MASTER

MATHEMATICAL FINANCE

MASTER'S FINAL WORK

DISSERTATION

RESIDUE SUM FORMULA FOR PRICING OPTIONS UNDER THE VARIANCE GAMMA MODEL

PEDRO MARIA ULISSES DOS SANTOS JALHAY FEBRER

NOVEMBER - 2020

LISBON SCHOOL OF ECONOMICS & MANAGEMENT
UNIVERSIDADE DE LISBOA

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ABSTRACT

The main result of this dissertation is the proof of the triple sum series formula for the price of an European call option driven by the Variance Gamma process. With this intention, we present some notions and properties of Lévy processes and multidimensional complex analysis, with emphasis on the application of residue calculus to the Mellin-Barnes Integral. Subsequently, we construct the Mellin-Barnes integral representation, in $\mathbb{C}^3$, for the price of the option and, buttressed with the aforementioned residue calculus, we deduce the triple sum series representation for the price of the European option and its corresponding greeks. Finally, with the use of the new formula, some values for a particular case study are computed and discussed.

KEYWORDS: Lévy Process; Variance Gamma Process; Multidimensional Complex Analysis; Mellin Transform; Option Pricing.

RESUMO

O resultado principal desta dissertação é a demonstração da fórmula de serie de soma tripla para o preço de uma opção Europeia induzido por um processo Variance Gamma. Com esta intenção, apresentamos certas propriedades e noções sobre processos de Lévy e análise complexa multidimensional, dando ênfase à aplicação do cálculo de resíduos ao integral Mellin-Barnes. Subsequentemente, iremos construir a representação na forma do integral Mellin-Barnes, em $\mathbb{C}^3$, para o preço de uma opção e, apoiados pelo anteriormente mencionado cálculo de resíduos, deduziremos a representação em serie de soma tripla para o preço de uma opção Europeia e os seus correspondentes gregos. Para terminar, dando uso à nova formula, serão computados e discutidos alguns valores para um caso de estudo particular.

PALAVRAS-CHAVE: Processo de Lévy; Processo Variance Gamma; Análise Complexa Multidimensional; Transformada de Mellin; Valorização de Opções.

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1 Introduction

One of pivotal tasks of mathematical finance is the pricing of financial derivatives, such as options, yet it can be an arduous task to develop a model that is both consistent with the empirical evidence, soluble and its numerical estimation neither erroneous nor time consuming. One of the first attempts to solve this quandary, was the Gaussian model first introduced by Fischer Black and Myron Scholes in [7], and latter expanded by Robert Merton in [21], aptly named the Black-Scholes model, where the nondeterministic variable in the underlying asset is modeled by a geometric Brownian motion. Its simplicity and the admission of a close formula for the option price, are the main reason why, till this day, it remains the most used model by market practitioners. Still the model fails to account for either sudden price drops or raises that can be expressed as discontinuous price jumps; moreover it assumes the volatility to remain constant for changes in relation to the strike price and time to maturity contrary to the empirical data and, furthermore the distributions of asset returns have been shown to be negatively skewed and exhibit fat-tails which is not captured by the symmetric Gaussian model.

Many generalizations of the Black-Scholes model have been introduced, such as models with stochastic volatility or regime switching multifractal models, but the one we will be examining assumes that the underlying asset price dynamics is described by a Lévy Process, namely the Variance Gamma process, first proposed by Dilip Madan and Eugene Seneta in [17].

The descriptive power of models based on Lévy processes for accurately portraying financial markets (not displaying the aforementioned problems present in the Black-Scholes model) has been known since the works of Benoît Mandelbrot [20] and Eugene Fama [11], and with the advent of technology and computer development have been gaining traction in recent decades. Yet, the Black-Scholes model remains mostly ubiquitous, the main reason for this state of affairs, is that pricing models based on Lévy processes, at best, admit a semi-closed pricing formula, or prices must be computed must be computed by numerical simulations.

In recent years, in order to circumvent this problem, a more theoretical approach has been undertaken directed at α-stable Lévy processes, $L^{\alpha,\beta}$ (see [24]). The first major breakthrough, presented by Peter Carr and Liuren Wu [9], was the restriction of parameter β, of $L^{\alpha,\beta}$, to -1, forcing negative skewness and the existence of conditional moments of all orders and thus guaranteeing the existence of a martingale measure and finite option values. Independently, research carried out by Rudolf Gorenflo and Francesco Mainardi among others into the space-time fractional diffusion equations $\left({}_0^*\mathcal{D}_t^\gamma - \mu^\theta \mathcal{D}_x^\alpha\right) g\left(x,t\right) = 0$, where ${}_0^*\mathcal{D}_t^\gamma$ is the Caputo fractional derivative and ${}^\theta\mathcal{D}_x^\alpha$ the

Riesz-Feller fractional derivative, has yield important results (see [19], [12], [18] and [13] for more details). Namely its solution, the Green Function $g^{\theta}_{\alpha,\gamma}(x,t)$, can be represented by a Mellin-Barnes integral of a Gamma fraction

$$g^{\theta}_{\alpha,\gamma}(x,t) = \frac{1}{\alpha x}\frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty} \frac{\Gamma(\frac{t_1}{\alpha})\Gamma(1-\frac{t_1}{\alpha})\Gamma(1-t_1)}{\Gamma(1-\frac{\gamma t_1}{\alpha})\Gamma(\frac{\alpha-\theta}{2\alpha}t_1)\Gamma(1-\frac{\alpha-\theta}{2\alpha}t_1)} \left(\frac{x}{(-\mu t^{\gamma})^{1/\alpha}}\right)^{t_1} dt_1 \quad (1)$$

and for the case where $\gamma = 1$, (1) describes the probability distribution of an α-stable Lévy process. The last piece of the puzzle, came from the works of Mikael Passare, August Tsikh and Oleg Zhadanov in [23], [22] and [27], where they ascertained that under certain conditions Residue Calculus can be applied to a Mellin-Barnes integral of a Gamma fraction converting it into a multiple sum series.

Finally, Jean-Phillipe Aguilar, Cyril Coste and Jan Korbel, in their works [1], [2] and [3] used the green function (1) to express the price of an European option as a Mellin-Barnes integral and applying the previously mentioned results developed by Passare et al, were able to arrive at a series representation for an European call option

$$\begin{aligned} C_{\alpha,\gamma}(S,K,r,\mu,\tau) = \frac{Ke^{-r\tau}}{\alpha} &\sum_{\substack{n=0\\m=0}}^{\infty} \frac{(-1)^n}{n!\Gamma(1-\gamma\frac{n-m-1}{\alpha})} \\ &\times \left(-\log\frac{S}{K} - r\tau - \mu\tau\right)^n (-\mu\tau^{\gamma})^{\frac{1+m-n}{\alpha}} \end{aligned} \quad (2)$$

This dissertation takes after these works. Due to the Variance Gamma process being expressed by the difference of two Gamma processes we will need to extended the Theorem for the representation of the Double Mellin-Barnes Integral by a sum of residues, presented in [27], to the Triple Mellin-Barnes Integral case. Subsequently, we will develop a Mellin-Barnes integral representation for the price of an European call option under the Variance Gamma model, and similarly to [1], will use the previous result to express the aforementioned call option price as a triple series representation, and complement it, with the corresponding formulas for the greek functions.

This dissertation is organized as follows: Section 2 will introduce preliminary concepts such as Lévy processes, the Variance Gamma process and Multi-dimensional Residue Theory and discuss some of their properties. In Section 3, we will prove the representation of the triple Mellin-Barnes Integral by a sum of residues Theorem. In Section 4 we present the main results of this thesis, the derivation of the triple series representation for an European call option under the Variance Gamma model, and the subsequential greek functions. In Section 5 taking advantage of the data in [25] we will test the accuracy of the Variance Gamma formula and its greeks. The last section is dedicated to conclusions.

2 PRELIMINARY THEORY

2.1 Lévy Processes

Fist we start by giving a brief summary of Lévy processes. We will enumerate, without proof, concepts, definitions and propositions, that are basal for the derivation of the main results of this thesis. A more in depth overview of this subject can be founded in the textbooks [4] and [10]. We start by formally defining a Lévy process.

Definition 1 (Lévy Process)**.** *Let $X = \{X_t : t \geq 0\}$ be stochastic process in the probability space $(\Omega, \mathcal{F}, P)$. We say X is a Levy process if:*

1. $X_0 = 0$, *(a.s.)*
2. *X has independent stationary increments, i.e. for any $n \in \mathbb{N}$ partition of time intervals $0 = t_0 < t_1 \leq ... < t_n < \infty$, the random variables $X_{t_{j+1}} - X_{t_j}$ are independent and $X_{t_{j+1}} - X_{t_j} \overset{d}{=} X_{t_{j+1}-t_j}$, for all $0 \leq j \leq n$.*
3. *X is stochastic continuous, i.e. for any $\alpha > 0$ and $s > 0$, we have:*

$$\lim_{t \to s} P\left(|X_t - X_s| > \alpha\right) = 0 \tag{3}$$

Note that the property of independent stationary increments, implies that Lévy processes are infinity divisible, and reflects the weak market efficiency hypothesis, that is, present and past values and historical trends cannot be used to predict the future value of assets. Also observe that Brownian motions are a particular case of Lévy processes, still Lévy processes do not impose trajectory continuity, this laxity will permit the replication of price jumps observed in the market.

To have a better grasp of a Lévy process behavior consider its decomposition:

Theorem 1 (Lévy-Itô decomposition)**.** *Let X be a Lévy process. There exists a vector $b \in \mathbb{R}^d$, a Brownian motion process B_A with covariance matrix $A \in \mathbb{R}^{d\times d}$ and an independent compensated Poisson measure $\tilde{N}$ on $\mathbb{R}^+ \times \mathbb{R}^d$, such that X_t can be decomposed as:*

$$X_t = bt + B_A(t) + \int_{|x|<1} x\tilde{N}(t, dx) + \int_{|x|\geq 1} x\tilde{N}(t, dx) \tag{4}$$

Theorem 1 states that every Lévy process can be view as the sum of a drift component bt, a diffusion component given by the Brownian motion $B_A(t)$, a small jumps component expressed by $\int_{|x|<1} x\tilde{N}(t, dx)$ and a big jump component given by $\int_{|x|\geq 1} x\tilde{N}(t, dx)$.

While the Lévy-Itô decomposition may thoroughly describe the behavior of a Lévy process, we may also want a more succinct description fitter for describing more complex processes.

Theorem 2 (Lévy Khintchine). *Let X be a Lévy process then there exists a vector $b \in \mathbb{R}^d$, a positive defined symmetric matrix $A \in \mathbb{R}^{d\times d}$ and a Lévy measure ν on $\mathbb{R}^d \setminus \{0\}$, such that, for all $u \in \mathbb{R}^d$, the characteristic function of X_t will be given by:*

$$\phi_{X_t}(u) = \exp\left\{ t\left(i\langle b, u\rangle - \frac{\langle u, Au\rangle}{2} + \int\limits_{\mathbb{R}^d\setminus\{0\}} [e^{i\langle u,y\rangle} - 1 - i\langle u, y\rangle \mathbb{1}_{|y|<1}(y)]\, \nu(dy) \right)\right\} \tag{5}$$

where a Lévy measure is a Borel measure defined in $\mathbb{R}^d \setminus \{0\}$ such that

$$\int_{\mathbb{R}^d\setminus\{0\}} \min\{|y|^2, 1\}\nu(dy) \leq \infty \tag{6}$$

Since any Lévy process can be completely described by the triplet (b, A, ν), its characteristic function becomes a ubiquitous way of presenting it. processes such as the CGMY, the Generalized Tempered Stable, the Finite Moment Log-Stable, among others, are generally described this way rather than by their Lévy-Itô decomposition.

We must now inquire about the computation of option prices using Lévy processes. The Black-Scholes model assumes the absence of arbitrage and market completeness. Further, since an equivalent martingale measure $\mathbb{Q}$ exists if and only if the market is arbitrage free, and $\mathbb{Q}$ is unique if and only if it the market is complete, then the Black-Scholes model ensures the existence and uniqueness of $\mathbb{Q}$. Recall that $\mathbb{Q}$ is equivalent to the real observed probability measure $\mathbb{P}$, and $S_t e^{-(r-q)t}$, is martingale under $\mathbb{Q}$. Therefore asset price dynamics under $\mathbb{Q}$ will be given by:

$$S_T = S_t e^{\left(r-q-\frac{\sigma^2}{2}\right)\tau + B_\tau^{\mathbb{Q}}}, \text{ where } \tau = T - t. \tag{7}$$

As might be expected, considering Lévy processes are much more encompassing than diffusion processes, neither market completeness nor absence of arbitrage are guaranteed. Since we will be using the Variance Gamma process which has both positive and negative jumps, as can be seen in (11), absence of arbitrage will be achieved, on the other hand, market completeness will not. Among the plurality of possible martingale measures, for simplicity and due to its verisimilitude with real market results, we will chose the mean correcting martingale measure. Under this measure asset price dynamics will be:

$$S_T = S_t e^{(r-q)\tau + X_\tau^{\mathbb{Q}} - \log\left(\phi_{X_\tau^{\mathbb{Q}}}(-i)\right)} \tag{8}$$

Note that these dynamics mirror the Black-Scholes case and that $S_t e^{-(r-q)t}$ is martingale under $\mathbb{Q}$. Therefore using the risk neutral valuation formula we can finally express the price for an European call option as:

$$C(S,K,r,\tau) = e^{-(r-q)\tau}\mathbb{E}^{\mathbb{Q}}\left[\left(Se^{(r-q)\tau + X_\tau^{\mathbb{Q}} - \log\left(\phi_{X_\tau^{\mathbb{Q}}}(-i)\right)} - K\right)^+\right] \tag{9}$$

Formula (9) will be the one used to prove our main result in Section 4.

2.2 Variance Gamma Process

The main result of this dissertation will be the computation of a triple series sum for the price of an European option driven by the Variance Gamma model. Therefore, our next course of action, will be to briefly introduce the two equivalent definitions of the Variance Gamma process and their respective properties. The proof of the properties are straightforward, yet for a more thorough overview of Variance Gamma process we recommend, [17], [16] and [25].

Definition 2. *The Variance Gamma process $X_{VG}(t;\sigma,\nu,\theta)$ is a Brownian motion $\theta t + \sigma B(t) \sim N(\theta t, \sigma^2 t)$ with drift $\theta \in \mathbb{R}$ and volatility $\sigma > 0$, coupled with an independent subordinator gamma process $G_t^\nu \sim Gamma(t/\nu, 1/\nu)$, where $t \geq 0$ and $\nu > 0$, that is:*

$$X_{VG}(t;\sigma,\nu,\theta) = \theta G_t^\nu + \sigma B(G_t^\nu) \tag{10}$$

Under this definition one can, by direct computation, easily arrive at the following properties:

Proposition 1. *The characteristic function, mean, variance, skewness and kurtosis of the Variance Gamma process $X_{VG}(t;\sigma,\nu,\theta)$ are respectively given by:*

- *Characteristic function:* $\phi_{VG}(u,t;\sigma,\nu,\theta) = \left(1 - iu\theta\nu + \sigma^2\nu u^2/2\right)^{-\frac{t}{\nu}}$
- *Mean:* $\mathbb{E}[X_{VG}(t)] = \theta$
- *Variance:* $Var[X_{VG}(t)] = \sigma^2 + \nu\theta^2$
- *Skewness:* $Skew[X_{VG}(t)] = \theta\nu(3\sigma^2 + 2\nu\theta^2)/(\sigma^2 + \nu\theta^2)^{3/2}$
- *Kurtosis:* $Kurt[X_{VG}(t)] = 3(1 + 2\nu - \nu\sigma^4(\sigma^2 + \nu\theta^2)^{-2})$

While expression (10) is the original definition of the Variance Gamma process, the following, due to its expedience, will be the one we will use to prove our main result in section 4.

Definition 3. *The Variance Gamma process $X_{VG}\,(t; C, G, M)$ is the difference of two independent gamma processes $G_t^1 \sim Gamma(Ct, 1/M)$ and $G_t^2 \sim Gamma(Ct, 1/G)$:*

$$X_{VG}(t; C, G, M) = G_t^1 - G_t^2 \tag{11}$$

Analogously to the previews definition one can easily arrive at the following properties by direct computation:

Proposition 2. *The characteristic function, mean, variance, skewness and kurtosis of the Variance Gamma process $X_{VG}\,(t; C, G, M)$ are respectively given by*

- *Characteristic function:* $\phi_{VG}(u, t; C, G, M) = \left[\dfrac{MG}{MG + iu(M - G) + u^2}\right]^{-Ct}$
- *Mean:* $\mathbb{E}[X_{VG}(t)] = C(G - M)/(MG)$
- *Variance:* $Var[X_{VG}(t)] = C(G^2 + M^2)/(MG)^2$
- *Skewness:* $Skew[X_{VG}(t)] = 2C^{-1/2}(G^3 - M^3)/(G^2 + M^2)^{3/2}$
- *Kurtosis:* $Kurt[X_{VG}(t)] = 3(1 + 2C^{-1}(G^4 + M^4)/(M^2 + G^2)^2)$

Lastly, observe that the characteristic functions for both definitions will the same after application of variable changes $C = \dfrac{1}{\nu}$, $G = \left(\sqrt{\dfrac{\theta^2\nu^2}{4} + \dfrac{\sigma^2\nu}{2}} - \dfrac{\theta\nu}{2}\right)^{-1}$ and $M = \left(\sqrt{\dfrac{\theta^2\nu^2}{4} + \dfrac{\sigma^2\nu}{2}} + \dfrac{\theta\nu}{2}\right)^{-1}$, that is, they are the same process.

2.3 One-Dimensional Residue Calculus

This subsection will summarize some results of one dimensional complex analysis, for the proofs or a more in depth theory overview we recommend the textbooks [15] and [5]. The first thing to recall is the definition of residue of a meromorphic function f on an isolated singularity a:

$$\operatorname{Res}_a f = \frac{1}{2\pi i}\int_\gamma f \tag{12}$$

where $\gamma : [0, 1] \to \mathbb{C}$ is the closed path $\gamma(t) = a + re^{2\pi i}$ for r small enough.

