$\sigma_\Lambda^{MV} = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j\, e^{\mathbb{E}[Z_i]+\mathbb{E}[Z_j]+\frac{1}{2}\left(\sigma_{Z_i}^2+\sigma_{Z_j}^2\right)}\, Cov[Z_i, Z_j] \right)^{\frac{1}{2}}. \tag{2.51}$$

2.4.2.3 Optimal choice of Λ according to the "TVaR$_\alpha$-based" Approximation

The "Maximum Variance" Approximation is a globally optimal choice to define the coefficients γ_i and Λ. However, sometimes one is interested in calculating the risk measures TVaR and LTVaR of S. Recall that, by Theorem 2.4.2, when $S_l \leq_{cx} S$ occurs one has that $\text{TVaR}_\alpha[S_l] \leq \text{TVaR}_\alpha[S]$. Based on this inequality and taking into account the same procedure developed for defining the "Maximum Variance" Approximation, Vanduffel et al. (2008) defined the "TVaR$_\alpha$-based" Approximation by seeking to maximize the TVaR of S. Thus, the larger the TVaR of S_l is, the closer S_l is to S locally on the right tail of the distribution of S and, as such, the better the approximation and bounding interval.

Similarly to the "Maximum Variance" Approximation, Vanduffel et al. (2008) obtained an approximation for the TVaR of S_l. This was carried out by expanding the correlations ρ_i around ρ_i^{MV}, obtained through the 'Maximum Variance" Approximation. For that they applied Theorem 2.4.4 and the first-order Taylor approximation around a, $f(x) \approx f(a) + f'(a)(x-a)$. The result is given by:

$$\begin{aligned}\text{TVaR}_\alpha(S_l) &= \sum_{i=1}^{n} \alpha_i \, e^{\mathbb{E}[Z_i]+\frac{1}{2}\sigma_{Z_i}^2} \cdot \frac{\Phi(\rho_i \sigma_{Z_i} - z_\alpha)}{1-\alpha} \approx \qquad (2.52)\\ &\approx \sum_{i=1}^{n} \alpha_i \, e^{\mathbb{E}[Z_i]+\frac{1}{2}\sigma_{Z_i}^2} \cdot \frac{\Phi(\rho_i^{MV}\sigma_{Z_i} - z_\alpha) + \phi(\rho_i^{MV}\sigma_{Z_i} - z_\alpha)\left(\rho_i - \rho_i^{MV}\right)\sigma_{Z_i}}{1-\alpha}.\end{aligned}$$

where ϕ is the density function of the standard normal distribution.

Maximizing the first-order approximation of TVaR of S_l is the same as maximizing the following expression:

$$\begin{aligned}\sum_{i=1}^{n} \alpha_i \, e^{\mathbb{E}[Z_i]+\frac{1}{2}\sigma_{Z_i}^2} \, \phi(\rho_i^{MV}\sigma_{Z_i} - z_\alpha)\rho_i \sigma_{Z_i} &= Corr\left[\sum_{i=1}^{n} \alpha_i \, e^{\mathbb{E}[Z_i]+\frac{1}{2}\sigma_{Z_i}^2} \, \phi(\rho_i^{MV}\sigma_{Z_i} - z_\alpha) Z_i, \Lambda\right]\\ &\times \left(\mathbb{V}\left[\sum_{i=1}^{n} \alpha_i \, e^{\mathbb{E}[Z_i]+\frac{1}{2}\sigma_{Z_i}^2} \, \phi(\rho_i^{MV}\sigma_{Z_i} - z_\alpha) Z_i\right]\right)^{\frac{1}{2}}.\end{aligned} \qquad (2.53)$$

Using this approximation of the TVaR of S_l, Vanduffel et al. (2008) defined the "TVaR$_\alpha$-based" Approximation as the lower bound S_l based on the coefficients γ_i, i.e. Λ, that maximize expression (2.53). Note that this is the same as maximizing the first-order approximation of $\text{TVaR}_\alpha(S_l)$.

Definition 2.4.8 Let S and S_l be random variables defined as in Theorem 2.4.3. The **"TVaR$_\alpha$-based" Approximation** is the lower bound S_l which approximates S, based on the coefficients γ_i^{TVaR} defined by:

$$\gamma_i^{\text{TVaR}} = \alpha_i \, e^{\mathbb{E}[Z_i]+\frac{1}{2}\sigma_{Z_i}^2} \, \phi(\rho_i^{MV}\sigma_{Z_i} - z_\alpha), \quad \forall i \in \{1, 2, \ldots, n\}. \qquad (2.54)$$

Therefore, the conditioning random variable Λ^{TVaR}, the correlation between Z_i and Λ^{TVaR}, ρ_i^{TVaR}, and the standard deviation of Λ^{TVaR}, $\sigma_\Lambda^{\text{TVaR}}$, are, respectively:

$$\Lambda^{\text{TVaR}} = \sum_{i=1}^{n} \alpha_i \, e^{\mathbb{E}[Z_i]+\frac{1}{2}\sigma_{Z_i}^2} \, \phi(\rho_i^{MV}\sigma_{Z_i} - z_\alpha) Z_i, \qquad (2.55)$$

$$\rho_i^{\text{TVaR}} = \frac{1}{\sigma_{Z_i}\sigma_\Lambda^{\text{TVaR}}} \sum_{j=1}^{n} \alpha_i \, e^{\mathbb{E}[Z_i]+\frac{1}{2}\sigma_{Z_i}^2} \, \phi(\rho_i^{MV}\sigma_{Z_i} - z_\alpha) Cov[Z_i, Z_j], \ \ \forall i \in \{1,2,\dots,n\}, \tag{2.56}$$

$$\sigma_\Lambda^{\text{TVaR}} = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j \, e^{\mathbb{E}[Z_i]+\mathbb{E}[Z_j]+\frac{1}{2}\left(\sigma_{Z_i}^2+\sigma_{Z_j}^2\right)} \, \phi(\rho_i^{MV}\sigma_{Z_i} - z_\alpha) \, \phi(\rho_j^{MV}\sigma_{Z_j} - z_\alpha) Cov[Z_i, Z_j] \right)^{\frac{1}{2}}. \tag{2.57}$$

Finally, by Theorem 2.4.2, when $S_l \leq_{cx} S$ one has that $\text{LTVaR}_\alpha[S_l] \geq \text{LTVaR}_\alpha[S]$. Similarly, to have a locally optimal choice of the coefficients γ_i, and Λ, it is necessary to minimize LTVaR of S_l. Vanduffel et al. (2008) proved that the coefficients γ_i that minimize the first-order approximation of $\text{LTVaR}_\alpha(S_l)$ are the same as those that maximize the first-order approximation of $\text{TVaR}_\alpha(S_l)$, that is, $\gamma_i^{\text{LTVaR}} = \gamma_i^{\text{TVaR}}$.

Chapter 3
Risk Adjustment for Longevity Risk

In this Chapter the Model for Risk Adjustment (RA) for Longevity Risk proposed in this study is presented. The Chapter is divided into two parts. The first part analyzes in detail the Model for RA for Surrender Risk presented by Carlehed (2023), as well as, comments on its advantages and disadvantages and possible improvements. The second part develops the new Model for RA for Longevity Risk, which is an original generalization of the work of Carlehed (2023), solving some of its weaknesses.

3.1 Background on the Risk Adjustment for Surrender Risk

The Surrender Risk is a non-financial risk as described in IFRS 17 (2023a, § B86). Namely, it is necessary to construct a RA that reflects "the compensation that the entity requires for bearing the uncertainty about the amount and timing of the cash flows that arises from non-financial risk" (IFRS 17, 2023a, § 37). The standard does not specify the method for determining the RA, however, it recommends following five criteria (IFRS 17, 2023a, § B91). In Chapter 1, the main methodologies for calculating the RA were introduced. Smith et al. (2019) stated that 53% of the companies in their sample planned to use the CoC technique, while 33% intended to use a quantile technique, highlighting the dominance of these procedures. Nevertheless, several companies still have difficulty in choosing methods that are quick to calculate and effective in determining a good RA. Taking that into account, combined with some scientific research on the subject, Carlehed (2023) developed a method for determining a RA for the surrender risk in life insurance, as presented in detail below.

Under IFRS 17 (2023a, §§ 32-37), the calculation of the RA is the 3rd BB, and so, it is preceded by the 1st and 2nd BB which together determine the present value of the future cash flows, as can be seen in IFRS 17 (2018b). Carlehed (2023) defined cash flow as the difference between premiums and insurance payments and expenses, so it has a positive value if the premium amount is greater than the liabilities, and it is negative otherwise. This definition is different to that conventionally used in IFRS 17, in which cash flow is defined by the difference between liabilities and premiums, being positive if the liability amount is greater than the premiums, and negative otherwise. In this work, the definition of cash flow used in IFRS 17 is considered. However, in order to avoid inconsistencies, the definition of Carlehed (2023) will be kept throughout the exposition of his Model for RA for Surrender Risk, in this section.

Consider a portfolio of life insurance contracts with similar risk properties where customers pay annual premiums and receive a predefined payment at their death if they continue to be in the portfolio. No new costumers can enter the portfolio but there is the possibility of surrenders. When analyzing the surrender risk assumptions, a company usually calculates a best estimate for the surrender rate on an annual basis. This estimate, together with a historical volatility, will be the starting points for the forecast of the future volatility for the surrender rates, and for their stochastic modeling.

Let T be the maximum contract boundary of the life insurance contracts belonging to the portfolio, where the contract boundary is the time, usually in years, until the end of the portfolio liabilities, according to IFRS 17 (2023a, § 34 & § B61). Assume that the present values of the future net cash flows, a_t, for each year $t \in \{1, 2, \ldots, T\}$, given a zero surrender rate, are calculated *a priori* by the company. Actuarial assumptions, such as those for mortality, are included in a_t. Then, the PVFCF given a zero surrender rate, denoted by S^*, is defined by:

$$S^* = \sum_{t=1}^{T} a_t. \tag{3.1}$$

Naturally, S^* does not evaluate the PVFCF correctly, since it does not consider the possibility of surrender. Therefore, assume that the best estimate for the surrender rate, for each year $t \in \{1, 2, \ldots, T\}$, is denoted by s_t. Also, let the best estimate for the remain rate, for each year $t \in \{1, 2, \ldots, T\}$, be denoted by r_t and defined as $r_t = 1 - s_t$. The modified total PVFCF under surrender risk, which is denoted by S, is defined by:

$$S = \sum_{t=1}^{T} b_t, \tag{3.2}$$

where b_t represents the present values of the future net cash flows, for each year $t \in \{1, 2, \ldots, T\}$, under surrender risk and it is defined by:

$$b_t = a_t \prod_{k=1}^{t} r_k = a_t \prod_{k=1}^{t} (1 - s_k), \quad \forall t \in \{1, 2, \ldots, T\}. \tag{3.3}$$

Note that, for each year t, $r_t \leq 1$ so that $b_t \leq a_t$ and, finally, $S \leq S^*$. By introducing the model as presented above, Carlehed (2023) considers the assumption of homogeneity, which states that all contracts belonging to the portfolio have the same remain rate, r_t, in each year t, regardless of whether they refer to different products. Also, it assumes that the expected remain rate is independent of time, i.e., $r = \mathbb{E}(r_t)$.

3.1.1 Lognormal Model for remain rates

In order to model the total PVFCF under surrender risk, it is necessary to predict the value of future remain rates or the range of possible outcomes that they may have. This allows to understand the impact of surrender risk and, consequently, to compute the RA to prevent it.

Carlehed (2023) introduced two possible models for the remain rates: the lognormal model and the sticky model. Here, only the lognormal model is considered, since the stochastic model of survival rates in the RA for Longevity Risk is based on it. Also, as stated in Carlehed (2023), both models can produce remain rates $r_t > 1$, however the frequency and stability of unreasonable outcomes in the sticky model, which in the paper is considered as a fact of little relevance to the final result, can have, in reality, a huge impact, especially, in the generalization of the RA for Longevity Risk.

Given X a lognormally distributed random variable, $X \sim \text{Lognormal}(\mu, \sigma^2)$, one has that $\mathbb{E}(X) = e^{\mu + \frac{\sigma^2}{2}}$, $\mathbb{V}(X) = \left(e^{\sigma^2} - 1\right) e^{2\mu + \sigma^2}$, $\text{VaR}_\alpha(X) = e^{\mu + \sigma z_\alpha}$, $\text{TVaR}_\alpha(X) = e^{\mu + \frac{\sigma^2}{2}} \cdot \dfrac{\Phi(\sigma - z_\alpha)}{1 - \alpha}$ and $\text{LTVaR}_\alpha(X) = e^{\mu + \frac{\sigma^2}{2}} \cdot \dfrac{\Phi(z_\alpha - \sigma)}{\alpha}$. From these properties of the lognormal distribution, it is possible to derive the following results.

Theorem 3.1.1 Let X be a lognormally distributed random variable defined by:

$$X = \alpha e^{Z}, \tag{3.4}$$

where $\alpha > 0$ and Z is a normally distributed random variable with parameters μ and σ^2 such that $\mu = -\frac{\sigma^2}{2}$, that is, $Z \sim N\left(-\frac{\sigma^2}{2}, \sigma^2\right)$ and $e^Z \sim \text{Lognormal}\left(-\frac{\sigma^2}{2}, \sigma^2\right)$. Then, X can be written as:

$$X = \mathbb{E}(X)e^{Z}, \tag{3.5}$$

where $\mathbb{E}(X) = \alpha$.

Proof To show the result, it is enough to prove that $\mathbb{E}(X) = \alpha$:

$$\mathbb{E}(X) = \mathbb{E}\left(\alpha e^{Z}\right) = \alpha e^{\mu + \frac{\sigma^2}{2}} = \alpha e^{-\frac{\sigma^2}{2} + \frac{\sigma^2}{2}} = \alpha e^{0} = \alpha.$$

■

Carlehed (2023) relied on the definition and properties of geometric Brownian motion to obtain the stochastic model for the remain rates in discrete time.

Definition 3.1.2 Let r be the historical remain rate calculated *a priori* by the company. It can be, for example, the last historical remain rate or an average of the last ones. The **Lognormal Model for remain rates based on a single historical remain rate** is defined by:

$$r_t = r e^{-\frac{\sigma^2}{2} + \sigma X_t}, \quad \forall t \in \{1, 2, \dots, T\}, \tag{3.6}$$

where r_t are the future remain rates obtained by the model, $\underline{X} = (X_1, X_2 \dots, X_T)$ is a T-dimensional multivariate standard normal random vector with mutually independent components, i.e., $X_t \overset{i.i.d.}{\sim} N(0,1)$, and $\sigma > 0$ is a constant parameter that will be associated with the volatility of the model and that will be estimated later.

