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REINSURANCE OPTIMIZATION: MINIMIZING THE RUIN PROBABILITY FOR BOTH CEDENT AND REINSURER

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ABSTRACT

This study considers a reinsurance market with two participants, one taking the role of first line insurer and the other taking the role of reinsurer. The two firms have conflicting interests, in the sense that they both seek to absorb the highest proportion of the insurance premium possible, while also taking on the least amount of risk they can. As such, we consider a game where the first line insurer and the reinsurer aim to minimize their respective ruin probabilities. We define the surplus processes for each involved party and derive a set of integro-differential equations that describe the behaviour of their ruin probabilities. Numerical illustrations for this model are provided. Then, we present the Pareto equilibrium conditions for this market and suggest one possible approach to implement the numerical solution of the problem.

KEYWORDS: Reinsurance; Ruin Probability; Pareto Equilibrium; Risk Management; Risk Theory.

JEL: C61; C63; G22.

RESUMO

Este estudo considera um mercado de resseguro com dois participantes, um com o papel de segurador direto e outro com o papel de ressegurador. Existe um conflito de interesses entre as duas seguradoras, no sentido em que ambas procuram absorver a maior proporção do prémio de seguro ao seu alcance, tomando, contudo, a menor quantidade de risco possível. Assim, consideramos um jogo em que o segurador direto e o ressegurador têm por objetivo a minimização das suas respetivas probabilidades de ruína. São definidos processos de riqueza para cada uma das partes envolvidas e é derivado um conjunto de equações integro-diferenciais que descrevem o comportamento das probabilidades de ruína. Ilustrações numéricas deste modelo são oferecidas. Posteriormente, apresentamos as condições de equilíbrio de Pareto para este mercado e sugerimos uma possível abordagem para a implementação numérica da solução para este problema.

PALAVRAS-CHAVE: Resseguro; Probabilidade de Ruína; Equilíbrio de Pareto; Gestão de Risco; Teoria do Risco.

JEL: C61; C63; G22.

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I dedicate this work to my late Grandmother Irene, who I never got to meet, but whose yearning for schooling, albeit unfulfilled, fostered a respect for education that persevered through generations and motivates me every day to seize all the learning opportunities that I have the privilege of being able to access today.

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α	Ceded proportion of each claim
$\bar{V}_i$	Probability of ruin of the first line insurer in a infinite time horizon, as a function of the initial wealths of the first line insurer and the reinsurer
$\bar{V}_r$	Probability of ruin of the reinsurer in a infinite time horizon, as a function of the initial wealths of the first line insurer and the reinsurer
γ	Reinsurance premium
λ	Intensity of the Poisson process for counting claims per unit of time
τ^i	Time of ruin of the first line insurer
τ^r	Time of ruin of the reinsurer
c	Direct insurance premium
$F(\cdot)$	Probability distribution function of the severity of claims
$f(\cdot)$	Probability density function of the severity of claims
N_t	Poisson process for counting claims
r	Minimum income demanded by the reinsurer
T_i^*	Time between arrival of the of the $(i-1)-th$ and the $i-th$ claim
T_i	Time of arrival of the $i-th$ claim
x^i	Initial wealth of the first line insurer
X_t^i	First line insurer surplus process
x^r	Initial wealth of the reinsurer
X_t^r	Reinsurer surplus process
Y_i	Severity of the $i-th$ claim

1 INTRODUCTION

A reinsurance treaty is a contract established between two insurance firms, where the direct insurer - also called cedent or first line insurer - transfers part of its risk, obtained through underwriting policies for other companies and individuals, to a separate insurer, called the reinsurer. To do so, the cedent pays a premium to the reinsurer.
There are several reasons that an insurer and a reinsurer would find it beneficial to engage in a such a contract. By transferring part of the risk to an outside entity, the insurer is able to substitute future uncertain losses by a certain fixed payment, thus reducing their risk exposure and consequently stabilizing business results and reducing capital requirements. This frees up capacity for the insurer (Albrecher, 2017; Cummins et al, 2021). It can also be the case that solvency capital requirements or business goals lead the insurer to need to transfer part of its risk to multiple reinsurers (Zeng & Luo, 2013). In fact, the decision to enter into a reinsurance contract can be regarded as a capital structure decision by the insurer, since these solvency capital requirements can be achieved either through raising capital or reducing risk, which can be accomplished through risk transfer to a reinsurer (Cummins et al, 2021). Garven & Tennant (2003) have also pointed out that reinsurance may be used to reduce agency costs, by unlevering the insurance company. On the other hand, there may be some fiscal advantages in reinsurance, if reserve taxation is strong enough that it becomes attractive to use them to pay premiums (Albrecher, 2017).
Typically, the reinsurers are larger players in the market that are able to take on more complex and severe risks with proportionally lower amounts of required capital, because their dimension provides them with more diversification capacity and expertise in handling these kinds of risks. As such, insurers can use reinsurance to protect themselves against tail and model risk and also to benefit from some of the reinsurer know-how (Drexler & Rosen, 2022). In addition, reinsurance is a means of protection against catastrophic risk and of limitation of liability of the insurer against specific risks that they may not be equipped to take on (Cummins et al, 2021). Furthermore, Cummins et al (2021) point out that reinsurers are more capable of raising capital in a timely fashion, compared to the relatively small insurers, thus granting primary insurers the opportunity to transition out of hard markets more swiftly and dampening the insurance market cycles. Overall, the privileged position that a reinsurer holds in the market allows it to profit from underwriting risks that first line insurers may find too dangerous to hold themselves. This may have a positive social impact by making insurance more accessible and affordable and by allowing more risks to actually be insurable (Albrecher, 2017).
However, reinsurance also comes with associated costs. The insurer has to pay a premium for the risk transfer, therefore reducing the net income from the policies it has underwritten. This premium depends not only on the distribution of the risk itself, from which the

actuarial price can be derived, but also on the safety loading that the reinsurer charges, as well as the market conditions (Bühlmann, 1980, 1984). This often results in reinsurance premiums quite above the actuarial price of the risk, if there is a shortage of capital for reinsurers or agency problems (Cummins et al 2021). On the other hand, these treaties require sharing of information that may pose issues if the cedent and the reinsurer at some point become competitors (Borch, 1960).

If the benefits of reinsurance outweigh its costs both for first line insurer and reinsurer, they may choose to craft a policy for a risk transfer between them. This contract will define all conditions under which the reinsurer would partially pay for claims that arise from policies originally underwritten by the cedent. It will also specify the reinsurance premium and the form of reinsurance provided, namely if it is a proportional or non-proportional type of reinsurance, and the respective associated parameters and/or thresholds. The simplest type of proportional reinsurance is the quota-share treaty, where for each claim, the direct insurer pays a certain agreed upon proportion and the reinsurer pays the remaining part. On the non-proportional types of reinsurance, the most simple case is the excess of loss reinsurance, where the reinsurer will pay the amount of a claim above a minimal threshold. This kind of agreement can be complemented with a ceiling, that will consist of the maximum value that the reinsurer will pay for a claim. In a stop-loss agreement, the individual claims are taken in aggregate terms and the reinsurer covers the amount above a specified level for the aggregate quantity.

In order to specify the terms and conditions of a reinsurance treaty, the involved parties will engage in bargaining, where both first line insurer and reinsurer will attempt to optimize their own results, so that they can seize the best possible deal. This point will be further developed in Section 2, where a brief literature review is presented.

In this work, we consider a market with two participants - a direct insurer and a reinsurer - that will enter into a quota-share reinsurance treaty and that will negotiate conditions such that each participant minimizes its own probability or ruin. In order to do so, we define the surplus processes for each competitor in the market and then study the probability of these processes achieving negative values. In Section 3, the models for the surpluses of the first line insurer and reinsurer, which are obviously dependent, are presented and a set of three integro-differential equations that describe the behaviour of their ruin probabilities are derived, one of them having an explicit boundary condition and the other two having implicit boundary conditions. The last part of this section is dedicated to the numerical implementation of the model and presents some numerical examples. Section 4 offers some equilibrium conditions for an optimal Pareto strategy, where neither participant is able to improve its ruin probability without increasing the ruin probability of the other part. A possible algorithm for the solution of these conditions is put forward. The last section includes some conclusions, limitations of the model and discusses future work.

2 LITERATURE REVIEW

Early studies on optimal reinsurance tend to consider only the interests of the first line insurer when designing policies. However, a reinsurance treaty is a contract between two parties that must agree on its terms and conditions. As such, the (conflicting) interests of both direct insurer and reinsurer should be considered when suggesting an optimal policy. The idea to do so seems to come originally from Borch (1960), who argued that, under the assumption that no party is forced to enter an agreement, any reinsurance treaty should be such that both the cedent and the reinsurer benefit from it. The inclusion of the reinsurer as a more active partner in the theoretic models, creates a necessity for considering bargaining between direct insurer and reinsurer. Though both have common objectives for the management of the shared risk, there will also be some degree of conflict as both will aim to capture the highest amount of the premium possible while taking on the least risk they can (Kaishev, 2004; Yang & Chen, 2022). Furthermore, Kaishev & Dimitrova (2006) state that the increase in severity and frequency of losses due to catastrophic events over the years means that the insolvency risk of the reinsurer cannot be ignored.

There are several ways to measure and represent the conflicting interests of cedent and reinsurer, the most common being utility functions, probability of ruin, and risk variance (Cai et al, 2013).

Utility theory is widely used as a way to represent preferences in financial and economic problems through utility functions. Zeng & Luo (2013) study a problem where the optimal reinsurance treaty maximizes a weighted sum of the utilities of both the first line insurer and the reinsurer, thus being able to take in consideration the interests of both participants while also attributing relative relevance to each one through their respective weights on the utility function. Chen & Shen (2018) create a Stackelberg game scenario where first line insurer and reinsurer maximize their respective utilities, given the actions that will be taken by their competitor. Suijs et al (1998) argue that insurance games, in the non-life lines of business, can be modelled using a cooperative games' framework, that they apply also using utility theory. Borch (1960) tackles the optimal reinsurance problem by maximizing the joint gain, measured by the product of the increase of utility with reinsurance compared to the initial situation for both parties. While utility functions encapsulate not only preferences but also the attitude towards risk of market participants, allowing for power dynamics in the market to be reflected by the differences in this attitude between the cedent and the reinsurer, Borch (1960) recognizes that a limitation of the model he presents is that it doesn't consider the possibility of ruin of the reinsurer.

Ruin theory stems from considerations about the solvency of insurers and studies the level of surplus associated to a portfolio of insurance policies (Dickson, 2017). It is said that the company is ruined if the surplus reaches a negative value. Some studies aim to minimize some measure reflecting the joint ruin probability of insurers and reinsurers or, equivalently, maximize their joint survival probability. Fang & Qu (2014) look for a reinsurance treaty combining quota-share and stop-loss policies that maximizes the probability of both contract participants surviving. A second optimization measure is required so that the two retention parameters are uniquely estimated. For this second criteria, the authors suggest the minimization of a loss function of insurer and reinsurer, that is defined using their respective VaR. Kaishev & Dimitrova (2006) provide explicit solutions for an excess of loss reinsurance policy with a limiting level that maximizes the probability of joint survival of the cedent and the reinsurer up until a finite time horizon. They take two alternative approaches that also consider the split of the first line insurance premium, besides the parameters of the risk transfer portion of the reinsurance treaty. Either the split is fixed and the parameters are optimized or, conversely, the parameters are fixed and the split is optimized. In the 2010 sequel to this paper, Dimitrova & Kaishev (2010) add a performance measure, given by the expected profits at a finite time horizon given the joint survival until then, with the goal of bringing together the conflicting goals of profit maximization and risk management. On this line of thought, Cai et al. (2013) maximize the joint survival probability and the joint profitable probability, defined as the probability that the portion of the claim paid by each participant is less than their respective premium income.

The individual probability of ruin of the first line insurer or of the reinsurer can also be optimized, rather than a joint probability. Chen et al (2019) create a setup where the cedent seeks to minimize its ruin probability while the reinsurer maximizes the present value of its profits until the time of ruin of the insurer. This type of approach stems from the argument that first line insurers focus on risk management, whereas reinsurers focus on profitability. This analysis suggests that the reinsurance premium is a feedback controller that switches between two states, depending on the level of cash reserves of the reinsurer. Other approaches include, for instance, the maximization of the expected value of a terminal payoff combined with the minimization of its variance for both participants, which is called the mean-variance criterium (Yang & Chen, 2022); and the minimization of a weighted Conditional Tail Expectation (Bazaz & Najafabadi, 2015). Zeng (2010) studies a problem where direct insurer and reinsurer compete by one trying to minimize an expected payoff that the other tries to maximize. Balbas et al (2013) take a risk sharing approach in a market composed of n insurers that enter in reinsurance contracts among themselves in order to reduce non-systemic risk, through vector optimization under a general risk measure. One limitation of the proposed model is that the method can give solutions that result in a significant increase of the probability of ruin. To address this concern, the authors suggest that their method is complemented with some control of the ruin probability in order to assess if a more conservative risk measure should be used in the optimization procedure.

As we have seen, even when the problem of optimal reinsurance is analysed through lenses other than those of risk theory, several authors emphasize that the probability of ruin should not be overlooked. Dimitrova & Kaishev (2010) go as far as suggesting that reinsurance treaties should maximize the likelihood of survival for both cedent and reinsurer. As such, this work examines a market composed of two participants that aim to minimize their respective probability of ruin when entering a reinsurance contract. Furthermore, since we are modelling a market where each participant behaves in a way that attempts to optimize a measure of their interest, thus creating some level of conflict between them, we take a game theory approach to the problem. By doing this we hope to be able to find some market equilibrium that brings together the conflicting interests present in the market.

The use of game theory frameworks is common for the study of optimal reinsurance. Chen et al (2019) and Zeng (2010) search for Nash equilibria, *i.e.*, a state of the world where no competitor has any interest in changing their strategy while the other participants maintain their behaviour. Yang & Chen (2022), as well as Chen & Shen (2018), model this problem in the form of a Stackelberg game. This is a game where one player is considered the leader - in this case the reinsurer - and the others are considered followers - the first line insurer. The reinsurer determines its pricing strategy based on the reaction function of the first line insurer, who will choose a retention level. Knowing this, the cedent will adjust its reinsurance strategy according to the decisions made by the reinsurer. This type of approach is justified by the observation that the reinsurance market is dominated by very large companies, while the first line insurance market presents higher levels of competitiveness. Morozov (1998) also uses a Stakelberg game framework, where the reinsurer selects the price of reinsurance but the cedent's strategy is based on a loss-ratio limit. Suijs et al (1998) opt to analyse the problem using cooperative games' theory. Cooperative games focus on the gains that can be obtained through coalitions, thus being more applicable in larger markets than the simplified version that we propose to study here. Nonetheless, these authors argue that non-life reinsurance problems can be well modelled by cooperative games with stochastic payoffs, when premiums are subadditive, and they seek for Pareto optimal allocations of risk and respective premiums. Another proponent of the game theory approach to reinsurance optimization is Aase (2002), who argues that bid-ask spreads on premiums - which are considered to derive from transaction costs and information asymmetry - can be derived from looking at the core of a reinsurance game.

In this study we investigate Pareto optimal strategies, where the insurer chooses the percentage of each claim it wants to cede to the reinsurer, while the latter selects a pricing strategy.

3 PROBLEM SETTING

This work considers a simplified insurance market with only two participants: a first line insurer and a reinsurer. In this section, we define the problem of optimal reinsurance in a setting where both participants wish to minimize their ruin probability or, equivalently, maximize their survival probability. In this context, a participant is in a situation of ruin when its respective surplus process goes below zero, regardless of the value of the surplus deficit. As Albrecher (2017) points out, this is not necessarily a bankruptcy situation, but is a useful and intuitive criterion of risk. The surplus processes of the reinsurer and of the insurer are defined as follows, respectively:

$$X_t^r = \begin{cases} X_t^{0,r} = x^r + (\gamma - r)t - \alpha \sum_{i=1}^{N_t} Y_i, & t \leq \tau^i, \\ X_t^{1,r} = X_{\tau^i}^{0,r} - r(t - \tau^i), & t > \tau^i, \end{cases} \tag{1}$$

$$X_t^i = \begin{cases} X_t^{0,i} = x^i + (c - \gamma)t - (1 - \alpha) \sum_{i=1}^{N_t} Y_i, & t \leq \tau^r, \\ X_t^{1,i} = X_{\tau^{0,r}}^{0,i} + X_{\tau^{0,r}}^{0,r} + c(t - \tau^{0,r}) - \sum_{\tau^{0,r} < T_i \leq t} Y_i, & t > \tau^r, \end{cases} \tag{2}$$

where $x^r, x^i \geq 0$ are the values of the initial surplus of reinsurer and direct insurer, respectively; γ is the premium received by the reinsurer and paid by the insurer; r is a minimum income demanded by the reinsurer to remain in the business, including fixed costs and remuneration of capital; c is the premium received by the insurer; α is the proportion of each claim that the insurer cedes to the reinsurer ($0 \leq \alpha \leq 1$), and τ^r, τ^i are the times of ruin of the reinsurer and direct insurer, respectively. The premiums γ and c are assumed to be continuously received, for simplicity. N_t is the counting process of claims, assumed to be a Poisson process with intensity λ, and Y_i is the severity of the $i-th$ claim. We assume that the random variables Y_i are independent and identically distributed, and that $E[Y]$ is finite. In this context $S_t = \sum_{i=1}^{N_t} Y_i$ is a compound Poisson process. The time of arrival of the $i-th$ claim to the system is a random variable T_i and the times between claim arrivals are independent exponentially distributed random variables, T_i^*, with parameter $\frac{1}{\lambda}$.
These processes interact with each other reflecting the dynamic of this 2-participant market. At time 0 both insurer and reinsurer are present in the market and a reinsurance agreement is in place such that there is a continuous income for both participants, given by the net premium that each one receives. For each claim that arises, the insurer covers a proportion $1 - \alpha$, while the reinsurer covers the remaining proportion α of the claim.