Consider the Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)$ on the singularity a. The previous definition is equivalent to $\operatorname{Res}_a f = c_{-1}$. The residue computation for some functions is facilitated by this last result, for example, the residues of the gamma function $\Gamma(z) = \int_0^\infty x^{z-1}e^z dx$ can easily be seen to be:

$$\operatorname{Res}_{-n}\Gamma(z) = \lim_{z\to\infty}(z+n)\Gamma(z) = \frac{(-1)^n}{n!} \tag{13}$$

The logical next step will be to generalize the expression (12) for any open set A. This leads us to the Cauchy's Residue theorem:

Theorem 3 (Cauchy's Residue Theorem)**.** *Let f be a meromorphic function in an open set U, and γ a closed chain in $U \setminus A$ homologous to 0, where A is the set of poles of f in U, then*

$$\frac{1}{2\pi i}\int_\gamma f = \sum_{a\in A} W_\gamma(a)\cdot \operatorname{Res}_a f \tag{14}$$

where $W_\gamma(a) = \frac{1}{2\pi i}\int_\gamma \frac{dw}{w-a}$ is the winding number, that is the number of times the path γ circumvents counterclockwise the pole a.

Now say we want to compute the integral on the real axis $\int_{-\infty}^{\infty} f(z)dz = \lim_{R\to\infty}\int_{-R}^{R} f(z)dz$. Consider γ to be a path consisting of a segment on the real line $[-R,R]$ and an upper semi-circle $S_R^+ := \{z\in\mathbb{C} : |z| = R, \operatorname{Im}(z) > 0\}$, then the path integral can be written as the sum $\int_{-R}^{R} f(z)dz + \int_{S_R^+} f(z)dz$. If $\int_{S_R^+} f(z)dz \to 0$ as $R\to\infty$, we can take advantage of the Cauchy Residue theorem and conclude the integral $\int_{-\infty}^{\infty} f(z)dz$ will be given by $2\pi i$ times the sum of the residues of f on the set of its singularities, N, in the upper plane $\Pi_+ = \{z\in\mathbb{C} : \operatorname{Im}(z) > 0\}$. Formally:

Theorem 4 (Jordan Theorem)**.** *Let f be a meromorphic function in $\mathbb{C}$ continuous on $\mathbb{R}$ (hence, it lacks any singularities in $\mathbb{R}$). If there exists a constant c such that for $|z|$ big enough we have $|f(z)| \le \frac{c}{|z|^\alpha}$, for some $\alpha > 1$, then $\lim_{R\leftarrow\infty}\int_{S_R} f = 0$ which in turn implies that*

$$\int_{-\infty}^{\infty} f(z)dz = 2\pi i \sum_{a\in N\cap\Pi_+} \operatorname{Res}_a f \tag{15}$$

2.4 Multidimensional Residue Calculus

Now let us extrapolate the previous results and definitions to the general multidimensional case. A more in depth theoretical overview of multidimensional complex analysis can be found in the textbooks [14] and [6].

Definition 4 (Grothendieck Residue)**.** *Let h and f_i, for any index $i \in \{1, ..., n\}$, be functions in $\mathbb{C}^n$, where h is holomorphic. Consider the meromorphic differential n-form:*

$$\omega = \frac{h(z)dz}{f_1(z)...f_n(z)}, \quad dz = dz_1 \wedge ... \wedge dz_n, \tag{16}$$

which has the singularities $D_j = \{z \in \mathbb{C} : f_j(z) = 0\}$, such that the intersection $\bigcap_{j=1}^n D_j$ is discrete. The Grothendieck residue on a singularity $a \in \bigcap_{j=1}^n D_j$ is defined as

$$\operatorname{Res}_a \omega = \frac{1}{(2\pi i)^n} \int_{C_a} \omega \tag{17}$$

where $C_a = \{z \in U_a : |f_j(z)| = \epsilon, j = 1, ..., n\}$ is a cycle in a small neighborhood U_a of the singularity a with the orientation $d(\arg f_1) \wedge ... \wedge d(\arg f_n) \geq 0$.

Before proceeding we will formalize the concept of a multidimensional polyhedron in $\mathbb{C}^n$. The two following definitions will underpin most of the theorems from sections 3 and 4.

Definition 5 (Polyhedron)**.** *Consider a proper (the inverse images of a compact set are compact) holomorphic map $g : \mathbb{C}^n \to G$ where $G = G_1 \times ... \times G_n$ is a domain (connected open subset of a finite-dimensional vector space) where, for each $j = 1, ..., n$, $G_j \subset \mathbb{C}$ is a domain with piecewise smooth boundary. We define a polyhedron Π as the inverse image:*

$$\Pi := g^{-1}(G) \tag{18}$$

and for a multi-index $K = \{k_1, ..., k_p\} \subset \{1, ..., n\}$ we define the polyhedron's faces as

$$\sigma_K := \{z : g_k(z) \in \partial G_k \textit{ for } k \in K,\ g_j(z) \in G_j \textit{ for } j \in K^C\} \tag{19}$$

Definition 6 (Compatible divisors)**.** *Consider the polyhedron Π and the family of divisors $\{D_i\}_{i \in \{1,...,n\}}$, they are said to be compatible if for any $i \in \{1, ..., n\}$ we get:*

$$\sigma_i \cap D_i = \emptyset \tag{20}$$

Analogously to the one dimensional integral on the real-axis (15), we may want to compute an integral $\int_\sigma \omega$ where ω is the meromorphic form (16) and σ is the boundary of an polyhedron Π. For an unbounded polyhedron we need the integrand to vanish as it goes to infinity. To achieve this goal let us introduce the auxiliary functions

$$\rho_j = \frac{|f_j|^2}{\|f\|^2}, \quad \text{for any } j \in \{1, ..., n\} \tag{21}$$

where $\|f\|^2 = |f_1|^2 + ... + |f_n|^2$. Using the functions (21) we define the differential $(n, p-1)$-forms as

$$\xi_J = \sum_{j \in J} (-1)^{(j,J)-1} \rho_j \overline{\partial} \rho_J[j] \wedge \omega \tag{22}$$

where $J = \{j_1, ..., j_s\} \subset \{1, ..., n\}$, for $1 < s \leq n$, is a multi-index, (j, J) is the position of j in set J and $\overline{\partial} \rho_J[j] = \overline{\partial} \rho_{j_1} \wedge ...[j]... \wedge \overline{\partial}_{j_s}$. We now define the multidimensional condition analogous to $\lim_{R \to \infty} \int_{S_R} f(z) dz = 0$.

Definition 7 (Jordan condition)**.** *Consider the sphere $S_R = \{z \in \sigma : \|z\| = R\}$, where $\sigma = \sigma_{12...n}$ is the boundary of the polyhedron Π. A differential form ξ_J satisfies the Jordan condition on face σ_{J^o}, where $J^o = \{1, ..., n\} \setminus J$, if there exists a sequence of positive real numbers R_k that goes to infinity, such that*

$$\lim_{k \to \infty} \int_{S_{R_k} \cap \sigma_{J^o}} \xi_J = 0 \tag{23}$$

Note that for $n = 1$, there exists only one form $\xi = \omega$ and thus the condition (23) corresponds to unidimensional condition $\lim_{z \to \infty} \int_{S_R} f(z) dz = 0$. For the multidimensional case consider the set $N = \{z \in \mathbb{C}^n : \|f(z)\| = 0\} = \bigcap_{i=1}^n D_i$, we thus gave:

Theorem 5 (The Jordan Lemma)**.** *Let ω be a meromorphic form with the polar divisors $\{D_i\}_{i \in \{1,...,n\}}$ compatible with polyhedron Π. If for every multi-index J the form ξ_J satisfies the Jordan condition on the face σ_{J^o}, we get*

$$\int_\sigma \omega = (2\pi i)^n \sum_{a \in N \cap \Pi} \operatorname{Res}_a \omega \tag{24}$$

The Jordan's Lemma proof can be found in [23] with a step taken from [14].

3 MELLIN-BARNES INTEGRAL

We will start by enumerating concepts and properties for the one-dimensional and three-dimensional Mellin-Barnes integral, which will be crucial when proving the main result of this dissertation. For a more in dept look at Fourier, Laplace and Mellin Transforms and their corresponding properties the book [26] is recommended.

3.1 One-Dimensional Mellin-Barnes Integral

Definition 8 (Mellin-Barnes Integral)**.** *The Mellin-Barnes Integral is given by a ratio of products of Gamma functions of linear arguments*

$$\Phi(t) := \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\prod_{j=1}^{m}\Gamma(a_j z + b_j)}{\prod_{k=1}^{p}\Gamma(c_k z + d_k)} t^{-z} dz \tag{25}$$

where its characteristic quantity, Δ, is defined by

$$\Delta = \sum_{j=1}^{m} a_j - \sum_{k=1}^{p} c_k \tag{26}$$

Before proceeding let us state the Stirling's approximation of the gamma function

$$\Gamma(z) = \sqrt{2\pi} z^{z-1/2} e^{-z} O\left(1+\frac{1}{z}\right) \xrightarrow[|z|\to\infty]{} \sqrt{2\pi} z^{z-1/2} e^{-z} \tag{27}$$

Equipped with this expression one can easily see how Δ governs the behavior of the ratio of Gammas as $|z| \to \infty$, and therefore which residues, of the singularities to left or to right of the strip, one will sum to compute the integral (25):

$$\Phi(t) = \begin{cases} \sum\limits_{\mathrm{Re}(s_n)<\gamma} \mathrm{Res}_{s_n}\left(\frac{\prod_{j=1}^{m}\Gamma(a_j z+b_j)}{\prod_{k=1}^{p}\Gamma(c_k z+d_k)}\right) t^{-s_n} & \text{if } \Delta > 0 \\ -\sum\limits_{\mathrm{Re}(s_n)>\gamma} \mathrm{Res}_{s_n}\left(\frac{\prod_{j=1}^{m}\Gamma(a_j z+b_j)}{\prod_{k=1}^{p}\Gamma(c_k z+d_k)}\right) t^{-s_n} & \text{if } \Delta < 0 \end{cases} \tag{28}$$

For example, one can express an exponential term e^x as a Mellin-Barnes integral:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \int_{\gamma-i\infty}^{\gamma+i\infty} (-1)^{-t}\Gamma(t) x^{-t} \frac{dt}{2\pi i} \tag{29}$$

where $\gamma > 0$.

3.2 Multidimensional Mellin-Barnes Integral

Let us now extend the Mellin-Barnes integral to the general multidimensional case.

Definition 9 (Multiple Mellin-Barnes Integral)**.** *The Multiple Mellin-Barnes integral is given by a ratio of products of Gamma functions*

$$\Phi(t) = \frac{1}{(2\pi i)^n} \int\limits_{\gamma+i\mathbb{R}^n} \frac{\prod_{j=1}^{m} \Gamma(s_j(z))}{\prod_{k=1}^{p} \Gamma(q_k(z))} t^{-z} dz \tag{30}$$

where $s_j(z) := \sum_{\nu=1}^{n} a_{j\nu} z_\nu + b_j$ *and* $q_k(z) := \sum_{\nu=1}^{n} c_{k\nu} z_\nu + d_k$ *are multi-linear functions and the terms* t^{-z} *and* dz *represent* $t_1^{-z_1}...t_n^{-z_n}$ *and* $dz = dz_1...dz_n$*, respectively.*

In order to simplify the notation for complex numbers, from now on we will use the notation $x_\nu := \mathrm{Re}(z_\nu)$, $y_\nu := \mathrm{Im}(z_\nu)$, for any $\nu = 1, ..., n$, and we will denote the vectors with coefficients $a_{j,\nu}$ and $c_{j,\nu}$ by $\underline{a}_j$ and $\underline{c}_j$ respectively.

Theorem 6. *Let* $S_1 = \{y \in \mathbb{R}^n : |y| = 1\}$ *be the unit sphere in* $\mathbb{R}^n$*. Consider the constant*

$$\alpha := \min_{y \in S_1} \left(\sum_{j=1}^{m} |\langle \underline{a}_j, y \rangle| - \sum_{j=1}^{p} |\langle \underline{c}_j, y \rangle| \right) \tag{31}$$

and the domain set

$$U = \{t \in (\mathbb{C}^n \setminus 0)^n : |\arg t_\nu| < \pi,\ \nu = 1, ..., n, \|\arg t\| < (\pi/2)\alpha\} \tag{32}$$

The Multi Mellin-Barnes integral in (30) *converges absolutely for* $t \in U$.

Proof. See Appendix. □

3.3 Three-Dimensional Mellin-Barnes Integral

In this section, similarly to what we did previously for the unidimensional case, we will present the triple Mellin-Barnes integral and deduce its formula as a sum of residues. To achieve this end let us first consider its integral form:

$$\Phi(t) = \frac{1}{(2\pi i)^3} \int\limits_{\gamma + i\mathbb{R}^3} \frac{\prod_{j=1}^{m} \Gamma(a_{j1}z_1 + a_{j2}z_2 + a_{j3}z_3 + b_j)}{\prod_{k=1}^{p} \Gamma(c_{k1}z_1 + c_{k2}z_2 + c_{k3}z_3 + d_j)} t^{-z_1} t^{-z_2} t^{-z_3} dz_1 \wedge dz_2 \wedge dz_3 \quad (33)$$

Henceforth, for brevity, we will denote the 3-form integrand of (33) by ω. Its zeroes will be the complex planes $L_j^{\nu} = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : a_{j1}z_1 + a_{j2}z_2 + a_{j3}z_3 + b_j = -\nu\}$, for any $\nu \in \mathbb{N}$ and $j \in \{1, ..., m\}$, which represent each singularity given by the gamma functions present in the numerator of the form ω. We will also denote the vectors $\underline{a}_j := \begin{bmatrix} a_{j1} \\ a_{j2} \\ a_{j3} \end{bmatrix}$, $\underline{c}_k := \begin{bmatrix} a_{k1} \\ a_{k2} \\ a_{k3} \end{bmatrix}$ and, most importantly, define the characteristic vector as

$$\Delta = \sum_{j=1}^{m} \underline{a}_j - \sum_{k=1}^{p} \underline{c}_k \quad (34)$$

Suppose that Δ is a nonzero vector. In this case, we can define the plane P_Δ where its real part intersects the point γ and has Δ as its normal vector, i.e. $P_\Delta := \{z \in \mathbb{C}^3 : \mathrm{Re}(\langle \Delta, z \rangle) = \mathrm{Re}(\langle \Delta, \gamma \rangle)\}$, and thereupon we can define the *admissible-polyhedra*, Π_Δ, as the real volume "below" P_Δ, i.e. $\Pi_\Delta := \{z \in \mathbb{C}^3 : \mathrm{Re}(\langle \Delta, z \rangle)) \leq \mathrm{Re}(\langle \Delta, \gamma \rangle)\}$.

Taking into account all these previous demarcations, we can construct an admissible polyhedron $\Pi \subset \Pi_\Delta$, that will be uniquely defined by the linear function $g : \mathbb{C}^3 \to G$, where $\Pi = g^{-1}(G)$ and

$$g(z) = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} (\mathrm{Re}(z) - \gamma) + i\,\mathrm{Im}(z), \quad (35)$$

with the image $G = \{z \in C^3 : \mathrm{Re}(z_1) \geq 0, \mathrm{Re}(z_2) \geq 0, \mathrm{Re}(z_3) \geq 0\}$, i.e the first octant.

From (19) and (35) we can ascertain that its only vertex is $\sigma_{1,2,3} = \gamma$ and that n_1, n_2 and n_3 are the normal vectors of the faces σ_1, σ_2 and σ_3 of the polyhedron:

$$\begin{aligned}\sigma_1 = \{z \in \mathbb{C}^3 : \mathrm{Re}(\langle n_1, z\rangle) = \mathrm{Re}(\langle n_1, z\rangle),\\ \mathrm{Re}(\langle n_2, z\rangle) \geq \mathrm{Re}(\langle n_2, z\rangle), \mathrm{Re}(\langle n_3, z\rangle) \geq \mathrm{Re}(\langle n_3, z\rangle)\}\end{aligned} \tag{36}$$

$$\begin{aligned}\sigma_2 = \{z \in \mathbb{C}^3 : \mathrm{Re}(\langle n_1, z\rangle) \geq \mathrm{Re}(\langle n_1, z\rangle),\\ \mathrm{Re}(\langle n_2, z\rangle) = \mathrm{Re}(\langle n_2, z\rangle), \mathrm{Re}(\langle n_3, z\rangle) \geq \mathrm{Re}(\langle n_3, z\rangle)\}\end{aligned} \tag{37}$$

$$\begin{aligned}\sigma_3 = \{z \in \mathbb{C}^3 : \mathrm{Re}(\langle n_1, z\rangle) \geq \mathrm{Re}(\langle n_1, z\rangle),\\ \mathrm{Re}(\langle n_2, z\rangle) \geq \mathrm{Re}(\langle n_2, z\rangle), \mathrm{Re}(\langle n_3, z\rangle) = \mathrm{Re}(\langle n_3, z\rangle)\}\end{aligned} \tag{38}$$

If the polyhedron was providently constructed, we can balkanize the singularities, L_j^ν, into three distinct sets, such that they are compatible with the polyhedron, i.e.:

$$D_1 = \bigcup_{\substack{j\in\{1,\dots,m\}\\ \nu\in\mathbb{N}\\ L_j^\nu\cap\sigma_1=\emptyset}} L_j^\nu, \qquad D_2 = \bigcup_{\substack{j\in\{1,\dots,m\}\\ \nu\in\mathbb{N}\\ L_j^\nu\cap\sigma_2=\emptyset}} L_j^\nu, \qquad D_3 = \bigcup_{\substack{j\in\{1,\dots,m\}\\ \nu\in\mathbb{N}\\ L_j^\nu\cap\sigma_3=\emptyset}} L_j^\nu \tag{39}$$

Theorem 7 (Residue formula for the Triple Mellin-Barnes integral)**.** *Let ω be the 3-form integrand of* (33) *with characteristic vector $\Delta \neq 0$ and divisors D_1, D_2 and D_3, defined in* (39)*, compatible with the admissible polyhedron $\Pi \subset \Pi_\Delta$, then the sum formula holds:*

$$\frac{1}{(2\pi i)^3} \int_{\gamma+i\mathbb{R}^3} \omega = \sum_{t\in\Pi\cap D_1\cap D_2\cap D_3} \mathrm{Res}_t\, \omega \tag{40}$$

where the series on the right-hand side converges absolutely for any $\underline{t} \in U$, for U defined in (32)*.*

Proof. See Appendix. □

4 Option Pricing Driven by a Variance Gamma Process

Equipped with the results of the previous chapter we will now center our attention in deducing the main result of this thesis, the formula for the price of an European call option under the Variance Gamma model. Similar to the derivation present in [1], we will arrive at this result in two steps. First we will derive the Mellin-Barnes representation for the aforementioned call option and secondly we will use residue calculus to derive the triple sum series formula.