Using the lognormal model introduced in Definition 3.1.2, Carlehed (2023) rewrote the present values of the future net cash flows, b_t, as follows:

$$b_t = a_t \prod_{s=1}^{t} r_s = a_t \prod_{s=1}^{t} r e^{-\frac{\sigma^2}{2} + \sigma X_s} = a_t r^t e^{-\frac{\sigma^2 t}{2} + \sigma V_t}, \tag{3.7}$$

where $V_t = \sum_{s=1}^{t} X_s$, so $V_t \sim N(0, t)$.

To proceed further in the construction of the model, Carlehed (2023) sets the condition that $a_t \in \mathbb{R}_0^+$, i.e., the present values of the future net cash flows needs to be non-negative, for each year $t \in \{1, , 2, \dots, T\}$, given a zero surrender rate. This assumption is very restrictive considering that a_t corresponds to the difference between premiums and insurance payments and expenses. In most scenarios, it is not possible to guarantee that this condition is fulfilled, which is a clear disadvantage of this method. This will be one of the points taken into account when developing the new Model for RA for Longevity Risk.

Assuming that $a_t \in \mathbb{R}_0^+$, it is possible to simplify the expression (3.7) for the present values of the future net cash flows, b_t, by taking into account that $e^{-\frac{\sigma^2 t}{2} + \sigma V_t} \sim \text{Lognormal}\left(-\frac{\sigma^2 t}{2}, \sigma^2 t\right)$ and that:

$$\mathbb{E}(b_t) = \mathbb{E}\left(a_t r^t e^{-\frac{\sigma^2 t}{2} + \sigma V_t}\right) = a_t r^t \, \mathbb{E}\left(e^{-\frac{\sigma^2 t}{2} + \sigma V_t}\right) = a_t r^t. \tag{3.8}$$

Therefore, applying Theorem 3.1.1, b_t can be reduced to:

$$b_t = \mathbb{E}(b_t)e^{-\frac{\sigma^2 t}{2}+\sigma V_t}, \quad \forall t \in \{1, 2, \ldots, T\}, \tag{3.9}$$

where $\mathbb{E}(b_t) = a_t r^t$ and $V_t = \sum_{s=1}^{t} X_s$, so $V_t \sim N(0, t)$.

Finally, using the properties of the lognormal distribution, it follows that the VaR, TVaR and LTVaR of b_t are given by:

$$\mathrm{VaR}_\alpha(b_t) = \mathbb{E}(b_t)e^{-\frac{\sigma^2 t}{2}+\sigma z_\alpha \sqrt{t}}, \tag{3.10}$$

$$\mathrm{TVaR}_\alpha(b_t) = \mathbb{E}(b_t)\,\frac{\Phi\left(\sigma\sqrt{t} - z_\alpha\right)}{1-\alpha}, \tag{3.11}$$

$$\mathrm{LTVaR}_\alpha(b_t) = \mathbb{E}(b_t)\,\frac{\Phi\left(z_\alpha - \sigma\sqrt{t}\right)}{\alpha}. \tag{3.12}$$

3.1.1.1 Term Structure for remain rates

In order to develop the Model for RA for Surrender Risk, Carlehed (2023) established a quite restrictive initial assumption that the expected remain rate is independent of time, as mentioned earlier. However, throughout the paper, it is demonstrated that using a term structure of remain rates, $v(t)$, instead of a single historical remain rate, r, does not complicate the lognormal model. This improvement is very advantageous for increasing the effectiveness of the lognormal model for remain rates.

The definition below is an update of Definition 3.1.2, considering the use of a term structure of remain rates.

Definition 3.1.3 Let $v(t)$ be a term structure of expected remain rates calculated *a priori* by the company. The **Lognormal Model for remain rates based on a term structure of expected remain rates** is defined by:

$$r_t = v(t)e^{-\frac{\sigma^2}{2}+\sigma X_t}, \tag{3.13}$$

where r_t is the future remain rates obtained by the model, $\underline{X} = (X_1, X_2 \ldots, X_T)$ is a T-dimensional multivariate standard normal random vector with mutually independent components, i.e., $X_t \overset{i.i.d.}{\sim} N(0, 1)$, and $\sigma > 0$ is a constant parameter associated with the volatility of the model (that will be estimated later).

Analogously to what was developed to obtain expressions (3.7) and (3.9), from Definition 3.1.3, the present values of the future net cash flows, b_t, is now given by:

$$b_t = a_t \prod_{s=1}^{t} r_s = a_t \prod_{s=1}^{t} v(s)\, e^{-\frac{\sigma^2}{2}+\sigma X_s} = a_t \left(\prod_{s=1}^{t} v(s)\right) e^{-\frac{\sigma^2 t}{2}+\sigma V_t} = \mathbb{E}(b_t)e^{-\frac{\sigma^2 t}{2}+\sigma V_t}, \tag{3.14}$$

where $\mathbb{E}(b_t) = a_t \left(\prod_{s=1}^{t} v(s)\right)$ and $V_t = \sum_{s=1}^{t} X_s$, so $V_t \sim N(0, t)$.

From Theorem 3.1.1, the final expression of b_t by Definition 3.1.3 is the same as that by Definition 3.1.2, differing only in the expected value of b_t, $\mathbb{E}(b_t)$. Therefore, expressions

(3.10), (3.11) and (3.12) concerning the risk measures VaR, TVaR and LTVaR of b_t, respectively, can be applied in this case, considering now $\mathbb{E}(b_t) = a_t \left(\prod_{s=1}^{t} v(s)\right)$.

Using a term structure for the lognormal model for the remain rates increases the model quality, but, in reality, it is hard to apply due to the complexity in obtaining a term structure for remain rates. By contrast, in the case of longevity risk, it is very simple to obtain a term structure of survival rates by applying a mortality table or a method that models the expected survival rates. This will be the basis for building the new Model for RA for Longevity Risk.

3.1.1.2 Reliability of the Lognormal Model for remain rates

The lognormal model is a very robust model in fitting future remain rates, if the assumptions of the model are verified. Otherwise, the results can be weak. As such, these assumptions must first be tested on historical data to ascertain the applicability of the model.

One of the key assumptions is the normality of the logarithm of the time series of remain rates, i.e., one must ensure that there are $\mu \in \mathbb{R}$ and $\sigma^2 \in \mathbb{R}^+$ such that $\log(r_t) \sim N(\mu, \sigma^2)$. In the case of using a term structure, it is necessary to ensure the normality of the logarithm of the ratio between the remain rates and the term structure, that is, there must be $\mu \in \mathbb{R}$ and $\sigma^2 \in \mathbb{R}^+$ such that $\log\left(\frac{r_t}{v(t)}\right) \sim N(\mu, \sigma^2)$.

To test the normality, a goodness-of-fit test should be perform on historical data. Carlehed (2023) proposes using the Shapiro-Wilk test or constructing a Q-Q plot of $\log(r_t)$ against the normal distribution, or of $\log\left(\frac{r_t}{v(t)}\right)$ in the case of using a term structure, $v(t)$. These two approaches make it possible not only to check the adequacy of the lognormal model to the case study, but also to decide which is the best option between using a single historic remain rate, as in Definition 3.1.2, or a term structure, as in Definition 3.1.3.

It is essential to have a sample of historical data of significant length and representative of the reality, otherwise both the normality test and the estimation of the volatility parameter, σ, which will be seen later, may be compromised and lead to unreliable results.

3.1.2 PVFCF for the Total Portfolio

So far, the present values of the future net cash flows, for each year $t \in \{1, , 2, \ldots, T\}$, under surrender risk, b_t, has been studied. However, since the final objective is to calculate a RA for the modified total PVFCF under surrender risk, it is necessary to look at the sum of b_t, S, as defined in (3.2), and not to each b_t, for each year t. It is relevant to note that, from expressions (3.9) and (3.14), one concludes that regardless of which definition, 3.1.2 or 3.1.3, is used, the present values of the future net cash flows, for each year t, is always $b_t = \mathbb{E}(b_t) e^{-\frac{\sigma^2 t}{2} + \sigma V_t}$, where $V_t \sim N(0, t)$. The difference is in the expected value of b_t, $\mathbb{E}(b_t)$. Therefore, the analysis and development of the Model for RA for Surrender Risk that will be presented next is valid for both definitions, since all the expressions introduced depend only on the definition of $\mathbb{E}(b_t)$:

$$S = \sum_{t=1}^{T} b_t = \sum_{t=1}^{T} \mathbb{E}(b_t) e^{-\frac{\sigma^2 t}{2} + \sigma V_t}. \tag{3.15}$$

This expression corresponds to the sum of lognormally distributed random variables, and it needs to be simplified in order to be used in the study of the RA under surrender risk, using the stated in Chapter 2. Carlehed (2023) performs this analysis but only for the case in Definition 3.1.2. The modified total PVFCF under surrender risk, denoted by S, can be defined by:

$$S = \sum_{t=1}^{T} \mathbb{E}(b_t) e^{-\frac{\sigma^2 t}{2}} e^{\sigma V_t} = \sum_{t=1}^{T} \alpha_t \, e^{Z_t}, \tag{3.16}$$

where $\alpha_t = \mathbb{E}(b_t) e^{-\frac{\sigma^2 t}{2}}$ and $Z_t = \sigma V_t$, for all $t \in \{1, , 2, \ldots, T\}$.

The expected value, $\mathbb{E}[Z_t]$, variance, $\mathbb{V}[Z_t]$, and covariance, $Cov[Z_s, Z_t]$, of Z_t are given by:

$$\mathbb{E}[Z_t] = \mathbb{E}[\sigma V_t] = \sigma \sum_{s=1}^{t} \mathbb{E}[X_s] = 0, \quad \forall t \in \{1, 2, \ldots, T\}, \tag{3.17}$$

$$\sigma_{Z_t}^2 = \mathbb{V}[Z_t] = \mathbb{V}[\sigma V_t] = \sigma^2 \sum_{s=1}^{t} \mathbb{V}[X_s] = \sigma^2 t, \quad \forall t \in \{1, 2, \ldots, T\}, \tag{3.18}$$

$$Cov[Z_s, Z_t] = Cov[\sigma V_s, \sigma V_t] = \sigma^2 \sum_{t=1}^{\min(s,t)} \mathbb{V}[X_s] = \sigma^2 \min(s, t), \quad \forall s, t \in \{1, 2, \ldots, T\}. \tag{3.19}$$

Notice that $\mathbb{E}(b_t) \in \mathbb{R}_0^+$, regardless of whether the lognormal model is used according to Definition 3.1.2 or 3.1.3, given that $r, v(t) \in [0, 1]$ and Carlehed (2023) restricts all a_t to non-negative real numbers, i.e., $a_t \in \mathbb{R}_0^+$. Consequently, it follows that $\alpha_t \in \mathbb{R}_0^+$ for all $t \in \{1, 2, \ldots, T\}$. On the other hand, $V_t \sim N(0, t)$, and hence $Z_t \sim N(0, \sigma^2 t)$, for all $t \in \{1, 2, \ldots, T\}$.

This shows that the assumptions of Theorem 2.4.3 are guaranteed. Then, for any given choice of the γ_t such that the conditioning random variable Λ is given by $\Lambda = \sum_{t=1}^{T} \gamma_t Z_t$, it follows that the random variables S, S_l and S_u have the following convex order:

$$S_l \leq_{cx} S \leq_{cx} S_u, \tag{3.20}$$

where the random variables S_l and S_u are defined through expressions (3.17) and (3.18), as follows:

$$S_l = \sum_{t=1}^{T} \alpha_t \, e^{\frac{1}{2}\left(1-\rho_t^2\right)\sigma^2 t + \rho_t \sigma \sqrt{t}\, \Phi^{-1}(U)}, \tag{3.21}$$

$$S_u = \sum_{t=1}^{T} \alpha_t \, e^{\sigma \sqrt{t}\, \Phi^{-1}(U)}, \tag{3.22}$$

with U being a uniformly distributed random variable on the unit interval $]0, 1[$, Φ the cumulative distribution function of the standard normal distribution and ρ_t the correlation between Z_t and Λ.

As mentioned in Chapter 2, the upper bound S_u is fixed, unlike the lower bound S_l which depends on Λ and, thus, on the correlation, ρ_t, between Z_t and Λ. Definitions 2.4.6, 2.4.7 and 2.4.8 introduce three possible optimal choices for Λ and its correlation, ρ_t. Nevertheless, it is possible to come up with a general expression for ρ_t that is applicable for any choice of Λ and the respective γ_t. Kaas et al. (2000) presented a simplified version of this expression for the optimal choice of Λ according to the "Taylor-based" Approximation, that was later used by Carlehed (2023) in the Model for RA for Surrender Risk. Vanduffel et al. (2005) used the expression when applying the optimal choice of Λ according to the "Maximal Variance" Approximation. However, to the best of our knowledge, a general version of this result has not been developed in the relevant literature, so it will be presented in the following original theorem.

Theorem 3.1.4 Let $\underline{Z} = (Z_1, Z_2 \ldots, Z_T)$ be a T-dimensional multivariate normal random vector such that $Z_t \sim N(0, \sigma^2 t)$, for all $t \in \{1, 2, \ldots, T\}$, where $\sigma \in \mathbb{R}^+$ is a constant parameter. Consider also that for some given choice of γ_t, the conditioning random variable Λ, is given by $\Lambda = \sum_{t=1}^{T} \gamma_t Z_t$. Then, the correlation between Z_t and Λ, ρ_t, is defined by:

$$\rho_t = \frac{\sum_{s=1}^{t} \sum_{k=s}^{T} \gamma_k}{\sqrt{t \sum_{s=1}^{T} \left(\sum_{k=s}^{T} \gamma_k \right)^2}} = \frac{\sum_{s=1}^{t} \beta_s}{\sqrt{t \sum_{s=1}^{T} \beta_s^2}}, \tag{3.23}$$

where $\beta_s = \sum_{k=s}^{T} \gamma_k$.

Further, let $\underline{X} = (X_1, X_2 \ldots, X_T)$ be a T-dimensional multivariate standard normal random vector with mutually independent components, i.e., $X_t \overset{i.i.d.}{\sim} N(0, 1)$. Then, it follows that $\Lambda = \sum_{t=1}^{T} \beta_t Y_t = \sum_{t=1}^{T} \gamma_t Z_t$, where $Y_t = \sigma X_t$, for all $t \in \{1, 2, \ldots, T\}$.