The implications of an eventual claim that causes one of the participants to reach ruin are different, depending on which participant has its surplus going below 0. If the cedent is the first to reach ruin, then the reinsurer no longer receives its premium income and they cease to provide any risk coverage. It uses its accumulated surplus to provide the mandatory minimal return until it also reaches a ruin situation, which, under these conditions, happens with probability 1. Alternatively, this situation can be viewed as one where the reinsurer loses its market and therefore closes its business, keeping what remains of its capital to invest in some other project or asset. For the sake of simplicity, in this work, "ruin of the reinsurer" designates the event where the reinsurer quits the market, be it by ruin in the proper sense or by impossibility of meeting the minimum return requirement. If the reinsurer is the first to attain a surplus level below 0, the direct insurer first covers the part of the ceded risk that the reinsurer is not able to pay and then continues to provide risk coverage by taking on the entirety of every new claim that happens from that moment on.

Chen et al (2019) argue that, when entering a reinsurance contract, the insurer is motivated by risk mitigation, which is consistent with a minimization of probability of ruin approach, but the reinsurer is motivated by profitability motives. Although this work focuses on risk mitigation for both participants, the profitability of the reinsurer is considered through the inclusion of the parameter r. Note as well that this is reflected on the different consequences of ruin events for insurer and reinsurer. When the insurer becomes ruined first, the reinsurer no longer provides any risk coverage, while when the reinsurer is the first to hit negative surplus, the insurer still takes on the risk of further claims.

Some conditions are required, in order to guarantee that each participant is somewhat solvent, in the sense that their ruin probability is not 1 regardless of their initial surplus. In order to achieve this, the net premium of each participant must exceed the expected payment of claims (Dickson, 2017), so the following must be satisfied:

$$\gamma - r > \alpha \lambda E[Y], \tag{3}$$

$$c - \gamma > (1 - \alpha) \lambda E[Y]. \tag{4}$$

Conditions (3) and (4) are assumed to be verified from this point on. As a consequence, $c - r > \lambda E[Y]$, intuitively meaning that, under these conditions, it is guaranteed that the market as whole has a non-zero probability of survival. Furthermore, the original insurance contract should be profitable on its own, without risk sharing through reinsurance, and, as such, $c > \lambda E[Y]$.

3.1 Ruin Probability of Insurer and Reinsurer

We are interested in studying the probability that there is a moment when the surplus processes go below the level 0. It is of interest to define those times for both surplus processes and for each of their branches, to account for ruin in every possible situation. The general times of ruin for reinsurer and insurer are, respectively:

$$\tau^r = inf\{t > 0 : X_t^r < 0\}, \tag{5}$$

$$\tau^i = inf\{t > 0 : X_t^i < 0\}. \tag{6}$$

However, ruin for one participant can happen before or after the ruin of the other participant. If the cedent achieves ruin before the reinsurer, the time of ruin of the cedent is $\tau^{0,i} = inf\{t > 0 : X_t^{0,i} < 0\}$ and the time of ruin of the reinsurer is $\tau^{1,r} = \tau^{0,i} + \frac{X^{0,r}_{\tau^{0,i}}}{r}\chi_{\tau^{0,i}<\infty}$. In this event, the reinsurer certainly hits negative surplus in finite time, given that it no longer receives premium income and so it cannot go on fulfilling the minimal return r indefinitely. If the reinsurer achieves ruin before the direct insurer, the time of ruin for the reinsurer is $\tau^{0,r} = inf\{t > 0 : X_t^{0,r} < 0\}$ and for the direct insurer is $\tau^{1,i} = inf\{t > \tau^{0,r} : X_t^{1,i} < 0\}$.

It is possible to study the probability of ruin in a finite horizon, *i.e.* before a specific time t, or in an infinite horizon, *i.e.* at some time $t < \infty$. We choose to study the problem in an infinite time horizon, as the optimization of probability of ruin in a finite time may result in solutions that ensure safety until the finite horizon but that are unsustainable thereafter. The event of ruin is complementary to the event of survival. Thus, the ruin probability can be expressed in terms of the probability of survival events, which motivates the following definitions.

Definition 3.1. (Relevant quantities for the calculation of the ruin probabilities): Let V_1 be a function such that $V_1 : \mathbb{R} \to [0, 1]$ and

$$V_1(z) = P\{\tau^{1,i} = +\infty | X^{1,i}_{\tau^r} = z\}. \tag{7}$$

Furthermore, let V_2, V_3 be functions such that $V_i : \mathbb{R} \times \mathbb{R} \to [0, 1], i = 2, 3$, and

$$V_2(x^i, x^r) = P\{\tau^{0,i} = +\infty, \tau^{0,r} = +\infty | X_0^{0,i} = x^i, X_0^{0,r} = x^r\}, \tag{8}$$

$$V_3(x^i, x^r) = P\{\tau^{0,r} < \tau^{0,i}, \tau^{1,i} = +\infty | X_0^{0,i} = x^i, X_0^{0,r} = x^r\}. \tag{9}$$

The probabilities of ruin for the reinsurer and for the insurer can be expressed in terms of V_2 and V_3, as expressed in the next proposition.

Proposition 3.1. (Probabilities of ruin): The probability of ruin of the reinsurer is a function $\bar{V}_r : \mathbb{R} \times \mathbb{R} \to [0,1]$ such that

$$\bar{V}_r(x^i, x^r) = 1 - V_2(x^i, x^r). \tag{10}$$

Moreover, the probability of ruin of the insurer is a function $\bar{V}_i : \mathbb{R} \times \mathbb{R} \to [0,1]$ such that

$$\bar{V}_i(x^i, x^r) = 1 - V_2(x^i, x^r) - V_3(x^i, x^r). \tag{11}$$

Proof. There are five different possible ruin events, listed below:

- Case 1: Both participants reach ruin in finite time and the insurer does so first
- Case 2: Both participants reach ruin in finite time and they do so at the same time
- Case 3: Both participants reach ruin in finite time and the reinsurer does so first
- Case 4: Only the reinsurer reaches ruin
- Case 5: No participant reaches ruin

The probability of case 4 is given by V_3 and the probability of case 5 is given by V_2. The reinsurer reaches ruin in all cases but case 5 and the insurer reaches ruin in all cases but 4 and 5. □

Note that, in our simplified market model, the reinsurer always faces higher or equal probability of ruin than the first line insurer. This results from the fact that the ruin of the direct insurer determines the ruin of the reinsurer. From the above proposition it is possible to see that the probability of ruin of the reinsurer is equal to the probability of ruin of the direct insurer plus a term, $V_3(x^i, x^r)$, that represents the probability of the reinsurer attaining a negative surplus while the insurer's surplus remains positive for all $t < \infty$.

In order to study the behaviour of $\bar{V}_r$ and $\bar{V}_i$, it is necessary to study the behaviour of V_2 and V_3. Then it will become apparent why we introduced the function V_1. We will prove a set of propositions establishing a system of integro-differential equations satisfied by the survival probabilities. The corresponding boundary conditions will be derived in Section 3.2. and then the system will be well posed. First of all it is necessary to prove that $V_i, i = 1, 2, 3$, are differentiable. To do so, we prove absolute continuity and the differentiability follows.

Proposition 3.2. (Absolute continuity for V_1): V_1 is absolutely continuous.

Proof. By definition,

$$V_1(z) = P\{\tau^{1,i} = +\infty | X^{1,i}_{\tau^r} = z\} = 1 - P\{\tau^{1,i} < +\infty | X^{1,i}_{\tau^r} = z\} = \\ = 1 - \sum_{n=1, T_n > \tau^r}^{+\infty} P\{\tau^{1,i} = T_n | X^{1,i}_{\tau^r} = z\}.$$

If we consider $\tau^r = T_m$, this is equal to

$$1 - \sum_{n=1}^{+\infty} P\{\tau^{1,i} = T_{m+n} | X^{1,i}_{T_m} = z\}.$$

This expression can be written as

$$1 - \sum_{n=1}^{+\infty} P\{X^{1,i}_{T_{m+1}} \geq 0, ..., X^{1,i}_{T_{m+n-1}} \geq 0, X^{1,i}_{T_{m+n}} < 0 | X^{1,i}_{T_m} = z\} =$$

$$= 1 - \sum_{n=1}^{+\infty} P\{X^{1,i}_{T_{m+1}} = z + c \times T^*_{m+1} - Y_{T_{m+1}} \geq 0, ..., \\ ..., X^{1,i}_{T_{m+n-1}} = z + c \sum_{i=m+1}^{m+n-1} T^*_i - \sum_{i=m+1}^{m+n-1} Y_i \geq 0, \\ X^{1,i}_{T_{m+n}} = z + c \sum_{i=m+1}^{m+n} T^*_i - \sum_{i=m+1}^{m+n} Y_i < 0 | X^{1,i}_{T_m} = z\}.$$

Writing the above expression in integral form, we get

$$1 - \sum_{n=1}^{+\infty} \int_{[0,+\infty[^n} \int_0^{z + c \times t^*_{m+1}} ... \int_0^{z + c \sum_{i=m+1}^{m+n-1} t^*_i - \sum_{i=m+1}^{m+n-2} Y_i} \int_{z + c \sum_{i=m+1}^{m+n} t^*_i - \sum_{i=m+1}^{m+n-1} Y_i}^{+\infty} \\ dF(Y_{m+n}) ... dF(Y_{m+1}) \lambda^n e^{-\lambda \sum_{i=1}^n t^*_{m+i}} dt^*_{m+n} ... dt^*_{m+1},$$

where $F(\cdot)$ is the distribution function of claim severities. Proceeding with the substitution $s = z + c \times t^*_{m+1}$, we obtain

$$1 - \sum_{n=1}^{+\infty} \int_z^{+\infty} \int_{[0,+\infty[^{n-1}} \int_0^s ... \int_0^{s + c \sum_{i=m+2}^{m+n-1} t^*_i - \sum_{i=m+1}^{m+n-2} Y_i} \int_{s + c \sum_{i=m+2}^{m+n} t^*_i - \sum_{i=m+1}^{m+n-1} Y_i}^{+\infty} \\ dF(Y_{m+n}) ... dF(Y_{m+1}) \lambda^n e^{-\lambda(\frac{s-z}{c} + \sum_{i=2}^n t^*_{m+i})} dt^*_{m+n} ... ds$$

$$= 1 - \sum_{n=1}^{+\infty} e^{\frac{\lambda z}{c}} \int_z^{+\infty} \int_{[0,+\infty[^{n-1}} \int_0^s ... \int_0^{s + c \sum_{i=m+2}^{m+n-1} t^*_i - \sum_{i=m+1}^{m+n-2} Y_i} \\ (1 - F(s + c \sum_{i=m+2}^{m+n} t^*_i - \sum_{i=m+1}^{m+n-1} Y_i)) dF(Y_{m+n}) ... dF(Y_{m+1}) \\ \lambda^n e^{-\lambda(\frac{s}{c} + \sum_{i=2}^n t^*_{m+i})} dt^*_{m+n} ... ds.$$

Since $e^{\frac{\lambda z}{c}}$ is an absolutely continuous function and the product and sum of absolutely continuous functions are absolutely continuous, the result is proved. □

Proposition 3.3. (Absolute continuity for V_2): Let

$$f_2(u) = V_2(z^i + (c - \gamma)u, z^r + (\gamma - r)u).$$

Then, $f_2(u)$ is absolutely continuous.

Proof. The proof for this proposition can be found in Appendix A. □

Proposition 3.4. (Absolute continuity for V_3): Let

$$f_3(u) = V_3(z^i + (c-\gamma)u, z^r + (\gamma - r)u).$$

Then, $f_3(u)$ is absolutely continuous.

Proof. By definition,

$$f_3(u) = P_u\{\tau^{0,r} < \tau^{0,i}, \tau^{1,i} = +\infty\} = \sum_{n=1}^{+\infty} P_u\{\tau^{0,r} = T_n < \tau^{0,i}, \tau^{1,i} = +\infty\},$$

where $P_u\{\cdot\} = P\{\cdot | X_0^{0,i} = z^i + (c-\gamma)u, X_0^{0,r} = z^r + (\gamma - r)u)\}$. The probability inside the sum can be written as

$$P_u\{\tau^{0,r} = T_n < \tau^{0,i}\} - \sum_{m=0}^{+\infty} P_u\{\tau^{0,r} = T_n < \tau^{0,i}, \tau^{1,i} = T_{n+m}\}.$$

On the proof of proposition 3.3., we showed that $P_u\{\tau^{0,r} = T_n < \tau^{0,i}\}$ is absolutely continuous. To prove the result, it is only necessary to prove that

$$P_u\{\tau^{0,r} = T_n < \tau^{0,i}, \tau^{1,i} = T_{n+m}\}$$

is also absolutely continuous.

$$P_u\{\tau^{0,r} = T_n < \tau^{0,i}, \tau^{1,i} = T_{n+m}\} =$$
$$P_u\{X_{T_1}^{0,i} \geq 0, X_{T_1}^{0,r} \geq 0, ..., X_{T_{n-1}}^{0,i} \geq 0, X_{T_{n-1}}^{0,r} \geq 0, X_{T_n}^{0,i} \geq 0, X_{T_n}^{0,r} < 0, X_{T_{n+1}}^{1,i} \geq$$
$$0, ..., X_{T_{n+m-1}}^{1,i} \geq 0, X_{T_{n+m}}^{1,i} < 0\}$$

Notice that $X_{\tau^{0,r}}^{0,i} + X_{\tau^{0,r}}^{0,r}$ equals, in this case, $z^i + z^r + (c-r)(u + \sum_{i=1}^n t_i^*) - \sum_{i=1}^n Y_i$. Defining $k(p,q,v) = z^i + z^r + (c-r)p + c \times q - v$, and using the variable substitution $s = u + t_1^*$, the previous expression is equal to,

$$\int_u^{+\infty} \int_{[0,+\infty[^{n-+m-1}} \int_0^{g(s,0)} \int_0^{g(s+t_2^*,Y_1)} \dots \int_0^{g(s+\sum_{i=2}^{n-1} t_i^*, \sum_{i=1}^{n-2} Y_i)}$$
$$\int_{h_r(s+\sum_{i=2}^n t_i^*, \sum_{i=1}^{n-1} Y_i)}^{h_i(s+\sum_{i=2}^n t_i^*, \sum_{i=1}^{n-1} Y_i)} \int_0^{k(s+\sum_{i=2}^n t_i^*, t_{n+1}^*, \sum_{i=1}^n Y_i)} \int_0^{k(s+\sum_{i=2}^n t_i^*, \sum_{i=n+1}^{n+m-1} t_i^*, \sum_{i=1}^{n+m-2} Y_i)}$$
$$\int_{k(s+\sum_{i=2}^n t_i^*, \sum_{i=n+1}^{n+m} t_i^*, \sum_{i=1}^{n+m-1} Y_i)}^{+\infty} dF(Y_{n+m})...dF(Y_1)\lambda^{n+m} e^{-\lambda(s-u+\sum_{i=2}^{n+m} t_i^*)} dt_{n+m}...ds =$$

$$= e^{\lambda u} \int_u^{+\infty} \int_{[0,+\infty[^{n-+m-1}} \int_0^{g(s,0)} \int_0^{g(s+t_2^*,Y_1)} \dots \int_0^{g(s+\sum_{i=2}^{n-1} t_i^*, \sum_{i=1}^{n-2} Y_i)}$$
$$\int_{h_r(s+\sum_{i=2}^n t_i^*, \sum_{i=1}^{n-1} Y_i)}^{h_i(s+\sum_{i=2}^n t_i^*, \sum_{i=1}^{n-1} Y_i)} \int_0^{k(s+\sum_{i=2}^n t_i^*, t_{n+1}^*, \sum_{i=1}^n Y_i)} \int_0^{k(s+\sum_{i=2}^n t_i^*, \sum_{i=n+1}^{n+m-1} t_i^*, \sum_{i=1}^{n+m-2} Y_i)}$$
$$(1 - F(k(s + \sum_{i=2}^n t_i^*, \sum_{i=n+1}^{n+m} t_i^*, \sum_{i=1}^{n+m-1} Y_i)))dF(Y_{n+m-1})...dF(Y_1)$$
$$\lambda^{n+m} e^{-\lambda(s+\sum_{i=2}^{n+m} t_i^*)} dt_{n+m}...ds,$$

similarly to results obtained in the proof of proposition 3.3.. Given that the first integrand does not depend on u, the function is absolutely continuous. □

Using the fact that these functions are absolutely continuous and therefore they are primitives of derivative functions defined almost everywhere, it is possible to construct a set of integro-differential equations that describe the behaviour of the relevant components of the ruin probabilities of the insurer and reinsurer. These equations are presented in propositions 3.5. to 3.7. and the respective boundary conditions will be derived in the next section.