4.1 Mellin-Barnes Representation for a Call Option

Before computing the call option price let us recall that from (11) the Variance Gamma process can be defined as the difference between two independent gamma processes:

$$X_{VG}(\tau; C, G, M) = G^1_\tau - G^2_\tau \tag{41}$$

where $G^1_\tau \sim Gamma(C\tau, 1/M)$ and $G^2_\tau \sim Gamma(C\tau, 1/G)$ for $G > 0$, $M > 0$ and $G > 0$. The probability density function of a gamma process $G \sim Gamma(\alpha, \beta)$ is given by

$$f_{G(\alpha,\beta)}(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \tag{42}$$

Given these definitions we can now state and derive the Mellin-Barnes integral representation in $\mathbb{C}^3$ for an European call option under the Variance Gamma model.

Proposition 3 (Mellin-Barnes representation for a Call Option under Variance Gamma)**.** *Let us denote* $[\log] := \log \frac{S}{K} + (r-q)\tau - \mu\tau$ *and consider the polyhedra* $P_1,\ P_2 \subset \mathbb{C}^3$ *defined by:*

$$\begin{aligned} P_1 := \{\underline{z} \in \mathbb{C}^3 : 0 < \mathrm{Re}(z_1) < 1,\ 0 < \mathrm{Re}(z_2),\ 0 < \mathrm{Re}(z_3) < C\tau, \\ \mathrm{Re}(z_1) + \mathrm{Re}(z_2) + \mathrm{Re}(z_3) > 1 + 2C\tau\} \end{aligned} \tag{43}$$

$$P_2 := \{\underline{z} \in \mathbb{C}^3 : 0 < \mathrm{Re}(z_1) < 1,\ 0 < \mathrm{Re}(z_2) < C\tau,\ 0 < \mathrm{Re}(z_3) < C\tau\} \tag{44}$$

Then, the price of an European call option driven by the Variance Gamma process is given by the formula:

$$\begin{aligned} C_{VG}(S, K, r, \mu, \tau) = \frac{K(GM)^{C\tau} e^{-(r-q)\tau}}{\Gamma(C\tau)^2} (I^1_{VG}(S, K, r, \mu, \tau) \\ + \mathbb{1}_{[\log]>0} I^2_{VG}(S, K, r, \mu, \tau)) \end{aligned} \tag{45}$$

where, for any $\underline{c}_1 \in P_1$ *and* $\underline{c}_2, \in P_2$*, we define* I^1_{VG} *and* I^2_{VG} *as:*

$$I^1_{VG}(S,K,r,\mu,\tau) := \int_{\underline{c}_1 + i\mathbb{R}^3} (-1)^{-t} \frac{\Gamma(t)\Gamma(t_x)\Gamma(t_y)\Gamma(1-t)\Gamma(C\tau - t_y)\Gamma(-1-2C\tau+t+t_x+t_y)}{\Gamma(1-C\tau+t_x)} \times M^{-t_x} G^{-t_y} (-[\log])^{1+2C\tau-t-t_x-t_y} \, \frac{dt}{2\pi i} \wedge \frac{dt_x}{2\pi i} \wedge \frac{dt_y}{2\pi i} \tag{46}$$

$$I^2_{VG}(S,K,r,\mu,\tau) := \int_{\underline{c}_2 + i\mathbb{R}^3} (-1)^{C\tau-t-t_x} \frac{\Gamma(t)\Gamma(t_x)\Gamma(t_y)\Gamma(1-t)\Gamma(C\tau - t_x)\Gamma(C\tau - t_y)}{\Gamma(2+2C\tau-t-t_x-t_y)} \times M^{-t_x} G^{-t_y} [\log]^{1+2C\tau-t-t_x-t_y} \, \frac{dt}{2\pi i} \wedge \frac{dt_x}{2\pi i} \wedge \frac{dt_y}{2\pi i} \tag{47}$$

Proof. For the mean correcting martingale measure $\mathbb{Q}$ in (7) the price of an European Call Option under the Variance Gamma process (41), according to (9), is defined as:

$$C_{VG}(S,K,r,\mu,\tau) = e^{-(r-q)\tau}\mathbb{E}^{\mathbb{Q}}[(S_T - K)^+] \tag{48}$$

$$= e^{-(r-q)\tau}\mathbb{E}^{\mathbb{Q}}[(Se^{(r-q)\tau-\mu\tau+X_{VG}(\tau;C,G,M)} - K)^+] \tag{49}$$

$$= e^{-(r-q)\tau} \int_0^{+\infty}\int_0^{+\infty} (Se^{(r-q)\tau-\mu\tau+x-y} - K)^+ g_{G^1_\tau}(x) g_{G^2_\tau}(y) dx\, dy \tag{50}$$

$$= Ke^{-(r-q)\tau} \int_0^{+\infty}\int_0^{+\infty} \left(e^{[\log]+x-y} - 1\right)^+ g_{G^1_\tau}(x) g_{G^2_\tau}(y) dx\, dy \tag{51}$$

By definition $G^1_\tau \sim Gamma(C\tau, 1/M)$ and $G^2_\tau \sim Gamma(C\tau, 1/G)$, hence from (42) their probability density functions are given respectively by $g_{G^1_\tau}(x) = \frac{M^{C\tau}}{\Gamma(C\tau)} x^{C\tau-1}e^{-Mx}$ and $g_{G^2_\tau}(y) = \frac{G^{C\tau}}{\Gamma(C\tau)} y^{C\tau-1}e^{-Gy}$, thus:

$$C_{VG}(S,K,r,\mu,\tau) = \frac{K(GM)^{C\tau}e^{-(r-q)\tau}}{\Gamma(C\tau)^2} \times \int_0^{+\infty}\int_0^{+\infty} \left(e^{[\log]+x-y} - 1\right)^+ x^{C\tau-1}e^{-Mx} y^{C\tau-1}e^{-Gy} dx\, dy \tag{52}$$

In order for the term $(e^{[\log]+x-y} - 1)^+$ to be different from zero we must have $y \leq x + [\log]$. Also notice, that by definition, both x and y are non-negative, hence the acceptable values of x and y are constrained to the green area of Figure-1.

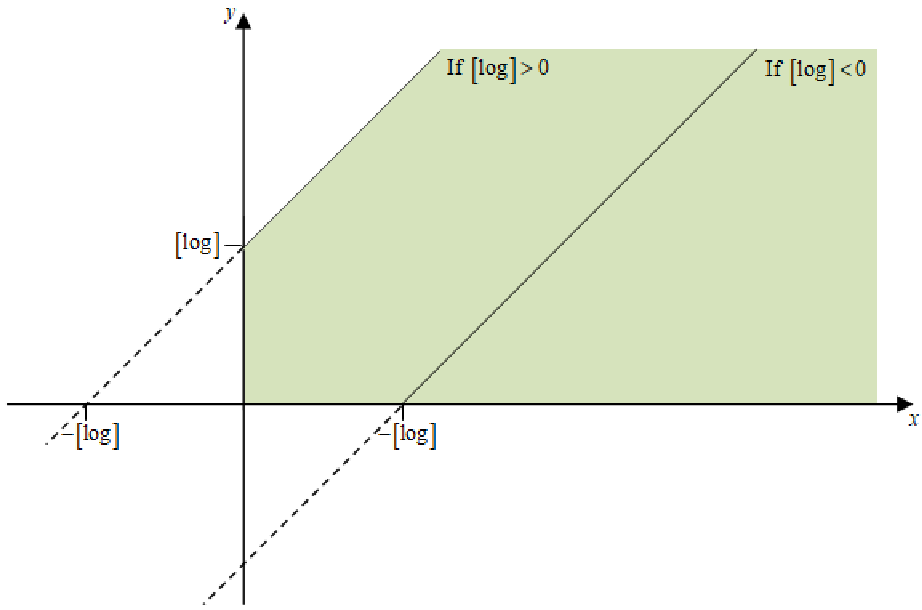


FIGURE 1: Area of variable (x, y) constraint

According to Figure-1, the values for which the integral is not zero are given by the set $\{(x, y) \in \mathbb{R}^2 : -[\log] < x,\ y < x + [\log]\}$ with the removal of the values in the set $\{(x, y) \in \mathbb{R}^2 : -[\log] < x < 0,\ y < x + [\log]\}$, in the cases where $[\log] > 0$. Thus C_{VG} can be expressed as the sum:

$$C_{VG}(S, K, r, \mu, \tau) = \frac{K(GM)^{C\tau} e^{-(r-q)\tau}}{\Gamma(C\tau)^2} (I^1_{VG}(S, K, r, \mu, \tau) + \mathbb{1}_{[\log]>0} I^2_{VG}(S, K, r, \mu, \tau)) \tag{53}$$

where I^1_{VG} and I^2_{VG} are defined as:

$$I^1_{VG}(S, K, r, \mu, \tau) = \int_{-[\log]}^{+\infty} \int_0^{x+[\log]} \left(e^{[\log]+x-y} - 1\right) x^{C\tau-1} e^{-Mx} y^{C\tau-1} e^{-Gy} dx\, dy \tag{54}$$

$$I^2_{VG}(S, K, r, \mu, \tau) = -\int_{-[\log]}^{0} \int_0^{x+[\log]} \left(e^{[\log]+x-y} - 1\right) x^{C\tau-1} e^{-Mx} y^{C\tau-1} e^{-Gy} dx\, dy \tag{55}$$

By the Mellin-Barnes representation of the exponential given by equation (29), for $\underline{c}_1 = \begin{bmatrix} c_{11} \\ c_{12} \end{bmatrix}$ where $c_{11}, c_{12} < 0$, we can represent both terms e^{-Mx} and e^{-Gy} as the integrals $\int_{c_{11}-i\infty}^{c_{11}+i\infty} \Gamma(t_x) M^{-t_x} x^{-t_x} \frac{dt_x}{2\pi i}$ and $\int_{c_{12}-i\infty}^{c_{12}+i\infty} \Gamma(t_y) G^{-t_y} x^{-t_y} \frac{dt_y}{2\pi i}$, respectively.

Implementing these new representations on the integral of equation (54) results in:

$$I_{VG}^{1}(S,K,r,\mu,\tau) = \int\limits_{\underline{c}_1+i\mathbb{R}^2} \Gamma(t_x)\Gamma(t_y)M^{-t_x}G^{-t_y} \times \int_{-[\log]}^{+\infty}\int_{0}^{x+[\log]} \left(e^{[\log]+x-y}-1\right)x^{C\tau-1-t_x}y^{C\tau-1-t_y}dx\,dy\,\frac{dt_x}{2\pi i}\frac{dt_y}{2\pi i} \quad (56)$$

$$= \int\limits_{\underline{c}_1+i\mathbb{R}^2} \Gamma(t_x)\Gamma(t_y)M^{-t_x}G^{-t_y} \times \int_{-[\log]}^{+\infty}\int_{0}^{x+[\log]} \left(e^{[\log]+x-y}-1\right)y^{C\tau-1-t_y}dy\,x^{C\tau-1-t_x}dx\,\frac{dt_x}{2\pi i}\frac{dt_y}{2\pi i} \quad (57)$$

Applying integration by parts over the y variable to the equation (57), produces:

$$\int\limits_{\underline{c}_1+i\mathbb{R}^2} \frac{\Gamma(t_x)\Gamma(t_y)}{C\tau-t_y}M^{-t_x}G^{-t_y} \times \int_{-[\log]}^{+\infty}\int_{0}^{x+[\log]} e^{[\log]+x-y}y^{C\tau-t_y}dy\,x^{C\tau-1-t_x}dx\,\frac{dt_x}{2\pi i}\frac{dt_y}{2\pi i} \quad (58)$$

$$= \int\limits_{\underline{c}_1+i\mathbb{R}^2} \frac{\Gamma(t_x)\Gamma(t_y)\Gamma(C\tau-t_y)}{\Gamma(C\tau-t_y+1)}M^{-t_x}G^{-t_y} \times \int_{-[\log]}^{+\infty}\int_{0}^{x+[\log]} e^{[\log]+x-y}y^{C\tau-t_y}dy\,x^{C\tau-1-t_x}dx\,\frac{dt_x}{2\pi i}\frac{dt_y}{2\pi i} \quad (59)$$

The Mellin-Barnes exponential representation for $e^{[\log]+x-y}$ is given by the integral $\int_{c_{13}-i\infty}^{c_{13}+i\infty}\Gamma(t)([\log]+x-y)^{-t}\frac{dt_y}{2\pi i}$, where $c_{1,3}<0$. This substitution of terms in (59) will result in:

$$\int\limits_{\underline{c}_1+i\mathbb{R}^3} (-1)^{-t}\frac{\Gamma(t)\Gamma(t_x)\Gamma(t_y)\Gamma(C\tau-t_y)}{\Gamma(C\tau-t_y+1)}M^{-t_x}G^{-t_y} \times \int_{-[\log]}^{+\infty}\int_{0}^{x+[\log]} ([\log]+x-y)^{-t}y^{C\tau-t_y}dy\,x^{C\tau-1-t_x}dx\,\frac{dt}{2\pi i}\frac{dt_x}{2\pi i}\frac{dt_y}{2\pi i} \quad (60)$$

where $\underline{c}_1$ was extended to the third dimension, i.e. $\underline{c}_1 = \begin{bmatrix} c_{11} \\ c_{12} \\ c_{13} \end{bmatrix}$.

For the integral $\int_0^{x+[\log]}([\log]+x-y)^{-t}y^{C\tau-t_y}dy$ in (60) consider the variable change $y := ([\log]+x)s$, this alteration of variables will result in the expression:

$$\begin{aligned}
&\int_0^{x+[\log]}([\log]+x-y)^{-t}y^{C\tau-t_y}dy \\
&\qquad = ([\log]+x)^{1+C\tau-t-t_y}\int_0^1([\log]+x-s)^{-t}s^{C\tau-t_y}ds \qquad (61)\\
&\qquad = ([\log]+x)^{1+C\tau-t-t_y}\frac{\Gamma(1-t)\Gamma(C\tau-t_y+1)}{\Gamma(2+C\tau-t-t_y)} \qquad (62)
\end{aligned}$$

Replacing the expression (62) in the original integral (60) we obtain:

$$\begin{aligned}
&\int\limits_{\underline{c}_1+i\mathbb{R}^3}(-1)^{-t}\frac{\Gamma(t)\Gamma(t_x)\Gamma(t_y)\Gamma(1-t)\Gamma(C\tau-t_y)}{\Gamma(2+C\tau-t-t_y)}M^{-t_x}G^{-t_y} \\
&\qquad \times\int_{-[\log]}^{+\infty}([\log]+x)^{1+C\tau-t-t_y}x^{C\tau-1-t_x}dx\,\frac{dt}{2\pi i}\,\frac{dt_x}{2\pi i}\,\frac{dt_y}{2\pi i} \qquad (63)
\end{aligned}$$

Similarly, for the integral $\int_{-[\log]}^{\infty}([\log]+x)^{1+C\tau-t-t_y}x^{C\tau-1-t_x}dx$ in (63) consider the variable change $x := [\log]\frac{1}{s}$, this exchange will result in:

$$\begin{aligned}
&\int_{-[\log]}^{+\infty}([\log]+x)^{1+C\tau-t-t_y}x^{C\tau-1-t_x}dx \\
&= [\log]^{1+2C\tau-t-t_x-t_y}\int_1^0\left(1-\frac{1}{s}\right)^{1+C\tau-t-t_y}\left(-\frac{1}{s}\right)^{C\tau-1-t_x}d\left(\frac{1}{s}\right) \qquad (64)\\
&= (-[\log])^{1+2C\tau-t-t_x-t_y}\int_0^1(1-s)^{1+C\tau-t-t_y}s^{-2-2C\tau+t+t_x+t_y}ds \qquad (65)\\
&= (-[\log])^{1+2C\tau-t-t_x-t_y}\frac{\Gamma(2+C\tau-t-t_y)\Gamma(-1-2C\tau+t+t_x+t_y)}{\Gamma(1-C\tau+t_x)} \qquad (66)
\end{aligned}$$

Replacing the expression (66) on the integral (63) and subsequently inserting the resulting term on the original expression (54) will finally achieve the desired formula (46):

$$\begin{aligned}
&I_{VG}^1(S,K,r,\mu,\tau) := \\
&\quad\int\limits_{\underline{c}_1+i\mathbb{R}^3}(-1)^{-t}\frac{\Gamma(t)\Gamma(t_x)\Gamma(t_y)\Gamma(1-t)\Gamma(C\tau-t_y)\Gamma(-1-2C\tau+t+t_x+t_y)}{\Gamma(1-C\tau+t_x)} \\
&\qquad\times M^{-t_x}G^{-t_y}(-[\log])^{1+2C\tau-t-t_x-t_y}\,\frac{dt}{2\pi i}\wedge\frac{dt_x}{2\pi i}\wedge\frac{dt_y}{2\pi i} \qquad (67)
\end{aligned}$$

The integral formula (67) converges if all the arguments of the Gamma functions in the numerator are positive, this happens when $\mathrm{Re}(t), \mathrm{Re}(t_x), \mathrm{Re}(t_y) > 0$, $\mathrm{Re}(t) < 1$, $\mathrm{Re}(t_y) < C\tau$ and $\mathrm{Re}(t) + \mathrm{Re}(t_x) + \mathrm{Re}(t_y) > 2C\tau + 1$, i.e. $\mathrm{Re}(\underline{t}) \in P_1$.