Proof The theorem is divided into two parts: the calculation of the correlation between Z_t and Λ, ρ_t, and the relationship between the two definitions of Λ.

Regarding the computation of the correlation, one has that $\rho_t = Corr(Z_t, \Lambda) = \dfrac{Cov(Z_t, \Lambda)}{\sigma_{Z_t} \sigma_\Lambda}$.

From the definition of Z_t, it automatically follows that the variance of Z_t is given by $\sigma_{Z_t}^2 = \mathbb{V}[Z_t] = \sigma^2 t$, for all $t \in \{1, 2, \ldots, T\}$. By using the expression of the covariance of Z_t, $Cov[Z_s, Z_t]$, introduced in the expression (3.19), one has that:

$$Cov(Z_t, \Lambda) = \sum_{s=1}^{T} \gamma_s Cov(Z_t, Z_s) = \sigma^2 \sum_{s=1}^{T} \gamma_s \min(t, s) = \sigma^2 \sum_{s=1}^{t} \sum_{k=s}^{T} \gamma_k,$$

$$\sigma_\Lambda^2 = \mathbb{V}(\Lambda) = \sum_{t=1}^{T} \sum_{s=1}^{T} \gamma_t \gamma_s Cov(Z_t, Z_s) = \sigma^2 \sum_{t=1}^{T} \sum_{s=1}^{T} \gamma_t \gamma_s \min(t, s) = \sigma^2 \sum_{s=1}^{T} \left(\sum_{k=s}^{T} \gamma_k \right)^2.$$

Thus, the correlation between Z_t and Λ, ρ_t, is given by:

$$\rho_t = Corr(Z_t, \Lambda) = \frac{Cov(Z_t, \Lambda)}{\sigma_{Z_t}\sigma_\Lambda} = \frac{\sigma^2 \sum_{s=1}^{t} \sum_{k=s}^{T} \gamma_k}{\sqrt{\sigma^4 t \sum_{s=1}^{T} \left(\sum_{k=s}^{T} \gamma_k \right)^2}} = \frac{\sum_{s=1}^{t} \sum_{k=s}^{T} \gamma_k}{\sqrt{t \sum_{s=1}^{T} \left(\sum_{k=s}^{T} \gamma_k \right)^2}} = \frac{\sum_{s=1}^{t} \beta_s}{\sqrt{t \sum_{s=1}^{T} \beta_s^2}}.$$

Finally, it is necessary to prove the relationship between the two definitions of Λ. Indeed:

$$\Lambda = \sum_{t=1}^{T} \beta_t Y_t = \sum_{t=1}^{T} \sum_{k=t}^{T} \gamma_k Y_t = \sum_{t=1}^{T} \sum_{s=1}^{t} \gamma_t Y_s = \sum_{t=1}^{T} \gamma_t \sigma \sum_{s=1}^{t} X_s = \sum_{t=1}^{T} \gamma_t \sigma V_t = \sum_{t=1}^{T} \gamma_t Z_t.$$

■

There are two significant advantages associated with Theorem 3.1.4. On the one hand, the expression for computing the correlation between Z_t and Λ, ρ_t, is a simple formula to implement and generic for any choice of Λ and hence γ_t, including the three optimal choices presented in Chapter 2. On the other hand, the relationship between the two definitions for Λ allows to unify the methodologies developed by different authors. More specifically, those of Kaas et al. (2000), Dhaene et al. (2006) and Carlehed (2023), that use the formulation of Λ through the β_t, in contrast to those of Vanduffel et al. (2005) and Vanduffel et al. (2008), where the definition of Λ is done through γ_t, as used in this document.

3.1.3 Risk Adjustment for Surrender Risk

In the previous section, Theorem 2.4.3 was applied to the modified total PVFCF under surrender risk, S, making it possible to use the remaining theorems and definitions introduced in Chapter 2. More precisely, Theorems 2.4.4 and 2.4.5 allow the calculation of the expected value and the risk measures of S, S_l and S_u, and Definitions 2.4.6, 2.4.7 and 2.4.8, introduce optimal choices of Λ and, therefore, improvements in the lower bound S_l. This will be the starting point to develop a RA for Surrender Risk with good properties, allowing to better model this non-financial risk.

Before calculating the RA for the modified total PVFCF under surrender risk, S, it is appropriate to calculate the RA for the present values of the future net cash flows, b_t, for each year $t \in \{1, , 2, \dots, T\}$. Remember that, to study the RA that should be reserved for b_t, one is interested in the left tail of the distribution, since Carlehed (2023) defines cash flow as the difference between premiums and liabilities, and so severe scenarios occur when b_t has smaller values. As such, Carlehed (2023) has introduced a definition for RA for b_t, through the risk measure VaR, using expression (3.10), as presented below:

$$\mathrm{RA}_\alpha^{\mathrm{VaR}}(b_t) = \mathbb{E}(b_t) - \mathrm{VaR}_\alpha(b_t) = \mathbb{E}(b_t) \left(1 - e^{-\frac{\sigma^2 t}{2} + \sigma z_\alpha \sqrt{t}} \right), \tag{3.24}$$

for all $\alpha \in]0, 1[$, where z_α is the α-quantile of the standard normal distribution.

The definition for RA presented by Carlehed (2023) verifies one of the fundamental conditions described in the standard: the need to disclose the confidence level of the RA according to IFRS 17 (2023a, § 119). Nevertheless, this definition has two significant disadvantages: the possibility of taking negative values depending on the confidence level used; the inability to use the results associated with convex order relations introduced in Chapter 2. Therefore, given that one is interested in analyzing the risk associated with the left tail of the distribution, it would be better to define the RA for b_t using the risk measure LTVaR, as shown below:

$$\begin{aligned} \mathrm{RA}_\alpha(b_t) &= \mathbb{E}(b_t) - \mathrm{LTVaR}_\alpha(b_t) = \\ &= \mathbb{E}(b_t)\left(1 - \frac{\Phi\left(z_\alpha - \sigma\sqrt{t}\right)}{\alpha}\right) = \mathbb{E}(b_t)\cdot\frac{\alpha - \Phi\left(z_\alpha - \sigma\sqrt{t}\right)}{\alpha}, \end{aligned} \tag{3.25}$$

for all $\alpha \in]0,1[$, where z_α is the α-quantile of the standard normal distribution.

Carlehed (2023) mentions in his paper the possibility of defining RA as in (3.25), but he does not use it, because the confidence level applied here to define the RA is distinct from α, since $\mathrm{LTVaR}_\alpha(b_t) \leq \mathrm{VaR}_\alpha(b_t)$. This fact can be seen as a disadvantage of this new definition of RA. However, although the exact value of the confidence level is unknown, it is known that it is better than α by the definition of LTVaR. On the other hand, this definition of RA fixes the two disadvantages associated with that of Carlehed (2023). If RA is defined using the expression (3.25), its value will always be non-negative, i.e. $\mathrm{RA}_\alpha(b_t) \geq 0$, for all $\alpha \in]0,1[$; and it will also be possible to apply the results associated with convex order relations stated in Chapter 2.

It is now possible to develop a calculation method for the RA for the modified total PVFCF under surrender risk, S. It is known that S is the sum of lognormally distributed random variables, so there is no simple analytical formula for its risk measures VaR, TVaR and LTVaR. However, through convex order relations, it is possible to develop an interval where the RA of S is guaranteed to be. To this end, given that there are advantages to applying the risk measure LTVaR, one begins by using Theorem 2.4.2. Then, by applying Theorems 2.4.2 and 2.4.4, expressions (3.17) and (3.18) and knowing that $\alpha_t = \mathbb{E}(b_t)e^{-\frac{\sigma^2 t}{2}}$, it follows that $\mathrm{LTVaR}_\alpha(S)$ is bounded by the next interval:

$$\begin{aligned} \mathrm{LTVaR}_\alpha(S) &\in \Big[\mathrm{LTVaR}_\alpha(S_u)\,;\mathrm{LTVaR}_\alpha(S_l)\Big] \\ &\in \left[\sum_{t=1}^{T}\alpha_t\, e^{\frac{\sigma^2 t}{2}}\cdot\frac{\Phi\left(z_\alpha - \sigma\sqrt{t}\right)}{\alpha}\,;\sum_{t=1}^{T}\alpha_t\, e^{\frac{\sigma^2 t}{2}}\cdot\frac{\Phi\left(z_\alpha - \rho_t\sigma\sqrt{t}\right)}{\alpha}\right] \\ &\in \left[\sum_{t=1}^{T}\mathbb{E}(b_t)\cdot\frac{\Phi\left(z_\alpha - \sigma\sqrt{t}\right)}{\alpha}\,;\sum_{t=1}^{T}\mathbb{E}(b_t)\cdot\frac{\Phi\left(z_\alpha - \rho_t\sigma\sqrt{t}\right)}{\alpha}\right], \end{aligned} \tag{3.26}$$

for all $\alpha \in]0,1[$, where z_α is the α-quantile of the standard normal distribution.

Knowing the interval that constrains $\mathrm{LTVaR}_\alpha(S)$, it is reasonable to define the RA using this risk measure, similarly to what was done in expression (3.25) for the calculation of the RA of b_t. Therefore, the definition that is considered to be the most appropriate for the RA for the modified total PVFCF under surrender risk, S, is presented below, together with a theorem for the $\mathrm{RA}_\alpha(S)$ bounds.

Definition 3.1.5 Let the modified total PVFCF under surrender risk, S, be defined as in (3.16), where b_t are the present values of the future net cash flows as introduced by Carlehed (2023) (the difference between premiums and liabilities). The **RA for the modified total PVFCF under Surrender Risk**, denoted by $\mathrm{RA}_\alpha(S)$, is defined by:

$$\mathrm{RA}_\alpha(S) = \mathbb{E}(S) - \mathrm{LTVaR}_\alpha(S). \tag{3.27}$$

Let S_l and S_u be defined by expressions (3.21) and (3.22), respectively, according to the application of Theorem 2.4.3. Then, $\mathrm{RA}_\alpha(S_l)$ and $\mathrm{RA}_\alpha(S_u)$ are also defined by expression (3.27), replacing S by S_l and S_u, respectively.

As mentioned before, there is no simple analytical formula for $\mathrm{LTVaR}_\alpha(S)$, so it is necessary to construct an interval to limit it. Analogously, it is necessary to construct an interval for $\mathrm{RA}_\alpha(S)$, given the complexity of calculating its exact value analytically. The next theorem provides an interval that bounds $\mathrm{RA}_\alpha(S)$, by using Definition 3.1.5, Theorem 2.4.5 and expression (3.26).

Theorem 3.1.6 Let the RA for the modified total PVFCF under Surrender Risk, $\mathrm{RA}_\alpha(S)$, be defined according to Definition 3.1.5. Then:

$$\begin{aligned}
\mathrm{RA}_\alpha(S) &\in \Big[\mathrm{RA}_\alpha(S_l)\,;\mathrm{RA}_\alpha(S_u)\Big] \\
&\in \left[\sum_{t=1}^{T}\mathbb{E}(b_t)\cdot\left(1-\frac{\Phi\left(z_\alpha-\rho_t\sigma\sqrt{t}\right)}{\alpha}\right);\sum_{t=1}^{T}\mathbb{E}(b_t)\cdot\left(1-\frac{\Phi\left(z_\alpha-\sigma\sqrt{t}\right)}{\alpha}\right)\right] \\
&\in \left[\sum_{t=1}^{T}\mathbb{E}(b_t)\cdot\frac{\alpha-\Phi\left(z_\alpha-\rho_t\sigma\sqrt{t}\right)}{\alpha}\,;\sum_{t=1}^{T}\mathbb{E}(b_t)\cdot\frac{\alpha-\Phi\left(z_\alpha-\sigma\sqrt{t}\right)}{\alpha}\right].
\end{aligned} \tag{3.28}$$

It should be noted that the upper bound of $\mathrm{RA}_\alpha(S)$ in (3.28), is actually the sum of the RA for the present values of the future net cash flows, b_t, for each year $t \in \{1,,2,\ldots,T\}$, i.e., $\mathrm{RA}_\alpha(S_u) = \sum_{t=1}^{T}\mathrm{RA}_\alpha(b_t)$, as can be seen from expression (3.25). This shows that the maximum value that should be considered for the RA for the modified total PVFCF under surrender risk is actually the sum of all $\mathrm{RA}_\alpha(b_t)$, with $t \in \{1,,2,\ldots,T\}$.

Using the bounds in Theorem 3.1.6 can be seen as a disadvantage of the Model for RA for Surrender Risk, as $\mathrm{RA}_\alpha(S)$ is not exactly computed. However, given the nature of the comonotonic bounds S_l and S_u, the RA for the modified total PVFCF under surrender risk will be closer to that of its lower bound, that is, $\mathrm{RA}_\alpha(S)$ will be closer to $\mathrm{RA}_\alpha(S_l)$. Therefore, the RA of the lower bound S_l can be seen as an approximation of the RA for the modified total PVFCF under surrender risk, i.e., $\mathrm{RA}_\alpha(S) \approx \mathrm{RA}_\alpha(S_l)$.

Therefore, the choice of Λ is essential to obtain the best possible approximation of $\mathrm{RA}_\alpha(S)$. At this point the three optimal choices of Λ, defined in Chapter 2, become especially relevant. Given the construction of the optimal choices introduced in Definitions 2.4.6, 2.4.7 and 2.4.8 and the way the RA for the modified total PVFCF under surrender risk, $\mathrm{RA}_\alpha(S)$, is defined in Definition 3.1.5, it is not surprising that the optimal choice of Λ, leading to the best approximation of $\mathrm{RA}_\alpha(S)$, is the "TVaR_α-based" Approximation. Although $\mathrm{RA}_\alpha(S) \approx \mathrm{RA}_\alpha(S_l)$, this approximation may underestimate the value of the RA for the modified total PVFCF under surrender risk, derived from the short experience or uncertainties associated with the estimation of the volatility parameter, σ. In case of uncertainty, it is preferable to use a value in the interval, rather than its lower bound to approximate $\mathrm{RA}_\alpha(S)$.

Also note that the use of the risk measure VaR to calculate the RA of the total modified PVFCF under surrender risk, $\mathrm{RA}_\alpha(S)$, is not considered a better alternative, since convex order relations do not preserve the order of this risk measure, making it not straightforward to construct an interval that bounds $\mathrm{RA}_\alpha(S)$.