Proposition 3.5. (Integro-differential equation for V_1): The function V_1 solves the following integro-differential equation:

$$V_1'(z) = \tfrac{\lambda}{c}(V_1(z) - \textstyle\int_0^z V_1(z-y)dF(y)), \qquad a.e.\ z \geq 0.$$

Proof. By definition, $V_1(z) = P\{\tau^{1,i} = +\infty | X^{1,i}_{\tau^r} = z\}$. The event of survival in an infinite time frame means that the firm survives each claim that arises in the system. Let T_i be the time of arrival of the $i-th$ claim. We have, for a general increment h,

$$V_1(z) = P\{\tau^{1,i} = +\infty, T_1 > h | X^{1,i}_{\tau^r} = z\} + P\{\tau^{1,i} = +\infty, T_1 \leq h < T_2 | X^{1,i}_{\tau^r} = z\} + o(h)$$

Using the memoryless property of the exponential distribution and noticing that if $T_1 > h$ the first claim has not yet occurred, this is equal to

$$\begin{gathered} P\{\tau^{1,i} = +\infty | X^{1,i}_{\tau^r} = z + ch\}e^{-\lambda h} + \\ + E[P\{\tau^{1,i} = +\infty | X^{1,i}_{\tau^r} = z + cT_1 - Y_1\}\chi_{z+cT_1-Y_1 \geq 0} \cdot \chi_{T_1 \leq h < T_2}] + o(h). \end{gathered}$$

Using $e^{-\lambda h} = 1 - \lambda h + o(h)$, the above expression is given by

$$\begin{aligned} &V_1(z+ch)(1-\lambda h) + \textstyle\int_0^h \int_{h-t_1}^{+\infty} \int_0^{z+ct_1} V_1(z+ct_1-y)dF(y)\lambda e^{-\lambda t_2}dt_2 \lambda e^{-\lambda t_1}dt_1 + o(h) = \\ &= V_1(z+ch) - \lambda V_1(z)h + \textstyle\int_0^h \int_0^{z+ct_1} V_1(z+ct_1-y)dF(y)e^{-\lambda(h-t_1)}\lambda e^{-\lambda t_1}dt_1 + o(h). \end{aligned}$$

For small values of h, we can write

$$V_1(z) = V_1(z+ch) - \lambda V_1(z)h + \lambda h \textstyle\int_0^z V_1(z-y)dF(y) + o(h).$$

Taking the limit as $h \to 0^+$,

$$\begin{aligned} 0 &= \lim_{h\to 0^+} \tfrac{V_1(z+ch)-V_1(z)}{h} - \lambda V_1(z) + \lambda \textstyle\int_0^z V_1(z-y)dF(y) = \\ &= cV_1'(z) - \lambda V_1(z) + \lambda \textstyle\int_0^z V_1(z-y)dF(y), \end{aligned}$$

which is equivalent to the proposed condition,

$$V_1'(z) = \frac{\lambda}{c}(V_1(z) - \int_0^z V_1(z-y)dF(y)),$$

whenever z is a Lebesgue point of V_1. $\square$

Proposition 3.6. (Integro-differential equation for V_2): The function $V_2(x^i, x^r)$ solves the following integro-differential equation

$$(\frac{\partial V_2}{\partial x^i}(c-\gamma) + \frac{\partial V_2}{\partial x^r}(\gamma - r))_{x^i,x^r} =$$
$$\lambda(V_2(x^i,x^r) - \int_0^{\min\{\frac{x^r}{\alpha},\frac{x^i}{1-\alpha}\}} V_2(x^i-(1-\alpha)Y, x^r-\alpha Y)dF(y)).$$

Proof. The reasoning for this proof is similar to the one used to prove Proposition 3.5. For a general increment h, we have

$$V_2(x^i,x^r) = P\{\tau^{0,i} = +\infty, \tau^{0,r} = +\infty | X_0^{0,i} = x^i, X_0^{0,r} = x^r\} =$$

$$= P\{\tau^{0,i} = +\infty, \tau^{0,r} = +\infty, T_1 > h | X_0^{0,i} = x^i, X_0^{0,r} = x^r\} + P\{\tau^{0,i} = +\infty, \tau^{0,r} = +\infty, T_1 \leq h < T_2 | X_0^{0,i} = x^i, X_0^{0,r} = x^r\} + o(h) =$$

$$= V_2(x^i + (c-\gamma)h, x^r + (\gamma - r)h)e^{-\lambda h} + E[P\{\tau^{0,i} = +\infty, \tau^{0,r} = +\infty | X_0^{0,i} = x^i + (c-\gamma)T_1 - (1-\alpha)Y_1, X_0^{0,r} = x^r + (\gamma - r)T_1 - \alpha Y_1\}\chi_{x^i+(c-\gamma)T_1-(1-\alpha)Y_1\geq 0} \cdot \chi_{x^r+(\gamma-r)T_1-\alpha Y_1 \geq 0} \cdot \chi_{T_1 \leq h < T_2}] + o(h) =$$

$$= V_2(x^i + (c-\gamma)h, x^r + (\gamma-r)h)(1-\lambda h) + \int_0^h \int_{h-t_1}^{+\infty} \int_0^{\min\{\frac{x^i+(c-\gamma)t_1}{1-\alpha}, \frac{x^r+(\gamma-r)t_1}{\alpha}\}} V_2(x^i + (c-\gamma)t_1 - (1-\alpha)y, x^r + (\gamma-r)t_1 - \alpha y)dF(y)\lambda e^{-\lambda t_2}dt_2 \lambda e^{-\lambda t_1}dt_1 + o(h).$$

For a small value of h, the latter right hand side expression approximates to

$$V_2(x^i + (c-\gamma)h, x^r + (\gamma-r)h) - \lambda V_2(x^i,x^r)h + \lambda h \int_0^{\min\{\frac{x^i}{1-\alpha},\frac{x^r}{\alpha}\}} V_2(x^i - (1-\alpha)y, x^r - \alpha y)dF(y).$$

Taking the limit, as $h \to 0^+$, we obtain

$$0 = \lim_{h\to 0^+} \frac{V_2(x^i+(c-\gamma)h, x^r+(\gamma-r)h) - V_2(x^i,x^r)}{h} - \lambda V_2(x^i,x^r) + \lambda \int_0^{\min\{\frac{x^i}{1-\alpha},\frac{x^r}{\alpha}\}} V_2(x^i-(1-\alpha)y, x^r-\alpha y)dF(y) =$$

$$= (\frac{\partial V_2}{\partial x^i}(c-\gamma) + \frac{\partial V_2}{\partial x^r}(\gamma-r))(x^i,x^r) - \lambda V_2(x^i,x^r) + \lambda \int_0^{\min\{\frac{x^i}{1-\alpha},\frac{x^r}{\alpha}\}} V_2(x^i - (1-\alpha)y, x^r - \alpha y)dF(y),$$

which is equivalent to the proposed condition,

$$(\frac{\partial V_2}{\partial x^i}(c-\gamma) + \frac{\partial V_2}{\partial x^r}(\gamma - r))_{x^i,x^r} =$$
$$\lambda(V_2(x^i,x^r) - \int_0^{\min\{\frac{x^r}{\alpha},\frac{x^i}{1-\alpha}\}} V_2(x^i-(1-\alpha)y, x^r-\alpha y)dF(y)).$$

□

Proposition 3.7. (Integro-differential equation for V_3): The function $V_3(x^i, x^r)$ solves the following integro-differential equation:

$$(\frac{\partial V_3}{\partial x^i}(c-\gamma)+\frac{\partial V_3}{\partial x^r}(\gamma-r))_{x^i,x^r}=\lambda(V_3(x^i,x^r)-$$
$$-\int_0^{\min\{\frac{x^r}{\alpha},\frac{x^i}{1-\alpha}\}} V_3(x^i-(1-\alpha)Y, x^r-\alpha Y)dF(y)-\int_{\frac{x^r}{\alpha}}^{x^i+x^r} V_1(x^i+x^r-y)dF(y)).$$

Proof. This proof follows similar reasoning to the one employed in the proofs of the propositions 3.5. and 3.6. and can be found in Appendix B. □

It is worth noticing that the equations found to be solved by V_2 and V_3 represent the dynamics of their respective directional derivatives on the plane of initial wealths, along the vector $(c-\gamma, \gamma-r)$.
Propositions 3.5. through 3.7. establish the following set of integro-differential equations, that describe the behaviour of the survival probabilities, lacking boundary conditions:

$$V_1^{'}(z)=\frac{\lambda}{c}\left(V_1(z)-\int_0^z V_1(z-y)dF(y)\right) \tag{12}$$

$$\left(\frac{\partial V_2}{\partial x^i}(c-\gamma)+\frac{\partial V_2}{\partial x^r}(\gamma-r)\right)_{x^i,x^r}=\lambda\Bigg(V_2(x^i,x^r)-$$
$$-\int_0^{\min\{\frac{x^r}{\alpha},\frac{x^i}{1-\alpha}\}} V_2(x^i-(1-\alpha)y, x^r-\alpha y)dF(y)\Bigg) \tag{13}$$

$$\left(\frac{\partial V_3}{\partial x^i}(c-\gamma)+\frac{\partial V_3}{\partial x^r}(\gamma-r)\right)_{x^i,x^r}=\lambda\Bigg(V_3(x^i,x^r)-$$
$$-\int_0^{\min\{\frac{x^r}{\alpha},\frac{x^i}{1-\alpha}\}} V_3(x^i-(1-\alpha)y, x^r-\alpha y)dF(y)-\int_{\frac{x^r}{\alpha}}^{x^i+x^r} V_1(x^i+x^r-y)dF(y)\Bigg) \tag{14}$$

3.2 Boundary Conditions

For the system of integro-differential conditions proposed in the last section to be well posed it is necessary to establish boundary conditions for each equation. We were able to obtain an explicit condition for V_1 and implicit conditions for V_2 and V_3. First of all, we should understand how each one of these functions behaves when the initial reserves, tend to infinity. That is the aim of propositions 3.8 to 3.10.

Proposition 3.8. (Limit behaviour of V_1):. $\lim_{z\to+\infty} V_1(z) = 1$.

Proof. This result follows directly from the proof of Grandell (1991) - Section 1.1., page 5 - and the memoryless property of the compound Poisson process. □

Proposition 3.9. (Limit behaviour of V_2):. When $x^i \to +\infty$ and $x^r \to +\infty$, the survival probability given by V_2 tends to 1.

Proof. We want to prove that $\lim_{x^i\to+\infty, x^i\to+\infty} V_2(x^i, x^r) = 1$. By definition,

$$\begin{gathered} V_2(x^i, x^r) = P\{\tau^{0,i} = +\infty, \tau^{0,r} = +\infty | X_0^{0,i} = x^i, X_0^{0,r} = x^r\} = \\ = 1 - P\{\tau^{0,i} < +\infty \vee \tau^{0,r} < +\infty | X_0^{0,i} = x^i, X_0^{0,r} = x^r\} = \\ = 1 - P\{\tau^{0,i} < +\infty | X_0^{0,i} = x^i, X_0^{0,r} = x^r\} - P\{\tau^{0,r} < +\infty | X_0^{0,i} = x^i, X_0^{0,r} = \\ x^r\} + P\{\tau^{0,i} < +\infty, \tau^{0,r} < +\infty | X_0^{0,i} = x^i, X_0^{0,r} = x^r\} \geq \\ \geq 1 - P\{\tau^{0,i} < +\infty | X_0^{0,i} = x^i, X_0^{0,r} = x^r\} - P\{\tau^{0,r} < +\infty | X_0^{0,i} = x^i, X_0^{0,r} = x^r\}. \end{gathered}$$

Since these quantities are all probabilities, if we prove that $P\{\tau^{0,i} < +\infty | X_0^{0,i} = x^i, X_0^{0,r} = x^r\}$ and $P\{\tau^{0,r} < +\infty | X_0^{0,i} = x^i, X_0^{0,r} = x^r\}$ tend to 0 when x^i and x^r tend to infinity, then it is proved that $\lim_{x^i\to+\infty, x^i\to+\infty} V_2(x^i, x^r) = 1$. The remainder of this proof is based on a similar proof by Grandell (1991) - Section 1.1., page 5.
Let $A_i = \{\omega \in \Omega : \lim_{t\to+\infty} \frac{X_t^{0,i}(\omega)}{t} = (c-\gamma) - (1-\alpha)\lambda E[Y]\}$, where Ω is the sample space. Since we assume $(c-\gamma) > (1-\alpha)\lambda E[Y]$, we have that

$$\exists k > 0 : \forall t > k, \frac{X_t^{0,i}}{t} > 0,$$

which is equivalent to saying

$$\exists k > 0 : \forall t > k, X_t^{0,i} > 0.$$

Let $B_k^i = \{\omega \in \Omega : t > k \implies X_t^{0,i} > 0\}$. If any sample path belongs to A_i, then there is a value k such that the sample path belongs to B_k^i. Therefore A_i is contained in $\cup_{k=1}^{\infty} B_k^i$. Since by the law of large numbers $P(A^i) = 1$, then $P(\cup_{k=1}^{\infty} B_k^i) = 1$.
Defining T^i as

$$T^i = \begin{cases} \sup\{t > 0 : X_t^{0,i} < 0\} & \text{if } \exists t > 0 : X_t^{0,i} < 0 \\ 0 & \text{if } \forall t > 0, X_t^{0,i} \geq 0, \end{cases}$$

the fact that $P(\cup_{k=1}^{\infty} B_k^i) = 1$ means that $P(T^i < +\infty) = 1$. Let $Z_t^{0,i} = (c-\gamma)t - (1-\alpha)\sum_{i=1}^{N_t} Y_i$. Then,

$$\begin{gathered} P\{\tau^{0,i} < +\infty | X_0^{0,i} = x^i, X_0^{0,r} = x^r\} = \\ = P\{\exists 0 \leq t < +\infty : Z_t^{0,i} < -x^i\} = \end{gathered}$$

$$= P\{inf_{t\geq 0} Z_t^{0,i} < -x^i\} =$$
$$= P\{inf_{t\in[0,T^i]} Z_t^{0,i} < -x^i\}$$
$$= \sum_{n=1}^{+\infty} P\{inf_{t\in[0,T^i]} Z_t^{0,i} < -x^i, N_t = n\} \leq$$
$$\leq \sum_{n=1}^{+\infty} P\{T^i < +\infty, (c-\gamma)t - (1-\alpha)\sum_{n=1}^{n} Y_i \leq -x^i\}.$$

Since $\sum_{n=1}^{+\infty} P\{T^i < +\infty, (c-\gamma)t-(1-\alpha)\sum_{n=1}^{n} Y_i \leq x^i\} \to 0$ as $x^i \to +\infty, \forall x^r \in \mathbb{R}$, then $P\{\tau^{0,i} < +\infty | X_0^{0,i} = x^i, X_0^{0,r} = x^r\}$ also tends to 0.
A similar proof can be done for the case of $P\{\tau^{0,r} < +\infty | X_0^{0,i} = x^i, X_0^{0,r} = x^r\}$ tending to 0 and can be found in Appendix C. □

Proposition 3.10. (Limit behaviour of V_3): When $x^i \to +\infty$ and $x^r \to +\infty$, V_3 tends to 0.

Proof. We want to show that $\lim_{x^i\to+\infty, x^r\to+\infty} V_3(x^i, x^r) = 0$. By definition,

$$V_3(x^i, x^r) = P\{\tau^{0,r} < \tau^{0,i}, \tau^{1,i} = +\infty | X_0^{0,i} = x^i, X_0^{0,r} = x^r\}.$$

Therefore,

$$V_3(x^i, x_r) \leq P\{\tau^{0,r} < +\infty | X_0^{0,i} = x^i, X_0^{0,r} = x^r\}.$$

We have proved that $P\{\tau^{0,r} < +\infty | X_0^{0,i} = x^i, X_0^{0,r} = x^r\}$ tends to zero as x^i and x^r tend to infinity in the proof of Proposition 3.9., thus the result follows directly. □

The intuition behind propositions 3.8. to 3.10. is that when the initial reserves are extremely large compared to the magnitude of claims and premiums, survival is virtually guaranteed, which is the point of the following corollary.

Corollary 3.1. When the initial reserves, x^i and x^r, tend to infinity, the probability of ruin of the reinsurer, $\bar{V}_r(x^i, x^r)$, and of the insurer, $\bar{V}_i(x^i, x^r)$, both tend to 0.

Proof. Propositions 3.9. and 3.10. establish that $\lim_{x^i\to+\infty, x^r\to+\infty} V_2(x^i, x_r) = 1$ and $\lim_{x^i\to+\infty, x^r\to+\infty} V_3(x^i, x_r) = 0$, respectively. Since we also proved, on Proposition 3.1. that $\bar{V}_r(x^i, x^r) = 1 - V_2(x^i, x_r)$ and $\bar{V}_i(x^i, x^r) = 1 - V_2(x^i, x_r) - V_3(x^i, x_r)$, the result follows directly. □

Propositions 3.8. to 3.10., besides providing some intuition into the limit behaviour of the ruin probabilities of the insurer and the reinsurer, will also be useful to derive the boundary conditions necessary so that the system presented in Section 3.1. is well posed. The derivation of said boundary conditions is the object of the propositions 3.11 through 3.13.

Proposition 3.11. (Boundary condition for V_1): $V_1(z)$ is such that $V_1(0) = \frac{c-\lambda E[Y]}{c}$.