Conversely, we can charter the same steps for the integral (55) of I^2_{VG} as we did for (54) of I^1_{VG}; apply the Mellin-Barnes representation of the exponential term to both e^{-Mx} and e^{-Gy}, subsequently, use integration by parts over the variable y, again apply the Mellin-Barnes representation of $e^{[\log]+x-y}$, and finally apply the change of variables $x := ([\log]+x)s$ to arrive at:

$$\int\limits_{\underline{c}_2+i\mathbb{R}^2} (-1)^{1-t}\frac{\Gamma(t)\Gamma(t_x)\Gamma(t_y)\Gamma(1-t)\Gamma(C\tau-t_y)}{\Gamma(2+C\tau-t-t_y)}M^{-t_x}G^{-t_y} \times \int_{-[\log]}^{0}([\log]+x)^{1+C\tau-t-t_y}x^{C\tau-1-t_x}dx\,\frac{dt}{2\pi i}\,\frac{dt_x}{2\pi i}\,\frac{dt_y}{2\pi i} \tag{68}$$

Notice that the integral (68) is simply the expression (63), with the difference that, the variable x in the integral $\int_{-[\log]}^{0}([\log]+x)^{1+C\tau-t-t_y}x^{C\tau-1-t_x}dx$ ranges between $-[\log]$ and 0 instead of $-[\log]$ and $+\infty$. Therefore, for this case, we will apply the variable change $x := -[\log]s$, which will result in:

$$\begin{aligned}\int_{-[\log]}^{0}&([\log]+x)^{1+C\tau-t-t_y}x^{C\tau-1-t_x}dx\\ &= [\log]^{1+2C\tau-t-t_x-t_y}(-1)^{1+C\tau-t_x}\int_0^1(1-s)^{1+C\tau-t-t_y}s^{C\tau-1-t_x}ds \qquad (69)\\ &= [\log]^{1+2C\tau-t-t_x-t_y}(-1)^{1+C\tau-t_x}\frac{\Gamma(2+C\tau-t-t_y)\Gamma(C\tau-t_x)}{\Gamma(2+2C\tau-t-t_x-t_y)} \qquad (70)\end{aligned}$$

Replace the expression (70) on the integral (68) and subsequently implanting the resulting expression (47) we will arrive at the desired formula:

$$I^2_{VG}(S,K,r,\mu,\tau) := \int\limits_{\underline{c}_2+i\mathbb{R}^3}(-1)^{C\tau-t-t_x}\frac{\Gamma(t)\Gamma(t_x)\Gamma(t_y)\Gamma(1-t)\Gamma(C\tau-t_x)\Gamma(C\tau-t_y)}{\Gamma(2+2C\tau-t-t_x-t_y)} \times M^{-t_x}G^{-t_y}[\log]^{1+2C\tau-t-t_x-t_y}\,\frac{dt}{2\pi i}\wedge\frac{dt_x}{2\pi i}\wedge\frac{dt_y}{2\pi i} \tag{71}$$

The integral formula (71) converges if all the arguments of the Gamma functions in the numerator are positive, this happens when $\mathrm{Re}(t),\mathrm{Re}(t_x),\mathrm{Re}(t_y) > 0$, $\mathrm{Re}(t) < 1$ and $\mathrm{Re}(t_x),\mathrm{Re}(t_y) < C\tau$, i.e. $\mathrm{Re}(\underline{t}) \in P_2$. $\square$

4.2 Residue Summation Formula for a Call Option

We will now state and proof the main result of this thesis, the triple representation formula for an European call option under the Variance Gamma model.

Theorem 8 (European Call Option Price under the Variance Gamma Process)**.** *The price for an European Call-Option under the Variance Gamma process $X_{VG}(\tau; C, G, M)$ is given by the formula:*

$$C_{VG}(S,K,r,\mu,\tau) = \frac{K(GM)^{C\tau}e^{-(r-q)\tau}}{\Gamma(C\tau)^2}\Big(C^1_{VG}(S,K,r,\mu,\tau) + C^2_{VG}(S,K,r,\mu,\tau) + \mathbb{1}_{[\log]>0}C^3_{VG}(S,K,r,\mu,\tau)\Big) \tag{72}$$

where C^1_{VG}, C^2_{VG} and C^3_{VG} are defined as:

$$C^1_{VG}(S,k,\tau,\mu,r) := \sum_{\substack{k=0\\n=0\\m=0}}^{\infty}(-1)^{n+m}\frac{\Gamma(C\tau+m)\Gamma(-1-2C\tau-k-n-m)}{n!m!\Gamma(1-C\tau-n)} \times M^nG^m(-[\log])^{1+2C\tau+k+n+m} \tag{73}$$

$$C^2_{VG}(S,k,\tau,\mu,r) := \sum_{\substack{k=0\\n=0\\m=0}}^{\infty}(-1)^{m}\frac{\Gamma(C\tau+m)\Gamma(1+2C\tau+k-n+m)}{n!m!\Gamma(2+C\tau+k-n+m)} \times M^{-1-2C\tau-k+n-m}G^m[\log]^n \tag{74}$$

$$C^3_{VG}(S,k,\tau,\mu,r) := \sum_{\substack{k=0\\n=0\\m=0}}^{\infty}(-1)^{C\tau+m}\frac{\Gamma(C\tau+n)\Gamma(C\tau+m)}{n!m!\Gamma(2+2C\tau+k+n+m)} \times M^nG^m[\log]^{1+2C\tau+k+n+m} \tag{75}$$

Proof. The formula (46) for I^1_{VG} can be written as:

$$I^1_{VG}(S,K,r,\mu,\tau) = \int_{\underline{c}_1+i\mathbb{R}^3}\omega^1_{VG} \tag{76}$$

where $\underline{c}_1$ is a three dimensional point $\begin{bmatrix}c_{11}\\c_{12}\\c_{13}\end{bmatrix} \in P_1$ and ω^1_{VG} is a complex differential

3-form defined by:

$$\omega^1_{VG} := (-1)^{-t}\frac{\Gamma(t)\Gamma(t_x)\Gamma(t_y)\Gamma(1-t)\Gamma(C\tau-t_y)\Gamma(-1-2C\tau+t+t_x+t_y)}{\Gamma(1-C\tau+t_x)} \times M^{-t_x}G^{-t_y}(-[\log])^{1+2C\tau+t+t_x+t_y}\frac{dt}{2\pi i}\wedge\frac{dt_x}{2\pi i}\wedge\frac{dt_x}{2\pi i} \tag{77}$$

The divisors of ω^1_{VG} (where we used the notation $\underline{t} = (t, t_x, t_y)$) are:

$$L^1_1 := \{\underline{t}\in\mathbb{C}^3 : t = -n,\ n\in\mathbb{N}\},\ \ L^1_2 := \{\underline{t}\in\mathbb{C}^3 : 1-t = -n,\ n\in\mathbb{N}\}, \tag{78}$$

$$L^1_3 := \{\underline{t}\in\mathbb{C}^3 : t_y = -n,\ n\in\mathbb{N}\},\ \ L^1_4 := \{\underline{t}\in\mathbb{C}^3 : C\tau - t_y = -n,\ n\in\mathbb{N}\}, \tag{79}$$

$$L^1_5 := \{\underline{t}\in\mathbb{C}^3 : t_x = -n,\ n\in\mathbb{N}\}, \tag{80}$$

$$L^1_6 := \{\underline{t}\in\mathbb{C}^3 : -1-C\tau+t+t_x+t_y = -n,\ n\in\mathbb{N}\} \tag{81}$$

By (34) we can compute the characteristic vector of ω_1 as:

$$\Delta_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix} + \begin{bmatrix}-1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\-1\end{bmatrix} + \begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix} \tag{82}$$

Given the characteristic vector (82) for ω_1 we can thus constrict the values of $\underline{t}$ to the space where convergence is obtained:

$$\mathrm{Re}(\Delta_1\cdot\underline{t}) < \mathrm{Re}(\Delta_1\cdot\underline{c}) \Leftrightarrow \mathrm{Re}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}t\\t_x\\t_y\end{bmatrix}\right) < \mathrm{Re}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}\right) \tag{83}$$

$$\Leftrightarrow \mathrm{Re}(t)+\mathrm{Re}(t_x)+\mathrm{Re}(t_y) < c_1+c_2+c_3 \tag{84}$$

$$\Leftrightarrow \mathrm{Re}(t_x) < c_1+c_2+c_3-\mathrm{Re}(t)-\mathrm{Re}(t_y) \tag{85}$$

Therefore, the admissible half-space Π_{Δ_1} is the one located under the plane (85), i.e.:

$$\Pi_{\Delta_1} := \left\{\underline{t}\in\mathbb{C}^3 : \mathrm{Re}(t_x) < c_1+c_2+c_3-\mathrm{Re}(t)-\mathrm{Re}(t_y)\right\} \tag{86}$$

Given the half-space Π_{Δ_1} defined by (86), we will now construct an admissible polyhedron $\Pi_1 = \sigma^1 := g_1^{-1}(G)$, as in the case (35), where G is the first octant, which will be uniquely defined by the linear function:

$$g_1(\underline{t}) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} \mathrm{Re}(t) - c_1 \\ \mathrm{Re}(t_x) - c_2 \\ \mathrm{Re}(t_y) - c_3 \end{bmatrix} + i \begin{bmatrix} \mathrm{Im}(t) \\ \mathrm{Im}(t_x) \\ \mathrm{Im}(t_y) \end{bmatrix} \tag{87}$$

Under the linear function (87), the polyhedron Π_1 will be admissible, that is,

$$\Pi_1 = \{\underline{t} \in \mathbb{C}^3 : \mathrm{Re}(t) < c_1,\ \mathrm{Re}(t_y) < c_3, \\ \mathrm{Re}(t) + \mathrm{Re}(t_x) + \mathrm{Re}(t_y) < c_1 + c_2 + c_3\} \subset \Pi_{\Delta_1} \tag{88}$$

and its faces σ_1^1, σ_2^1 and σ_3^1 and vertex $\sigma_{\{1,2,3\}}^1$ will be respectively:

$$\sigma_1^1 = \{\underline{t} \in \mathbb{C}^3 : \mathrm{Re}(t) = c_1,\ \mathrm{Re}(t_y) \leq c_3, \\ \mathrm{Re}(t) + \mathrm{Re}(t_x) + \mathrm{Re}(t_y) \leq c_1 + c_2 + c_3\} \tag{89}$$

$$\sigma_2^1 = \{\underline{t} \in \mathbb{C}^3 : \mathrm{Re}(t) \leq c_1,\ \mathrm{Re}(t_y) = c_3, \\ \mathrm{Re}(t) + \mathrm{Re}(t_x) + \mathrm{Re}(t_y) \leq c_1 + c_2 + c_3\} \tag{90}$$

$$\sigma_3^1 = \{\underline{t} \in \mathbb{C}^3 : \mathrm{Re}(t) \leq c_1,\ \mathrm{Re}(t_y) \leq c_3 \\ \mathrm{Re}(t) + \mathrm{Re}(t_x) + \mathrm{Re}(t_y) = c_1 + c_2 + c_3\} \tag{91}$$

$$\sigma_{\{1,2,3\}}^1 = \{\underline{t} \in \mathbb{C}^3 : \mathrm{Re}(t) = c,\ \mathrm{Re}(t_y) = c_3, \\ \mathrm{Re}(t) + \mathrm{Re}(t_x) + \mathrm{Re}(t_y) = c_1 + c_2 + c_3\} = \underline{c} \tag{92}$$

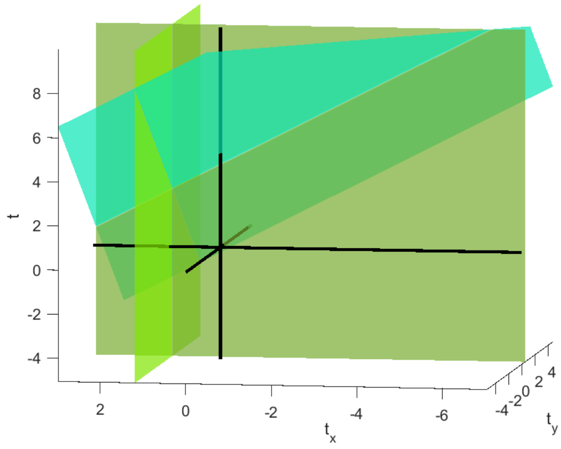
Finally let's group the divisors into three families:

$$D_1^1 = L_1^1 \cup L_2^1, \qquad D_2^1 = L_3^1 \cup L_4^1, \qquad D_3^1 = L_5^1 \cup L_6^1 \tag{93}$$

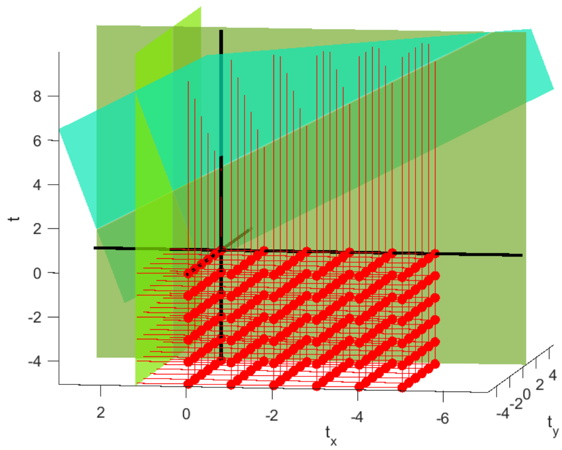
Notice that conditions needed to apply the Jordan lemma are all satisfied, since $\Pi_1 \subset \Pi_{\Delta_1}$ and the family of divisors is compatible with the polyedron Π_1:

$$\sigma_1^1 \cap D_1^1 = \emptyset, \qquad \sigma_2^1 \cap D_2^1 = \emptyset, \qquad \sigma_3^1 \cap D_3^1 = \emptyset \tag{94}$$

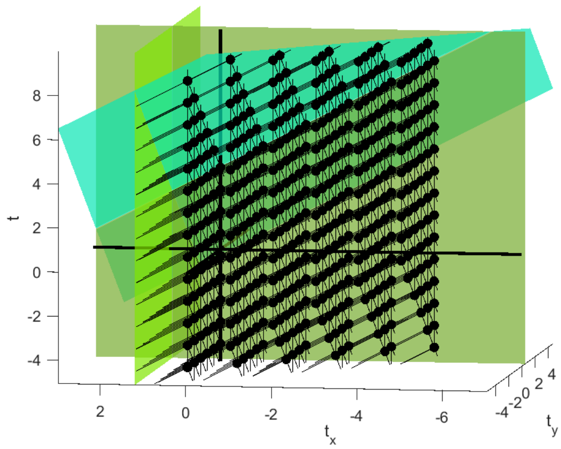
Before applying the residue summation notice that the form ω_{VG}^1 can be considered as having two sets of discontinuity points under the polyhedron Π_1; the first set is defined as $S_1 := \{\underline{t} \in \mathbb{C}^3 : t = -k,\ t_x = -n,\ t_y = -m,\ (k, n, m) \in \mathbb{N}^3\}$ which are the singularity points given by the functions $\Gamma(t)$, $\Gamma(t_x)$ and $\Gamma(t_y)$, the second set is defined as $S_2 := \{\underline{t} \in \mathbb{C}^3 : t = -k,\ t_y = -m,\ -1-2C\tau+t+t_x+t_y = -n,\ (k, n, m) \in \mathbb{N}^3\}$ which are the discontinuity points given by the functions $\Gamma(t)$, $\Gamma(t_x)$ and $\Gamma(-1-2C\tau t+t_x+t_y)$.



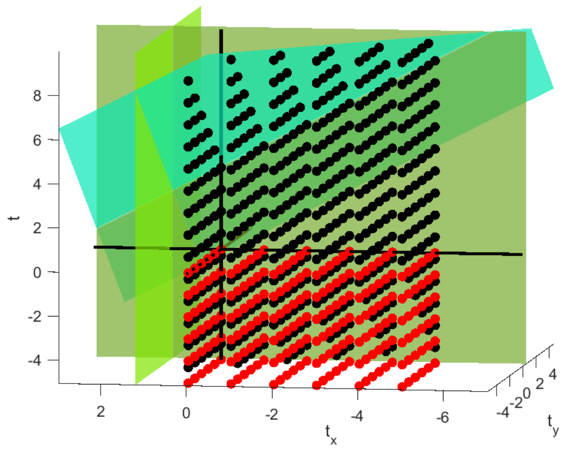
(a) The real part of the planes σ_1^1, σ_2^1 and σ_3^1, and their intersection at the point $\sigma_{\{1,2,3\}}^1$



(b) The discontinuity points of the set S_1 illustrated as red points



(a) The discontinuity points of set S_2, illustrated as black points



(b) The points of the sets S_1 and S_2 illustrated respectively, as red points and black points

FIGURE 3: Discontinuity points under polyhedron Π_1

Given this delineated partition we can now express equation (76) as:

$$I_{VG}^1 = C_{VG}^1 + C_{VG}^2 \tag{95}$$

where we define the terms C_{VG}^1 and C_{VG}^2, respectively as:

$$C_{VG}^1 := \sum_{s \in S_1} \mathrm{Res}_s\, \omega_{VG}^1, \qquad C_{VG}^2 := \sum_{s \in S_2} \mathrm{Res}_s\, \omega_{VG}^1 \tag{96}$$

The computation of the residues for the first set S_1 present in the first series C^1_{VG} of (96) is straightforward:

$$\begin{aligned}\operatorname{Res}_{(-k,-n,-m)}\,\omega^1_{VG} &= (-1)^{-k}\frac{(-1)^k}{k!}\frac{(-1)^n}{n!}\frac{(-1)^m}{m!} \\ &\quad\times\frac{\Gamma(1+k)\Gamma(C\tau+m)\Gamma(-1-2C\tau+k+n+m)}{\Gamma(1-C\tau-n)} \\ &\quad\times M^n G^m(-[\log])^{1+2C\tau+k+n+m} \qquad (97)\\ &= (-1)^{n+m}\frac{\Gamma(C\tau+m)\Gamma(-1-2C\tau+k+n+m)}{n!m!\Gamma(1-C\tau-n)} \\ &\quad\times M^n G^m(-[\log])^{1+2C\tau+k+n+m} \qquad (98)\end{aligned}$$