Finally, it is necessary to discuss the adequacy of the model and the fulfillment of the five criteria established by IFRS 17 (2023a, § B91) as described in Chapter 1. Regarding criterion (a), surrenders are usually characterized as high frequency but low severity scenarios, as stated by Carlehed (2023). So, in these cases, the model will fit reality well. The case of a mass lapse (a low frequency but high severity situation) is a weakness of this model, unless the historical data includes such situations. In that case, the historical information is incorporated in the estimation of the volatility parameter, σ, leading to an increment of the $\mathrm{RA}_\alpha(S)$. With respect to criteria (b) and (c), as can be seen from the interval created to bound the RA for the modified total PVFCF under surrender risk, in Theorem 3.1.6, their limits increase when the volatility parameter, σ, increases or the maximum contract boundary, T, increases, so the criteria are satisfied. Finally, for criteria (d) and (e), which concern the uncertainty of the parameter estimations, one should consider values contained in the interval defined in Theorem 3.1.6, rather than its lower limit to approximate $\mathrm{RA}_\alpha(S)$, in order to compensate for the lack of experience or the fragility of the estimations.

3.1.4 Volatility Estimation

To conclude the study of the Model for RA for Surrender Risk, it is necessary to estimate the volatility parameter, σ, the most important parameter of the model. As mentioned, one of the key assumptions for the lognormal model to work well is the normality of the logarithm of the time series of remain rates, that is, $\log(r_t) \sim N(\mu, \sigma^2)$, or in case of using a term structure, the normality of the logarithm of the ratio between the remain rates and the term structure, i.e., $\log\left(\dfrac{r_t}{v(t)}\right) \sim N(\mu, \sigma^2)$. This fact is directly related to how the estimate for the volatility parameter, σ, is computed. Namely, the application of the Maximum Likelihood Estimate (MLE) for the volatility parameter, σ, can be applied.

Theorem 3.1.7 Let Z be a normally distributed random variable with parameters μ and σ^2 such that $\mu = -\frac{\sigma^2}{2}$, that is, $Z \sim N\left(-\frac{\sigma^2}{2}, \sigma^2\right)$. Then, the **Maximum Likelihood Estimate (MLE) for the volatility parameter**, σ, is given by:

$$\hat{\sigma} = \sqrt{-2 + 2\sqrt{1 + \frac{1}{n}\sum_{k=1}^{n} z_k^2}}, \tag{3.29}$$

where $(z_1, \ldots, z_n)$ are n observations of a random sample $(Z_1, \ldots, Z_n)$ of size n, such that, $Z_k \overset{i.i.d.}{\sim} N\left(-\frac{\sigma^2}{2}, \sigma^2\right)$, for all $k \in \{1, 2, \ldots, n\}$.

Proof Consider $Z_k \overset{i.i.d.}{\sim} N\left(-\frac{\sigma^2}{2}, \sigma^2\right)$ for all $k \in \{1, 2, \ldots, n\}$, then the likelihood function is given by:

$$L(\sigma) = \prod_{k=1}^{n} f_{Z_k}(z_k \,|\, \sigma) = \prod_{k=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}\left(z_k + \frac{\sigma^2}{2}\right)^2}.$$

In order to maximize the likelihood function, consider the log-likelihood function and its first and second derivatives:

$$l(\sigma) = -\frac{n}{2}\log(2\pi) - n\log\sigma - \frac{1}{2\sigma^2}\sum_{k=1}^{n}\left(z_k + \frac{\sigma^2}{2}\right)^2,$$

$$l'(\sigma) = -n\left(\frac{1}{\sigma} + \frac{\sigma}{4}\right) + \frac{1}{\sigma^3}\sum_{k=1}^{n} z_k^2,$$

$$l''(\sigma) = -\frac{n}{4} - \frac{n}{\sigma^2}\left(\frac{3}{n\sigma^2}\sum_{k=1}^{n} z_k^2 - 1\right).$$

Let $\hat{\sigma}$ be defined as given in expression (3.29), it is easy to see that it is the zero of the first derivative of the log-likelihood function. Let y be given by $y = \frac{1}{n}\sum_{k=1}^{n} z_k^2$, which is always positive, i.e., $y > 0$. Then, $\hat{\sigma} = \sqrt{-2 + 2\sqrt{1+y}}$ and to show that the second derivative of the log-likelihood function is negative at that point, note that:

$$y > 0 \implies \frac{3y}{-2 + 2\sqrt{1+y}} - 1 > 0 \implies \frac{3y}{\hat{\sigma}^2} - 1 > 0 \implies l''(\hat{\sigma}) < 0.$$

■

From Definitions 3.1.2 and 3.1.3, both the Lognormal Models, the one with remain rates based on a single historical remain rate and the one based on a term structure of expected remain rates, use a T-dimensional multivariate standard normal random vector with mutually independent components, $\underline{X} = (X_1, X_2 \ldots, X_T)$ where $X_t \overset{i.i.d.}{\sim} N(0,1)$. Considering $Z = -\frac{\sigma^2}{2} + \sigma X_t$ where $\sigma > 0$ is the volatility parameter, regardless of the t chosen in the interval $\{1,,2,\ldots,T\}$, it can be concluded that $Z \sim N\left(-\frac{\sigma^2}{2}, \sigma^2\right)$. On the other hand, it should be noted that according to Definition 3.1.2, $Z = \log(r_t)$, while according to Definition 3.1.3, $Z = \log\left(\frac{r_t}{v(t)}\right)$. This fact proves the importance of the key assumption that $\log(r_t) \sim N(\mu, \sigma^2)$ or $\log\left(\frac{r_t}{v(t)}\right) \sim N(\mu, \sigma^2)$. Given $Z \sim N\left(-\frac{\sigma^2}{2}, \sigma^2\right)$, it is possible to apply Theorem 3.1.7 so that the MLE for the volatility parameter, σ, is given by the expression (3.29). Carlehed (2023) used this approach in constructing the Model for RA for Surrender Risk. In practical terms, to calculate the MLE for σ according to expression (3.29), consider that $(z_1, \ldots, z_n)$ are n observations, such that, $z_k = \log(r_{(t,k)})$, if applying Definition 3.1.2, and $z_k = \log\left(\frac{r_{(t,k)}}{v(t)}\right)$, when applying Definition 3.1.3, where $r_{(t,k)}$ are historical remain rates, for all $k \in \{1,2,\ldots,n\}$. Note that the data on historical remain rates are considered independently of t, since the volatility parameter, σ, does not vary with t.

Through the MLE of the volatility parameter, σ, it becomes possible to apply the Model for RA for Surrender Risk. However, as mentioned, it is essential that the sample of historical data used to calculate the MLE of σ has significant length and is representative of reality, otherwise it may compromise the results obtained.

3.2 Generalization of the Model for Risk Adjustment for Longevity Risk

In this section, our original generalization of the work of Carlehed (2023) is described, more precisely, the Model for RA for Longevity Risk.

The Longevity Risk is an insurance risk with a strong impact on many types of life insurance, particularly in the annuity contracts. As described in IAA (2018), in this kind of products, mortality and survival rates are the key assumptions to study the expected decrements of an insured population. An incorrect estimation of future mortality or survival rates may result in an unrealistic calculation of the present value of the future cash flows, i.e. the 1$^{\text{st}}$ and 2$^{\text{nd}}$ BB's under IFRS 17, as shown in IFRS 17 (2018b). This effect may have a negative impact on the management of reserves to cover insurance liabilities. According to IFRS 17 (2023a, § B86), insurance risks require the calculation of a RA, the 3$^{\text{rd}}$ BB, so, since Longevity Risk is one of them, it is necessary to build a good and appropriate Model for RA for this risk.

Let us first introduce a proposed change to the definition of cash flow, different from the one used by Carlehed (2023). As described in IFRS 17 (2023a, § B87), the RA for an insurance contract is the compensation that a company would demand to become indifferent between fulfilling a liability that has a wide range of possible outcomes or a liability that will produce fixed cash flows with the same expected present value as the insurance contract. Therefore, given the dependency between the RA and the liabilities arising from insurance contracts as established in the standard, the cash flow will be defined as the sum of insurance payments and expenses. There are two main reasons for choosing this definition. First, guarenteeing a positive value of the liabilities, an actuarial perspective is chosen in which severe scenarios are in the right tail of the distribution, contrary to the financial perspective developed by Carlehed (2023). This will have a major impact on the risk measures chosen when constructing the RA for Longevity Risk. On the other hand, in this new definition, premiums will not be subtracted, so that the amounts are always non-negative. This detail is of utmost importance to ensure that all assumptions of Theorem 2.4.3 are fulfilled.

Consider a portfolio of life insurance contracts with similar risk properties where, after paying the premium, customers receive predefined (constant or variable) payments each year until their death. There is a huge variety of insurance products in these circumstances, such as whole life, term and deferred annuities, as well as annuities with predefined variable payments, such as arithmetically or geometrically increasing annuities. All these products are described in detail in Dickson et al. (2009), as well as their expected present values and other properties. No new costumers can enter the portfolio and there is no possibility of surrender, so the only cause for leaving the portfolio is by the death of insurance policyholders. When analyzing the longevity risk assumptions, the company usually calculates the best estimate for the mortality rate for each age on an annual basis, which is summarized in a mortality table. This estimate, together with the historical volatility will be the initial inputs for developing a forecast of the future volatility for the mortality rates and their stochastic modeling.

Let T be the maximum contract boundary of the life insurance contracts belonging to the portfolio, according to IFRS 17 (2023a, § 34 & § B61), and ω the maximum age that a customer can reach, which, as such, is the highest age included in the mortality table applied by the company. Let us assume that the present values of the future net cash flows of liabilities, $a_{(x,t)}$, for each year $t \in \{1, 2, \ldots, T\}$ and initial age $x \in \{1, 2, \ldots, \omega\}$, given a zero mortality rate, are calculated *a priori* by the company. Initial age is defined as the age of the customer at the present date, i.e. $t = 0$. Consider, also, that the present values of future net cash flows of liabilities, a_t, for each year $t \in \{1, 2, \ldots, T\}$, are given by the sum of all present values in that year for each initial age x, i.e. $a_t = \sum_{x=0}^{\omega} a_{(x,t)}$. Then, the PVFCF given a zero mortality rate, denoted by S^*, is defined by:

$$S^* = \sum_{t=1}^{T} a_t = \sum_{t=1}^{T} \sum_{x=0}^{\omega} a_{(x,t)}. \tag{3.30}$$

Clearly, S^* does not evaluate the PVFCF correctly, since it does not consider the longevity risk. Assume that the best estimate for the mortality rate, for each year $t \in \{1, 2, \ldots, T\}$ and initial age $x \in \{1, 2, \ldots, \omega\}$, is denoted by q^*_{x+t}. Also, let the best estimate for the survival rate, for each year $t \in \{1, 2, \ldots, T\}$ and initial age $x \in \{1, 2, \ldots, \omega\}$, be denoted by p^*_{x+t} and defined by $p^*_{x+t} = 1 - q^*_{x+t}$. The symbol * in the mortality and survival rates, q^*_{x+t} and p^*_{x+t}, respectively, is introduced to distinguish from the probability of mortality and survival, q_{x+t} and p_{x+t}, respectively, which can be obtained from the mortality table applied by the company, and which will be particularly relevant later on, when defining the term structure for survival rates. Thus, the modified total PVFCF under longevity risk, denoted by S, is defined by:

$$S = \sum_{t=1}^{T} b_t = \sum_{t=1}^{T} \sum_{x=0}^{\omega} b_{(x,t)}, \tag{3.31}$$

where b_t are the present values of future net cash flows of liabilities, for each year $t \in \{1, 2, \ldots, T\}$, under longevity risk, and $b_{(x,t)}$ are the present values of future net cash flows of liabilities, for each year $t \in \{1, 2, \ldots, T\}$ and initial age $x \in \{1, 2, \ldots, \omega\}$, under longevity risk, defined by:

$$b_{(x,t)} = a_{(x,t)} \prod_{s=1}^{t} p^*_{x+s-1} \quad \forall t \in \{1, 2, \ldots, T\} \quad \forall x \in \{1, 2, \ldots, \omega\}. \tag{3.32}$$

Similarly to the relation between the definitions of a_t and $a_{(x,t)}$, the present values of future net cash flows of liabilities, b_t, for each year t, under longevity risk are given by the sum of all present values under longevity risk in that year for each initial age x, i.e. $b_t = \sum_{x=0}^{\omega} b_{(x,t)}$, for all $t \in \{1, 2, \ldots, T\}$. Hence, note that for each year t and initial age x, $p^*_{x+t} \leq 1$ so that $b_{(x,t)} \leq a_{(x,t)}$. By the definition of a_t and b_t, it follows that, for each year t, $b_t \leq a_t$ and, finally, $S < S^*$.

By the definition of S and $b_{(x,t)}$, it can be assumed that there is an inconsistency with respect to the maximum age, ω, since S includes cases where $x+t > \omega$. Of course, for these scenarios, the company does not expect to have any liabilities, since the maximum age has already been surpassed and, for all year t and initial age x such that $x + t > \omega$, it follows that $a_{(x,t)} = 0$ and $b_{(x,t)} = 0$, having consequently no impact on S^* nor on S.

Finally, notice that by introducing the new model as presented above, one is using the same homogeneity assumption applied by Carlehed (2023) in his Model for RA for Surrender Risk, which states that all contracts belonging to the portfolio have the same survival rate, p^*_{x+t}, in each year t and initial age x, regardless of whether they refer to different products or not.

3.2.1 Term Structure and Lognormal Model for Longevity Risk

Similarly to the importance of the remain rates for measuring the total PVFCF in the Model for RA for Surrender Risk by Carlehed (2023), survival rates are also the key for creating the lognormal model inherent to the Model developed for RA for Longevity Risk. However, despite the parallels, there are some differences in the prediction of these rates using the lognormal model.

For the surrender risk presented above, it was necessary to use a single historical remain rate or to create a term structure for the remain rates, which significantly improved the quality of the lognormal model used, as described in Definitions 3.1.2 and 3.1.3, respectively. Nevertheless, in the case of longevity risk, survival rates do not depend exclusively on time but also on the age of the customers. It is therefore essential that both variables are taken into account when modeling survival rates.

As such, the following definition is proposed to replace Definitions 3.1.2 or 3.1.3 in the study of longevity risk, which uses survival probabilities calculated *a priori* by the company as a term structure of expected survival rates that depends on both the year, t, and the initial age of the customers, x.