Proof. V_1 can be written in its integral form as

$$V_1(z) = V_1(0) + \int_0^z V_1'(u)du.$$

Using equation (12), this becomes

$$V_1(z) = V_1(0) + \frac{\lambda}{c}\int_0^z V_1(u) - \int_0^u V_1(u-y)dF(y)du.$$

Using Fubini's theorem to change the integration order, the above expression is equal to

$$V_1(0) + \frac{\lambda}{c}(\int_0^z V_1(u)du - \int_0^z \int_y^z V_1(u-y)dudF(y)).$$

Proceeding with the substitution $s = u - y$, we get

$$V_1(0) + \frac{\lambda}{c}(\int_0^z V_1(u)du - \int_0^z \int_0^{z-y} V_1(s)dsdF(y)).$$

Applying once again Fubini's theorem, this becomes

$$V_1(0) + \frac{\lambda}{c}(\int_0^z V_1(u)du - \int_0^z V_1(s)\int_0^{z-s} dF(y)ds) =$$
$$= V_1(0) + \frac{\lambda}{c}\int_0^z V_1(u) - V_1(u)F(z-u)du.$$

Proceeding with a new substitution $y = z - u$, we are able to write V_1 in integral form as

$$V_1(z) = V_1(0) + \frac{\lambda}{c}\int_0^z V_1(z-y)(1-F(y))dy.$$

From proposition 3.8. we know that $\lim_{z\to+\infty} V_1(z) = 1$. Thus, $V_1(z-y)$ tends to 1 pointwise when z approaches infinity. Applying the monotone convergence theorem, we then obtain

$$1 = \lim_{z\to+\infty} V_1(z) = \lim_{z\to+\infty}[V_1(0) + \frac{\lambda}{c}\int_0^z V_1(z-y)(1-F(y))dy] =$$
$$= V_1(0) + \frac{\lambda}{c}\int_0^{+\infty} 1 \times (1-F(y))dy = V_1(0) + \frac{\lambda}{c}E[Y]$$
$$\Leftrightarrow$$
$$V_1(0) = \frac{c-\lambda E[Y]}{c},$$

as we aimed to prove. □

Given the result in proposition 3.11., we obtain the following equation that describes the behaviour of V_1

$$V_1(z) = \frac{c-\lambda E[Y]}{c} + \frac{\lambda}{c}\int_0^z V_1(z-y)(1-F(y))dy. \tag{15}$$

For V_1 it was possible to obtain an explicit boundary condition. However, for V_2 and V_3 only implicit boundary conditions were derived, as these functions are bivariate, which poses additional analytical difficulties. For V_2 and V_3, equations (13) and (14), respectively, describe the behaviour of their derivatives over a set of lines with slope $\frac{\gamma-r}{c-\gamma}$ on the plane (x^i, x^r). This set of lines can be represented by the following parameterization

$$(x^i, x^r) = (z^i, z^r) + u(c-\gamma, \gamma - r), \tag{16}$$

where $z^i, z^r, u \in \mathbb{R}_0^+$, with $z^i = 0$ or $z^r = 0$.
Using this parameterization, V_2 can be written in integral form as

$$V_2(x^i, x^r) = V_2(z^i + u(c-\gamma), z^r + u(\gamma - r)) =$$

$$= V_2(z^i, z^r) + \int_0^u \frac{d}{dt} V_2(z^i + t(c-\gamma), z^r + t(\gamma-r))dt =$$

$$= V_2(z^i, z^r) + \int_0^u ((c-\gamma)\frac{\partial V_2}{\partial x^i} + (\gamma - r)\frac{\partial V_2}{\partial x^r})(z^i + t(c-\gamma), z^r + t(\gamma-r))dt.$$

Using equation (13), we conclude that

$$\begin{aligned} V_2(x^i, x^r) = V_2(z^i, z^r) + \lambda \int_0^u V_2(z^i + t(c-\gamma), z^r + t(\gamma - r)) - \\ - \int_0^{\min\{\frac{z^r + t(\gamma-r)}{\alpha}, \frac{z^i + t(c-\gamma)}{1-\alpha}\}} V_2(z^i + t(c-\gamma) - (1-\alpha)y, z^r + t(\gamma - r) - \alpha y)dF(y)dt. \end{aligned} \tag{17}$$

Through similar arguments, it is also possible to prove that

$$\begin{aligned} V_3(x^i, x^r) = & V_3(z^i, z^r) + \lambda \int_0^u V_3(z^i + t(c-\gamma), z^r + t(\gamma - r)) - \\ & - \int_0^{\min\{\frac{z^r + t(\gamma-r)}{\alpha}, \frac{z^i + t(c-\gamma)}{1-\alpha}\}} V_3(z^i + t(c-\gamma) - (1-\alpha)y, z^r + t(\gamma - r) - \alpha y)dF(y) - \\ & - \int_{\frac{z^r + t(\gamma-r)}{\alpha}}^{z^i + z^r + (c-r)t} V_1(z^i + z^r + (c-r)t - y)dF(y)dt. \end{aligned} \tag{18}$$

From equations (17) and (18) it is possible to derive implicit boundary conditions for V_2 and V_3, based on the linearity of the integral operator.

Proposition 3.12 (Boundary condition for V_2): V_2 has a boundary condition of the type

$$V_2(z^i, z^r) = \frac{1}{\lim_{u\to+\infty} w^1_{z^i,z^r}(u)},$$

where $w^1_{z^i,z^r}(u)$ is the solution of the linear equation (17), substituting $V_2(z^i, z^r)$ by 1.

Proof. Equation (17) can be written in the following form

$$v(u) = A + \Psi_{z^i,z^r} v(u), \qquad u \geq 0,$$

where Ψ_{z^i,z^r} is a linear operator with domain $L_{\infty,loc}[0,+\infty[$ continuous in each $L_\infty[0,b]$. As such, this equation has a unique solution for each pair (z^i, z^r) and each $A \in \mathbb{R}$. Furthermore, if we consider $w^A_{z^i,z^r}$ the solution for $w(u) = A + \Psi_{z^i,z^r} w(u)$, the function $A \mapsto w^A_{z^i,z^r}$ is linear. In particular, $w^A_{z^i,z^r} = A w^1_{z^i,z^r}$ for any $A \in \mathbb{R}$. Given that we proved that V_2 is absolutely continuous and that it tends to 1 when the initial reserves tend to infinity, the result follows. □

Proposition 3.13. (Boundary condition for V_3): V_3 has the following boundary condition:

$$V_3(z^i, z^r) = -\frac{c-\lambda E[Y]}{c} \lim_{u\to+\infty} \frac{V_{3,(1,0)}(z^i+(c-\gamma)u, z^r+(\gamma-r)u)}{V_{3,(0,1)}(z^i+(c-\gamma)u, z^r+(\gamma-r)u)},$$

where $V_{3,(v_1,v_3)}$ refers to equation (18) together with the initial conditions $V_1(0) = v_1$ and $V_3(z^i, z^r) = v_3$.

Proof. From equations (15) and (18) and using parameterization (16), we obtain the following equations' system:

$$V_1(z) = V_1(0) + \frac{\lambda}{c} \int_0^z V_1(z-y)(1-F(y))dy$$

$$\begin{aligned} V_3(z^i + (c-\gamma)u, z^r + (\gamma-r)u) = V_3(z^i, z^r) + \lambda \int_0^u V_3(z^i + (c-\gamma)s, z^r + (\gamma-r)s) - \\ - \int_0^{\min\{\frac{z^r+(\gamma-r)s}{\alpha}, \frac{z^i+(c-\gamma)s}{1-\alpha}\}} V_3(z^i + (c-\gamma)s - (1-\alpha)y, z^r + (\gamma-r)s - \alpha y)dF(y) - \\ - \int_{\frac{z^r+(\gamma-r)s}{\alpha}}^{z^i+z^r+(c-r)s} V_1(z^i + z^r + (c-r)t - y)dF(y)ds \end{aligned}$$

Let $(V_{1,v_1}, V_{3,(v_1,v_3)})$ be the solution to this system with the initial conditions $V_1(0) = v_1$ and $V_3(z^i, z^r) = v_3$. The function $(v_1, v_3) \mapsto (V_{1,v_1}, V_{3,(v_1,v_3)})$ is linear. Therefore,

$$\begin{aligned} (V_{1,v_1}, V_{3,(v_1,v_3)}) = v_1(V_{1,1}, V_{3,(1,0)}) + v_3(V_{1,0}, V_{3,(0,1)}) = \\ v_1(V_{1,1}, V_{3,(1,0)}) + v_3(0, V_{3,(0,1)}). \end{aligned}$$

From proposition 3.11., we know that $v_1 = \frac{c-\lambda E[Y]}{c}$. In proposition 3.10. we have derived that $\lim_{u\to+\infty} V_3(z^i + (c-\gamma)u, z^r + (\gamma-r)u) = 0$. As such, we conclude that

$$v_3 = -\frac{c-\lambda E[Y]}{c} \lim_{u\to+\infty} \frac{V_{3,(1,0)}(z^i+(c-\gamma)u, z^r+(\gamma-r)u)}{V_{3,(0,1)}(z^i+(c-\gamma)u, z^r+(\gamma-r)u)}.$$

□

3.3 Numerical Examples

Up until this point we have obtained a formula for the probability given by $V_1(z)$ with an explicit boundary condition, in equation (15), and expressions for V_2 and V_3, given by the equations (17) and (18) with implicit boundary conditions, established in propositions 3.12. and 3.13. This setup allows for the numerical calculations of the survival probabilities of direct insurer and reinsurer, given the insurance premium c, the minimum return of the reinsurer r, a reinsurance policy contractually defining the reinsurance premium γ and the ceded portion of the claims α, the claim severity distribution, the intensity of the Poisson counting process λ, as well as the initial surplus of each participant.
To simplify the notation, let us introduce some linear operators.

Definition 3.2. (Some useful operators): Let Ψ, Θ and Λ be linear operators such that

$$\begin{aligned}\Psi v =&\lambda \int_0^u v(z^i + (c-\gamma)t, z^r + (\gamma - r)t) - \\ &- \int_0^{\min\{\frac{z^r + t(\gamma-r)}{\alpha}, \frac{z^i + t(c-\gamma)}{1-\alpha}\}} v(z^i + (c-\gamma)t - (1-\alpha)y, z^r + (\gamma - r)t - \alpha y) dF(y)dt,\end{aligned} \tag{19}$$

$$\Theta v = -\lambda \int_0^u \int_{\frac{z^r + (\gamma - r)t}{\alpha}}^{z^i + z^r + (c-r)t} v(z^i + z^r + (c-r)t - y) dF(y)dt, \tag{20}$$

and

$$\Lambda v(z) = \frac{\lambda}{c} \int_0^z v(z-y)(1-F(y))dy. \tag{21}$$

Under the above definition, we can express (15), (17) and (18) as follows

$$V_1 = \frac{c - \lambda E[Y]}{c} + \Lambda V_1, \tag{22}$$

$$V_2 = V_2(z^i, z^r) + \Psi V_2, \tag{23}$$

$$V_3 = V_3(z^i, z^r) + \Psi V_3 + \Theta V_1. \tag{24}$$

Since some boundary conditions are not explicitly known, there is a need to define some auxiliary functions for which we are able to do numerical computation.

Definition 3.3. (Auxiliary functions): Let $\tilde{V_2}$, $\tilde{V_3^1}$, $\tilde{V_3^2}$ be functions with domain in $\mathbb{R}^2$ and $\tilde{V_1^1}$ and $\tilde{V_1^2}$ be functions with domain in $\mathbb{R}$ such that

$$\tilde{V_2} = 1 + \Psi\tilde{V_2}, \tag{25}$$

$$\begin{pmatrix} \tilde{V_1^1} \\ \tilde{V_3^1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \Lambda\tilde{V_1^1} \\ \Theta\tilde{V_1^1} + \Psi\tilde{V_3^1} \end{pmatrix}, \tag{26}$$

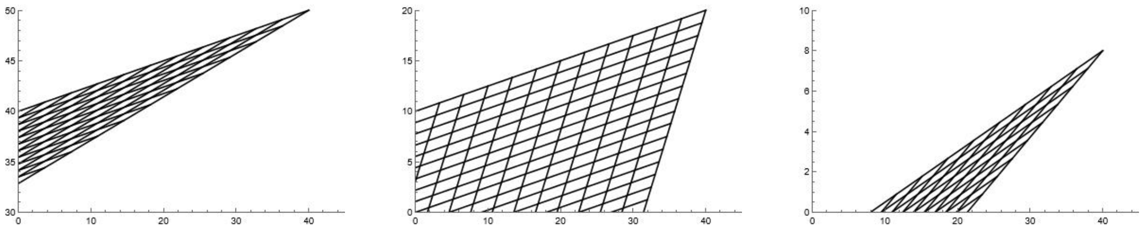
$$\begin{pmatrix} \tilde{V_1^2} \\ \tilde{V_3^2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \Lambda\tilde{V_1^2} \\ \Theta\tilde{V_1^2} + \Psi\tilde{V_3^2} \end{pmatrix}. \tag{27}$$

The numerical solution to equation (25) can be normalized, thus allowing us to obtain values for the probabilities given by V_2. A linear combination of the solutions for $\tilde{V_3^1}$ and $\tilde{V_3^2}$ will be used in order to derive values for V_3. Then, we are able to estimate the probability of ruin of the reinsurer and of the first line insurer, through the expressions (10) and (11), respectively.
We start by obtaining $\tilde{V_2}$ numerically, through equation (25). To solve this integral equation we opted to apply the trapezium rule, requiring the development of several integration grids depending on the parameters of the problem and the relations among them, as well as the initial surpluses. These grids represent a set of intersection points, with non-negative coordinates, between two families of lines. The first family of lines is given by the parameterization (16), over which the integration variable t will be discretized. The second family of lines will contain the points of discretization of the integration variable y and is represented by the following parameterization:

$$(x^i, x^r) = (z^i, z^r) + u(1-\alpha, \alpha), \qquad u \geq 0. \tag{28}$$

where $z^i, z^r, u \in \mathbb{R}_0^+$, and $z^i = 0$ or $z^r = 0$. In order to implement such grids, we created for each necessary case a matrix A that contains all of the points of intersection, with both positive coordinates, of the lines of the form (16) and (28), for a set of intercepts (z^i, z^r). The points of intersection of the lines with the first quadrant axes are stored in separated vectors, AC for lines represented by the points on the columns of A, and AL for those points stored along the lines of A. A graphical representation of some examples of such grids are presented in Figure 1.

Next, we present a possible approximated solution to equations of the form $g(x^i, x^r) = \Phi(x^i, x^r) + \Psi g(x^i, x^r)$. This result can be used to estimate $\tilde{V_2}(x^i, x^r)$, if we consider $\Phi(x^i, x^r) = 1$ and $g = \tilde{V_2}$.



(a) Grid with all lines crossing the vertical axis

(b) Grid with lines crossing both axes

(c) Grid with all lines crossing the horizontal axis

FIGURE 1: Geometrical representation of some possible integration grids.

Numerical Scheme 1 (Numerical solution of $g = \Phi + \Psi g$): Let g be a function such that $g : \mathbb{R}^2 \to [0, K], K \in \mathbb{R}$, and $g = \Phi + \Psi g$. If $\min\{\frac{z^r+t(\gamma-r)}{\alpha}, \frac{z^i+t(c-\gamma)}{1-\alpha}\} = \frac{(\gamma-r)t}{\alpha}$, such function can be approximated over a set of discrete points by the following expression:

$$
\begin{aligned}
g(x^i_{k,l}, x^r_{k,l}) \approx & \Bigg[\Phi(x^i_{k,l}, x^r_{k,l}) + \lambda \left(\int_0^{\frac{x^r_{k,l-1}}{\gamma-r}} g_{k,l-1}dt + g_{k,l-1} \times \frac{x^r_{k,l} - x^r_{k,l-1}}{2(\gamma-r)}\right) - \\
& - \lambda\Bigg(\int_0^{\frac{x^r_{k,n[k]}}{\alpha}} g(x^i_{k,n[k]} - (1-\alpha)y, x^r_{k,n[k]} - \alpha y) f(y)dy \times \frac{x^r_{k,n[k]}}{2(\gamma-r)} + \\
& + \sum_{j=n[k]}^{l-2} \Bigg(\Bigg(\int_0^{\frac{x^r_{k,j}}{\alpha}} g(x^i_{k,j} - (1-\alpha)y, x^r_{k,j} - \alpha y) f(y)dy + \\
& + \int_0^{\frac{x^r_{k,j+1}}{\alpha}} g(x^i_{k,j+1} - (1-\alpha)y, x^r_{k,j+1} - \alpha y) f(y)dy\Bigg) \times \frac{x^r_{k,j+1} - x^r_{k,j}}{2(\gamma-r)}\Bigg) + \\
& + \Bigg(\int_0^{\frac{x^r_{k,l-1}}{\alpha}} g(x^i_{k,l-1} - (1-\alpha)y, x^r_{k,l-1} - \alpha y) f(y)dy + f\left(\frac{x^r_{n^*[l],l}}{\alpha}\right) g_{k+1,l} \times \frac{x^r_{n^*[l],l}}{2\alpha} + \\
& + \sum_{s=0}^{n^*[l]-k-2} \left(\left(f\left(\frac{x^r_{n^*[l]-s,l}}{\alpha}\right) g_{k+1+s,l} + f\left(\frac{x^r_{n^*[l]-s-1,l}}{\alpha}\right) g_{k+2+s,l}\right) \times \frac{x^r_{n^*[l]-s-1,l} - x^r_{n^*[l]-s,l}}{2\alpha}\right) + \\
& + \left(f\left(\frac{x^r_{k+1,l}}{\alpha}\right) g_{n^*[l],l} + f\left(\frac{x^r_{k,l}}{\alpha}\right) g_{0,l}\right) \times \frac{x^r_{k,l} - x^r_{k+1,l}}{2\alpha}\Bigg) \frac{x^r_{k,l} - x^r_{k,l-1}}{2(\gamma-r)}\Bigg)\Bigg] \Bigg/ \\
& \left(1 - \frac{\lambda}{2} \frac{x^r_{k,l} - x^r_{k,l.1}}{\gamma - r}\left(1 - f(0)\frac{x^r_{n^*[l],l}}{2\alpha}\right)\right),
\end{aligned}
\tag{29}
$$

where $g_{k,l} = g(x^i_{k,l}, x^r_{k,l})$, $g_{0,l}$ is the value of the function g on the intersection of the column l of the integration grid with the horizontal axis, $g_{k,0}$ is the value of the function g on the intersection of line k of the integration grid with the horizontal axis, $n[k]$ is the first column that intersects line k of the integration grid in a point with both positive coordinates, $n^*[l]$ is the first line that intersects column l in a point with both positive coordinates, and using the following approximations:

$$\int_0^{\frac{x^r_{k,l}}{\gamma-r}} g_{k,l}dt \approx (g_{k,0}+g_{k,n[k]}) \times \tfrac{x^r_{k,n[k]}}{2(\gamma-r)} + \sum_{j=n[k]}^{l-1}\left((g_{k,j}+g_{k,j+1}) \times \tfrac{x^r_{k,j+1}-x^r_{k,j}}{2(\gamma-r)}\right)$$

$$\int_0^{\frac{x^r_{k,l}}{\alpha}} g(x^i_{k,l}-(1-\alpha)y, x^r_{k,l}-\alpha y)f(y)dy \approx \left(f(0)g_{k,l} + f\left(\tfrac{x^r_{n^*[l],l}}{\alpha}\right)g_{k+1,l}\right) \times \tfrac{x^r_{n^*[l],l}}{2\alpha} +$$
$$+\sum_{s=0}^{n[l]-k-2}\left(\left(f\left(\tfrac{x^r_{n^*[l]-s,l}}{\alpha}\right)g_{k+1+s,l}+f\left(\tfrac{x^r_{n^*[l]-s-1,l}}{\alpha}\right)g_{k+2+s,l}\right)\times\tfrac{x^r_{n^*[l]-s-1,l}-x^r_{n^*[l]-s,l}}{2\alpha}\right)+$$
$$+\left(f\left(\tfrac{x^r_{k+1,l}}{\alpha}\right)g_{n^*[l],l}+f\left(\tfrac{x^r_{k,l}}{\alpha}\right)g_{0,l}\right)\times\tfrac{x^r_{k,l}-x^r_{k+1,l}}{2\alpha}.$$