Embedding the result (98) on the first equation of (96) will produce:

$$\begin{aligned}C^1_{VG}(S,k,\tau,\mu,r) = \sum_{\substack{k=0\\n=0\\m=0}}^{\infty}(-1)^{n+m}\frac{\Gamma(C\tau+m)\Gamma(-1-2C\tau-k-n-m)}{n!m!\Gamma(1-C\tau-n)} \\ \times M^n G^m(-[\log])^{1+2C\tau+k+n+m} \qquad (99)\end{aligned}$$

For the computation of the residues of second set S_2 present in the second series of (96) let's consider the variable change:

$$\begin{cases} u := t \\ u_y := t_y \\ u_x := -1-2C\tau+t+t_x+t_y \end{cases} \Leftrightarrow \begin{cases} t = u \\ t_y = u_y \\ t_x = 1+2C\tau-u+u_x-u_y \end{cases} \qquad (100)$$

If we apply the variables changes in (100) to the expression (77), the form ω^1_{VG} wil be written as:

$$\begin{aligned}\omega^1_{VG} = (-1)^{-u}\frac{\Gamma(u)\Gamma(1+2C\tau-u+u_x-u_y)\Gamma(u_y)\Gamma(1-u)\Gamma(C\tau-u_y)\Gamma(u_x)}{\Gamma(2+C\tau-u+u_x+u_y)} \\ \times M^{-1-2C\tau+u-u_x+u_y}G^{-u_y}(-[\log])^{-u_x}\frac{du}{2\pi i}\wedge\frac{du_x}{2\pi i}\wedge\frac{du_y}{2\pi i} \qquad (101)\end{aligned}$$

Then the residues of the second series C^2_{VG} in (96) are given by:

$$\begin{aligned}\mathrm{Res}_{(-k,-n,-m)}\,\omega^1_{VG} &= (-1)^k \frac{\frac{(-1)^k}{k!}\Gamma(1+2C\tau+k-n+m)\frac{(-1)^m}{m!}\Gamma(C\tau+m)\frac{(-1)^n}{n!}}{\Gamma(2+C\tau+k-n+m)} \\ &\quad \times M^{-1-2C\tau-k+n-m} G^m (-[\log])^n \qquad (102) \\ &= (-1)^{n+m}\frac{\Gamma(1+2C\tau+k-n+m)\Gamma(C\tau+m)}{n!m!\Gamma(2+C\tau+k-n+m)} \\ &\quad \times M^{-1-2C\tau-k+n-m} G^m (-[\log])^n \qquad (103)\end{aligned}$$

Finally, replacing the expression (103) on the second equation of (96) results in the formula:

$$\begin{aligned}C^2_{VG}(S,k,\tau,\mu,r) = \sum_{\substack{k=0\\n=0\\m=0}}^{\infty} (-1)^{n+m}\frac{\Gamma(1+2C\tau+k-n+m)\Gamma(C\tau+m)}{n!m!\Gamma(2+C\tau+k-n+m)} \\ \times M^{-1-2C\tau-k+n-m} G^m (-[\log])^n \qquad (104)\end{aligned}$$

Analogously to what we did for I^1_{VG}, the expression (47) of I^2_{VG} can be written as:

$$I^2_{VG}(S,K,r,\mu,\tau) = \int_{\underline{c}_2+i\mathbb{R}} \omega^2_{VG} \qquad (105)$$

where $\underline{c}_2$ is a three dimensional point $\begin{bmatrix} c_{21} \\ c_{22} \\ c_{23} \end{bmatrix} \in P_2$ and ω^2_{VG} is a complex differential 3-form defined by:

$$\begin{aligned}\omega^2_{VG} := (-1)^{C\tau-t-t_x}\frac{\Gamma(t)\Gamma(t_x)\Gamma(t_y)\Gamma(1-t)\Gamma(C\tau-t_x)\Gamma(C\tau-t_y)}{\Gamma(2+2C\tau-t-t_x-t_y)} \\ \times M^{-t_x}G^{-t_y}(-[\log])^{1+2C\tau-t-t_x-t_y}\frac{dt}{2\pi i}\wedge\frac{dt_x}{2\pi i}\wedge\frac{dt_x}{2\pi i} \qquad (106)\end{aligned}$$

The divisors of ω^2_{VG} are:

$$L^2_1 := \{\underline{t}\in\mathbb{C}^3 : t=-n,\ n\in\mathbb{N}\},\quad L^2_2 := \{\underline{t}\in\mathbb{C}^3 : 1-t=-n,\ n\in\mathbb{N}\} \qquad (107)$$

$$L^2_3 := \{\underline{t}\in\mathbb{C}^3 : t_x=-n,\ n\in\mathbb{N}\},\quad L^2_4 := \{\underline{t}\in\mathbb{C}^3 : C\tau-t_x=-n,\ n\in\mathbb{N}\} \qquad (108)$$

$$L^2_5 := \{\underline{t}\in\mathbb{C}^3 : t_y=-n,\ n\in\mathbb{N}\},\quad L^2_6 := \{\underline{t}\in\mathbb{C}^3 : C\tau-t_y=-n,\ n\in\mathbb{N}\} \qquad (109)$$

Just like the (82) case, we use (34) to compute the characteristic vector of ω^2_{VG}:

$$\Delta_2 = \begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix} + \begin{bmatrix}-1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\-1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\-1\end{bmatrix} - \begin{bmatrix}-1\\-1\\-1\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix} \tag{110}$$

The resulting characteristic vector of Δ_2 expressed in (110), will determine the space for $\underline{t}$ where convergence is achieved:

$$\mathrm{Re}(\Delta_2 \cdot \underline{t}) < \mathrm{Re}(\Delta_2 \cdot \underline{c}) \Leftrightarrow \mathrm{Re}\left(\begin{bmatrix}1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}t\\t_x\\t_y\end{bmatrix}\right) < \mathrm{Re}\left(\begin{bmatrix}1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}\right) \tag{111}$$

$$\Leftrightarrow \mathrm{Re}(t) + \mathrm{Re}(t_x) + \mathrm{Re}(t_y) < c_1 + c_2 + c_3 \tag{112}$$

$$\Leftrightarrow \mathrm{Re}(t_x) < c_1 + c_2 + c_3 - \mathrm{Re}(t) - \mathrm{Re}(t_y) \tag{113}$$

We thus conclude that the admissible half-space Π_{Δ_2} is located under the plane (113):

$$\Pi_{\Delta_2} = \left\{\underline{t} \in \mathbb{C}^3 : \mathrm{Re}(t_x) < +c_1 + c_2 + c_3 - \mathrm{Re}(t) - \mathrm{Re}(t_y)\right\} \tag{114}$$

Similarly to what we did for Π_{Δ_1}, given the expression (114) for Π_{Δ_2}, we will now construct an admissible polyhedron $\Pi_2 = \sigma^2 := g_2^{-1}(G)$, as in the case (35), where G is the first octant and g_2 is the linear function:

$$g_2(\underline{t}) = -I(\mathrm{Re}(\underline{t}) - \underline{c}_2) + i\,\mathrm{Im}(\underline{t}) \tag{115}$$

where I is the identity matrix. Under the linear function prescribed in (115), the polyhedron Π_2 will be admissible:

$$\Pi_2 := \left\{\underline{t} \in \mathbb{C}^3 : \mathrm{Re}(t) < c_{21},\ \mathrm{Re}(t_x) < c_{22},\ \mathrm{Re}(t_y) < c_{23}\right\} \subset \Pi_{\Delta_2} \tag{116}$$

and its faces σ^2_1, σ^2_2 and σ^2_3 and vertex $\sigma^2_{\{1,2,3\}}$ will respectively be:

$$\sigma^2_1 = \left\{\underline{t} \in \mathbb{C}^3 : \mathrm{Re}(t) = c_{21},\ \mathrm{Re}(t_x) \le c_{22},\ \mathrm{Re}(t_y) \le c_{23}\right\} \tag{117}$$

$$\sigma^2_2 = \left\{\underline{t} \in \mathbb{C}^3 : \mathrm{Re}(t_x) = c_{22},\ \mathrm{Re}(t) \le c_{21},\ \mathrm{Re}(t_x) \le c_{23}\right\} \tag{118}$$

$$\sigma^2_3 = \left\{\underline{t} \in \mathbb{C}^3 : \mathrm{Re}(t_y) = c_{23},\ \mathrm{Re}(t) \le c_{21},\ \mathrm{Re}(t_x) \le c_{22}\right\} \tag{119}$$

$$\sigma^2_{\{1,2,3\}} = \left\{\underline{t} \in \mathbb{C}^3 : \mathrm{Re}(t) = c_{21},\ \mathrm{Re}(t_x) = c_{22},\ \mathrm{Re}(t_y) = c_{23}\right\} = \underline{c}_2 \tag{120}$$

Finally we will group the divisors into three sets:

$$D_1^2 = L_1^2 \cup L_2^2, \quad D_2^2 = L_3^2 \cup L_4^2, \quad D_3^2 = L_5^2 \cup L_6^2 \tag{121}$$

Given that $\Pi_2 \subset \Pi_{\Delta_2}$ and the previous partition, the Jordan lemma conditions are all satisfied, since the family of divisors is compatible with the polyhedron Π_2:

$$\sigma_1^2 \cap D_1 = \emptyset, \quad \sigma_2^2 \cap D_2 = \emptyset, \quad \sigma_3^2 \cap D_3 = \emptyset \tag{122}$$

Unlike the ω_{VG}^1 case, the form ω_{VG}^2 under the polyhedron Π_2 has only one set of residues; $S_3 = \{\underline{t} \in \mathbb{C}^3 : t = -k,\ t_x = -n,\ t_y = -m,\ (k,n,m) \in \mathbb{N}^3\}$ resulting from the functions $\Gamma(t)$, $\Gamma(t_x)$ and $\Gamma(t_y)$. Therefore equation (47) can be expressed as:

$$I_{VG}^2 = C_{VG}^3 := \sum_{s \in S_3} \mathrm{Res}_s\, \omega_{VG}^2 \tag{123}$$

The computation of the residues of set S_3 present in the series of (123) is straightforward:

$$\begin{aligned} \mathrm{Res}_{(-k,-n,-m)}\, \omega_{VG}^2 &= (-1)^{C\tau+k+n} \frac{\frac{(-1)^k}{k!}\frac{(-1)^n}{n!}\frac{(-1)^m}{m!}\Gamma(1+k)\Gamma(C\tau+n)\Gamma(C\tau+m)}{\Gamma(2+2C\tau+k+n+m)} \\ &\quad \times M^n G^m (-[\log])^{1+2C\tau+k+n+m} \qquad (124) \\ &= (-1)^{C\tau+m} \frac{\Gamma(C\tau+n)\Gamma(C\tau+m)}{n!m!\Gamma(2+2C\tau+k+n+m)} \\ &\quad \times M^n G^m (-[\log])^{1+2C\tau+k+n+m} \qquad (125) \end{aligned}$$

Swapping the term in (125) on equation (123) will result in:

$$C_{VG}^3(S,k,\tau,\mu,r) = \sum_{\substack{k=0\\n=0\\m=0}}^{\infty} (-1)^{C\tau+k+n} \frac{\Gamma(C\tau+n)\Gamma(C\tau+m)}{n!m!\Gamma(2+2C\tau+k+n+m)} \times M^n G^m (-[\log])^{1+2C\tau+k+n+m} \tag{126}$$

The derived expressions (99), (104) and (126) ascertain the equalities $I_{VG}^1 = C_{VG}^1 + C_{VG}^2$ and $I_{VG}^2 = C_{VG}^3$ which proves (72). □

The price formula for an European call option given by the expression (72) entails an easily derivable price for an European put option by the use of the Put-Call parity:

$$P_{VG}(S,K,r,\mu,\tau) = C_{VG}(S,K,r,\mu,\tau) - S(1 - e^{-\log\frac{S}{K} - (r-q)\tau}) \tag{127}$$

4.3 The Greeks

Given the simple formula for the European call option deduced previously, one may inquiry to the availability of an equally simple measure for risk exposure. The greeks quantify the sensibility of the option price to changes in the model parameters. In this chapter, we will show the existence of series formulas for the Δ, Γ, ρ and Θ measures, which will be obtained by a differentiation of (72) on the appropriate parameter.

Theorem 9 (The Greeks)**.** *The delta, gamma, rho and theta function for an European option under the Variance Gamma process $X_{VG}(\tau; C, G, M)$ are given by:*

- ***Delta** is defined as $\Delta_C := \frac{\partial C}{\partial S}$, hence:*

$$\Delta_C(S, K, \tau, \mu, r) = \frac{K(GM)^{C\tau} e^{-(r-q)\tau}}{S\Gamma(C\tau)^2} \sum_{\substack{k=0\\n=0\\m=0}}^{\infty} \Delta_1 + \Delta_2 + \mathbb{1}_{[\log]>0}\Delta_3 \tag{128}$$

where Δ_1, Δ_2 and Δ_3 are defined as:

$$\Delta_1 = (-1)^{n+m} \frac{\Gamma(C\tau + m)\Gamma(-2C\tau - k - n - m)}{n!m!\Gamma(1 - C\tau - n)} M^n G^m (-[\log])^{2C\tau+k+n+m} \tag{129}$$

$$\Delta_2 = (-1)^m \frac{\Gamma(C\tau + m)\Gamma(2C\tau + k - n + m)}{n!m!\Gamma(1 + C\tau + k - n + m)} M^{-2C\tau-k+n-m} G^m [\log]^n \tag{130}$$

$$\Delta_3 = (-1)^{C\tau+m} \frac{\Gamma(C\tau + n)\Gamma(C\tau + m)}{n!m!\Gamma(1 + 2C\tau + k + n + m)} M^n G^m [\log]^{2C\tau+k+n+m} \tag{131}$$

- ***Gamma** is defined as $\Gamma_C := \frac{\partial^2 C}{\partial S^2}$, hence:*

$$\Gamma_C(S, K, r, \mu, \tau) = \frac{K(GM)^{C\tau} e^{-(r-q)\tau}}{S^2\Gamma(C\tau)^2} \times \sum_{\substack{k=0\\n=0\\m=0}}^{\infty} (\Gamma_1 - \Delta_1) + (\Gamma_2 - \Delta_2) + \mathbb{1}_{[\log]>0}(\Gamma_3 - \Delta_3) \tag{132}$$

where Δ_1, Δ_2 and Δ_3 are described by (129)*,* (130) *and* (131)*, respectively, and Γ_1, Γ_2 and Γ_3 are defined as:*

$$\Gamma_1 = (-1)^{n+m} \frac{\Gamma(C\tau + m)\Gamma(1 - 2C\tau - k - n - m)}{n!m!\Gamma(1 - C\tau - n)} M^n G^m (-[\log])^{-1+2C\tau+k+n+m} \tag{133}$$

$$\Gamma_2 = (-1)^m \frac{\Gamma(C\tau + m)\Gamma(-1 + 2C\tau + k - n + m)}{n!m!\Gamma(C\tau + k - n + m)} M^{1-2C\tau-k+n-m} G^m [\log]^n \quad (134)$$

$$\Gamma_3 = (-1)^{C\tau+m} \frac{\Gamma(C\tau + n)\Gamma(C\tau + m)}{n!m!\Gamma(2C\tau + k + n + m)} M^n G^m [\log]^{-1+2C\tau+k+n+m} \quad (135)$$

- ***Rho*** *is defined as* $\rho_C := \frac{\partial C}{\partial r}$*, hence:*

$$\begin{aligned}\rho_C(S, K, r, \mu, \tau) &= \tau S \Delta_C(S, K, r, \mu, \tau) - \tau C(S, K, \tau, \mu, r) \\ &= \frac{K\tau (GM)^{C\tau} e^{-(r-q)\tau}}{\Gamma(C\tau)^2} \\ &\times \sum_{\substack{k=0\\n=0\\m=0}}^{\infty} (C^1_{VG} - \Delta_1) + (C^2_{VG} - \Delta_2) + \mathbb{1}_{[\log]>0}(C^3_{VG} - \Delta_3) \quad (136)\end{aligned}$$

where C^1_{VG}*,* C^2_{VG}*,* C^2_{VG}*,* Γ_1*,* Γ_2 *and* Γ_3 *are described by* (73)*,* (74)*,* (75)*,* (133)*,* (134) *and* (135)*, respectively.*

- ***Theta*** *is defined as* $\Theta_C := \frac{\partial C}{\partial t} = -\frac{\partial C}{\partial \tau}$*, hence:*

$$\begin{aligned}\Theta_C(S, K, r, \mu, \tau) = -\frac{K(GM)^{C\tau} e^{-(r-q)}}{\Gamma(C\tau)^2} \sum_{\substack{k=0\\n=0\\m=0}}^{\infty} (\theta_1 C^1_{VG} + (r - q - \mu)\Delta_1) \\ + (\theta_2 C^2_{VG} + (r - q - \mu)\Delta_2) + \mathbb{1}_{[\log]>0}(\theta_3 C^3_{VG} + (r - q - \mu)\Delta_3) \quad (137)\end{aligned}$$

where C^1_{VG}*,* C^2_{VG}*,* C^3_{VG}*,* Δ_1*,* Δ_2 *and* Δ_3 *are expressed in* (73)*,* (74)*,* (75)*,* (129)*,* (130) *and* (131) *respectively and* θ_1*,* θ_2 *and* θ_3 *are defined as:*

$$\begin{aligned}\theta_1 &= C\log(GM) - (r - q) - 2C\psi(C\tau) + C\psi(C\tau + m) \quad &(138)\\ &- 2C\psi(-1 - 2C\tau - k - n - m) + C\psi(1 - C\tau - n) + 2C\log(-[\log]) \quad &(139)\\ \theta_2 &= C\log(GM) - (r - q) - 2C\psi(C\tau) + C\psi(C\tau + m) \quad &(140)\\ &+ 2C\psi(1 + 2C\tau + k - n + m) - C\psi(2 + C\tau + k + n + m) - 2C\log(M) \quad &(141)\\ \theta_3 &= C\log(GM) - (r - q) - 2C\psi(C\tau) + C\psi(C\tau + n) + C\psi(C\tau + m) \quad &(142)\\ &- 2C\psi(2 + 2C\tau + k - n + m) + C\pi i + 2C\log([\log]) \quad &(143)\end{aligned}$$

where ψ *is the digamma function* $\psi(z) = \frac{d\log\Gamma(z)}{dz} = -\gamma + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+z}\right)$.