Definition 3.2.1 Let $v(x,t) = p_{x+t-1}$ be the survival probabilities calculated *a priori* by the company, which will be used as the term structure of the expected survival rates. The **Lognormal Model for survival rates based on a term structure of expected survival rates** is defined by:

$$p^*_{x+t-1} = v(x,t)e^{-\frac{\sigma^2}{2}+\sigma X_t} = p_{x+t-1}\, e^{-\frac{\sigma^2}{2}+\sigma X_t}, \tag{3.33}$$

where p^*_{x+t-1} is the future survival rate obtained by the model, $\underline{X} = (X_1, X_2 \ldots, X_T)$ is a T-dimensional multivariate standard normal random vector with mutually independent components, i.e., $X_t \overset{i.i.d.}{\sim} N(0,1)$, and $\sigma > 0$ is a constant parameter associated with the volatility.

Contrary to the complexity of obtaining a term structure for remain rates in the Model for RA for Surrender Risk, it is possible to develop a term structure for survival rates based on the survival probabilities from the mortality table applied by a company. In order to calculate the present value of future cash flows, i.e. the 1st and 2nd BB's under IFRS 17, it is necessary to apply a mortality table previously defined by the company as being best suited to the business or products under study. Therefore, this table can be used to build a term structure of expected survival rates, necessary for calculating the RA for Longevity Risk, the 3rd BB.

Applying the lognormal model introduced in Definition 3.2.1, and using the well-known multiplicative property of the survival probabilities stated in Dickson et al. (2009), that ${}_tp_x = \prod_{s=1}^{t} p_{x+s-1}$, the present value of the future net cash flows of liabilities under longevity risk, $b_{(x,t)}$, for each year $t \in \{1, 2, \ldots, T\}$ and initial age $x \in \{1, 2, \ldots, \omega\}$, can be rewritten as follows:

$$b_{(x,t)} = a_{(x,t)} \prod_{s=1}^{t} p^*_{x+s-1} = a_{(x,t)} \prod_{s=1}^{t} p_{x+s-1}\, e^{-\frac{\sigma^2}{2}+\sigma X_s} = a_{(x,t)}\, {}_tp_x\, e^{-\frac{\sigma^2 t}{2}+\sigma V_t}, \tag{3.34}$$

where $V_t = \sum_{s=1}^{t} X_s$, so $V_t \sim N(0,t)$.

Analogously to what was developed to obtain expressions (3.9) and (3.14), through Definitions 3.1.2 and 3.1.3, respectively, the formula for the present value of the future net cash flows of liabilities, b_t, for each year t, under the longevity risk is presented below, taking into account Definition 3.2.1 and applying Theorem 3.1.1:

$$b_t = \sum_{x=0}^{\omega} b_{(x,t)} = \sum_{x=0}^{\omega} a_{(x,t)}\, {}_tp_x\, e^{-\frac{\sigma^2 t}{2}+\sigma V_t} = \left(\sum_{x=0}^{\omega} a_{(x,t)}\, {}_tp_x\right) e^{-\frac{\sigma^2 t}{2}+\sigma V_t} = \mathbb{E}(b_t) e^{-\frac{\sigma^2 t}{2}+\sigma V_t}, \tag{3.35}$$

where $\mathbb{E}(b_t) = \sum_{x=0}^{\omega} a_{(x,t)}\, {}_tp_x$ and $V_t = \sum_{s=1}^{t} X_s$, so $V_t \sim N(0,t)$.

Again, from Theorem 3.1.1, the final formula of b_t by Definition 3.2.1 is the same as that by Definitions 3.1.2 and 3.1.3, differing only in the expected value of b_t, $\mathbb{E}(b_t)$. Therefore, expressions (3.10), (3.11) and (3.12) for the risk measures VaR, TVaR and LTVaR of b_t, respectively, can also be applied in this case, with the small change that now $\mathbb{E}(b_t) = \sum_{x=0}^{\omega} a_{(x,t)}\, {}_tp_x$.

Finally, it is important to discuss the applicability of the lognormal model for survival rates. Similarly to the Model for RA for Surrender Risk, it is essential to verify the normality of the logarithm of the ratio between the survival rates and the survival probabilities, which act as a term structure, in order to guarantee the quality of the fit of future survival rates. Namely, one must ensure that there are $\mu \in \mathbb{R}$ and $\sigma^2 > 0$ such that $\log\left(\frac{p^*_{x+t-1}}{p_{x+t-1}}\right) \sim N(\mu, \sigma^2)$. A goodness-of-fit test must be applied to the historical data, as for instance the Shapiro-Wilk test or the use of a Q-Q plot of $\log\left(\frac{p^*_{x+t-1}}{p_{x+t-1}}\right)$ against the normal distribution should be performed, as proposed by Carlehed (2023). Other tests, such as the Kolmogorov-Smirnov or Anderson-Darling tests can also be used.

There is an additional difficulty in obtaining the normality of the lognormal model for survival rates, which comes from the fact that there are two variables associated with the term structure, year, t, and initial age, x, unlike the Model for RA for Surrender Risk. The lognormal model for survival rates assumes the independence between the volatility parameter, σ, and the term structure variables, t and x. Regarding t, there is no dependence between the volatility parameter, σ, and year t, since the behavior of mortality is considered to be similar in each future year. However, it is sometimes difficult to verify the independency between the volatility parameter, σ, and the initial age, x, since this relationship is heavily influenced by the composition of the company's portfolio and the mortality table best suited to it.

Ideally, a volatility parameter should be constructed for each initial age, x, i.e., σ_x. However, this would require historical data for a large number of years in order to test the normality. Therefore, in order to build the lognormal model for survival rates, it is assumed that the volatility parameter, σ, and the initial age, x, are independent, which simplifies the model but also reduces its applicability. If normality is verified, then the model will provide a good fit for future survival rates. Otherwise, a possible solution is to subdivide the range of initial ages into subsets of consecutive initial ages, constructing a volatility parameter for each of these subsets and testing for normality separately. This process will not be presented in detail in this work, but can be left as a recommendation for possible future research.

3.2.2 PVFCF for the Total Portfolio

The objective now is to calculate the RA for the modified total PVFCF under longevity risk, S, as defined in (3.31), and not just its results for each year t. To this end, note that the expression (3.35) obtained for b_t using Definition 3.2.1 is analogous to the expressions (3.9) and (3.14), previously obtained using Definitions 3.1.2 and 3.1.3, respectively. The analysis previously developed in the Model for RA for Surrender Risk for calculating the modified total PVFCF is valid and can also be generalized to the Model for RA for Longevity Risk, since it only depends on the definition of $\mathbb{E}(b_t)$, which now is $\mathbb{E}(b_t) = \sum_{x=0}^{\omega} a_{(x,t)}\, {}_tp_x$.

Expression (3.35), obtained for b_t, can be used to rewrite expression (3.31). Furthermore, the notation can be changed so that the results presented in Chapter 2 can be immediately applied, as follows:

$$S = \sum_{t=1}^{T} b_t = \sum_{t=1}^{T} \mathbb{E}(b_t) e^{-\frac{\sigma^2 t}{2} + \sigma V_t} = \sum_{t=1}^{T} \alpha_t \, e^{Z_t}, \tag{3.36}$$

where $\alpha_t = \mathbb{E}(b_t) e^{-\frac{\sigma^2 t}{2}}$ and $Z_t = \sigma V_t$, for all $t \in \{1, , 2, \ldots, T\}$.

Given that the characteristics of S established in expression (3.36) are analogous to those present in expression (3.16) of the Model for RA for Surrender Risk, it is possible to conclude that the expected value, $\mathbb{E}[Z_t]$, variance, $\mathbb{V}[Z_t]$, and covariance, $Cov[Z_s, Z_t]$, of Z_t are given by expressions (3.17), (3.18) and (3.19), respectively.

It also follows that $\mathbb{E}(b_t) \in \mathbb{R}_0^+$, given that $a_{(x,t)}$ are non-negative real numbers, i.e., $a_{(x,t)} \in \mathbb{R}_0^+$, since these correspond to the present value of the future net cash flows of liabilities, and ${}_tp_x \in [0, 1]$. Consequently, it follows that $\alpha_t \in \mathbb{R}_0^+$ for all $t \in \{1, 2, \ldots, T\}$. On the other hand, $V_t \sim N(0, t)$, and hence $Z_t \sim N(0, \sigma^2 t)$, for all $t \in \{1, 2, \ldots, T\}$. So, given that the assumptions of Theorem 2.4.3 are verified, it is possible to apply it. Then, for any given choice of the γ_t such that the conditioning random variable Λ is given by $\Lambda = \sum_{t=1}^{T} \gamma_t Z_t$, it follows that the random variables S, S_l and S_u follow the abovementioned convex order relation, described by $S_l \leq_{cx} S \leq_{cx} S_u$, where the random variables S_l and S_u are defined as previously presented in expressions (3.21) and (3.22), respectively.

It should be remarked that if premiums were considered and subtracted in the definition of cash flow, it would not be possible to guarantee that $\alpha_t \in \mathbb{R}_0^+$ for all $t \in \{1, 2, \ldots, T\}$, which would compromise the application of Theorem 2.4.3.

Finally, regarding the bounds of S, the upper limit S_u is once again fixed, unlike the lower limit S_l, which depends significantly on the definition of Λ and the consequent correlation, ρ_t, between Z_t and Λ. To apply the three possible optimal choices for Λ and its correlation, ρ_t, established in Definitions 2.4.6, 2.4.7 and 2.4.8, one can apply the original Theorem 3.1.4 as in the Model for RA for Surrender Risk.

3.2.3 Risk Adjustment for Longevity Risk

First, let us calculate the RA for the present value of the future net cash flows of liabilities, b_t, for each year $t \in \{1, , 2, \ldots, T\}$. Note that one is interested in the right tail of the distribution, since severe cases occur when b_t has higher values, contrary to the Model for RA for Surrender Risk. Therefore, the RA for b_t is defined using the risk measure TVaR instead of the risk measure LTVaR:

$$\begin{aligned}\mathrm{RA}_\alpha(b_t) &= \mathrm{TVaR}_\alpha(b_t) - \mathbb{E}(b_t) = \\ &= \mathbb{E}(b_t)\left(\frac{\Phi\left(\sigma\sqrt{t}-z_\alpha\right)}{1-\alpha} - 1\right) = \mathbb{E}(b_t)\cdot\frac{\alpha - \Phi\left(z_\alpha - \sigma\sqrt{t}\right)}{1-\alpha}.\end{aligned} \tag{3.37}$$

The use of the risk measure TVaR to define the RA for b_t can be seen as a disadvantage, given the difficulty of disclosing the exact value of the confidence level, as it is required under IFRS 17 (2023a, § 119), since it is unknown. However, it is known that $\mathrm{TVaR}_\alpha(b_t) \geq \mathrm{VaR}_\alpha(b_t)$, so although the applied confidence level is distinct from α, it is known that it is better than α by the definition of TVaR. Furthermore, if the RA is defined using the expression (3.37), its value is always non-negative, i.e. $\mathrm{RA}_\alpha(b_t) \geq 0$, for all $\alpha \in]0,1[$; and it is also possible to apply the results associated with the convex order relations established in Chapter 2.

After having studied the RA for the present value of the future net cash flows of liabilities, b_t, it is now possible to develop a calculation method for the RA for the modified total PVFCF under longevity risk, S. Let us use the convex order relations to obtain an interval where the TVaR of S is guaranteed to be. Using Theorems 2.4.2 and 2.4.4, expressions (3.17) and (3.18) and knowing that $\alpha_t = \mathbb{E}(b_t)e^{-\frac{\sigma^2 t}{2}}$, it follows that $\mathrm{TVaR}_\alpha(S)$ is bounded by the interval:

$$\begin{aligned}\mathrm{TVaR}_\alpha(S) &\in \Big[\mathrm{TVaR}_\alpha(S_l)\,;\mathrm{TVaR}_\alpha(S_u)\Big] \\ &\in \left[\sum_{t=1}^{T}\alpha_t\, e^{\frac{\sigma^2 t}{2}}\cdot\frac{\Phi\left(\rho_t\sigma\sqrt{t}-z_\alpha\right)}{1-\alpha}\,;\sum_{t=1}^{T}\alpha_t\, e^{\frac{\sigma^2 t}{2}}\cdot\frac{\Phi\left(\sigma\sqrt{t}-z_\alpha\right)}{1-\alpha}\right] \\ &\in \left[\sum_{t=1}^{T}\mathbb{E}(b_t)\cdot\frac{\Phi\left(\rho_t\sigma\sqrt{t}-z_\alpha\right)}{1-\alpha}\,;\sum_{t=1}^{T}\mathbb{E}(b_t)\cdot\frac{\Phi\left(\sigma\sqrt{t}-z_\alpha\right)}{1-\alpha}\right].\end{aligned} \tag{3.38}$$

Knowing the interval that constrains $\mathrm{TVaR}_\alpha(S)$, it is reasonable to define the RA using this risk measure, as it was done in the construction of Definition 3.1.5 by using the expression (3.26).

Definition 3.2.2 Let the modified total PVFCF under longevity risk, S, be defined as given in expression (3.36), where b_t is the the present value of the future net cash flows of liabilities. The **RA for the modified total PVFCF under Longevity Risk**, denoted by $\mathrm{RA}_\alpha(S)$, is defined by:

$$\mathrm{RA}_\alpha(S) = \mathrm{TVaR}_\alpha(S) - \mathbb{E}(S). \tag{3.39}$$

Let S_l and S_u be defined by expressions (3.21) and (3.22), respectively, according to the application of Theorem 2.4.3. Then, $\mathrm{RA}_\alpha(S_l)$ and $\mathrm{RA}_\alpha(S_u)$ are also defined by expression (3.39), replacing S by S_l and S_u, respectively.

The theorem below shows an interval where $\mathrm{RA}_\alpha(S)$ is bounded, using Definition 3.2.2, Theorem 2.4.5 and expression (3.38).