On the other hand, if $\min\{\frac{z^r+t(\gamma-r)}{\alpha}, \frac{z^i+t(c-\gamma)}{1-\alpha}\} = \frac{(c-\gamma)t}{1-\alpha}$, such function can be approximated over a set of discrete points by the following expression:

$$g(x^i_{k,l},x^r_{k,l}) = \Bigg[\Phi(x^i_{k,l},x^r_{k,l}) + \lambda\left(\int_0^{\frac{x^i_{k,l-1}}{c-\gamma}} g_{k,l-1}dt + g_{k,l-1}\times\frac{x^i_{k,l}-x^i_{k,l-1}}{2(c-\gamma)}\right) -$$
$$-\lambda\Bigg(\int_0^{\frac{x^i_{k,n[k]}}{1-\alpha}} g(x^i_{k,n[k]}-(1-\alpha)y, x^r_{k,n[k]}-\alpha y)f(y)dy \times \frac{x^i_{k,n[k]}}{2(c-\gamma)}+$$
$$+\sum_{j=n[k]}^{l-2}\Bigg(\Bigg(\int_0^{\frac{x^i_{k,j}}{1-\alpha}} g(x^i_{k,j}-(1-\alpha)y, x^r_{k,j}-\alpha y)f(y)dy+$$
$$+\int_0^{\frac{x^i_{k,j+1}}{1-\alpha}} g(x^i_{k,j+1}-(1-\alpha)y, x^r_{k,j+1}-\alpha y)f(y)dy\Bigg)\times\frac{x^i_{k,j+1}-x^i_{k,j}}{2(c-\gamma)}\Bigg)+$$
$$+\Bigg(\int_0^{\frac{x^i_{k,l-1}}{1-\alpha}} g(x^i_{k,l-1}-(1-\alpha)y, x^r_{k,l-1}-\alpha y)f(y)dy + f\left(\frac{x^i_{n^*[l],l}}{1-\alpha}\right)g_{k+1,l}\times\frac{x^i_{n^*[l],l}}{2(1-\alpha)}+$$
$$+\sum_{s=0}^{n^*[l]-k-2}\left(\left(f\left(\frac{x^i_{n^*[l]-s,l}}{1-\alpha}\right)g_{k+1+s,l}+f\left(\frac{x^i_{n^*[l]-s-1,l}}{1-\alpha}\right)g_{k+2+s,l}\right)\times\frac{x^i_{n^*[l]-s-1,l}-x^i_{n^*[l]-s,l}}{2(1-\alpha)}\right)+$$
$$+\left(f\left(\frac{x^i_{k+1,l}}{1-\alpha}\right)g_{n^*[l],l}+f\left(\frac{x^i_{k,l}}{1-\alpha}\right)g_{0,l}\right)\times\frac{x^i_{k,l}-x^i_{k+1,l}}{2(1-\alpha)}\Bigg)\frac{x^i_{k,l}-x^i_{k,l-1}}{2(c-\gamma)}\Bigg)\Bigg]\Bigg/$$
$$\left(1-\frac{\lambda}{2}\frac{x^i_{k,l}-x^i_{k,l.1}}{c-\gamma}\left(1-f(0)\frac{x^i_{n^*[l],l}}{2(1-\alpha)}\right)\right), \tag{30}$$

using the same notation as in the previous case and using the following approximations:

$$\int_0^{\frac{x^i_{k,l}}{c-\gamma}} g_{k,l}dt \approx (g_{k,0}+g_{k,n[k]}) \times \tfrac{x^i_{k,n[k]}}{2(c-\gamma)} + \sum_{j=n[k]}^{l-1}\left((g_{k,j}+g_{k,j+1}) \times \tfrac{x^i_{k,j+1}-x^i_{k,j}}{2(c-\gamma)}\right)$$

$$\int_0^{\frac{x^i_{k,l}}{1-\alpha}} g(x^i_{k,l}-(1-\alpha)y, x^i_{k,l}-\alpha y)f(y)dy \approx \left(f(0)g_{k,l} + f\left(\frac{x^i_{n^*[l],l}}{1-\alpha}\right)g_{k+1,l}\right) \times \frac{x^i_{n^*[l],l}}{2(1-\alpha)} +$$
$$+\sum_{s=0}^{n[l]-k-2}\left(\left(f\left(\frac{x^i_{n^*[l]-s,l}}{1-\alpha}\right)g_{k+1+s,l}+f\left(\frac{x^i_{n^*[l]-s-1,l}}{1-\alpha}\right)g_{k+2+s,l}\right)\times\frac{x^i_{n^*[l]-s-1,l}-x^i_{n^*[l]-s,l}}{2(1-\alpha)}\right)+$$
$$+\left(f\left(\frac{x^i_{k+1,l}}{1-\alpha}\right)g_{n^*[l],l}+f\left(\frac{x^i_{k,l}}{1-\alpha}\right)g_{0,l}\right)\times\frac{x^i_{k,l}-x^i_{k+1,l}}{2(1-\alpha)}.$$

Proof. The proof of this proposition can be found in Appendix D. □

Using Numerical Scheme 1, we are able to numerically calculate $\tilde{V_2}(x^i, x^r)$ and to estimate the normalization constant that will be used to obtain $V_2(x^i, x^r)$, through the expression presented in Proposition 3.14.

Proposition 3.14. (Obtaining $V_2(x^i, x^r)$ from $\tilde{V_2}(x^i, x^r)$): $\tilde{V_2}(x^i, x^r)$ and $V_2(x^i, x^r)$ are related through the following expression:

$$V_2(u) = \frac{\tilde{V_2}(u)}{\tilde{V_2}(\infty)}, \tag{31}$$

where u is the superior limit on the integral over t in the definition of Ψ.

Proof. This follows from the fact that $\tilde{V_2}(u)$ is a limited increasing function. □

Example 3.1.: A reinsurance policy is in place such that the reinsurance premium, γ, is 7 and the ceded portion of each claim, α, is 30%. The insurance premium, c, is 15 and the minimum income for the reinsurer, r, is 5. The initial surplus of the cedent and the reinsurer are, respectively, 40 and 8. Supposing that the frequency parameter, λ, is 1 and that the claims follow a Gamma distribution with shape parameter 5 and scale parameter 1, the application of Numerical Scheme 1 results in an approximated value for $\tilde{V_2}(40, 8)$ of 3.69723.

In order to estimate $\tilde{V_2}(\infty)$, some examples are estimated with the stated parameters and initial surpluses along the line $x_r = \frac{\gamma-r}{c-\gamma}x^i$, corresponding to parameterization (16) with intercept on $(0, 0)$. Considering a sequence u_n such that $u_0 = 0$ and $u_n \to \infty$, we have

$$\begin{aligned}\tilde{V_2}(\infty) &= 1 + \textstyle\sum_{n=1}^{\infty}(\tilde{V_2}(u_n) - \tilde{V_2}(u_{n-1})) = \\ &= \tilde{V_2}(u_k) + \textstyle\sum_{n=k+1}^{\infty}(\tilde{V_2}(u_n) - \tilde{V_2}(u_{n-1})).\end{aligned}$$

Over the line $x_r = \frac{\gamma-r}{c-\gamma}x^i$, it is observed that $\frac{(\tilde{V_2}(u_{n+1})-\tilde{V_2}(u_n))}{(\tilde{V_2}(u_n)-\tilde{V_2}(u_{n-1}))}$ seems to be converging to a value of 0.9688. As such, we consider the following approximation of $\tilde{V_2}(\infty)$.

$$\tilde{V_2}(\infty) \approx \tilde{V_2}(u_k) + \tfrac{0.9688^k}{1-0.9688}.$$

This results in a final approximated value for $\tilde{V_2}(\infty)$ of 9.83647. It is worth noticing that the matrices for this process can rapidly become very large, thus making the necessary calculations require a lot of computational power. As such, the apparent stabilization of the quotient of successive increments could in reality be a very slow convergence process to some other constant. More computational power could help provide a more accurate estimation for the normalization constant $\tilde{V_2}(\infty)$.

Nevertheless, using the estimated normalization constant, we obtain an approximated value of 0.372903 for the probability V_2. Using equation (10), this translates in a probability of ruin of the reinsurer of around 62.7%. This would be a concerning value from a ruin theory standpoint and can be justified by the fact that the expected value of claims ceded to the reinsurer is $\alpha\lambda E[Y] = 1.5$, while the net income of the reinsurer is $\gamma - r = 2$, meaning that the reinsurer is operating on a rather slim margin. Furthermore, the low initial reserves of the reinsurer compared to the expected value of claims also plays a role in this rather high ruin probability.

For the set of parameters of this example, we are able to plot the probability of ruin of the reinsurer for the initial wealths considered along the integration grid and this is presented in Figure 2.

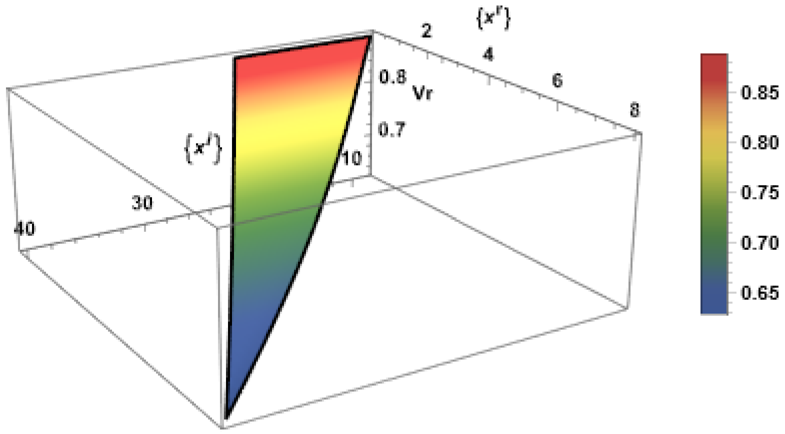


FIGURE 2: Probability of ruin of the reinsurer as a function of the initial surpluses, along the integration grid.

We can observe that, as the initial surpluses of the cedent and of the reinsurer grow, the probability of ruin of the reinsurer decreases. The positive effect of the direct insurer's wealth can be justified by the fact that the eventual ruin of the insurer, in the herein described market, would in time mandatorily lead to the reinsurer's ruin. □

We now present an expression to approximate $V_1(z)$. Since this is a univariate function, the discretization of the argument is made along a single vector.

Numerical Scheme 2 (Approximation of V_1): $V_1(z)$, for $z \in \mathbb{R}_0^+$, can be approximated by the following expression, where y_j are elements of a vector of k equidistant points, such that $y_1 = 0$ and $y_k = z$.

$$V_1(z) \approx \left[\frac{c-\lambda E[Y]}{c} + \frac{\lambda}{c}\left(V_1(z-y_2)(1-F(y_2))\frac{y_2}{2} + \sum_{j=2}^{k-1}((V_1(z-y_j)(1-F(y_j))+ + V_1(z-y_{j+1})(1-F(y_{j+1}))) \times \frac{y_{j+1}-y_j}{2})\right)\right] \Big/ \left[1 - \frac{\lambda}{c}(1-F(0))\frac{y_2}{2}\right]. \quad (32)$$

Proof. Follows from the application of the trapezium rule to equation (15). □

Example 3.2.: Using the same parameters as in example 3.1. ($c = 15, r = 5, \gamma = 7, \alpha = 0.3, \lambda = 1, Y \sim Gamma(5,1)$), we can plot $V_1(z)$ for several values of z. This is presented in Figure 3.

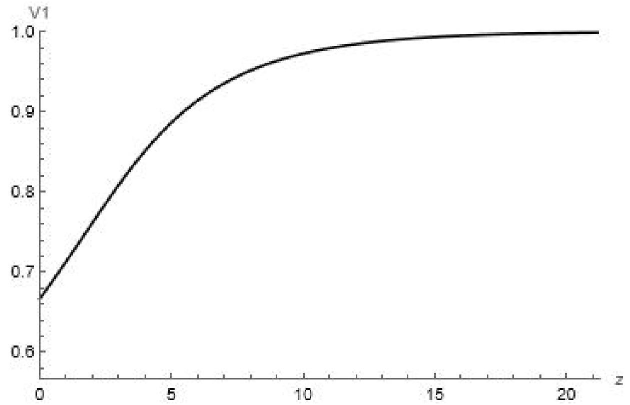


FIGURE 3: Probability of survival of the direct insurer, given the previous ruin of the reinsurer and the insurer's level of surplus at that time.

We can see that these approximations yield an increasing function plateauing near to 1, as expected, given that $V_1(z)$ represents the probability of survival of the insurer, given the ruin of the reinsurer and the level of surplus of the direct insurer immediately after the moment of said ruin. Therefore, we can observe that the higher that surplus, the higher the survival probability given by V_1. Notice as well, that $V_1(0) = \frac{2}{3}$, consistent with the boundary condition given in Proposition 3.11. □

In order to estimate V_3, it is necessary to solve the systems of equations (26) and (27). The simplification of these systems is the object of proposition 3.15.

Proposition 3.15. (Re-statement of systems (26) and (27)): The systems (26) and (27) can be respectively written as

$$\begin{pmatrix} \tilde{V_1^1} \\ \tilde{V_3^1} \end{pmatrix} = \begin{pmatrix} \frac{c}{c-\lambda E[y]} V_1 \\ \frac{c}{c-\lambda E[y]} \Theta V_1 + \Psi \tilde{V_3^1} \end{pmatrix}, \tag{33}$$

$$\tilde{V_3^2} = 1 + \Psi \tilde{V_3^2}. \tag{34}$$

Proof. Follows directly from the definition of the systems □

The equations on $\tilde{V_3^j}, j = 1, 2$ are similar to equation (25), for which we have a solution in Numerical Scheme 1, plus a term in V_1, in the case of the system (33). This term will be calculated in the points of the integration grid of $\tilde{V_3^1}$, that takes the same shape as the one described for $\tilde{V_2}$: crossing lines following parameterizations (16) and (28), exemplified in Figure 1. The next Numerical Scheme is devised to handle this term.

Numerical Scheme 3 (Approximation of ΘV_1): If $\min\{\frac{z^r+t(\gamma-r)}{\alpha}, \frac{z^i+t(c-\gamma)}{1-\alpha}\} = \frac{(\gamma-r)t}{\alpha}$, ΘV_1 can be approximated over the set of discrete points of the integration grid for $V_3(x^i, x^r)$ by the following expression:

$$-\lambda \left(\int_0^{z_k^i} V_1(z_k^i - y) f(y) dy + \int_{\frac{x^r_{k,n[k]}}{\alpha}}^{z_k^i + \frac{c-r}{\gamma-r} x^r_{k,n[k]}} V_1\left(z_k^i + \frac{c-r}{\gamma-r} x^r_{k,n[k]} - y \right) f(y) dy \right)$$
$$\times \frac{x^r_{k,n[k]}}{2(\gamma-r)} + \sum_{j=n[k]}^{l-1} \left(\left(\int_{\frac{x^r_{k,j}}{\alpha}}^{z_k^i + \frac{c-r}{\gamma-r} x^r_{k,j}} V_1\left(z_k^i + \frac{c-r}{\gamma-r} x^r_{k,j} - y \right) f(y) dy + \right.\right.$$
$$\left.\left. + \int_{\frac{x^r_{k,j+1}}{\alpha}}^{z_k^i + \frac{c-r}{\gamma-r} x^r_{k,j+1}} V_1\left(z_k^i + \frac{c-r}{\gamma-r} x^r_{k,j+1} - y \right) f(y) dy \right) \frac{x^r_{k,j+1} - x^r_{k,j}}{2(\gamma-r)} \right), \tag{35}$$

using the following approximation:

$$\int_{\frac{x^r_{k,l}}{\alpha}}^{G^i_{k,l}} V_1(G^i_{k,l} - y) f(y) dy \approx$$
$$\sum_{s=L}^{Q-1} \left((V_1(y_{k,l,Q+1-s}) f(y_{k,l,s}) + V_1(y_{k,l,Q-s}) f(y_{k,l,s+1})) \times \frac{y_{k,l,s+1} - yk,l,s}{2} \right),$$

where z_k^i is the intercept on the horizontal axis of line k of the integration grid for $\tilde{V_3^1}$, $G_{k,l} = z_k^i + \frac{c-r}{\gamma-r} x^r_{k,l}$, AY is the integration vector for V_1 respective to the point k, l in the integration grid for $\tilde{V_3^1}$, L is the position on AY of the lower bound of the integral and Q is the length of AY, such that $y_{k,l,Q} = G^i_{k,l}$.

If $\min\{\frac{z^r+t(\gamma-r)}{\alpha}, \frac{z^i+t(c-\gamma)}{1-\alpha}\} = \frac{(c-\gamma)t}{1-\alpha}$, ΘV_1 can be approximated over the set of discrete points of the integration grid for $V_3(x^i, x^r)$ by the following expression:

$$-\lambda\left(\int_{\frac{z^r}{\alpha}}^{z_k^r} V_1(z_k^r - y)f(y)dy + \int_{\frac{x^r_{k,n[k]}}{\alpha}}^{z_k^r+\frac{c-r}{c-\gamma}x^i_{k,n[k]}} V_1\left(z_k^r + \frac{c-r}{c-\gamma}x^i_{k,n[k]} - y\right)f(y)dy\right)$$
$$\times \frac{x^i_{k,n[k]}}{2(c-\gamma)} + \sum_{j=n[k]}^{l-1}\left(\left(\left(\int_{\frac{x^r_{k,j}}{\alpha}}^{z_k^r+\frac{c-r}{c-\gamma}x^i_{k,j}} V_1\left(z_k^r + \frac{c-r}{c-\gamma}x^i_{k,j} - y\right)f(y)dy+\right.\right.\right.$$
$$\left.\left.+\int_{\frac{x^r_{k,j+1}}{\alpha}}^{z_k^r+\frac{c-r}{c-\gamma}x^i_{k,j+1}} V_1\left(z_k^r + \frac{c-r}{c-\gamma}x^i_{k,j+1} - y\right)f(y)dy\right)\frac{x^i_{k,j+1}-x^i_{k,j}}{2(c-\gamma)}\right), \tag{36}$$

using the following approximation:

$$\int_{\frac{x^i_{k,l}}{1-\alpha}}^{G^r_{k,l}} V_1(G^r_{k,l} - y)f(y)dy \approx$$
$$\sum_{s=L}^{Q-1}\left((V_1(y_{k,l,Q+1-s})f(y_{k,l,s}) + V_1(y_{k,l,Q-s})f(y_{k,l,s+1})) \times \frac{y_{k,l,s+1} - yk,l,s}{2}\right),$$

where z_k^r is the intercept on the vertical axis of line k of the integration grid for V_3, $G^r_{k,l} = z_k^r + \frac{c-r}{c-\gamma}x^i_{k,l}$, AY is the integration vector for V_1 respective to the point k, l in the integration grid for V_3, L is the position on AY of the lower bound of the integral and Q is the length of AY, such that $y_{k,l,Q} = G^r_{k,l}$.