Proof. The previous results are easily obtained from a direct differentiation of the expressions (73), (74) and (75) for the terms C^1_{VG}, C^2_{VG} and C^3_{VG}, for the chosen parameter (i.e. S, S^2, r or t), and sequentially proper rearrangement of the terms. □

5 NUMERICAL RESULTS

The theorems of the previous section can be heuristically observed to be sound. To do this, firstly, we will compare the results obtained by the formula (72) with a both a Monte Carlo simulation for an European call option (under the Variance Gamma process) and the actual values observed on the Market. We also take these results to observe the speed of convergence of the new method. Secondly, we will see that it is well behaved, that is, its price for any initial stock value and its implied volatility smile is similar to the expected behavior observed in any stock. Thirdly we will study the behavior of the greek measures (128), (132), (136) and (137), derived in the previous chapter, and compared them to the ones in the Black-Scholes model.

The programs for the following subsection can be found in the GitHub page with URL: `https://github.com/pedrofebrer/Thesis-Programms`.

5.1 Variance Gamma Formula Values

For the aforementioned comparison we will use the values of an European call option with the S&P500 as its underlying asset, bought at the close of the market at April 18^{th} 2002. According to [25], at the close of the market on 18 April 2002, we had a risk free rate of return $r = 1.9\%$, a dividend of $q = 1,2\%$ and the stock price closed at $S_0 = 1124.47$, with volatility 0.1812 and risk neutral parameters $C = 1.3574$, $G = 5.8704$ and $M = 14.2699$ for the Variance Gamma model.

The results are presented in the Table-I below. We used the parameters $n = 22$, $m = 27$ and $k = 7$ for the direct Variance Gamma formula (72), denoted by "F", we simulated 10000 trajectories for the Monte Carlo method, denoted by "MC" and the observed market values are denoted by "Real":

Strike Price	Time of maturity: May 2002 F			June 2002 F			September 2002 F			December 2002 F			March 2003 F			June 2003 F			December 2003 F		
	F	MC	Real	F	MC	Real	F	MC	Real	F	MC	Real	F	MC	Real	F	MC	Real	F	MC	Real
975	152.90	151.97	-	157.13	157.53	-	167.68	167.54	161.60	177.56	176.02	173.30	186.82	185.68	-	195.53	191.31	-	211.61	209.91	-
995	133.54	132.68	-	138.46	138.69	-	150.40	150.30	144.80	161.27	160.02	157.00	171.28	170.35	-	180.59	176.55	182.10	197.56	196.05	-
1025	104.78	104.05	-	110.97	111.21	-	125.28	125.23	120.10	137.74	136.93	133.10	148.90	148.26	146.50	159.09	155.34	-	177.36	176.13	-
1050	81.16	80.54	-	88.68	88.93	84.50	105.24	105.24	100.70	119.08	118.64	114.80	131.19	130.75	-	142.08	138.54	143.00	161.35	160.37	171.40
1075	58.01	57.48	-	67.14	67.33	64.30	86.18	86.28	82.50	101.40	101.31	97.60	114.41	114.16	-	125.96	122.61	-	146.14	145.35	-
1090	44.42	43.99	43.10	54.69	54.89	-	75.31	75.46	-	91.32	91.44	-	104.83	104.67	-	116.73	113.50	-	137.40	136.73	-
1100	35.56	35.18	35.60	46.66	46.86	-	68.32	68.52	65.50	84.84	85.06	81.20	98.66	98.56	96.20	110.78	107.61	111.30	131.74	131.16	140.40
1110	26.90	26.57	-	38.87	39.08	39.50	61.56	61.78	-	78.56	78.86	-	92.66	92.62	-	104.98	101.89	-	126.22	125.73	-
1120	18.52	18.24	22.90	31.39	31.60	33.50	55.07	55.29	-	72.49	72.86	-	86.85	86.85	-	99.35	96.36	-	120.83	120.43	-
1125	14.48	14.25	20.20	27.80	28.00	30.70	51.92	52.16	51.00	69.54	69.94	66.90	84.02	84.04	81.70	96.60	93.64	97.00	118.19	117.83	-
1130	10.61	10.41	-	24.31	24.52	28.00	48.85	49.10	-	66.64	67.08	-	81.23	81.27	-	93.89	90.97	-	115.58	115.27	-
1135	7.10	6.92	-	20.97	21.17	25.60	45.86	46.11	45.50	63.81	64.27	-	78.49	78.56	-	91.22	88.35	-	113.00	112.74	-
1140	6.00	5.84	13.30	17.80	18.01	23.20	42.95	43.22	-	61.03	61.52	58.90	75.81	75.90	-	88.59	85.78	-	110.47	110.25	-
1150	4.66	4.56	-	12.58	12.79	19.10	37.39	37.70	38.10	55.67	56.20	53.90	70.58	70.73	68.30	83.47	80.75	83.30	105.49	105.36	112.80
1160	3.76	3.69	-	10.04	10.21	15.30	32.22	32.59	-	50.57	51.13	-	65.56	65.76	-	78.52	75.89	-	100.66	100.61	-
1170	3.08	3.05	-	8.26	8.42	12.10	27.51	27.91	-	45.74	46.34	-	60.76	61.00	-	73.75	71.21	-	95.97	96.00	-
1175	2.81	2.79	-	7.53	7.69	10.90	25.35	25.78	27.70	43.44	44.05	42.50	58.43	58.70	56.60	71.44	68.93	-	93.68	93.75	99.80
1200	1.82	1.84	-	4.94	5.09	-	17.24	17.71	19.60	33.06	33.80	33.00	47.66	48.02	46.10	60.54	58.32	60.90	82.76	83.02	-
1225	1.22	1.28	-	3.36	3.51	-	12.22	12.73	13.20	24.81	25.56	24.90	38.33	38.77	36.90	50.81	48.86	49.80	72.72	73.17	-
1250	0.84	0.94	-	2.35	2.49	-	8.84	9.41	-	18.74	19.42	18.30	30.51	31.03	29.30	42.26	40.55	41.20	63.56	64.19	66.90
1275	0.59	0.71	-	1.67	1.84	-	6.49	7.11	-	14.26	14.84	13.20	24.18	24.75	22.50	34.88	33.44	-	55.28	56.01	-
1300	0.42	0.55	-	1.21	1.38	-	4.81	5.43	-	10.92	11.44	-	19.17	19.71	17.20	28.64	27.42	27.10	47.85	48.63	49.50
1325	0.31	0.44	-	0.88	1.05	-	3.61	4.17	-	8.42	8.93	-	15.23	15.77	12.80	23.45	22.43	-	41.24	42.05	-
1350	0.23	0.35	-	0.65	0.81	-	2.73	3.23	-	6.52	7.03	-	12.12	12.66	-	19.18	18.38	17.10	35.41	36.28	35.70
1400	0.12	0.23	-	0.37	0.49	-	1.59	2.04	-	3.98	4.41	-	7.74	8.25	-	12.84	12.27	10.10	25.88	26.82	25.20
1450	0.07	0.17	-	0.21	0.30	-	0.95	1.32	-	2.47	2.81	-	5.00	5.50	-	8.63	8.22	-	18.79	19.73	17.00
1500	0.04	0.12	-	0.13	0.19	-	0.58	0.91	-	1.56	1.83	-	3.27	3.74	-	5.84	5.53	-	13.60	14.50	12.20

TABLE I: S&P 500 Call Option prices and estimations

We will take advantage of 75 actual recorded values presented in Table-I, and make an error estimation for each model, by calculating their respective root mean square error, which is given by the formula $RMSE = \sqrt{\sum_{i=1}^{n}(\text{market price}_i - \text{model price}_i)^2/n}$. Under this metric the deviations from the observed results are:

RMSE	
Black-Scholes	6.6692
Variance Gamma Monte Carlo	3.6959
Variance Gamma Formula	3.5183

TABLE II: Root Mean Square Error

Therefore not only is the formula (72) more expedient due to much lower computational time (in our tests it was between 72 and 92 times faster), but it also outperforms the Monte-Carlo method (and consequently the Black-Scholes by a wide margin). There may be rare events where the Monte-Carlo returns a better result, due to its randomness, but in all the simulation we executed this event was never realized.

5.2 Convergence of the Variance Gamma Formula

In order to study the numerical convergence and precision of the new formula we must first realize that observing the value of each isolated term in the triple sum (72) is fallacious since an unit increase of, for instance, parameter n will lead to the sum of an extra $m \times k$ terms, which may lead to an error of substantially higher magnitude than each individual term. For example, for all the strike prices K and times to maturity τ of Table-I, the values of the terms at $n, m, k = 10$ are always zero, under a two decimal numerical precision. On the other hand, the values reached for the double sum series when $n = 10$ and $0 \leq m, k \leq 10$ can be as low as -67.679 (for $K = 975$ and $\tau = 81 \times 7/365$), as can seen in Table-III.

As one would expect from the definition $[\log] = \log \frac{S}{K} + (r-q)\tau - \mu\tau$, the higher the time to maturity τ the higher will be the double series sum value, as can be seen in Table-III where the strike price is 975. Similarly, the more S differs from K the higher will be the sum value, as can be observed in Table-IV where the time of maturity is September 2002. In both of these cases we will have to choose bigger values for the parameters n, m and k in order to assure convergence. That is, to ensure that the triple sum series in (72) from 0 to some fixed n_{max}, m_{max} and k_{max} values, for each respective n, m and k parameter, will yield a truncation error lower than some chosen numerical precision.

Time of Maturity	May 2002	June 2002	September 2002	December 2002	March 2003	June 2003	December 2003
Error	0.000	0.000	0.145	0.296	2.197	-22.501	-67.679

TABLE III: Values for the double series where $n = 10$, $0 \leq m, k \leq 10$ and $K = 975$

Strike	975	995	1025	1050	1075	1100	1200	1300	1325	1350	1400	1450	1500
Error	0.145	0.049	0.008	0.001	0.000	0.000	0.000	0.000	0.001	0.004	0.049	0.333	1.579

TABLE IV: Values for the double series where $n = 10$, $0 \leq m, k \leq 10$ and $\tau = 81 \times 7/365$

Let us denote each term of the sum (72) by $C_{VG}(n, m, k)$, given this notation we can write $C_{VG} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} C_{VG}(n, m, k)$. To determine the values n_{max}, m_{max} and k_{max} for which the sum C_{VG} converges, for all values K and τ of Table-I, we will apply the euclidean norm to the 189 resulting values from the three possible double sum series: the series computed by summing the terms of C_{VG} for a fixed n and $0 \leq m, k \leq n$, i.e. $C_{VG}^{\text{n const}}(n) = \sum_{m=0}^{n} \sum_{k=0}^{n} C_{VG}(n, m, k)$, the series computed by summing the terms of C_{VG} for a fixed m and $0 \leq n, k \leq m$, i.e. $C_{VG}^{\text{m const}}(m) = \sum_{n=0}^{m} \sum_{k=0}^{m} C_{VG}(n, m, k)$ and the series computed by summing the terms of C_{VG} for a fixed k and $0 \leq n, m \leq k$, i.e. $C_{VG}^{\text{k const}}(k) = \sum_{n=0}^{k} \sum_{m=0}^{k} C_{VG}(n, m, k)$.

	Double series sum				Double series sum		
	$C_{VG}^{\text{n const}}(m)$	$C_{VG}^{\text{m const}}(m)$	$C_{VG}^{\text{k const}}(k)$		$C_{VG}^{\text{n const}}(n)$	$C_{VG}^{\text{m const}}(m)$	$C_{VG}^{\text{k const}}(k)$
0	1584.541	1584.541	1584.541	14	3.521	5.567	0.000
1	947.283	1654.716	93.741	15	1.289	2.707	0.000
2	1490.255	2180.378	14.829	16	0.439	1.303	0.000
3	400.789	2229.189	0.663	17	0.140	0.621	0.000
4	536.914	1871.335	0.124	18	0.041	0.294	0.000
5	355.390	1354.925	0.008	19	0.012	0.138	0.000
6	364.206	880.180	0.001	20	0.003	0.064	0.000
7	291.953	528.778	0.000	21	0.001	0.030	0.000
8	218.112	300.352	0.000	22	0.000	0.014	0.000
9	142.454	163.798	0.000	23	0.000	0.006	0.000
10	83.357	86.648	0.000	24	0.000	0.003	0.000
11	43.583	44.762	0.000	25	0.000	0.001	0.000
12	20.626	22.685	0.000	26	0.000	0.001	0.000
13	8.891	11.314	0.000	27	0.000	0.000	0.000

TABLE V: Convergence of the three double series

From the values above we can ascertain, that for any K and τ of Table-I, we can assure convergence with a two decimal precision, when the sum of the three double sum series has a result lower than 0.005, for instance $n = 22$, $m = 27$ and $k = 7$.

5.3 *Option Price Behavior for the Variance Gamma Formula*

Hitherto we only applied the formula for the same initial stock value. Still the formulas for the European options under the Variance Gamma (72) and (127), present the typical behavior for different initial stock values. In fact, if we take $K = 1100$ for the same parameters of the previous subsection and vary the initial stock price S_0, we get the typical call and put option behavior. This can be seen by the difference given by our formula and the Black-Scholes formula for the price of an European Option:

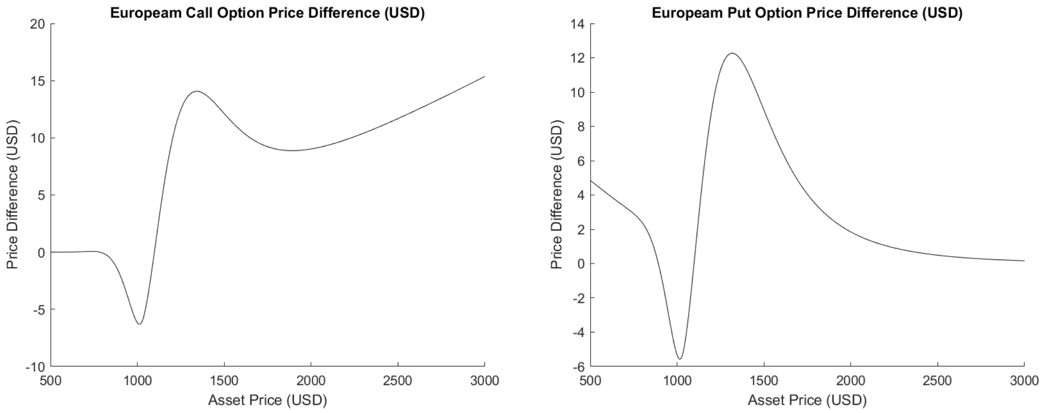


FIGURE 4: Variance Gamma and Black Scholes Formulas Price Differences

One of the most important tool used in finance is called the implied volatility, which consists in finding the volatility for which the model employed (typically the Black-Scholes model) in determining the option prices is congruent with the observed values C, that is, the values σ_I such that $C_{BS}(S, K, r, \sigma_I, \tau) = C$ holds. In our particular case we fixed the time to maturity at $\tau = 35 \times 7/365$, while the rest of the variables remain equal to the previews section, and applied the Newton algorithm to compute σ_I.

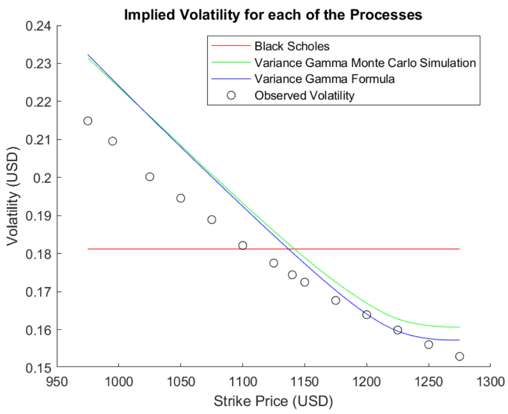


FIGURE 5: Implied Volatility

As can be observed from Figure-5, the formula (72) displays a volatility smile typically present in most assets, including the present asset.

5.4 The Greek Formulas Behavior

Greeks are extremely important for financial institutions and their endeavors such as hedging against market uncertainty. Therefore we terminate this section by visualizing the behavior of greek functions under the Variance Gamma model and contrasting them with the ones under the Black-Scholes model.