Theorem 3.2.3 Let the RA for the modified total PVFCF under Longevity Risk, $\mathrm{RA}_\alpha(S)$, be defined according to Definition 3.2.2. Then:

$$
\begin{aligned}
\mathrm{RA}_\alpha(S) &\in \Big[\mathrm{RA}_\alpha(S_l)\,;\mathrm{RA}_\alpha(S_u)\Big] \\
&\in \left[\sum_{t=1}^{T}\mathbb{E}(b_t)\cdot\left(\frac{\Phi\left(\rho_t\sigma\sqrt{t}-z_\alpha\right)}{1-\alpha}-1\right);\sum_{t=1}^{T}\mathbb{E}(b_t)\cdot\left(\frac{\Phi\left(\sigma\sqrt{t}-z_\alpha\right)}{1-\alpha}-1\right)\right] \\
&\in \left[\sum_{t=1}^{T}\mathbb{E}(b_t)\cdot\frac{\alpha-\Phi\left(z_\alpha-\rho_t\sigma\sqrt{t}\right)}{1-\alpha};\sum_{t=1}^{T}\mathbb{E}(b_t)\cdot\frac{\alpha-\Phi\left(z_\alpha-\sigma\sqrt{t}\right)}{1-\alpha}\right].
\end{aligned}
\tag{3.40}
$$

Note that, as in the Model for RA for Surrender Risk, the upper bound of the interval, $\mathrm{RA}_\alpha(S_u)$, is actually the sum of the RA for the present values of the future net cash flows of liabilities, b_t, for each year $t \in \{1, 2, \ldots, T\}$, i.e., $\mathrm{RA}_\alpha(S_u) = \sum_{t=1}^{T}\mathrm{RA}_\alpha(b_t)$. This can be seen from expression (3.37), which shows that this sum should be the maximum value to be considered for RA.

Definition 3.2.2 and Theorem 3.2.3 establish how the RA is calculated for the modified total PVFCF under longevity risk, $\mathrm{RA}_\alpha(S)$. Again, although the model for calculating the $\mathrm{RA}_\alpha(S)$ is not an exact value, there is certainty that the interval defined by Theorem 3.2.3 contains $\mathrm{RA}_\alpha(S)$. Furthermore, given the nature of the comonotonic bounds S_l and S_u, the RA for the modified total PVFCF under longevity risk will be closer to its lower bound, that is, $\mathrm{RA}_\alpha(S)$ will be closer to $\mathrm{RA}_\alpha(S_l)$. Therefore, the RA of the lower bound S_l can be seen as an approximation of the RA for the modified total PVFCF under longevity risk, i.e., $\mathrm{RA}_\alpha(S) \approx \mathrm{RA}_\alpha(S_l)$.

This fact justifies the importance of the choice of Λ to obtain the best possible approximation of $\mathrm{RA}_\alpha(S)$. Given the construction of the three optimal choices of Λ introduced in the Definitions 2.4.6, 2.4.7 and 2.4.8 and how the RA for the modified total PVFCF under longevity risk, $\mathrm{RA}_\alpha(S)$, is defined in Definition 3.2.2, it is expected that the optimal choice of Λ that will lead to the best approximation of $\mathrm{RA}_\alpha(S)$ is the "TVaR_α-based" Approximation. However, like in the Model for RA for Surrender Risk, this approximation may underestimate the value of RA for the total modified PVFCF under longevity risk, leading to some error, derived from short experience or the uncertainties associated with the estimation of the volatility parameter, σ. Because of this, considering an interval that contains $\mathrm{RA}_\alpha(S)$, as introduced in Theorem 3.2.3, becomes more relevant given that, in case of uncertainty, it would be preferable to use a value in the interval different from its lower bound to approximate $\mathrm{RA}_\alpha(S)$ and compensate for the underestimated approximation of it.

It should also be remarked that the use of the risk measure VaR to calculate the RA of the modified total PVFCF under longevity risk, $\mathrm{RA}_\alpha(S)$, is not considered to be a better alternative, since the convex order relations do not preserve the order of this risk measure, making it not straightforward to construct an interval that bounds the $\mathrm{RA}_\alpha(S)$.

Finally, regarding the discussion of the adequacy of the model and the fulfillment of the five criteria established by IFRS 17 (2023a, § B91) as described in Chapter 1, this debate will be left to Chapter 4 to be complemented by the practical example.

3.2.4 Volatility Estimation

To conclude the study of the Model for RA for Longevity Risk, it is necessary to calculate the fundamental parameter of the model: the volatility parameter, σ. As already mentioned, one of the key assumptions for the lognormal model to work well is the normality of the logarithm of the ratio between the survival rates and the survival probabilities, i.e., $\log\left(\frac{p^*_{x+t-1}}{p_{x+t-1}}\right) \sim N(\mu, \sigma^2)$. Similarly to the Model for RA for Surrender Risk, this fact is directly connected to the method used to estimate the volatility parameter, σ, through the MLE established in Theorem 3.1.7.

The Lognormal Model for survival rates based on a term structure of expected survival rates presented in Definition 3.2.1 uses a T-dimensional multivariate standard normal random vector with mutually independent components, $\underline{X} = (X_1, X_2 \dots, X_T)$ where $X_t \overset{i.i.d.}{\sim} N(0,1)$. When considering $Z = -\frac{\sigma^2}{2} + \sigma X_t$ where $\sigma > 0$ is the volatility parameter, regardless of the t chosen in the interval $\{1, , 2, \dots, T\}$, it follows that $Z \sim N\left(-\frac{\sigma^2}{2}, \sigma^2\right)$. Note that according to Definition 3.2.1, $Z = \log\left(\frac{p^*_{x+t-1}}{p_{x+t-1}}\right)$. Given that $Z \sim N\left(-\frac{\sigma^2}{2}, \sigma^2\right)$, it is possible to apply expression (3.29). To calculate the MLE for σ consider that $(z_1, \dots, z_n)$ are n observations, such that, $z_k = \log\left(\frac{p^*_{x+t-1\,;\,k}}{p_{x+t-1}}\right)$, where $p^*_{x+t-1\,;\,k}$ are the historical survival rates and p_{x+t-1} are the survival probabilities, which are used as the term structure of the expected survival rates. It should be noted that the data on historical survival rates are considered independently of the year t and the initial age x, since the volatility parameter, σ, does not vary with t and x.

In conclusion, through the MLE of the volatility parameter, σ, it is possible to apply the Model for RA for Longevity Risk, not forgetting, as mentioned above, that it is essential that the sample of historical data used to calculate the MLE of σ has significant length and is representative of reality, otherwise it may compromise the results obtained.

Chapter 4

Practical Example

This Chapter illustrates a practical example of the application of the Model for Risk Adjustment (RA) for Longevity Risk developed in Chapter 3, using a database inspired by a portfolio of products susceptible to longevity risk belonging to one of the leading insurance companies operating in Portugal. The Chapter is divided into two parts: the first part that describes the database used and the software programs needed for implementing the model; the second part where the results are presented and discussed, namely through comparison with other methodologies.

4.1 Database Description and Calculating Tools

The practical example in this Chapter uses simulated data inspired by a database of an annuity portfolio provided by EY. For confidentiality reasons, the data has been modified, as well as some of the assumptions and additional information about it. Some possible inconsistencies in the results may arise from these changes.

The database refers to an annuity product in which the policyholder, after paying the premium, receives, at the end of each year, an annuity between the ages of 60 and 80, if he/she is still alive. This amount paid annually is not fixed, i.e., each client receives an annual amount defined at the beginning of the contract which can vary each year and be different from that received by other clients.

The portfolio consists of 11474 policies still in force at the beginning of 2022, from an annuity product that started in 2015 with 13630 policies linked to individuals aged between 60 and 80. It should be noted that no new customers are allowed to enter the portfolio, so the 2156 policies that were closed between 2015 and 2022 relate only to the death of the insurance policyholders. Moreover, it should also be remarked that at the beginning of 2022, all the individuals still in the portfolio are aged 67 or more, so the insurance company expects to have liabilities with this portfolio for a maximum of 14 more years, which means until the end of 2035. For this reason, 14 years is taken as the maximum contract boundary, i.e., $T = 14$.

In terms of the assumptions made, the Mortality Table for Portugal (2010 - 2012) developed by INE (2013) was used to calculate the present value of the future net cash flows of liabilities, as well as to build the term structure of the expected survival rates to be used. This table was considered instead of other more recent versions published by INE for two main reasons: the first is that this table was a common choice used by insurance companies for assessing mortality at the time the portfolio was created; the second is that by considering an older and outdated mortality table and taking into account the trend of growth in average life expectancy in Portugal, there will be a greater volatility associated with the Model for RA for Longevity Risk, which is more interesting from the analysis point of view. Obviously, other mortality tables could naturally be used, as long as there is consistency in their continuous application, i.e., as long as no more than one mortality table is used. Regarding the interest rate, a fixed annual rate around 3% of was applied for all the years in which the portfolio is in force, according to the choice used by the company. In fact, that rate is in line with the decreasing pattern of the UFR, which since 2017 has fallen from 4.2% to 3.3%, as published by EIOPA (2023) for the year 2024.

For the implementation of the Model for RA for Longevity Risk, two software programs were used. The data inputs were cleaned and modified in Microsoft Excel. The data was afterwards uploaded into R Studio tool, where a computer program was developed to compute the Model for RA for Longevity Risk and generate the outputs.

The computational tool is designed to receive portfolios different from the one discussed in this practical example, and requires the following 4 inputs:

- a table with the present value of the future net cash flows of liabilities given a zero mortality rate, for each year and initial age considered. This table represents what was defined in Chapter 3 by $a_{(x,t)}$, for each year $t \in \{1, 2, \ldots, T\}$ and initial age $x \in \{1, 2, \ldots, \omega\}$;
- the mortality table applied by the insurance company for the portfolio under study. This table will be used to calculate the survival probabilities, ${}_tp_x$;
- a table of the portfolio's historical exposure, i.e., the number of individuals by age for each historical year in which the portfolio was in force and of which there is a record. Naturally, the more years of historical portfolio behavior, the better the results, unless the assumptions suffer major changes during that period and no longer represent reality;
- a table of the portfolio's historical deaths, i.e., the number of individuals by age who died in each historical year in which the portfolio is in force and for which there is a record. The number of years considered should be the same as in the exposure data.

Finally, it should be noted that the inputs described above are essential for calculating the bounding interval for the RA of the portfolio, $\mathrm{RA}_\alpha(S)$, according to Theorem 3.2.3. The first and second inputs are used to calculate the expected present values of future net cash flows of liabilities, $\mathbb{E}(b_t)$, while the third and fourth inputs are necessary for the MLE of the volatility parameter, σ, of the model. If the first input cannot be obtained, a table of present values of future net cash flows of liabilities under longevity risk, $b_{(x,t)}$, for each year $t \in \{1, 2, \ldots, T\}$ and initial age $x \in \{1, 2, \ldots, \omega\}$, can be used instead.

4.2 Results and Discussion

In order to illustrate the application of the Model for RA for Longevity Risk, four key aspects were considered so as to obtain results and discuss some of the relevant ones:

- a comparison of the 3 Optimal Choices for the Lower Bound presented in Definitions 2.4.6, 2.4.7 and 2.4.8, in order to conclude which one results in the best outcomes for bounding the RA in the Model for RA for Longevity Risk;
- an analysis of the 5 criteria established by IFRS 17 (2023a, § B91) and their fulfillment in the shown example;
- the verification of the assumption of the normality of the logarithm of the ratio between survival rates and survival probabilities, which is crucial for achieving good results with the new Model for RA for Longevity Risk;
- a comparison of the 2 classic techniques for calculating the RA versus the original method here developed, including a discussion of the advantages and disadvantages of each.

4.2.1 Comparison of the 3 Optimal Choices for the Lower Bound

In Chapter 2, 3 optimal choices for the lower bound, S_l, were presented, according to Definitions 2.4.6, 2.4.7 and 2.4.8. Any of these 3 choices can be used to calculate the RA using the Model for RA for Longevity Risk. However, due to how the RA is enunciated in Definition 3.2.2 and the way in which the interval that bounds $\text{RA}_\alpha(S)$ is created in Theorem 3.2.3, the "TVaR_α-based" Approximation is expected to better approximate the RA of the lower bound, S_l, to the RA for the modified total PVFCF under longevity risk, S, i.e., $\text{RA}_\alpha(S) \approx \text{RA}_\alpha(S_l)$.

In order to demonstrate this fact in the practical example, the Monte Carlo Method is used to simulate the RA for the modified total PVFCF under longevity risk, S, for 5 possible values for the volatility parameter, σ, and with $\alpha = 0.95$. Recall that the confidence level associated to the Model for RA for Longevity Risk is not equal but higher than α, given the definition of TVaR, as mentioned in Chapter 3. For each of these 5 volatility values, σ, 10^7 simulations were carried out in order to calculate the RA, which is denoted by $\text{RA}_\alpha(S_{\text{MC}})$. The analytical calculation of the RA of the upper and lower bounds was also carried out, according to the Model for RA for Longevity Risk, using Theorem 3.2.3. The RA of the upper bound, S_u, will be denoted by $\text{RA}_\alpha(S_u)$, while the RA of the 3 optimal choices for the lower bound, S_l, presented in Definitions 2.4.6, 2.4.7 and 2.4.8 will be denoted by $\text{RA}_\alpha(S_l^{\text{TB}})$, $\text{RA}_\alpha(S_l^{\text{MV}})$ and $\text{RA}_\alpha(S_l^{\text{TVaR}})$, respectively.

The results of the Monte Carlo Simulation and the analytical calculations using the Model for RA for Longevity Risk are shown in Table 1.

Table 1: RA of the 3 Optimal Choices for S_l, depending on σ

	$\sigma = 0.006$	$\sigma = 0.05$	$\sigma = 0.15$	$\sigma = 0.25$	$\sigma = 0.35$
$RA_\alpha(S_u)$	3781328	32442986	115717176	226688113	366920042
$RA_\alpha(S_{MC})$	3198646	27195634	95278997	184253573	296091873
$RA_\alpha(S_l^{TVaR})$	3197451	27178495	95021110	183100397	293012814
$RA_\alpha(S_l^{MV})$	3197440	27177111	95006276	183055250	292921499
$RA_\alpha(S_l^{TB})$	3197440	27177099	95006145	183054855	292920705

It can be seen that, the value obtained by Monte Carlo simulation, $\text{RA}_\alpha(S_{\text{MC}})$, is between the RA of the upper and lower bounds, $\text{RA}_\alpha(S_u)$ and $\text{RA}_\alpha(S_l)$, respectively. Furthermore, whatever the volatility value, the simulated $\text{RA}_\alpha(S_{\text{MC}})$ is closer to the lower bound, $\text{RA}_\alpha(S_l)$.