Proof. Both results derive directly from the application of the trapezium rule to ΘV_1. □

Given the Numerical Schemes 1 and 3, we are able to derive an expression to approximate $\tilde{V_3^1}$.

Numerical Scheme 4 (Approximation of $\tilde{V_3^1}$): If $\min\{\frac{z^r+t(\gamma-r)}{\alpha}, \frac{z^i+t(c-\gamma)}{1-\alpha}\} = \frac{(\gamma-r)t}{\alpha}$, $\tilde{V_3^1}$ can be approximated over the set of discrete point by the following expression:

$$\tilde{V_3^1}(x^i_{k,l}, x^r_{k,l}) \approx \left[1 + \lambda\left(\int_0^{\frac{x^r_{k,l-1}}{\gamma-r}} \tilde{V_3^1}_{k,l-1}dt + \tilde{V_3^1}_{k,l-1} \times \frac{x^r_{k,l} - x^r_{k,l-1}}{2(\gamma-r)}\right) -\right.$$
$$-\lambda\left(\int_0^{\frac{x^r_{k,n[k]}}{\alpha}} \tilde{V_3^1}(x^i_{k,n[k]} - (1-\alpha)y, x^r_{k,n[k]} - \alpha y)f(y)dy \times \frac{x^r_{k,n[k]}}{2(\gamma-r)}+\right.$$
$$+\sum_{j=n[k]}^{l-2}\left(\left(\int_0^{\frac{x^r_{k,j}}{\alpha}} \tilde{V_3^1}(x^i_{k,j} - (1-\alpha)y, x^r_{k,j} - \alpha y)f(y)dy+\right.\right.$$
$$\left.\left.+\int_0^{\frac{x^r_{k,j+1}}{\alpha}} \tilde{V_3^1}(x^i_{k,j+1} - (1-\alpha)y, x^r_{k,j+1} - \alpha y)f(y)dy\right) \times \frac{x^r_{k,j+1} - x^r_{k,j}}{2(\gamma-r)}\right)+$$
$$+\left(\int_0^{\frac{x^r_{k,l-1}}{\alpha}} \tilde{V_3^1}(x^i_{k,l-1} - (1-\alpha)y, x^r_{k,l-1} - \alpha y)f(y)dy + f\left(\frac{x^r_{n*[l],l}}{\alpha}\right)\tilde{V_3^1}_{k+1,l} \times \frac{x^r_{n*[l],l}}{2\alpha}+\right.$$

$$\sum_{s=0}^{n^*[l]-k-2}\left(\left(\left(f\left(\frac{x^r_{n^*[l]-s,l}}{\alpha}\right)\tilde{V_3^1}_{k+1+s,l}+f\left(\frac{x^r_{n^*[l]-s-1,l}}{\alpha}\right)\tilde{V_3^1}_{k+2+s,l}\right)\times\frac{x^r_{n^*[l]-s-1,l}-x^r_{n^*[l]-s,l}}{2\alpha}\right)\right.$$
$$+\left(\left(f\left(\frac{x^r_{k+1,l}}{\alpha}\right)\tilde{V_3^1}_{n^*[l],l}+f\left(\frac{x^r_{k,l}}{\alpha}\right)\tilde{V_3^1}_{0,l}\right)\times\frac{x^r_{k,l}-x^r_{k+1,l}}{2\alpha}\right)\frac{x^r_{k,l}-x^r_{k,l-1}}{2(\gamma-r)}\Bigg)$$
$$-\lambda\left(\int_0^{z^r_k}V_1(z^r_k-y)f(y)dy+\int_{\frac{x^r_{k,n[k]}}{\alpha}}^{z^r_k+\frac{c-r}{c-\gamma}x^r_{k,n[k]}}V_1\left(z^r_k+\frac{c-r}{c-\gamma}x^r_{k,n[k]}-y\right)f(y)dy\right)$$
$$\times\frac{x^r_{k,n[k]}}{2(\gamma-r)}+\sum_{j=n[k]}^{l-1}\left(\left(\left(\int_{\frac{x^r_{k,j}}{\alpha}}^{z^r_k+\frac{c-r}{c-\gamma}x^r_{k,j}}V_1\left(z^r_k+\frac{c-r}{c-\gamma}x^r_{k,j}-y\right)f(y)dy+\right.\right.\right.$$
$$+\int_{\frac{x^r_{k,j+1}}{\alpha}}^{z^r_k+\frac{c-r}{c-\gamma}x^r_{k,j+1}}V_1\left(z^r_k+\frac{c-r}{c-\gamma}x^r_{k,j+1}-y\right)f(y)dy\right)\frac{x^r_{k,j+1}-x^r_{k,j}}{2(c-\gamma)}\Bigg)\Bigg]\Bigg/$$
$$\left(1-\frac{\lambda}{2}\frac{x^r_{k,l}-x^r_{k,l.1}}{\gamma-r}\left(1-f(0)\frac{x^r_{n^*[l],l}}{2\alpha}\right)\right). \tag{37}$$

If $\min\{\frac{z^r+t(\gamma-r)}{\alpha},\frac{z^i+t(c-\gamma)}{1-\alpha}\}=\frac{(c-\gamma)t}{1-\alpha}$, $\tilde{V_3^1}$ can be approximated over the set of discrete points by the following expression:

$$\tilde{V_3^1}(x^i_{k,l},x^r_{k,l})\approx\left[1+\lambda\left(\int_0^{\frac{x^i_{k,l-1}}{c-\gamma}}\tilde{V_3^1}_{k,l-1}dt+\tilde{V_3^1}_{k,l-1}\times\frac{x^i_{k,l}-x^i_{k,l-1}}{2(c-\gamma)}\right)-\right.$$
$$-\lambda\left(\int_0^{\frac{x^i_{k,n[k]}}{1-\alpha}}\tilde{V_3^1}(x^i_{k,n[k]}-(1-\alpha)y,x^r_{k,n[k]}-\alpha y)f(y)dy\times\frac{x^i_{k,n[k]}}{2(c-\gamma)}+\right.$$
$$+\sum_{j=n[k]}^{l-2}\left(\left(\left(\int_0^{\frac{x^i_{k,j}}{1-\alpha}}\tilde{V_3^1}(x^i_{k,j}-(1-\alpha)y,x^r_{k,j}-\alpha y)f(y)dy+\right.\right.\right.$$
$$+\int_0^{\frac{x^i_{k,j+1}}{1-\alpha}}\tilde{V_3^1}(x^i_{k,j+1}-(1-\alpha)y,x^r_{k,j+1}-\alpha y)f(y)dy\right)\times\frac{x^i_{k,j+1}-x^i_{k,j}}{2(c-\gamma)}\Bigg)+$$
$$+\left(\left(\int_0^{\frac{x^i_{k,l-1}}{1-\alpha}}\tilde{V_3^1}(x^i_{k,l-1}-(1-\alpha)y,x^r_{k,l-1}-\alpha y)f(y)dy+f\left(\frac{x^i_{n^*[l],l}}{1-\alpha}\right)\tilde{V_3^1}_{k+1,l}\times\frac{x^i_{n^*[l],l}}{2(1-\alpha)}+\right.\right.$$
$$\sum_{s=0}^{n^*[l]-k-2}\left(\left(\left(f\left(\frac{x^i_{n^*[l]-s,l}}{1-\alpha}\right)\tilde{V_3^1}_{k+1+s,l}+f\left(\frac{x^i_{n^*[l]-s-1,l}}{1-\alpha}\right)\tilde{V_3^1}_{k+2+s,l}\right)\times\frac{x^i_{n^*[l]-s-1,l}-x^i_{n^*[l]-s,l}}{2(1-\alpha)}\right)\right.$$
$$+\left(\left(f\left(\frac{x^i_{k+1,l}}{1-\alpha}\right)\tilde{V_3^1}_{n^*[l],l}+f\left(\frac{x^i_{k,l}}{1-\alpha}\right)\tilde{V_3^1}_{0,l}\right)\times\frac{x^i_{k,l}-x^i_{k+1,l}}{2(1-\alpha)}\right)\frac{x^i_{k,l}-x^i_{k,l-1}}{2(c-\gamma)}\Bigg)-$$
$$-\lambda\left(\int_{\frac{z^r_k}{\alpha}}^{z^r_k}V_1(z^r_k-y)f(y)dy+\int_{\frac{x^r_{k,n[k]}}{\alpha}}^{z^r_k+\frac{c-r}{c-\gamma}x^i_{k,n[k]}}V_1\left(z^r_k+\frac{c-r}{c-\gamma}x^i_{k,n[k]}-y\right)f(y)dy\right)$$

$$\times \frac{x^i_{k,n[k]}}{2(c-\gamma)} + \sum_{j=n[k]}^{l-1}\Bigg(\Bigg(\Bigg(\int_{\frac{x^r_{k,j}}{\alpha}}^{z^r_k+\frac{c-r}{c-\gamma}x^i_{k,j}} V_1\left(z^r_k + \frac{c-r}{c-\gamma}x^i_{k,j} - y\right) f(y)dy +$$
$$+ \int_{\frac{x^r_{k,j+1}}{\alpha}}^{z^r_k+\frac{c-r}{c-\gamma}x^i_{k,j+1}} V_1\left(z^r_k + \frac{c-r}{c-\gamma}x^i_{k,j+1} - y\right) f(y)dy\Bigg)\frac{x^i_{k,j+1} - x^i_{k,j}}{2(c-\gamma)}\Bigg)\Bigg] \Bigg/ \tag{38}$$
$$\left(1 - \frac{\lambda}{2}\frac{x^i_{k,l} - x^i_{k,l.1}}{c-\gamma}\left(1 - f(0)\frac{x^i_{n^*[l],l}}{2(1-\alpha)}\right)\right).$$

Proof. Follows directly from the Numerical Schemes 1 and 3, and their respective proofs. □

The Numerical Schemes 1 and 4 allow us to obtain numerical results for $\tilde{V_3^1}$ and $\tilde{V_3^2}$. A linear combination of these functions is employed to derive values of V_3. However, this requires the definition of a new function.

Definition 3.4. ($V_4(x^r)$**)**: Let V_4 be a function such that $V_4 : \mathbb{R} \to [0, 1]$ and

$$V_4(x^r) = P\{\tau^{0,r} = +\infty | X_0^{0,r} = x^r\}. \tag{39}$$

Through analogous procedures to the ones applied to V_1 throughout Section 3, we are able to conclude that V_4 can be written in integral form as

$$V_4(x^r) = 1 - \frac{\lambda\alpha E[Y]}{\gamma - r} + \frac{\lambda\alpha}{\gamma - r}\int_0^{\frac{x^r}{\alpha}} V_4(x^r - \alpha y)(1 - F(y))dy, \tag{40}$$

and that this function can be approximated on the points of the integration grid by the following expression

$$V_4(x^r_{k,l,j}) = \Bigg[1 - \frac{\lambda\alpha E[Y]}{\gamma - r} + \frac{\lambda\alpha}{\gamma - r}\Bigg(V_4(x^r_{k,l,j-1})(1 - F(y_2)) \times \frac{y_2}{2} +$$
$$+ \sum_{s=1}^{j-2}\left((V_4(x^r_{k,l,j-s})(1 - F(y_{s+1})) + V_4(x^r_{k,l,j-s-1})(1 - F(y_{s+2}))) \times \frac{y_{s+2} - y_{s+1}}{2}\right)\Bigg)\Bigg]$$
$$\Bigg/\left[1 - \frac{\lambda\alpha}{\gamma - r}(1 - F(0))\frac{y_2}{2}\right], \tag{41}$$

where y_j are the elements of an integration vector with dimension Q such that $y_1 = 0$ and $y_Q = \frac{x^r_{k,l}}{\alpha}$.

We are now able to present the formula that will allow for the calculation of the probabilities given by V_3.

Proposition 3.16. (Approximation of $V_3(x^i, x_r)$): $V_3(x^i, x_r)$ can be obtained through the following linear combination

$$V_3(x^i, x_r) = A\tilde{V_3^1}(x^i, x_r) + B(x^r)\tilde{V_3^2}(x^i, x_r), \tag{42}$$

where $A = \frac{c-\lambda E[Y]}{c}$ and $B = \frac{1-V_4(x^r)-\frac{c-\lambda E[Y]}{c}\tilde{V_3^1}(+\infty,x^r)}{\tilde{V_3^2}(+\infty,x^r)}$.

Proof. Condition (42) follows directly from the definition of the systems of equations (26) and (27), if A and $B(x^r)$ are such that

$$\begin{pmatrix} V_1 \\ V_3 \end{pmatrix} = \begin{pmatrix} A \\ B(x^r) \end{pmatrix} + \begin{pmatrix} \Lambda V_1 \\ \Theta V_1 + \Psi V_3 \end{pmatrix}.$$

From Proposition 3.11. we know that $A = \frac{c-\lambda E[Y]}{c}$.
In order to derive an expression for $B(x^r)$, we note that

$$\lim_{x^i \to \infty} V_3(x^i, x^r) = P\{\tau^{0,r} < +\infty | X_0^{0,r} = x^r\} = 1 - V_4(x^r).$$

As such,

$$\frac{c-\lambda E[Y]}{c}\tilde{V_3^1}(+\infty, x^r) + B(x^r)\tilde{V_3^2}(+\infty, x^r) = 1 - V_4(x^r),$$

and the expression of $B(x^r)$ follows. □

Example 3.3.: Under the same setting as the previous examples $A = \frac{2}{3}$, and we estimate $V_4(8) = 0.924636$, $\tilde{V_3^1}(40, 8) = -2.26259$ and $\tilde{V_3^2}(40, 8) = 3.70911$. As for $\tilde{V_3^1}(+\infty, 8)$ and $\tilde{V_3^2}(+\infty, 8)$ we estimate values of -4 and 5.83, respectively. The reservations expressed regarding computational power in example 3.1. are also applicable regarding these estimations. Nonetheless, these values result in a 0.470331 estimate of B and resulting final value for $V_3(40, 8)$ of 0.236121. This, in conjunction with the results found in Example 3.1., and applying equation (9), gives an estimation of around 43.6% for the probability of ruin of the direct insurer in an infinite time horizon. □

4 EQUILIBRIUM CONDITIONS

We present a game with two participants, insurer and reinsurer, that interact by entering a reinsurance contract, and have conflicting interests, that are expressed by each party's goal of minimizing the respective ruin probabilities. Given the system of equations describing the behaviour of the ruin probability of insurer and reinsurer given in Section 3, we now aim to find an optimal strategy, in a Pareto sense, for both parties. This means that the optimal solution is such that there is no other feasible strategy that improves the ruin probability of one competitor without negatively affecting the probability of ruin of the other, thus achieving a market equilibrium. The reinsurer selects the reinsurance premium, while the insurer chooses a ceded portion of claims. As such, a strategy is given by the pair (α, γ). We now formally define the concept of equilibrium point within the context of the problem and derive an optimality condition.

Definition 4.1. (Equilibrium point): A strategy $(\hat{\alpha}, \hat{\gamma})$ is an equilibrium point of the game if and only if the following conditions are met:

1. $\bar{V}_r(\hat{\alpha} + h_1, \hat{\gamma} + h_2, x^i, x^r) < \bar{V}_r(\hat{\alpha}, \hat{\gamma}, x^i, x^r) \Rightarrow \bar{V}_i(\hat{\alpha} + h_1, \hat{\gamma} + h_2, x^i, x^r) > V_i(\hat{\alpha}, \hat{\gamma}, x^i, x^r)$
2. $\bar{V}_i(\hat{\alpha} + h_1, \hat{\gamma} + h_2, x^i, x^r) < \bar{V}_i(\hat{\alpha}, \hat{\gamma}, x^i, x^r) \Rightarrow \bar{V}_r(\hat{\alpha} + h_1, \hat{\gamma} + h_2, x^i, x^r) > V_r(\hat{\alpha}, \hat{\gamma}, x^i, x^r)$

This formalizes the idea of the Pareto equilibrium, where it is impossible to find a solution where all competitors are better off.

Proposition 4.1. (Equilibrium condition I): The equilibrium condition is an equation of the type

$$b \nabla_{\alpha,\gamma} \bar{V}_r = \nabla_{\alpha,\gamma} \bar{V}_i, \tag{43}$$

where $b < 0$.

Proof. Follows directly from definition 4.1. □

The previous proposition means that, in equilibrium, the gradients of the probabilities of ruin of insurer and reinsurer are collinear but point to opposite directions, which directly follows from the concept of Pareto optimality.

Proposition 4.2. (Equilibrium condition II): The equilibrium condition is

$$\nabla_{\alpha,\gamma} V_3 = d \nabla_{\alpha,\gamma} V_2, \tag{44}$$

where $d < -1$.

Proof. From proposition 4.1. and using equations (10) and (11), we know that

$$b \nabla_{\alpha,\gamma}(1 - V_2) = \nabla_{\alpha,\gamma}(1 - V_2 - V_3),$$

with $b < 0$.
This means that

$$-b \nabla_{\alpha,\gamma} V_2 = -\nabla_{\alpha,\gamma} V_2 - \nabla_{\alpha,\gamma} V_3 \Leftrightarrow \nabla_{\alpha,\gamma} V_3 = (b - 1) \nabla_{\alpha,\gamma} V_2.$$

Since $b < 0$, $b - 1$ can be substituted by a constant $d < -1$. □

4.1 Algorithm for Numerical Optimization

In order to be able to derive the optimal solution for this reinsurance problem, it is necessary to create an algorithm that can obtain such solution based on the implementation of the equilibrium condition presented in Proposition 4.2.. Since neither V_2 or V_3 have explicit boundary conditions, we will be once again resorting to the auxiliary functions $\tilde{V_2}$ and $\tilde{V_3}$ in the algorithm. To obtain $\nabla_{\alpha,\gamma} V_2$ for a fixed set of parameters we have Algorithm 1.