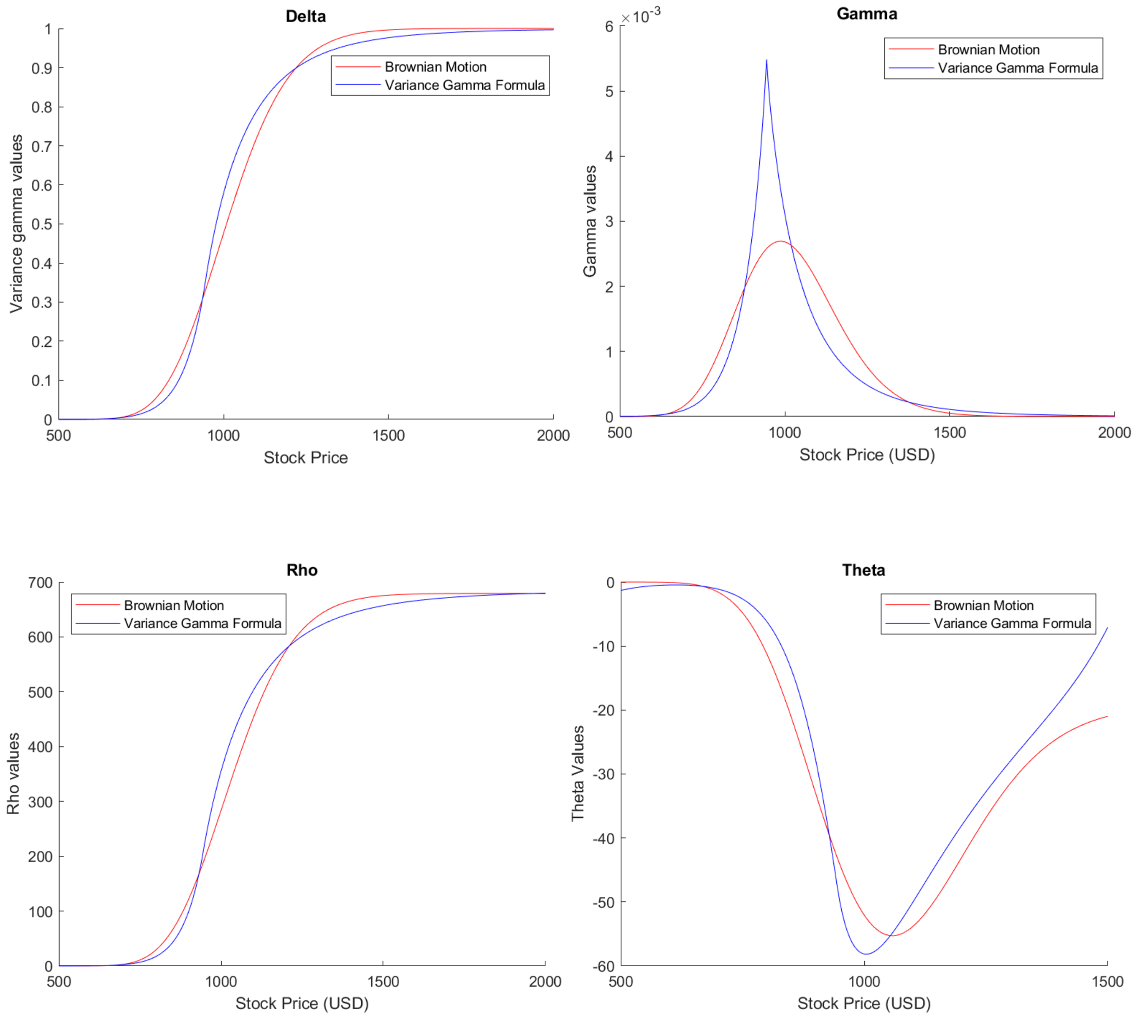


FIGURE 6: Greeks

As can be seen, the greek measures for the Variance Gamma model, seem similar enough to the ones of the Black-Scholes, yet they exhibit enough discrepancies to be worthy of note, primarily in the Gamma and Theta functions, presumably due to its higher similitude with the empirical data. These seemingly more accurate new greek functions only involve the simple computation of triple sum series, which is a stark contrast with the much more ponderous old-school scheme method used for Lévy processes.

6 CONCLUSIONS

In this dissertation, we have derived a triple Mellin-Barnes integral representation for the price of an European call option driven by a Variance Gamma process (45). Subsequently we applied multidimensional residue calculus to the aforementioned integral and computed a triple sum series for the the European call option (72). Triple sum series for the delta, gamma, rho and theta greeks were also found by direct differentiation. When tested, (72) exhibited the behavior typically observed in the market for European options, for instance the volatility smile, and it outperformed the Monte-Carlo simulation method in both accuracy and computational time. The greeks also displayed their conventional behavior.

For practical applications, the simplicity present in the aforementioned formulas (such as the lack of necessity of simulations for pricing European options, or of schemes to compute the greeks), coupled with their higher rates of precision and much lower computation time makes them ideal for financial practitioners, without the necessity for more theoretical concepts such as schemes, complex calculus and fractional calculus. For example formulas (128) and (132) can be directly used to generate a portfolio with optimal delta and gamma hedge strategies.

In terms of future research the most obvious course of action would be to compare the formula (72), in terms of accuracy and computational time, to other semi-closed formulas such as the Bessel functions representation formula or the Fast Fourier Transform for the price of an European call option under the Variance Gamma model (which their definition and proof are presented in the papers [16] and [8] respectively).

For more theoretical results, the more pressing question will be the ability to use a similar reasoning as presented in Section 4, to arrive at a sum series for the more general CGMY process and Generalized Tempered Stable process, even if this necessitates a higher dimensional Mellin-Barnes integral. One also might inquire further to the pricing of more complex financial instruments, like American or Barrier options, and specially instruments like Asian option, where the integral involved in their definition seems to make them a prime candidate for residue calculus.

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A Proofs

A.1 Proof of Theorem 7

Proof. Before tackling the proof, note that for any $z \in \mathbb{C}$ we have the inequality:

$$\left|\sqrt{2\pi} z^{z-1/2} e^{-z}\right| = \sqrt{2\pi}(|z|e^{i\arg(z)})^{\mathrm{Re}(z)+i\,\mathrm{Im}(z)-1/2} e^{-\mathrm{Re}(z)-i\,\mathrm{Im}(z)\arg(z)} \tag{144}$$

$$= \sqrt{2\pi}|z|^{\mathrm{Re}(z)-1/2} e^{-\arg(z)\,\mathrm{Im}(z)-\mathrm{Re}(z)} \tag{145}$$

$$\geq c_1 |z|^{\mathrm{Re}(z)-1/2} e^{-\frac{\pi}{2}|\mathrm{Im}(z|)-|\mathrm{Re}(z)|} \tag{146}$$

for some constant c_1 and also the inequality:

$$\left|\sqrt{2\pi} z^{z-1/2} e^{-z}\right| = \sqrt{2\pi}|z|^{\mathrm{Re}(z)-1/2} e^{-\arg(z)\,\mathrm{Im}(z)-\mathrm{Re}(z)} \tag{147}$$

$$\leq c_2 |z|^{\mathrm{Re}(z)-1/2} e^{-\frac{\pi}{2}|\mathrm{Im}(z)|+|\mathrm{Re}(z)|} \tag{148}$$

The last inequality (148) is verified if and only if the inequality $-\arg(z)\,\mathrm{Im}(z) - \mathrm{Re}(z) \leq -\frac{\pi}{2}|\,\mathrm{Im}(z)| + |\,\mathrm{Re}(z)|$ is valid. Inasmuch as we can write $s_j = re^{i\theta}$, where $r \in \mathbb{R}_0^+$ and $\theta \in [-\pi, \pi]$, the inequality can be written as $r\theta\sin\theta - r\cos\theta \leq -\frac{\pi}{2}|r\sin\theta| + |r\cos\theta|$ which hols for any for any $r \in \mathbb{R}^+$ and $\theta \in]-\pi, \pi[$. In fact the inequality is equivalent to $0 \leq (\theta - \pi/2)\sin\theta + 2\cos\theta$ for $\theta \in [0, \pi/2]$, $0 \leq (\theta - \pi/2)\sin\theta$ for $\theta \in [\pi/2, \pi]$, $0 \leq (\theta + \pi/2)\sin\theta$ for $\theta \in [-\pi/2, 0]$ and $0 \leq (\theta + \pi/2)\sin\theta + 2\cos\theta$ for $\theta \in [-\pi, -\pi/2]$ which are all true, therefore the inequality (148) will hold.

If z does not intersect the set $\mathbb{Z}_0^- + i\{0\}$, then as $|z| \to \infty$ we can apply the Stirling Formula (27) and from the expressions (146) and (148) we know there exists constants c_1 and c_2 such that:

$$c_1|z|^{\mathrm{Re}(z)-1/2} e^{-\frac{\pi}{2}|\mathrm{Im}(z)|-|\mathrm{Re}(z)|} < |\Gamma(z)| < c_2(\epsilon)|z|^{\mathrm{Re}(z)-1/2} e^{-\frac{\pi}{2}|\mathrm{Im}(z)|+|\mathrm{Re}(z)|} \tag{149}$$

On the other hand, if we constrict the real value, x, to a compact set $K \subset \mathbb{R} \setminus \mathbb{Z}_0^-$, x will be bounded and the gamma function will be continuous in the domain $K + i\mathbb{R}$. We can thus denote its supremum as $M = \sup_{x \in K}|x| < \infty$ and infimum as $m = \inf_{x \in K}|x|$. Under this notation we have $e^{-M} < e^{-|x|}$ and $e^{|x|} < e^M$ and also as $|y| \to \infty$ we have $|x + iy| = \sqrt{x^2 + y^2} \sim (1 + |y|)$. Applying these properties to (149) results in the inequalities

$$k_1(|y|+1)^{x-1/2} e^{-\frac{\pi}{2}|y|} < |\Gamma(x+iy)| < k_2(\epsilon)(|y|+1)^{x-1/2} e^{-\frac{\pi}{2}|y|} \tag{150}$$

for some constants k_1 and k_2. Taking advantage of the inequalities (150), we can bound the integrand of expression (30) by:

$$\left|\frac{\prod_{j=1}^{m}\Gamma(s_j(z))}{\prod_{k=1}^{p}\Gamma(q_k(z))}t^{-z}\right| \leq C\frac{\prod_{j=1}^{m}\left(|\langle \underline{a}_j, x\rangle|\right)^{\langle \underline{a}_j,x\rangle+b_j-1/2}}{\prod_{j=1}^{p}\left(|\langle \underline{c}_j, x\rangle|\right)^{\langle \underline{c}_j,x\rangle+b_j-1/2}} \tag{151}$$

$$\times \exp\left\{|\langle y, \arg t\rangle| - \frac{\pi}{2}\left(\sum_{j=1}^{m}|\langle \underline{a}_j, y\rangle| - \sum_{j=1}^{p}|\langle \underline{c}_j, y\rangle|\right)\right\} \tag{152}$$

for some constant C. If for all $y \in \mathbb{R}^n$ and $t \in (\mathbb{C}^n \setminus \{0\})^n$ the inequality

$$|\langle y, \arg t\rangle| < \frac{\pi}{2}\left(\sum_{j=1}^{m}|\langle \underline{a}_j, y\rangle| - \sum_{j=1}^{m}|\langle \underline{a}_j, y\rangle|\right) \tag{153}$$

is satisfied, then the integrand in (30) decreases exponentially as $\|y\| \to \infty$, making the integral converge absolutely. Taking into account (31) the inequality (153) will hold if

$$\max_{y\in S_1}|\langle y, \arg t\rangle| < \frac{\pi}{2}\alpha \tag{154}$$

By the Cauchy-Schwartz inequality

$$\max_{y\in S_1}|\langle y, \arg t\rangle| \leq \max_{y\in S_1}\|y\|\|\arg t\| = \|\arg t\| \tag{155}$$

and since we are working in U, $\|\arg t\| < \frac{\pi}{2}\alpha$ which concludes our proof. □

A.2 Proof of Theorem 8

Proof. We will extrapolate the proof present in [27], to the three dimensional case. We begin by separating the gammas in the numerator of the form ω into three groups Γ_1, Γ_2, Γ_3, such that, for the singularities in (35) the zeroes of f_1 do not intersect D_1, the ones of f_2, D_2, and the ones of f_3, D_3, .i.e. $\forall_{i\in\{1,2,3\}}\mathrm{Ker}(f_i) \cap D_i = \emptyset$. Similarly we also denote the multiple gammas in the denominator of ω by Γ_4. Taking this new notation into account we can write the form ω in the standard form (30), i.e:

$$\omega = \frac{h dz_1 \wedge dz_2 \wedge dz_3}{f_1 f_2 f_3} \tag{156}$$

where f_1, f_2, f_3 and h are defined by:

$$f_1 = \frac{1}{\Gamma_1}, \quad f_2 = \frac{1}{\Gamma_2}, \quad f_3 = \frac{1}{\Gamma_3}, \quad h = \frac{t_1^{-z_1}t_2^{-z_2}t_3^{-z_3}}{\Gamma_4} \tag{157}$$

The proof of the theorem follows once we are able to validate compatibility and the Jordan conditions, i.e. (23), under the polyhedron Π. The first thing to note is that, by definition, the zeroes D_1, D_2 and D_3 are compatible with the polyhedron Π. Secondly, observe that there exists a linear transformation, g^{-1}, the inverse of (35), that simplifies the proof. Thus, we can, without lost of generality, apply this linear change of variables, which will result in the real part of σ being the first octant, the real part of σ_1, σ_2 and σ_3 the $\{y,z\}$-plane, $\{x,z\}$-plane and the $\{x,y\}$-plane, respectively, the real part of $\sigma_{\{1,2\}}$, $\sigma_{\{1,3\}}$ and $\sigma_{\{2,3\}}$ the z-axis, y-axis and the x-axis, respectively and $\mathrm{Re}(\gamma) = \mathrm{Re}(\sigma_{\{1,2,3\}}) = 0$. This linear transformation will obviously also be applied to the ω-form. Under this new change of variables it will suffice for us to prove the Jordan lemma for the differential forms

$$\xi_1 = \frac{h\overline{f}_1 dz}{f_2 f_3 \|f\|}, \quad \xi_2 = \frac{h\overline{f}_2 dz}{f_1 f_3 \|f\|}, \quad \xi_3 = \frac{h\overline{f}_3 dz}{f_1 f_2 \|f\|}, \tag{158}$$

$$\xi_{\{2,3\}} = h\frac{\overline{f}_2 d\overline{f}_3 - \overline{f}_3 d\overline{f}_2}{f_1 ||f||^4} dz, \quad \xi_{\{1,3\}} = h\frac{\overline{f}_1 d\overline{f}_3 - \overline{f}_3 d\overline{f}_1}{f_2 ||f||^4} dz, \tag{159}$$

$$\xi_{\{1,2\}} = h\frac{\overline{f}_1 d\overline{f}_2 - \overline{f}_2 d\overline{f}_1}{f_3 ||f||^4} dz, \quad \xi_{\{1,2,3\}} = \xi^1_{\{1,2,3\}} + \xi^2_{\{1,2,3\}} + \xi^3_{\{1,2,3\}} \tag{160}$$

on the corresponding half-spaces:

$$\sigma_{\{2,3\}} = l_1 + i\mathbb{R}^3, \quad \sigma_{\{1,3\}} = l_2 + i\mathbb{R}^3, \quad \sigma_{\{1,2\}} = l_3 + i\mathbb{R}^3, \tag{161}$$

$$\sigma_1 = P_{\{2,3\}} + i\mathbb{R}^3, \quad \sigma_2 = P_{\{1,3\}} + i\mathbb{R}^3, \quad \sigma_3 = P_{\{1,2\}} + i\mathbb{R}^3, \quad \sigma_\emptyset = V + i\mathbb{R}^3 \tag{162}$$

where, $\xi^1_{\{1,2,3\}}$, $\xi^2_{\{1,2,3\}}$ and $\xi^3_{\{1,2,3\}}$ are defined by

$$\xi^1_{\{1,2,3\}} = h\frac{\overline{f}_1 d\overline{f}_2 d\overline{f}_3 - \overline{f}_2 d\overline{f}_1 d\overline{f}_3 + \overline{f}_3 d\overline{f}_1 d\overline{f}_2}{||f||^6} dz, \tag{163}$$

$$\xi^2_{\{1,2,3\}} = -2h\frac{\overline{f}_1 \overline{f}_3 d\overline{f}_2 \left(f_1 d\overline{f}_1 + f_2 d\overline{f}_2 + f_3 d\overline{f}_3\right)}{||f||^8} dz, \tag{164}$$

$$\xi^3_{\{1,2,3\}} = h\frac{\overline{f}_1 \overline{f}_2 \overline{f}_3 \left(f_1 d\overline{f}_1 + f_2 d\overline{f}_2 + f_3 d\overline{f}_3\right)^2}{||f||^{10}} dz \tag{165}$$

and l_1, l_2, l_3, $P_{\{2,3\}}$, $P_{\{1,3\}}$ and $P_{\{1,2\}}$ are subsequently defined by $l_1 = \{x \in \mathbb{R}^3 : x_1 \geq 0, x_2 = 0, x_3 = 0\}$, $l_2 = \{x \in \mathbb{R}^3 : x_1 = 0, x_2 \geq 0, x_3 = 0\}$, $l_3 = \{x \in \mathbb{R}^3 : x_1 = 0, x_2 = 0, x_3 \geq 0\}$, $P_{2,3} = \{x \in \mathbb{R}^3 : x_1 = 0, x_2 \geq 0, x_3 \geq 0\}$, $P_{1,3} = \{x \in \mathbb{R}^3 : x_1 \geq 0, x_2 = 0, x_3 \geq 0\}$, $P_{1,2} = \{x \in \mathbb{R}^3 : x_1 \geq 0, x_2 \geq 0, x_3 = 0\}$ and $V = \{x \in \mathbb{R}^3 : x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}$.

Further, the computation of the integral ξ_1 over $\sigma_{\{2,3\}}$ will be analogous to ξ_2 over $\sigma_{\{1,3\}}$ and ξ_3 over $\sigma_{\{1,2\}}$, as will the computation of $\xi_{\{2,3\}}$ over σ_1 to $\xi_{\{1,3\}}$ over σ_2 and $\xi_{\{1,2\}}$ over σ_3. Thus it will suffice to examine just three cases: (211), (215) and (220).

Now let's consider the sequence of sets $\mathcal{U}_k = \{x + iy \in \mathbb{C}^3 : \|x\| \leq R_k, \|y\| \leq R_k\}$, where $R_k \to \infty$ as $k \to \infty$. Define the surface $S_k = \partial \mathcal{U}_k$ and let ξ be one of the seven integrands of (158), (159) or (160) defined on its corresponding σ of (161) or (162), it is a well known property of Lebesgue integrals that there exists a constant $c \in \mathbb{R}$, such that:

$$\int_{\sigma \cap S_k} \xi \leq \|S_k\| \sup_{(\sigma \cap S_k) \setminus Z} \|\xi\| \leq c R_k^5 \sup_{(\sigma \cap S_k) \setminus Z} \|\xi\| \tag{166}$$

where Z is a set with zero Lebesgue measure. Our job will be to prove that as $k \to \infty$, $\|\xi\| \to 0$ at an exponential rate, resulting in the integral in (166), and consequently all (211), (215) and (220) being zero. In order to achieve this, we will divide $\sigma \cap S_k$ in to two sets:

$$B_k = \{x + iy \in \sigma : \|x\| \leq R_k, \ \|y\| = R_k\} \tag{167}$$

$$O_k = \{x + iy \in \sigma : \|x\| = R_k, \|y\| \leq R_k\} \tag{168}$$

Depending on the σ we are working with, this separation will yield different results. For instance, for σ_1 we have $B_k = \{x + iy \in \mathbb{C}^3 : 0 \leq x_1 \leq R_k, \|y\| = R_k\}$ and $O_k = \{x + iy \in \mathbb{C}^3 : x_1 = R_k, \|y\| \leq R_k\}$, for $\sigma_{1,2}$ we have $B_k = \{x + iy \in \mathbb{C}^3 : x_1^2 + x_2^2 \leq R_k^2, \ \|y\| = R_k\}$ and $O_k = \{x + iy \in \mathbb{C}^3 : x_1^2 + x_2^2 = R_k^2, \|y\| \leq R_k\}$ and for $\sigma_{1,2,3}$ we have $B_k = \{x + iy \in \mathbb{C}^3 : x_1^2 + x_2^2 + x_3^2 \leq R_k^2, \ \|y\| = R_k\}$ and $O_k = \{x + iy \in \mathbb{C}^3 : x_1^2 + x_2^2 + x_3^2 = R_k, \|y\| \leq R_k\}$.