Table 2: Percentage of RA of the 3 Optimal Choices for S_l different from $\text{RA}_\alpha(S_{\text{MC}})$, depending on σ

	$\sigma = 0.006$	$\sigma = 0.05$	$\sigma = 0.15$	$\sigma = 0.25$	$\sigma = 0.35$
$RA_\alpha(S_u)$	18.216516 %	19.294831 %	21.450876 %	23.030511 %	23.921011 %
$RA_\alpha(S_{MC})$	0 %	0 %	0 %	0 %	0 %
$RA_\alpha(S_l^{TVaR})$	-0.037356 %	-0.063021 %	-0.270664 %	-0.625864 %	-1.0399 %
$RA_\alpha(S_l^{MV})$	-0.037699 %	-0.068111 %	-0.286234 %	-0.650366 %	-1.07074 %
$RA_\alpha(S_l^{TB})$	-0.037705 %	-0.068157 %	-0.286371 %	-0.65058 %	-1.071008 %

In Table 2, it can be observed the relative differences between the RA of the upper bound and the 3 optimal choices of the lower bound, $\text{RA}_\alpha(S_u)$, $\text{RA}_\alpha(S_l^{\text{TB}})$, $\text{RA}_\alpha(S_l^{\text{MV}})$ and

$\mathrm{RA}_\alpha(S_l^{\mathrm{TVaR}})$, respectively, and the simulation of the RA for the modified total PVFCF under longevity risk, $\mathrm{RA}_\alpha(S_{\mathrm{MC}})$.

Looking at Table 2, it is clear how much higher, between 18% and 24% higher, the RA of the upper bound, $\mathrm{RA}_\alpha(S_u)$, is in comparison to the simulated RA for the modified total PVFCF under longevity risk, $\mathrm{RA}_\alpha(S_{\mathrm{MC}})$, and with an increasing trend as the associated volatility increases. These are expected results, due to how the RA of the upper bound is constructed. Indeed, the upper bound seeks to obtain the maximum value for the RA, which is the sum of the RA for the present values of the future net cash flows of liabilities, b_t, for each year $t \in \{1, , 2, \ldots, T\}$, i.e., $\mathrm{RA}_\alpha(S_u) = \sum_{t=1}^{T} \mathrm{RA}_\alpha(b_t)$.

Regarding the lower bound, S_l, the values obtained for the RA from the 3 optimal choices of the lower bound, $\mathrm{RA}_\alpha(S_l^{\mathrm{TB}})$, $\mathrm{RA}_\alpha(S_l^{\mathrm{MV}})$ and $\mathrm{RA}_\alpha(S_l^{\mathrm{TVaR}})$, are 0.037% to 1.072% lower than $\mathrm{RA}_\alpha(S_{\mathrm{MC}})$, and they have a decreasing behavior with the increase in the associated volatility. These results support the fact that the RA of the lower bound, $\mathrm{RA}_\alpha(S_l)$, can be used to approximate the RA for the modified total PVFCF under longevity risk, $\mathrm{RA}_\alpha(S)$. Notice that these percentages could be even more reduced as the number of simulations increases.

As expected, of the 3 optimal choices for the lower bound, S_l, the one that gives the best approximation to the simulated value, $\mathrm{RA}_\alpha(S_{\mathrm{MC}})$, regardless of the value of the volatility parameter, σ, is the "TVaR_α-based" Approximation. Therefore, from now on, only this approximation will be considered for calculating the RA of the lower bound.

4.2.2 Adequacy of the Model and Fulfillment of IFRS 17

As mentioned, when discussing the Model for RA for Surrender Risk, a proposed method for calculating the RA has to verify the 5 criteria established in IFRS 17 (2023a, § B91) as described in Chapter 1. The adequacy of the Model for RA for Longevity Risk and its fulfillment of the 5 criteria will be discussed through the analysis of tables and figures associated with the practical example under study.

According to criterion (a), risks with low frequency and high severity need to have a higher RA than those risks with high frequency and low severity. This implies that, not only should there be an increase in the RA when considering a higher α, but also this increase should be more pronounced for values of α closer to 1, since this situation corresponds to risks with low frequency and high severity.

In order to demonstrate this fact in the practical example, the RA of the upper and lower bounds, $\mathrm{RA}_\alpha(S_u)$ and $\mathrm{RA}_\alpha(S_l^{\mathrm{TVaR}})$, respectively, were calculated, as well as the simulated RA for the modified total PVFCF under longevity risk, $\mathrm{RA}_\alpha(S_{\mathrm{MC}})$. The results are shown in Table 3 for 5 different values of α, as well as in Figure 4 for the whole range of possible values for α, i.e., for $\alpha \in]0, 1[$.

Table 3: RA for S_{MC}, S_l^{TVaR} and S_u, depending on α

	α = 70 %	α = 80 %	α = 90 %	α = 95 %	α = 99.5 %
$RA_\alpha(S_u)$	2108308	2551966	3209168	3781328	5335621
$RA_\alpha(S_{MC})$	1841675	2244546	2844133	3349283	4555727
$RA_\alpha(S_l^{TVaR})$	1784431	2159363	2714463	3197451	4508231

As expected, $\mathrm{RA}_\alpha(S_{\mathrm{MC}})$ is between the RA of the upper and lower bounds, $\mathrm{RA}_\alpha(S_u)$ and $\mathrm{RA}_\alpha(S_l^{\mathrm{TVaR}})$, respectively. Furthermore, the difference between $\mathrm{RA}_\alpha(S_{\mathrm{MC}})$ and $\mathrm{RA}_\alpha(S_l^{\mathrm{TVaR}})$ is small.

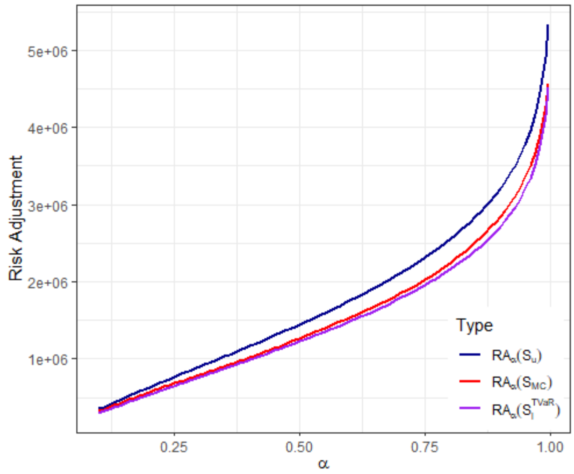


Figure 4: Graphical display of the RA for S_{MC}, S_l^{TVaR} and S_u, depending on α

There is an increasing behavior w.r.t. α of the 3 values of RA, and this growth is faster closer to 1. Therefore, the compliance of the Model for RA for Longevity Risk with criterion (a) is verified.

Moving on to criterion (b), risks with a longer duration, in other words, with a higher maximum contract boundary, must have a higher RA than risks with a shorter duration. The results are presented in Table 4 for 5 different values of the maximum contract boundary, T, as well as in Figure 5 for the whole range of possible values for T in this example, i.e., for $T \in \{1, \dots, 14\}$, and with $\alpha = 0.95$.

Table 4: RA for S_{MC}, S_l^{TVaR} and S_u, depending on T

	T = 2	T = 5	T = 7	T = 10	T = 14
$RA_\alpha(S_u)$	776732.5	2180234	2954457	3610795	3781328
$RA_\alpha(S_{MC})$	737230.3	2010077	2675993	3217855	3349283
$RA_\alpha(S_l^{TVaR})$	718166.6	1908841	2549039	3071523	3197451

The $\mathrm{RA}_\alpha(S_{\mathrm{MC}})$ remains bounded by the RA of the upper and lower bounds, $\mathrm{RA}_\alpha(S_u)$ and $\mathrm{RA}_\alpha(S_l^{\mathrm{TVaR}})$, respectively, showing again the convex order relation between them. The proximity between $\mathrm{RA}_\alpha(S_{\mathrm{MC}})$ and $\mathrm{RA}_\alpha(S_l^{\mathrm{TVaR}})$ is also maintained.

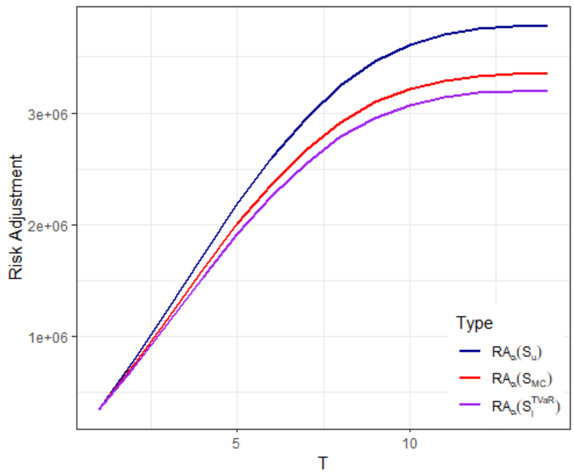


Figure 5: Graphical display of the RA for S_{MC}, S_l^{TVaR} and S_u, depending on T

The 3 values of RA show an increasing behavior, regardless of the value of T, demonstrating the fulfillment of criterion (b). The drop in the slope of the curves for higher years is justified by the type of product under study. As the annuity is only paid for individuals up to the age of 80, as the years go by more customers surpass this threshold and are no longer a liability for the insurance company, even if they are still alive.

Criterion (c) states that risks with higher volatility are associated with higher RA than risks with lower volatility. The results for 5 different values of the volatility parameter, σ, are shown in Table 5, as well as in Figure 5 for a range of possible values of σ up to 0.4, i.e., for $\sigma \in]0, 0.4]$, and with $\alpha = 0.95$.

Table 5: RA for S_{MC}, S_l^{TVaR} and S_u, depending on σ

	$\sigma = 0.006$	$\sigma = 0.05$	$\sigma = 0.15$	$\sigma = 0.25$	$\sigma = 0.35$
$RA_\alpha(S_u)$	3781328	32442986	115717176	226688113	366920042
$RA_\alpha(S_{MC})$	3349283	28549037	100502499	194922552	313491827
$RA_\alpha(S_l^{TVaR})$	3197451	27178495	95021110	183100397	293012814

Once again, the convex order relation is preserved between the 3 values of RA, regardless of the value of σ, highlighting the closeness between S_{MC} and S_l^{TVaR}.

The behavior of the RA w.r.t. the volatility parameter, σ, is similar to that of the variation of RA with α. For all 3 values of RA, there is an increasing trend, which becomes more pronounced as σ increases. This is explained mainly by the fact that higher levels of volatility are directly associated with a higher frequency of cases with greater severity. It can be concluded that the Model for RA for Longevity Risk complies with criterion (c).

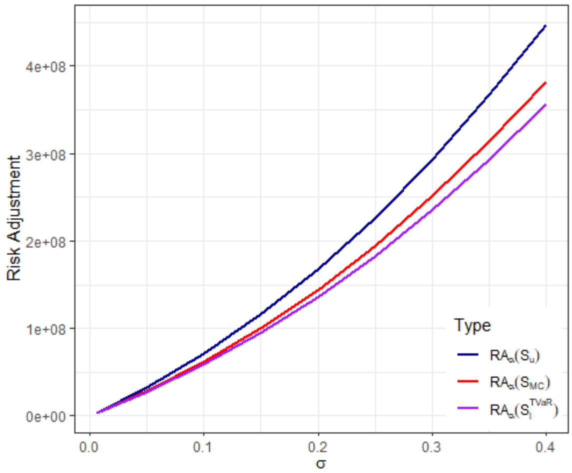


Figure 6: Graphical display of the RA for S_{MC}, S_l^{TVaR} and S_u, depending on σ

Finally, regarding to criteria (d) and (e), concerning the uncertainty of parameter estimation due to lack of information or experience, there is no way of graphically representing these situations in the practical example. However, one of the advantages of the Model for RA for Longevity Risk is the construction of an interval that bounds the RA for the modified total PVFCF under longevity risk, $\mathrm{RA}_\alpha(S)$, instead of presenting just a single exact value. It is known, and became clear from Figures 4, 5 and 6, that $\mathrm{RA}_\alpha(S_l^{\mathrm{TVaR}})$ serves as an approximation for $\mathrm{RA}_\alpha(S)$ but, this approximation may underestimate the value of the RA for the modified total PVFCF under longevity risk due to the lack of experience and uncertainty associated with estimating the parameters. In these situations, it is preferable to use a value within the interval defined in Theorem 3.2.3 that is higher than the lower bound, in order to better approximate $\mathrm{RA}_\alpha(S)$.

According to the details presented above for each criterion, it can be concluded that the Model for RA for Longevity Risk complies with the 5 criteria required in IFRS 17 (2023a, § B91).

From the analysis of Figures 4, 5 and 6, it can be incorrectly inferred that, although $\mathrm{RA}_\alpha(S_l^{\mathrm{TVaR}})$ approximates $\mathrm{RA}_\alpha(S)$, it greatly underestimates its real value. These figures were constructed on the basis of 10^3 simulations. Figure 7 shows the case in which 10^6 simulations are carried out to calculate the simulated RA, $\mathrm{RA}_\alpha(S_{\mathrm{MC}})$, for different values of the volatility parameter, σ. Figure 7 is there-

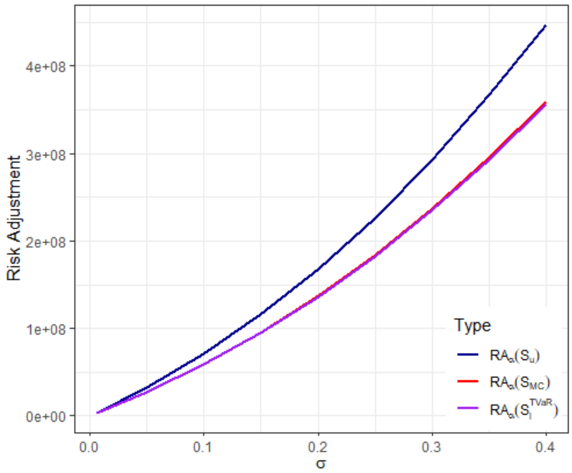


Figure 7: Graphical display of the upgrade of the RA for S_{MC}, depending on σ

fore a version of Figure 6, but with 1000 times more simulations to calculate $\mathrm{RA}_\alpha(S_{\mathrm{MC}})$.

It can be seen that the curves of $\mathrm{RA}_\alpha(S_{\mathrm{MC}})$ and $\mathrm{RA}_\alpha(S_l^{\mathrm{TVaR}})$ are practically overlapping, corroborating the conclusion that the lower bound applying the "TVaR_α-based" Approximation is the one that best approximates $\mathrm{RA}_\alpha(S)$.