Algorithm 1 Calculating $\nabla_{\alpha,\gamma} V_2$

DERIVATIVES OF $\tilde{V_2}$ IN ORDER TO THE INITIAL SURPLUSES

Solve $\frac{\partial \tilde{V_2}}{\partial x^i} = \frac{\partial \Psi}{\partial x^i}\tilde{V_2} + \Psi \frac{\partial \tilde{V_2}}{\partial x^i}$

Solve $\frac{\partial \tilde{V_2}}{\partial x^r} = \frac{\partial \Psi}{\partial x^r}\tilde{V_2} + \Psi \frac{\partial \tilde{V_2}}{\partial x^r}$

DERIVATIVE OPERATORS OF Ψ IN ORDER TO THE TARGET PARAMETERS

Define $\frac{\partial \Psi}{\partial \alpha}\tilde{V_2} = \frac{\partial \bar{\Psi}}{\partial \alpha}(\tilde{V_2}, \frac{\partial \tilde{V_2}}{\partial x^r}, \frac{\partial \tilde{V_2}}{\partial x^i})$

Define $\frac{\partial \Psi}{\partial \gamma}\tilde{V_2} = \frac{\partial \bar{\Psi}}{\partial \gamma}(\tilde{V_2}, \frac{\partial \tilde{V_2}}{\partial x^r}, \frac{\partial \tilde{V_2}}{\partial x^i})$

DERIVATIVES OF $\tilde{V_2}$ IN ORDER TO THE TARGET PARAMETERS

Solve $\frac{\partial \tilde{V_2}}{\partial \alpha} = \frac{\partial \Psi}{\partial \alpha}\tilde{V_2} + \Psi \frac{\partial \tilde{V_2}}{\partial \alpha}$

Solve $\frac{\partial \tilde{V_2}}{\partial \gamma} = \frac{\partial \Psi}{\partial \gamma}\tilde{V_2} + \Psi \frac{\partial \tilde{V_2}}{\partial \gamma}$

DERIVATIVES OF V_2 IN ORDER TO THE TARGET PARAMETERS

Do $\frac{\partial V_2}{\partial \alpha}(u) = \left[\frac{\partial \tilde{V_2}}{\partial \alpha}(u)\tilde{V_2}(\infty) - \tilde{V_2}(u)\frac{\partial \tilde{V_2}}{\partial \alpha}(\infty)\right] \Big/ \tilde{V_2}(\infty)^2$

Do $\frac{\partial V_2}{\partial \gamma}(u) = \left[\frac{\partial \tilde{V_2}}{\partial \gamma}(u)\tilde{V_2}(\infty) - \tilde{V_2}(u)\frac{\partial \tilde{V_2}}{\partial \gamma}(\infty)\right] \Big/ \tilde{V_2}(\infty)^2$

OBTAIN $\nabla_{\alpha,\gamma} V_2$

Do $\nabla_{\alpha,\gamma} V_2(x^i, x^r) = (\frac{\partial V_2}{\partial \alpha}(x^i, x^r), \frac{\partial V_2}{\partial \gamma}(x^i, x^r))$

Notice that all equations that must be solved in Algorithm 1 have the form

$$g = \Phi + \Psi g.$$

Thus, if the different functions Φ are estimated on the appropriate points, Numerical Scheme 1 can be employed to solve the necessary equations.

Algorithm 2 presents a similar scheme to obtain $\nabla_{\alpha,\gamma} V_3$.

Algorithm 1 and Algorithm 2 are able to obtain the value of the necessary derivatives to calculate the gradient vector featured in the equilibrium condition II. The spirit of the overall optimization algorithm would be to start with some initial guesses for the target parameters, α and γ, that define the reinsurance treaty, and calculate $\nabla_{\alpha,\gamma} V_2$ and $\nabla_{\alpha,\gamma} V_3$.

Then, a stopping criteria would be used in order to decide if this solution is such that the equilibrium condition is closely enough verified. If not, the algorithm would reiterate after re-adjusting the parameters, until the optimization criteria is met.

Algorithm 2 Calculating $\nabla_{\alpha,\gamma} V_3$

DERIVATIVES OF V_1 IN ORDER TO THE TARGET PARAMETERS

Solve $\frac{\partial V_1}{\partial \alpha} = \frac{\lambda}{c} \int_0^z \frac{\partial V_1}{\partial \alpha}(z-y)(1-F(y))dy$

Solve $\frac{\partial V_1}{\partial \gamma} = \frac{\lambda}{c} \int_0^z \frac{\partial V_1}{\partial \gamma}(z-y)(1-F(y))dy$

DERIVATIVE OPERATORS OF Θ IN ORDER TO THE TARGET PARAMETERS

Calculate $\frac{\partial \Theta}{\partial \alpha} V_1$

Calculate $\frac{\partial \Theta}{\partial \gamma} V_1$

DERIVATIVE OPERATORS OF Θ IN ORDER TO THE INITIAL SURPLUSES

Calculate $\frac{\partial \Theta}{\partial x^i} V_1$

Calculate $\frac{\partial \Theta}{\partial x^r} V_1$

DERIVATIVES OF $\tilde{V_3}$ IN ORDER TO THE INITIAL SURPLUSES

Solve $\frac{\partial \tilde{V_3}}{\partial x^i} = \frac{\partial \Psi}{\partial x^i} \tilde{V_3} + \frac{\partial \Theta}{\partial x^i} V_1 + \Psi \frac{\partial \tilde{V_3}}{\partial x^i}$

Solve $\frac{\partial \tilde{V_3}}{\partial x^r} = \frac{\partial \Psi}{\partial x^r} \tilde{V_3} + \frac{\partial \Theta}{\partial x^r} V_1 + \Psi \frac{\partial \tilde{V_3}}{\partial x^r}$

DERIVATIVE OPERATORS OF Ψ IN ORDER TO THE TARGET PARAMETERS

Define $\frac{\partial \Psi}{\partial \alpha} \tilde{V_3} = \frac{\partial \bar{\Psi}}{\partial \alpha}(\tilde{V_3}, \frac{\partial \tilde{V_3}}{\partial x^r}, \frac{\partial \tilde{V_3}}{\partial x^i})$

Define $\frac{\partial \Psi}{\partial \gamma} \tilde{V_3} = \frac{\partial \bar{\Psi}}{\partial \gamma}(\tilde{V_3}, \frac{\partial \tilde{V_3}}{\partial x^r}, \frac{\partial \tilde{V_3}}{\partial x^i})$

DERIVATIVES OF $\tilde{V_3}$ IN ORDER TO THE TARGET PARAMETERS

Solve $\frac{\partial \tilde{V_3}}{\partial \alpha} = \frac{\partial \Psi}{\partial \alpha} \tilde{V_3} + \frac{\partial \Theta}{\partial \alpha} V_1 + \Psi \frac{\partial \tilde{V_3}}{\partial \alpha} + \Theta \frac{\partial V_1}{\partial \alpha}$

Solve $\frac{\partial \tilde{V_3}}{\partial \gamma} = \frac{\partial \Psi}{\partial \gamma} \tilde{V_3} + \frac{\partial \Theta}{\partial \gamma} V_1 + \Psi \frac{\partial \tilde{V_3}}{\partial \gamma} + \Theta \frac{\partial V_1}{\partial \gamma}$

DERIVATIVES OF V_3 IN ORDER TO THE TARGET PARAMETERS

Do $\frac{\partial V_3}{\partial \alpha}(u) = \left[\frac{\partial \tilde{V_3}}{\partial \alpha}(u) \tilde{V_3}(\infty) - \tilde{V_3}(u) \frac{\partial \tilde{V_3}}{\partial \alpha}(\infty) \right] \Big/ \tilde{V_3}(\infty)^2$

Do $\frac{\partial V_3}{\partial \gamma}(u) = \left[\frac{\partial \tilde{V_3}}{\partial \gamma}(u) \tilde{V_3}(\infty) - \tilde{V_3}(u) \frac{\partial \tilde{V_3}}{\partial \gamma}(\infty) \right] \Big/ \tilde{V_3}(\infty)^2$

OBTAIN $\nabla_{\alpha,\gamma} V_3$

Do $\nabla_{\alpha,\gamma} V_3(x^i, x^r) = (\frac{\partial V_3}{\partial \alpha}(x^i, x^r), \frac{\partial V_3}{\partial \gamma}(x^i, x^r))$

5 CONCLUSIONS AND FUTURE RESEARCH

In this thesis, we created a market with two insurers, one taking the role of direct insurer to some risk pool present in society, and the other taking the role of the reinsurer. Both firms then engage in a proportional reinsurance contract. Following a strand of literature that argues that such policy needs to benefit both parties involved, we seeked to optimize the reinsurance parameters, *i.e.*, the ceded proportion of each claim and the reinsurance premium, in order to have direct insurer and reinsurer minimizing their ruin probability and achieving a Pareto optimum in the market. To the best of our knowledge, this is an original approach.

To do so, we defined surplus processes for the involved parties, inspired by the Lundberg process typically used in risk theory. From there, we derived a set of integro-differential equations and the respective boundary conditions, describing the behaviour of the ruin probabilities of the cedent and of the reinsurer as a function of their initial surpluses. We assumed that, if the first line insurer reaches ruin before the reinsurer, then the latter no longer provides risk coverage. We were able to prove that, under this assumption, the ruin probability of the reinsurer is always at least as large as the ruin probability of the direct insurer. Furthermore, we found that as the initial surpluses of both firms tend to infinity, their respective ruin probabilities tend to zero, which is a fairly intuitive result.

Approximated solutions for the derived set of integro-differential equations were presented and some numerical illustrations were put forward, implementing said solutions. It is worth noticing that the integration grids required to obtain good solutions for this problem grow rapidly as a function of the initial surpluses and are fairly sensitive to the parameters, so this can require a lot of computational power.

We then derived Pareto equilibrium conditions for the problem at hand and proposed a possible algorithm to numerically obtain optimal solutions.

The model presented is conceptually simple and has several limitations, such as not considering investment decisions of the cedent and the reinsurer. Also, the pricing decision of the reinsurer does not include any consideration of market conditions. In addition, Schlesinger & Doherty (1985) observe that when taking risk, an insurer should not only take into account the statistical properties of the risk itself, but also any existing correlations between that risk and the firm's wealth. In this study we consider income to be deterministic, so this effect cannot be contemplated. Further research could be developed to address these limitations as well as expanding the theoretical market to include more players. It would also be of interest to continue the development of the numerical solutions and schemes proposed and to study the sensitivity of the ruin probabilities to the parameters of the problem.

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Appendix A. Proof of Proposition 3.3.

By definition,

$$f_2(u) = P\{\tau^{0,i} = +\infty, \tau^{0,r} = +\infty | X_0^{0,i} = z^i + (c-\gamma)u, X_0^{0,r} = z^r + (\gamma - r)u)\}$$
$$= 1 - P\{\tau^{0,i} < +\infty \vee \tau^{0,r} < +\infty | X_0^{0,i} = z^i + (c-\gamma)u, X_0^{0,r} = z^r + (\gamma - r)u)\}.$$

To simplify the notation, let $P\{\cdot | X_0^{0,i} = z^i + (c-\gamma)u, X_0^{0,r} = z^r + (\gamma - r)u)\} = P_u\{\cdot\}$. We have

$$1 - P_u\{\tau^{0,i} < +\infty \vee \tau^{0,r} < +\infty\} =$$
$$= 1 - \sum_{n=1}^{+\infty} (P_u\{\tau^{0,i} = T_n, \tau^{0,r} > T_{n-1}\} + P_u\{\tau^{0,r} = T_n < \tau^{0,i}\}).$$

Using Fubini's theorem the sum can be separated into two series and the above expression becomes

$$1 - \sum_{n=1}^{+\infty} P_u\{\tau^{0,i} = T_n, \tau^{0,r} > T_{n-1}\} - \sum_{n=1}^{+\infty} P_u\{\tau^{0,r} = T_n < \tau^{0,i}\}.$$

The sum of absolutely continuous functions is absolutely continuous, so to prove the desired result it is enough to prove that the summed probabilities are absolutely continuous with respect to u.

$P_u\{\tau^{0,i} = T_n, \tau^{0,r} > T_{n-1}\}$ can be written as

$$P_u\{X_{T_1}^{0,i} \geq 0, X_{T_1}^{0,r} \geq 0, ..., X_{T_{n-1}}^{0,i} \geq 0, X_{T_{n-1}}^{0,r} \geq 0, X_{T_n}^{0,i} < 0\}$$

$$= P_u\{X_{T_1}^{0,i} = z^i + (c-\gamma)(u + t_1^*) - (1-\alpha)Y_1 \geq 0,$$
$$X_{T_1}^{0,r} = z^r + (\gamma - r)(u + t_1^*) - \alpha Y_1 \geq 0, ...,$$
$$..., X_{T_{n-1}}^{0,i} = z^i + (c-\gamma)(u + \textstyle\sum_{i=1}^{n-1} t_i^*) - (1-\alpha)\sum_{i=1}^{n-1} Y_i \geq 0,$$
$$X_{T_{n-1}}^{0,r} = z^r + (\gamma - r)(u + \textstyle\sum_{i=1}^{n-1} t_i^*) - \alpha \sum_{i=1}^{n-1} Y_i \geq 0,$$
$$X_{T_n}^{0,i} = z^i + (c-\gamma)(u + \textstyle\sum_{i=1}^{n} t_i^*) - (1-\alpha)\sum_{i=1}^{n} Y_i < 0\} =$$

$$= P_u\{Y_1 \leq \min\{\tfrac{z^i + (c-\gamma)(u+t_1^*)}{(1-\alpha)}, \tfrac{z^r + (\gamma - r)(u+t_1^*)}{\alpha}\}, ...,$$
$$..., Y_{n-1} \leq \min\{\tfrac{z^i + (c-\gamma)(u + \sum_{i=1}^{n-1} t_i^*)}{(1-\alpha)} - \textstyle\sum_{i=1}^{n-2} Y_i, \tfrac{z^r + (\gamma - r)(u + \sum_{i=1}^{n-1} t_i^*)}{\alpha} - \sum_{i=1}^{n-2} Y_i\},$$
$$Y_n > \tfrac{z^i + (c-\gamma)(u + \sum_{i=1}^{n} t_i^*)}{(1-\alpha)} - \textstyle\sum_{i=1}^{n-1} Y_i\}.$$

Let $h_i(p,q) = \frac{z^i + (c-\gamma)p}{(1-\alpha)} - q$, $h_r(p,q) = \frac{z^r + (\gamma - r)p}{\alpha} - q$ and $g(p,q) = \min\{h_i(p,q), h_r(p,q)\}$. This probability can be expressed in integral form as

$$\int_{[0,+\infty[^n} \int_0^{g(u+t_1^*,0)} \int_0^{g(u+t_1^*+t_2^*,Y_1)} ... \int_0^{g(u+\sum_{i=1}^{n-1} t_i^*, \sum_{i=1}^{n-2} Y_i)}$$
$$\int_{h_i(u+\sum_{i=1}^{n} t_i^*, \sum_{i=1}^{n-1} Y_i)}^{+\infty} dF(Y_n)...dF(Y_1)\lambda^n e^{-\lambda \sum_{i=1}^{n} t_i^*} dt_n^* ... dt_1^* =$$

$$= \int_{[0,+\infty[^n} \int_0^{g(u+t_1^*,0)} \int_0^{g(u+t_1^*+t_2^*,Y_1)} ... \int_0^{g(u+\sum_{i=1}^{n-1} t_i^*, \sum_{i=1}^{n-2} Y_i)} (1 - F(h_i(u +$$

$$\sum_{i=1}^{n} t_i^*, \sum_{i=1}^{n-1} Y_i)))dF(Y_n)...dF(Y_1)\lambda^n e^{-\lambda \sum_{i=1}^{n} t_i^*} dt_n^*...dt_1^*.$$

If we proceed to the substitution $s = u + t_1^*$, the above expression becomes

$$e^{\lambda u} \int_u^{+\infty} \int_{[0,+\infty[^{n-1}} \int_0^{g(s,0)} \int_0^{g(s+t_2^*,Y_1)} ... \int_0^{g(s+\sum_{i=2}^{n-1} t_i^*, \sum_{i=1}^{n-2} Y_i)} (1 - F(h_i(s + \sum_{i=2}^{n} t_i^*, \sum_{i=1}^{n-1} Y_i)))dF(Y_{n-1})...dF(Y_1)\lambda^n e^{-\lambda(s+\sum_{i=2}^{n} t_i^*)} dt_n^*...ds.$$

As for $P_u\{\tau^{0,r} = T_n < \tau^{0,i}\}$, this probability can be written as

$$P_u\{X_{T_1}^{0,r} \geq 0, X_{T_1}^{0,i} \geq 0, ..., X_{T_{n-1}}^{0,r} \geq 0, X_{T_{n-1}}^{0,i} \geq 0, X_{T_n}^{0,r} < 0, X_{T_n}^{0,i} \geq 0\}.$$

Analogously to the proof for the first probability, this is equal to

$$\int_{[0,+\infty[^n} \int_0^{g(u+t_1^*,0)} \int_0^{g(u+t_1^*+t_2^*,Y_1)} ... \int_0^{g(u+\sum_{i=1}^{n-1} t_i^*, \sum_{i=1}^{n-2} Y_i)} \int_{h_r(u+\sum_{i=1}^{n} t_i^*, \sum_{i=1}^{n-1} Y_i)}^{h_i(u+\sum_{i=1}^{n} t_i^*, \sum_{i=1}^{n-1} Y_i)} dF(Y_n)...dF(Y_1)\lambda^n e^{-\lambda \sum_{i=1}^{n} t_i^*} dt_n^*...dt_1^* =$$

$$\int_{[0,+\infty[^n} \int_0^{g(u+t_1^*,0)} \int_0^{g(u+t_1^*+t_2^*,Y_1)} ... \int_0^{g(u+\sum_{i=1}^{n-1} t_i^*, \sum_{i=1}^{n-2} Y_i)} (F(h_i(u + \sum_{i=1}^{n} t_i^*, \sum_{i=1}^{n-1} Y_i)) - F(h_r(u + \sum_{i=1}^{n} t_i^*, \sum_{i=1}^{n-1} Y_i)))dF(Y_{n-1})...dF(Y_1)\lambda^n e^{-\lambda \sum_{i=1}^{n} t_i^*} dt_n^*...dt_1^*.$$

Using the same substitution as before, $s = u + t_1^*$, a similar result is obtained:

$$P_u\{\tau^{0,r} = T_n < \tau^{0,i}\} = e^{\lambda u} \int_u^{+\infty} \int_{[0,+\infty[^{n-1}} \int_0^{g(s,0)} \int_0^{g(s+t_2^*,Y_1)} ... \int_0^{g(s+\sum_{i=2}^{n-1} t_i^*, \sum_{i=1}^{n-2} Y_i)} (F(h_i(s + \sum_{i=2}^{n} t_i^*, \sum_{i=1}^{n-1} Y_i)) - F(h_r(s + \sum_{i=2}^{n} t_i^*, \sum_{i=1}^{n-1} Y_i)))dF(Y_{n-1})...dF(Y_1)\lambda^n e^{-\lambda(s+\sum_{i=2}^{n} t_i^*)} dt_n^*...ds.$$

Since $e^{\lambda u}$ is an absolutely continuous function and the product of absolutely continuous functions is absolutely continuous, the result is proved.