It was decided not to use such a large number of simulations for the construction of Tables 3, 4 and 5 and Figures 4, 5 and 6 for the simple reason that it would be quite difficult to demonstrate the convex order relation inherent to the Model for RA for Longevity Risk due to an almost total overlap of curves, as is the case in Figure 7.

4.2.3 Volatility Estimation and Normality Tests

It is necessary to calculate the MLE of the volatility parameter, σ. For this purpose, the tables of the portfolio's historical exposure and historical deaths are used to obtain the historical survival rates and, consequently, the MLE of the volatility parameter, σ.

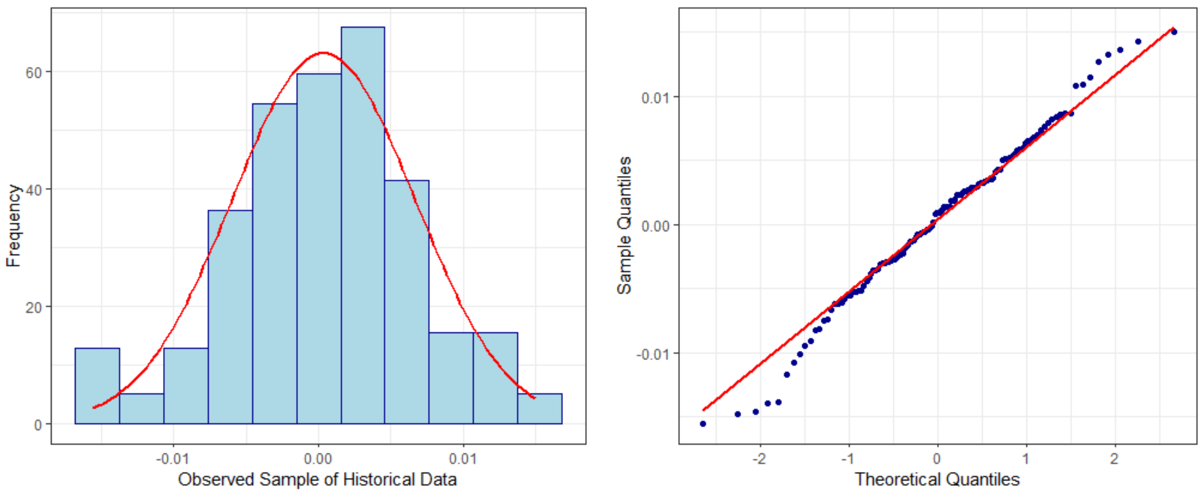
In the practical example, there is a 7-year history associated with the annuity under study, between 2015 and 2022, resulting in a sample of 126 ratios between the survival rates and the survival probabilities. Using Theorem 3.1.7, the MLE of the volatility parameter, σ, of the portfolio under study is $\hat{\sigma} = 0.006298827$. This estimate has already been used to construct Tables 3 and 4 and Figures 4 and 5, and it is one of the 5 values applied in Table 5 for the volatility parameter, σ.

However, it is not enough to estimate the volatility parameter, σ. As already mentioned in Chapter 3, to be realistic, the model must verify the normality assumption of the logarithm of the ratio between the survival rates and the survival probabilities. Only in this case, it is possible to be comfortable with the results obtained by the Model for RA for Longevity Risk.

To this end, several goodness-of-fit tests were carried out, as described in Chapter 3. The results are summarized in Table 6. As can be seen, the p-values of all the tests are significantly higher than 0.05, so the null hypothesis of normality is not rejected.

Table 6: p-values of the Sample Normality Tests

	p-value
Shapiro-Wilk's test	0.4721083
Anderson-Darling test	0.6014843
Kolmogorov-Smirnov test	0.6587264
Jarque-Bera test	0.7254813
D'Agostino Omnibus test	0.6527195
D'Agostino Skewness test	0.4062939
D'Agostino Kurtosis test	0.6858668



(a) Histogram of the Sample and Density of the Normal Distribution

(b) Q-Q plot of the Sample against the Normal Distribution

Figure 8: Additional Graphs to Prove Sample Normality

In addition to the goodness-of-fit tests, Figure 8 shows: the histogram of the sample overlapped by the probability density curve of a normal distribution; the Q-Q plot of the sample against the normal distribution. It can be seen that there is a significant fit between the sample and the normal distribution.

According to the goodness-of-fit tests and the additional graphs, the assumption of normality is satisfied, making it possible to apply the Model for RA for Longevity Risk.

4.2.4 Comparison of the Risk Adjustment Techniques

The 3 RA techniques will be applied to calculate the RA in the practical example under the same conditions. In this example, the expected modified total PVFCF under the longevity risk is equal to $\mathbb{E}[S] = 147911334$. In order to compare the value of the RA according to the 3 techniques, the RA of the upper and lower bounds, $\mathrm{RA}_\alpha(S_u)$ and $\mathrm{RA}_\alpha(S_l^{\mathrm{TVaR}})$, respectively, are calculated, as well as a total of $2 \cdot 10^7$ simulations are performed to calculate the RA for the modified total PVFCF under longevity risk using the Monte Carlo Method, $\mathrm{RA}_\alpha(S_{\mathrm{MC}})$. An instantaneous permanent shock of a decrease in 20% of the mortality rates used to calculate the PVFCF is also applied to assess the RA under SII, denoted by $\mathrm{RA}_\alpha(S_{\mathrm{SII}})$, as stated in SII (2015, § 138(1)). The VaR will not be studied in this example, since it is a quantile technique that is less sensitive to the tails of distributions when compared to the TVaR, as described in Chapter 1, and the convex order relations do not preserve the order of this risk measure, making it not straightforward to construct an interval that bounds it, as mentioned in Chapter 3.

The values of the RA are shown in Table 7 for $\alpha = 99.5\%$, as well as the percentage they represent of the expected modified total PVFCF under the longevity risk, $\mathbb{E}[S]$. For that, the MLE of the volatility parameter calculated above, $\hat{\sigma} = 0.006298827$, and the maximum contract boundary of the portfolio, i.e., $T = 14$ years, were applied. The calculation of the RA under SII assumes that $\alpha = 99.5\%$ and the methods used by the insurance sector to obtain the RA for other values of α are based on VaR and normal distribution characteristics. Hence, these values were not used to calculate the RA under SII because their results would not be comparable with those of the Model for RA for Longevity Risk and TVaR using the Monte Carlo Method.

Table 7: $\mathrm{RA}_\alpha(S)$ by the 3 Calculation Techniques

	α = 99.5 %	% of PVFCF
$RA_\alpha(S_{SII})$	6976992	4.71701 %
$RA_\alpha(S_u)$	5335621	3.60731 %
$RA_\alpha(S_{MC})$	4508296	3.047972 %
$RA_\alpha(S_l^{TVaR})$	4508231	3.047928 %

As expected, $\mathrm{RA}_\alpha(S_{\mathrm{MC}})$ remains between the RA of the upper and lower bounds, $\mathrm{RA}_\alpha(S_u)$ and $\mathrm{RA}_\alpha(S_l^{\mathrm{TVaR}})$, respectively. Once again, the proximity of $\mathrm{RA}_\alpha(S_{\mathrm{MC}})$ and $\mathrm{RA}_\alpha(S_l^{\mathrm{TVaR}})$ is good. However, the RA calculated under SII, $\mathrm{RA}_\alpha(S_{\mathrm{SII}})$, has a much higher value than the others, being over 1% higher than the RA of the upper bound, $\mathrm{RA}_\alpha(S_u)$, when comparing the percentage they represent of $\mathbb{E}[S]$, as shown in the second column of Table 7.

The RA calculated according to the quantile technique TVaR using the Monte Carlo Method, $\mathrm{RA}_\alpha(S_{\mathrm{MC}})$, is the most accurate result for the RA of the 3 techniques applied in the practical example and so it will be the reference technique. Because it is based on a very large number of simulations, this technique is more sensitive to skewness in the tails of the distributions. However, this technique is time consuming and requires more computational power as the number of simulations or the complexity of the model to be simulated increase, taking several times longer to obtain results when compared to the other 2 techniques.

On the other hand, the original Model for RA for Longevity Risk calculates an interval that bounds the RA for the modified total PVFCF under longevity risk, $\mathrm{RA}_\alpha(S)$. As can be seen in Table 7, the interval obtained by this technique contains $\mathrm{RA}_\alpha(S_{\mathrm{MC}})$. As already mentioned, the RA of the lower bound, $\mathrm{RA}_\alpha(S_l^{\mathrm{TVaR}})$, is very close to $\mathrm{RA}_\alpha(S_{\mathrm{MC}})$ and is therefore a good approximation of the exact value of $\mathrm{RA}_\alpha(S)$. The $\mathrm{RA}_\alpha(S_l^{\mathrm{TVaR}})$ has the advantage of being faster to calculate than the $\mathrm{RA}_\alpha(S_{\mathrm{MC}})$, thus proving the usefulness of the Model for RA for Longevity Risk for calculating the RA of the portfolio. As for the RA of the upper bound, $\mathrm{RA}_\alpha(S_u)$, it is significantly higher than $\mathrm{RA}_\alpha(S_l^{\mathrm{TVaR}})$ and $\mathrm{RA}_\alpha(S_{\mathrm{MC}})$, as was expected since it is the maximum value to be considered for the RA. In case the insurance company considers that the RA of the lower bound, $\mathrm{RA}_\alpha(S_l^{\mathrm{TVaR}})$, underestimates the value of the $\mathrm{RA}_\alpha(S)$, a higher value contained in the interval calculated by the Model for RA for Longevity Risk can be considered.

The RA under SII has a much higher value than the RA of the upper bound, $\mathrm{RA}_\alpha(S_u)$, which is the highest value that the RA for the modified total PVFCF under longevity risk, $\mathrm{RA}_\alpha(S)$, can have. This means that the RA under SII greatly overestimates the value that the $\mathrm{RA}_\alpha(S)$ can have in the worst case scenario, which is why it is not a good technique to apply in this portfolio. This practical example does not prove that the RA under SII overestimates the $\mathrm{RA}_\alpha(S)$ independently of the portfolio analyzed, but rather it shows that this technique is not sensitive to the historical characteristics and volatility of the portfolio. In this example, the RA under SII is much higher than the $\mathrm{RA}_\alpha(S)$ since the MLE for the volatility parameter is low, which means that the shock predefined by SII is excessive compared with the actual risk of the portfolio. If the value of the volatility parameter were very high, leaving the other conditions unchanged, the RA under SII would be the same, but much lower than $\mathrm{RA}_\alpha(S)$, meaning that the predefined shock would underestimate the actual risk of the portfolio. This phenomenon occurs because the RA under SII is a technique that only uses the first 2 inputs associated with the Model for RA for Longevity Risk, neglecting the importance that the historical behavior of exposure and deaths have for the uncertainty inherent to the portfolio. Therefore, it is questionable whether or not the technique of the RA under SII complies with criteria (c), (d) and (e) stated in IFRS 17 (2023a, § B91).

In conclusion, the technique that calculates the most accurate value for RA in this practical example is the quantile technique TVaR using the Monte Carlo Method, $\mathrm{RA}_\alpha(S_{\mathrm{MC}})$, due to its sensitivity to the tails of the distribution of the portfolio, as a result of the large number of simulations performed. However, the Model for RA for Longevity Risk is a much faster technique to calculate the RA when compared to the $\mathrm{RA}_\alpha(S_{\mathrm{MC}})$ and its result for the RA of the lower bound, $\mathrm{RA}_\alpha(S_l^{\mathrm{TVaR}})$, is nearly the same, as can be seen in Table 7. Furthermore, if the $\mathrm{RA}_\alpha(S_l^{\mathrm{TVaR}})$ value is considered to be an underestimate of $\mathrm{RA}_\alpha(S)$ by the insurance company, the Model for RA for Longevity Risk allows a higher value to be chosen as long as it is contained in the bounding interval. Thus, the Model for RA for Longevity Risk has a double advantage: it is a technique based on an analytical calculation which makes it faster to obtain a result for the RA; it produces an interval rather than an exact value, giving the company the possibility to choose a more prudent value depending on its risk aversion. Finally, the RA under SII leads to the worst performance of the 3 techniques in this example, mainly due to its lack of sensitivity to the historical behavior of the portfolio in terms of exposure and deaths.

Conclusion

The choice of technique and the calculation of the RA of the IFRS 17 is one of the major challenge for insurance companies. Since the standard does not prescribe the use of a specific technique for calculating the RA, companies are uncertain of which technique to implement, especially knowing from experience in application the disadvantages of the classical techniques. These uncertainties and disadvantages associated with existing methods create the opportunity to study new approaches or improvements to existing techniques for calculating the RA.

In this work, a Model for RA for Longevity Risk is developed as an original generalization of the Model for RA for Surrender Risk introduced by Carlehed (2023). The application of this new model produces an interval containing the RA of the portfolio that focuses on the volatility of the historical behavior of the portfolio in terms of exposure and deaths in comparison to what the insurance company expects. This approach provides the model with greater sensitivity to the characteristics of the portfolio when compared to the shocks predefined by the SII. On the other hand, the interval calculated by the model is obtained through an analytical formula, which makes it easier to apply and obtain a result when compared with the Quantile Techniques, that are time consuming and require computational power due to the use of simulation. The lower bound of this interval is a good approximation of the RA of the portfolio, yet the model constructs an interval giving the insurance company the possibility of choosing the most prudent value for the RA depending on its risk aversion.

The Model for RA for Longevity Risk has a number of advantages when compared to classical techniques, however, its application is quite restrictive and subject to strict assumptions. As such, this new model has a lot of space for improvement, so the following three recommendations are left for possible future research: extend the assumption of the normality of the logarithm of the ratio between survival rates and survival probabilities to other elliptical distributions, based on the work of Valdez et al. (2009); generalize the model when the interval of initial ages is subdivided into subsets of consecutive initial ages so as to obtain a volatility parameter for each subset, in cases where a single volatility parameter for all initial ages would compromise the verification of the assumption of the normality; build a Model for RA for Mortality Risk using the same methodology as for the Model for RA for Longevity Risk, and then obtain an Aggregate Model for RA for Mortality and Longevity Risks that reflects the level of diversification benefit.

Finally, the improvements in the techniques for calculating the RA presented in this work are particularly relevant for insurance companies, helping them deciding which technique to apply, as well as for future discussions on the review of the IFRS 17 or the development of supporting legislation by the authorities responsible for the sector worldwide.

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