Appendix B. Proof of Proposition 3.7.

For a general increment h, we have:

$$V_3(x^i, x^r) = P\{\tau^{0,r} < \tau^{0,i}, \tau^{1,i} = +\infty | X_0^{0,i} = x^i, X_0^{0,r} = x^r\} =$$

$$= P\{\tau^{0,r} < \tau^{0,i}, \tau^{1,i} = +\infty, T_1 > h | X_0^{0,i} = x^i, X_0^{0,r} = x^r\} + P\{\tau^{0,r} < \tau^{0,i}, \tau^{1,i} = +\infty, T_1 \leq h < T_2 | X_0^{0,i} = x^i, X_0^{0,r} = x^r\} + o(h) =$$

$$= V_3(x^i + (c-\gamma)h, x^r + (\gamma - r)h)e^{-\lambda h} + P\{\tau^{0,r} = T_1 < \tau^{0,i}, \tau^{1,i} = +\infty, T_1 \leq h < T_2 | X_0^{0,i} = x^i, X_0^{0,r} = x^r\} + P\{T_1 < \tau^{0,r} < \tau^{0,i}, \tau^{1,i} = +\infty, T_1 \leq h < T_2 | X_0^{0,i} = x^i, X_0^{0,r} = x^r\} + o(h) =$$

$$= V_3(x^i + (c-\gamma)h, x^r + (\gamma - r)h)(1 - \lambda h) + P\{x^r + (\gamma - r)T_1 - \alpha Y_1 < 0, x^i + (c - \gamma)T_1 - (1-\alpha)Y_1 + x^r + (\gamma - r)T_1 - \alpha Y_1 \geq 0, \tau^{1,i} = +\infty, T_1 \leq h < T_2 | X_0^{0,i} = x^i, X_0^{0,r} = x^r\} + P\{x^r + (\gamma - r)T_1 - \alpha Y_1 \geq 0, x^i + (c-\gamma)T_1 - (1-\alpha)Y_1 \geq$$

$$0, \tau^{0,r} < \tau^{0,i}, \tau^{1,i} = +\infty, T_1 \le h < T_2 | X_0^{0,i} = x^i, X_0^{0,r} = x^r\} + o(h) =$$

$$= V_3(x^i + (c-\gamma)h, x^r + (\gamma - r)h) - \lambda V_3(x^i, x^r)h + P\{x^r + (\gamma - r)T_1 - \alpha Y_1 < 0, x^i + x^r + (c-r)T_1 - Y_1 \ge 0, \tau^{1,i} = +\infty, T_1 \le h < T_2 | X_0^{0,i} = x^i, X_0^{0,r} = x^r\} + P\{x^r + (\gamma - r)T_1 - \alpha Y_1 \ge 0, x^i + (c-\gamma)T_1 - (1-\alpha)Y_1 \ge 0, \tau^{0,r} < \tau^{0,i}, \tau^{1,i} = +\infty, T_1 \le h < T_2 | X_0^{0,i} = x^i, X_0^{0,r} = x^r\} + o(h) =$$

$$= V_3(x^i + (c-\gamma)h, x^r + (\gamma - r)h) - \lambda V_3(x^i, x^r)h + E[P\{\tau^{1,i} = +\infty | X_{\tau^r}^{1,i} = x^i + x^r + (c-r)T_1 - Y_1\} \cdot \chi_{\frac{x^r + (\gamma - r)T_1}{\alpha} < Y_1 \le x^i + x^r + (c-r)T_1} \cdot \chi_{T_1 \le h < T_2}] + E[V_3(x^i + (c-\gamma)T_1 - (1-\alpha)Y_1, x^r + (\gamma - r)T_1 - \alpha Y_1) \cdot \chi_{Y_1 \le \frac{x^r + (\gamma - r)T_1}{\alpha}} \cdot \chi_{Y_1 \le \frac{x^i + (c-\gamma)T_1}{1-\alpha}} \cdot \chi_{T_1 \le h < T_2}] + o(h) =$$
$$= V_3(x^i + (c-\gamma)h, x^r + (\gamma - r)h) - \lambda V_3(x^i, x^r)h + \int_0^h \int_{h-t_1}^{+\infty} \int_{\frac{x^r + (\gamma - r)t_1}{\alpha}}^{x^i + x^r + (c-r)t_1} P\{\tau^{1,i} = +\infty | X_{\tau^r}^{1,i} = x^i + x^r + (c-r)t_1 - y\} dF(y) \lambda e^{-\lambda t_2} dt_2 \lambda e^{-\lambda t_1} dt_1 + \int_0^h \int_{h-t_1}^{+\infty} \int_0^{\min\{\frac{x^r + (\gamma - r)t_1}{\alpha}, \frac{x^i + (c-\gamma)t_1}{1-\alpha}\}} V_3(x^i + (c-\gamma)t_1 - (1-\alpha)y, x^r + (\gamma - r)t_1 - \alpha y) dF(y) \lambda e^{-\lambda t_2} dt_2 \lambda e^{-\lambda t_1} dt_1 + o(h).$$

For a small value of h, the above expression approximates

$$V_3(x^i + (c-\gamma)h, x^r + (\gamma - r)h) - \lambda V_3(x^i, x^r)h + \lambda h \int_{\frac{x^r}{\alpha}}^{x^i + x^r} P\{\tau^{1,i} = +\infty | X_{\tau^r}^{1,i} = x^i + x^r - y\} dF(y) + \lambda h \int_0^{\min\{\frac{x^r}{\alpha}, \frac{x^i}{1-\alpha}\}} V_3(x^i - (1-\alpha)y, x^r - \alpha y) dF(y) + o(h).$$

Noticing that $P\{\tau^{1,i} = +\infty | X_{\tau^r}^{1,i} = x^i + x^r - y\} = V_1(x^i + x^r - y)$, this is equal to

$$V_3(x^i + (c-\gamma)h, x^r + (\gamma - r)h) - \lambda V_3(x^i, x^r)h + \lambda h \int_{\frac{x^r}{\alpha}}^{x^i + x^r} V_1(x^i + x^r - y) dF(y) + \lambda h \int_0^{\min\{\frac{x^r}{\alpha}, \frac{x^i}{1-\alpha}\}} V_3(x^i - (1-\alpha)y, x^r - \alpha y) dF(y) + o(h).$$

Taking the limit, as $h \to 0^+$,

$$0 = \lim_{h \to 0^+} \frac{V_3(x^i + (c-\gamma)h, x^r + (\gamma - r)h) - V_3(x^i, x^r)}{h} - \lambda V_3(x^i, x^r) + \lambda \int_{\frac{x^r}{\alpha}}^{x^i + x^r} V_1(x^i + x^r - y) dF(y) + \lambda \int_0^{\min\{\frac{x^r}{\alpha}, \frac{x^i}{1-\alpha}\}} V_3(x^i - (1-\alpha)y, x^r - \alpha y) dF(y) =$$

$$= (\frac{\partial V_3}{\partial x^i}(c-\gamma) + \frac{\partial V_3}{\partial x^r}(\gamma - r))(x^i, x^r) - \lambda V_3(x^i, x^r) + \lambda \int_{\frac{x^r}{\alpha}}^{x^i + x^r} V_1(x^i + x^r - y) dF(y) + \lambda \int_0^{\min\{\frac{x^r}{\alpha}, \frac{x^i}{1-\alpha}\}} V_3(x^i - (1-\alpha)y, x^r - \alpha y) dF(y),$$

which is equivalent to the proposed condition

$$(\frac{\partial V_3}{\partial x^i}(c-\gamma) + \frac{\partial V_3}{\partial x^r}(\gamma - r))_{x^i, x^r} = \lambda (V_3(x^i, x^r) - \\ - \int_0^{\min\{\frac{x^r}{\alpha}, \frac{x^i}{1-\alpha}\}} V_3(x^i - (1-\alpha)Y, x^r - \alpha Y) dF(y) - \int_{\frac{x^r}{\alpha}}^{x^i + x^r} V_1(x^i + x^r - y) dF(y))$$

Appendix C. Continuation of the proof of Proposition 3.9.

Let $A_r = \{\omega \in \Omega : \lim_{t\to+\infty} \frac{X_t^{0,r}(\omega)}{t} = (\gamma - r) - \alpha\lambda E[Y]\}$, where Ω is the sample space. Since we assume $(\gamma - r) > \alpha\lambda E[Y]$, we have that

$$\exists k > 0 : \forall t > k, \frac{X_t^{0,r}}{t} > 0,$$

which equivalent to saying

$$\exists k > 0 : \forall t > k, X_t^{0,r} > 0.$$

Let $B_k^r = \{\omega \in \Omega : t > k \implies X_t^{0,r} > 0\}$. If any sample path belongs to A_r, then there is a value k such that the sample path belongs to B_k^r. Therefore A_r is contained in $\cup_{k=1}^{\infty} B_k^r$. Since by the law of large numbers $P(A^r) = 1$, then $P(\cup_{k=1}^{\infty} B_k^r) = 1$.
Defining T^r as

$$T^r = \begin{cases} sup\{t > 0 : X_t^{0,r} < 0\} & \text{if } \exists t > 0 : X_t^{0,r} < 0 \\ 0 & \text{if } \forall t > 0, X_t^{0,r} \geq 0, \end{cases}$$

the fact that $P(\cup_{k=1}^{\infty} B_k^r) = 1$ means that $P(T^r < +\infty) = 1$. Let $Z_t^{0,r} = (\gamma - r)t - \alpha \sum_{i=1}^{N_t} Y_i$. Then,

$$\begin{aligned} P\{\tau^{0,r} < +\infty | X_0^{0,i} = x^i, X_0^{0,r} = x^r\} = \\ = P\{\exists 0 \leq t < +\infty : Z_t^{0,r} < -x^r\} = \\ = P\{inf_{t\geq 0} Z_t^{0,r} < -x^r\} = \\ = P\{inf_{t\in[0,T^r]} Z_t^{0,r} < -x^r\} \\ = \sum_{n=1}^{+\infty} P\{inf_{t\in[0,T^r]} Z_t^{0,r} < -x^r, N_t = n\} \leq \\ \leq \sum_{n=1}^{+\infty} P\{T^r < +\infty, (\gamma - r)t - \alpha \sum_{n=1}^{n} Y_i \leq -x^r\}. \end{aligned}$$

Since $\sum_{n=1}^{+\infty} P\{T^r < +\infty, (\gamma - r)t - \alpha \sum_{n=1}^{n} Y_i \leq x^r\} \to 0$ as $x^r \to +\infty, \forall x^i \in \mathbb{R}$, then $P\{\tau^{0,r} < +\infty | X_0^{0,i} = x^i, X_0^{0,r} = x^r\}$ also tends to 0.

Appendix D. Proof for Numerical Scheme 1

If $\min\{\frac{z^r + t(\gamma - r)}{\alpha}, \frac{z^i + t(c-\gamma)}{1-\alpha}\} = \frac{(\gamma - r)t}{\alpha}$, we have that, given an integration grid that intersects lines from the parameterizations (16) and (28),

$$\begin{aligned} z^r = 0, \\ z^i = x^i - \tfrac{c-\gamma}{\gamma - r} x^r, \\ u = \tfrac{x^r}{\gamma - r}. \end{aligned}$$

Substituting this into the equation $g = \Phi(x^i, x^r) + \Psi g$, we obtain

$$\begin{aligned} \tilde{V_2}(x^i, x^r) = \Phi(x^i, x^r) + \lambda \int_0^{\frac{x^r}{\gamma - r}} \tilde{V_2}(x^i - \tfrac{c-\gamma}{\gamma - r} x^r + (c - \gamma)t, (\gamma - r)t) dt - \\ -\lambda \int_0^{\frac{x^r}{\gamma - r}} \int_0^{\frac{(\gamma - r)t}{\alpha}} \tilde{V_2}(x^i - \tfrac{c-\gamma}{\gamma - r} x^r + (c - \gamma)t - (1 - \alpha)y, (\gamma - r)t - \alpha y) dF(y) dt. \end{aligned}$$

Proceeding first with the discretization on t over the lines of the integration grid, this gives rise to the following approximation:

$$g_{k,l} \approx \Phi_{k,l} + \lambda\Bigg[(g_{k,0} + g_{k,n[k]}) \times \frac{x^r_{k,n[k]}}{2(\gamma-r)} + \sum_{j=n[k]}^{l-2}\Bigg((g_{k,j} + g_{k,j+1}) \times \frac{x^r_{k,j+1}-x^r_{k,j}}{2(\gamma-r)}\Bigg)+$$

$$+(g_{k,l-1} + g_{k,l}) \times \frac{x^r_{k,l}-x^r_{k,l-1}}{2(\gamma-r)}\Bigg] - \lambda\Bigg[\Bigg(\int_0^0 g(0-(1-\alpha)y, 0-\alpha y)f(y)dy+$$

$$+\int_0^{\frac{x^r_{k,n[k]}}{\alpha}} g(x^i_{k,n[k]}-(1-\alpha)y, x^r_{k,n[k]}-\alpha y)f(y)dy\Bigg) \times \frac{x^r_{k,n[k]}}{2(\gamma-r)}+$$

$$+\sum_{j=n[k]}^{l-2}\Bigg(\Bigg(\int_0^{\frac{x^r_{k,j}}{\alpha}} g(x^i_{k,j}-(1-\alpha)y, x^r_{k,j}-\alpha y)f(y)dy + \int_0^{\frac{x^r_{k,j+1}}{\alpha}} g(x^i_{k,j+1}-(1-$$

$$\alpha)y, x^r_{k,j+1}-\alpha y)f(y)dy\Bigg) \times \frac{x^r_{k,j+1}-x^r_{k,j}}{2(\gamma-r)}\Bigg)+$$

$$+\Bigg(\int_0^{\frac{x^r_{k,l-1}}{\alpha}} g(x^i_{k,l-1}-(1-\alpha)y, x^r_{k,l-1}-\alpha y)f(y)dy + \int_0^{\frac{x^r_{k,l}}{\alpha}} g(x^i_{k,l}-(1-\alpha)y, x^r_{k,l}-$$

$$\alpha y)f(y)dy\Bigg) \times \frac{x^r_{k,l}-x^r_{k,l-1}}{2(\gamma-r)}\Bigg].$$

Simplifying some expressions and seeking to isolate $g_{k,l}$, we can write

$$g_{k,l} \approx \Phi_{k,l} + \lambda\Bigg[\int_0^{\frac{x^r_{k,l-1}}{\gamma-r}} g_{k,l-1}dt + (g_{k,l-1} + g_{k,l}) \times \frac{x^r_{k,l}-x^r_{k,l-1}}{2(\gamma-r)}\Bigg]-$$

$$-\lambda\Bigg[\int_0^{\frac{x^r_{k,n[k]}}{\alpha}} g(x^i_{k,n[k]}-(1-\alpha)y, x^r_{k,n[k]}-\alpha y)f(y)dy \times \frac{x^r_{k,n[k]}}{2(\gamma-r)}+$$

$$+\sum_{j=n[k]}^{l-2}\Bigg(\Bigg(\int_0^{\frac{x^r_{k,j}}{\alpha}} g(x^i_{k,j}-(1-\alpha)y, x^r_{k,j}-\alpha y)f(y)dy + \int_0^{\frac{x^r_{k,j+1}}{\alpha}} g(x^i_{k,j+1}-(1-$$

$$\alpha)y, x^r_{k,j+1}-\alpha y)f(y)dy\Bigg) \times \frac{x^r_{k,j+1}-x^r_{k,j}}{2(\gamma-r)}\Bigg)+$$

$$\Bigg(\int_0^{\frac{x^r_{k,l-1}}{\alpha}} g(x^i_{k,l-1}-(1-\alpha)y, x^r_{k,l-1}-\alpha y)f(y)dy + \Bigg(f(0)g_{k,l} + f(\frac{x^r_{n[l],l}}{\alpha})g_{k+1,l}\Bigg)\frac{x^r_{n[l],l}}{2\alpha} +$$

$$\sum_{s=0}^{n[l]-k-2}\Bigg(\Bigg(f(\frac{x^r_{n[l]-s,l}}{\alpha})g_{k+1+s,l} + f(\frac{x^r_{n[l]-s-1,l}}{\alpha})g_{k+2+s,l}\Bigg)\frac{x^r_{n[l]-s-1,l}-x^r_{n[l]-s,l}}{2(\alpha)}\Bigg)$$

$$\Bigg(f(\frac{x^r_{k+1,l}}{\alpha})g_{n[l],l} + f(\frac{x^r_{k,l}}{\alpha})g_{0,l}\Bigg) \times \frac{x^r_{k,l}-x^r_{k+1,l}}{2\alpha}\Bigg)\frac{x^r_{k,l}-x^r_{k,l-1}}{2(\gamma-r))}\Bigg].$$

Moving the terms with $g_{k,l}$ to the left-hand side and dividing by the resulting coefficient, we obtain equation (29). An analogous proof can be done for equation (30) and is here omitted. The other integral approximations result directly from the application of the trapezium rule.