As a final tool for our proof consider the set $\mathcal{U}_\delta = \{z \in \mathbb{C}^3 : |s_j(z) + \nu| \geq \delta > 0, \ j = 1, ..., m; \ \nu \in \mathbb{N}\}$, which removes a neighborhood in $\mathcal{U}$ around the singularities present in the numerator of the ratio of products of gamma functions in ω. Hence we can write left most term of (166) as:

$$\int_{\sigma \cap S_k} \xi = \int_{O_k \cap \mathcal{U}_\delta} \xi + \int_{B_k \cap \mathcal{U}_\delta} \xi + \int_{\sigma \cap S_k \cap \mathcal{U}_\delta^c} \xi \tag{169}$$

Our proof will consist of three steps: firstly to estimate the value of $\|\omega\|\|_{O_k \cap \mathcal{U}_\delta}$, secondly to estimate the value of $\|\omega\|\|_{B_k \cap \mathcal{U}_\delta}$, thirdly to extend the two previous results to the value of the integral to the set of neighborhoods $\mathcal{U}_\delta^c$ (in the proof of Lemmas 1 2 and 3).

Step 1:

Recall that for any $z \in \mathbb{C}^3$ we have $s_j(z) = \langle a_j, x\rangle + b_j + i\langle a_j, y\rangle$, also for any point in the three-dimensional space, $x \in \mathbb{R}^3$, there exist $\theta \in [0, 2\pi[$ and $\phi \in [0, \pi[$ such that $x = \|x\|\hat{x} = \|x\|(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$, hence the real part of $s_j(z)$ for $z \in O_k$ is given by $\mathrm{Re}(s_j(z)) = R_k\langle a_j, \hat{x}\rangle + b_j$. Having fixed θ and ϕ, and given we are working in the space $O_k \cap \mathcal{U}_\delta$, we will chose the radii R_k such that $\mathrm{Re}(s_j(z))$ wont intersect the singularities of the form ω, (the latter case will be dealt in the lemmas). Thus, the numerator of the form ω can be segregated into two terms, the ones where $\langle a_j, \hat{x}\rangle > 0$, which we will order as $\mu + 1 \leq j \leq m$, and the ones where $\langle a_j, \hat{x}\rangle < 0$, which we will order as $1 \leq j \leq \mu$. Analogously, we can sort the denominator of ω , where $\langle c_j, \hat{x}\rangle > 0$, for $\chi + 1 \leq j \leq p$, and $\langle c_j, \hat{x}\rangle < 0$, for $1 \leq j \leq \chi$. In this case, as $R_k \to +\infty$ we have $\mathrm{Re}(s_j|_{O_k}) \to +\infty$ and $\mathrm{Re}(q_j|_{O_k}) \to +\infty$ for $\mu \leq j \leq m$ and $\chi \leq j \leq p$, respectively, and $\mathrm{Re}((1-s_j)|_{O_k}) \to +\infty$ and $\mathrm{Re}((1-q_j)|_{O_k}) \to +\infty$ for $1 \leq j \leq \mu$ and $1 \leq j \leq \chi$, respectively. Therefore, if we use the relation $\Gamma(s_j)\Gamma(1-s_j) = \pi/\sin(\pi s_j)$ and the apply the Stirling formula (27), we get:

$$\left|\frac{\Gamma_1\Gamma_2\Gamma_3}{\Gamma_4}\right|\Bigg|_{O_k} \leq k_1 \left|\frac{\prod_{j=1}^{\mu}\frac{1}{\sin(\pi s_j)(1-s_j)^{(1-s_j)-1/2}e^{-(1-s_j)}}\prod_{j=\mu+1}^{m} s_j^{s_j-1/2}e^{-s_j}}{\prod_{j=1}^{\chi}\frac{1}{\sin(\pi q_j)(1-q_j)^{(1-q_j)-1/2}e^{-(1-q_j)}}\prod_{j=\chi+1}^{p} q_j^{q_j-1/2}e^{-q_j}}\right| \tag{170}$$

$$\leq k_2 \left|\frac{\prod_{j=1}^{\mu}(1-s_j)^{s_j-1/2}\prod_{j=\mu+1}^{m} s_j^{s_j-1/2}e^{-\sum_{j=1}^{m}s_j}\prod_{j=1}^{\chi}\sin(\pi q_j)}{\prod_{j=1}^{\mu}(1-s_j)^{s_j-1/2}\prod_{j=\mu+1}^{m} s_j^{s_j-1/2}e^{-\sum_{j=1}^{m}s_j}\prod_{j=1}^{\chi}\sin(\pi q_j)}\right| \tag{171}$$

where k_1 and k_2 are undetermined constants independent of either k or y values.

Since all the linear equations s_j and q_j are on the set O_k, we have $\|x\| = R_k$ and $\|y\| \leq R_k$. Given this parametrization, the modulus of the functions s_j, $1 - s_j$, q_j and $1 - q_j$ are bounded below by A_1R_k and above by A_2R_k for some constants $A_1, A_2 > 0$, in fact:

$$|s_j| \leq \|\underline{a}_j\|\|x\| + |b_j| + \|\underline{a}_j\|\|y\| \leq \|\underline{a}_j\|R_k + \frac{|b_j|}{R_1}R_k + \|\underline{a}_j\|R_k \leq A_2R_k \tag{172}$$

$$|s_j| \geq ||a_{ji}||x_i| - |b_j|| \geq |A_1R_k - |b_j|| \geq A_1R_k \tag{173}$$

For the first step of (172) we applied the Cauchy-Schwartz inequality, for (173) we chosen a_{ji}, where $i \in \{1, 2, 3\}$, different from 0, and $A_1 \leq R_k|\hat{x}_i|$. The proofs for the linear equations $1 - s_j$, q_j and $1 - q_j$ are analogous.

Now if we take note that $R_k\langle a_i, \hat{x}\rangle < 0$ and $-R_k\|c_j\| \leq R_k\langle c_j, \hat{x}\rangle < 0$ for $1 \leq i \leq \mu$ and $\chi \leq j \leq p$ respectively and $R_k\|a_i\| \geq R_k\langle a_i, \hat{x}\rangle > 0$ and $R_k\|c_j\| \geq R_k\langle c_j, \hat{x}\rangle > 0$ for $1 \leq i \leq \mu$ and $\chi \leq j \leq p$, respectively, we conclude that:

$$
\begin{aligned}
\left|\frac{\prod_{j=1}^{\mu}(1-s_j)^{s_j-1/2}\prod_{j=\mu+1}^{m}s_j^{s_j-1/2}}{\prod_{j=1}^{\chi}(1-q_j)^{q_j-1/2}\prod_{j=\chi+1}^{p}q_j^{q_j-1/2}}\right| &\le \left|\frac{\prod_{j=1}^{\mu}(A_1R_ke^{i\arg(1-s_j)})^{R_k\langle\underline{a}_j,\hat{x}\rangle+b_j-1/2+i\operatorname{Im}(s_j)}}{\prod_{j=1}^{\chi}(A_2R_ke^{i\arg(1-q_j)})^{R_k\langle\underline{c}_j,\hat{x}\rangle+d_j-1/2+i\operatorname{Im}(q_j)}}\right| \\
&\quad\times\left|\frac{\prod_{j=\mu+1}^{m}(A_2R_ke^{i\arg(s_j)})^{R_k\langle\underline{a}_j,\hat{x}\rangle+b_j-1/2+i\operatorname{Im}(s_j)}}{\prod_{j=\chi+1}^{p}(A_1R_ke^{i\arg(1-q_j)})^{R_k\langle\underline{c}_j,\hat{x}\rangle+d_j-1/2+i\operatorname{Im}(q_j)}}\right| &(174)\\
&\le \frac{\prod_{j=1}^{\mu}(A_1R_k)^{R_k\langle\underline{a}_j,\hat{x}\rangle+b_j-1/2}\prod_{j=\mu+1}^{m}(A_2R_k)^{R_k\langle\underline{a}_j,\hat{x}\rangle+b_j-1/2}}{\prod_{j=1}^{\chi}(A_2R_k)^{R_k\langle\underline{c}_j,\hat{x}\rangle+d_j-1/2}\prod_{j=\chi+1}^{p}(A_1R_k)^{R_k\langle\underline{c}_j,\hat{x}\rangle+d_j-1/2}} \\
&\quad\times\frac{e^{-\sum_{j=1}^{\mu}\operatorname{Im}(s_j)\arg(1-s_j)-\sum_{j=\mu+1}^{m}\operatorname{Im}(s_j)\arg(s_j)}}{e^{-\sum_{j=1}^{\mu}\operatorname{Im}(q_j)\arg(1-q_j)-\sum_{j=\mu+1}^{m}\operatorname{Im}(q_j)\arg(q_j)}} &(175)\\
&\le c_0R_k^{c_1}c_2^{R_k}R_k^{c_3\|y\|}R_k^{R_k\langle\Delta,\hat{x}\rangle} &(176)
\end{aligned}
$$

where c_0, c_1, c_2 and c_3 are constants that we can define without any recourse to the angles θ and ψ, making the upper bound (176) hold for every $z\in O_k\cap\mathcal{U}_\delta$. More concretely we define:

$$
\begin{aligned}
c_0 &= \frac{\max\limits_{\mu\in\{0,\dots,m\}}A_1^{\sum\limits_{j=1}^{\mu}b_j-\mu/2}\max\limits_{\mu\in\{0,\dots,m\}}A_2^{\sum\limits_{j=\mu+1}^{m}b_j-(m-\mu)/2}}{\min\limits_{\chi\in\{0,\dots,p\}}A_1^{\sum_{j=1}^{\chi}b_j-\chi/2}\min\limits_{\chi\in\{0,\dots,p\}}A_2^{\sum_{j=\chi+1}^{p}b_j-(p-\chi)/2}} &(177)\\
&\ge \frac{A_1^{\sum_{j=1}^{\mu}b_j-\mu/2}A_2^{\sum_{j=\mu+1}^{m}b_j-(m-\mu)/2}}{A_1^{\sum_{j=1}^{\chi}b_j-\chi/2}A_2^{\sum_{j=\chi+1}^{p}b_j-(p-\chi)/2}} &(178)\\
c_1 &= \left\lceil\sum_{j=1}^{m}|b_j|\sum_{j=1}^{p}|d_j|+\frac{p}{2}\right\rceil \ge \sum_{j=1}^{m}b_j-\sum_{j=1}^{p}d_j+\frac{p-m}{2} &(179)\\
c_2 &= \max\left\{A_1^{\sum_{j=1}^{m}\|\underline{a}_j\|+\sum_{j=1}^{p}\|\underline{c}_j\|},A_1^{-\sum_{j=1}^{m}\|\underline{a}_j\|-\sum_{j=1}^{p}\|\underline{c}_j\|}\right\} &(180)\\
&\quad\times\max\left\{A_2^{\lceil\sum_{j=1}^{m}\|\underline{a}_j\|+\sum_{j=1}^{p}\|\underline{c}_j\|\rceil},A_2^{-\sum_{j=1}^{m}\|\underline{a}_j\|-\sum_{j=1}^{p}\|\underline{c}_j\|}\right\} &(181)\\
&\ge \frac{A_1^{\sum_{j=1}^{\mu}\langle\underline{a}_j,\hat{x}\rangle}A_2^{(1+\delta)\sum_{j=\mu+1}^{m}\langle\underline{a}_j,\hat{x}\rangle}}{A_1^{\sum_{j=1}^{\chi}\langle\underline{c}_j,\hat{x}\rangle}A_2^{\sum_{j=\chi+1}^{p}\langle\underline{c}_j,\hat{x}\rangle}} &(182)\\
c_3\|y\| &= \pi\left\lceil\sum_{j=1}^{m}\|a_j\|+\sum_{j=1}^{p}\|c_j\|\right\rceil\|y\| \ge \pi\sum_{j=1}^{m}\operatorname{Im}(s_j)+\pi\sum_{j=1}^{p}\operatorname{Im}(q_j) &(183)\\
&\ge \frac{e^{-\sum_{j=1}^{\mu}\operatorname{Im}(s_j)\arg(1-s_j)-\sum_{j=\mu+1}^{m}\operatorname{Im}(s_j)\arg(s_j)}}{e^{-\sum_{j=1}^{\mu}\operatorname{Im}(q_j)\arg(1-q_j)-\sum_{j=\mu+1}^{m}\operatorname{Im}(q_j)\arg(q_j))}} &(184)
\end{aligned}
$$

where in (183) we used the Cauchy-Schwartz inequality in conjunction with the fact that $\langle\underline{a}_j,y\rangle=\operatorname{Im}(s_j)$ and $\langle\underline{c}_j,y\rangle=\operatorname{Im}(q_j)$.

Now let's recall that $\Delta = \sum_{j=1}^{m} \underline{a}_j - \sum_{j=1}^{p} \underline{c}_j$, which we will use to define the value $\Delta_x := \langle \Delta, \hat{x} \rangle$. Observe that since we are working in $O_k \cap \mathcal{U}_\delta$, there doesn't exist a sequence $z_i \in O_k \cap \mathcal{U}_\delta$ such that $\sin\big(\pi(\langle \underline{a}_j, z\rangle + b_j)\big)$ goes to zero, otherwise inasmuch as sin and $\langle \underline{a}_j, z\rangle + b_j$ are continuous, we would have $z_i \in \mathcal{U}_\delta^c$, which is a contradiction, hence $0 < \inf_{z \in O_k \cap \mathcal{U}_\delta} \sin\big(\pi(\langle \underline{a}_j, z\rangle + b_j)\big)$ and $\sup_{z \in O_k \cap \mathcal{U}_\delta} 1/\prod_{j=1}^{\mu} \sin\big(\pi(\langle \underline{a}_j, z\rangle + b_j)\big) < \infty$. On the other hand, for the sin's in the denominator of ω, there exists a constant c such that:

$$\left|\prod_{j=1}^{\chi} \sin(\pi q_j)\right| = \prod_{j=1}^{\chi} \left|\frac{e^{i\pi \operatorname{Re}(q_j)} e^{-\pi \operatorname{Im}(q_j)} - e^{-i\pi \operatorname{Re}(q_j)} e^{\pi \operatorname{Im}(q_j)}}{2i}\right| \leq \prod_{j=1}^{\chi} e^{\pi |\operatorname{Im}(q_j)|} \leq e^{cR_k} \tag{185}$$

Similarly, by applying the Cauchy-Schwartz inequality to the modulus of t^{-z} we have:

$$\begin{aligned} \left|t_1^{-z_1} t_2^{-z_2} t_3^{-z_3}\right| &= |t_1|^{-x_1} |t_2|^{-x_2} |t_3|^{-x_3} e^{\langle y, \arg t\rangle} \\ &\leq |t_1|^{-x_1} |t_2|^{-x_2} |t_3|^{-x_3} e^{|y||\arg t|} \leq e^{-\|x\|\langle \log|t|, \hat{x}\rangle} e^{|\arg t| R_k} \end{aligned} \tag{186}$$

Therefore all the terms of ω not present in (170) grow at most exponentially as $R_k \to \infty$. If we combine the results from (176), (185) and (186) we get:

$$\left|\frac{\Gamma_1\Gamma_2\Gamma_3}{\Gamma_4} t^{-z}\right|\Bigg|_{O_k \cap \mathcal{U}_\delta} \leq c(t)^{R_k} R_k^{R_k \Delta_x} \tag{187}$$

where $c(t) > 0$ and is independent from k and y. Since $\Delta_x < 0$, from (187) as $k \to \infty$ the first integral of (169) vanishes.

Step 2:

Before tackling the second integral in (169), let's restate the inequality (149):

$$c_1 |z|^{\operatorname{Re}(z) - 1/2} e^{-\frac{\pi}{2}|\operatorname{Im}(z)| - |\operatorname{Re}(z)|} < |\Gamma(z)| < c_2(\epsilon) |z|^{\operatorname{Re}(z) - 1/2} e^{-\frac{\pi}{2}|\operatorname{Im}(z)| + |\operatorname{Re}(z)|} \tag{188}$$

In $B_k \cap \mathcal{U}_\delta$ we have $0 \leq \|x\| \leq R_k$ and $\|y\| = R_k$. analogously to (172) and (173), for R_k big enough, the modulus of the functions s_j, $1 - s_j$, q_j and $1 - q_j$ are bounded below by $A_1 R_k$ and above by $A_2 R_k$ for some constants $A_1, A_2 > 0$, in fact:

$$|s_j| \leq \|\underline{a}_j\|\|x\| + |b_j| + \|\underline{a}_j\|\|y\| \leq \|a_j\| R_k + \frac{|b_j|}{R_1} R_k + \|\underline{a}_j\| R_k \leq A_2 R_k \tag{189}$$

$$|s_j| \geq |\langle a_j, x\rangle - |b_j + \langle a_j, y\rangle|| \geq c|\langle a_j, x\rangle| \geq A_1 \|x\| \tag{190}$$

where we defined $A_1 = c \inf_{\substack{\theta \in [0, 2\pi[\\ \phi \in [0, \pi[}} \sqrt{a_{j1}^2 \sin^2\phi \cos^2\theta + a_{j2}^2 \cos^2\phi \cos^2\theta + a_{j3}^2 \sin^2\phi}$. The second step of (190) is valid when $R_k \to \infty$ because the value in $\left|b_j + \langle \underline{a}_j, y\rangle\right|$ will become significantly greater than $\left|\langle \underline{a}_j, x\rangle\right|$. These results remain for $1 - s_j$, q_j and $1 - q_j$. Combining the inequalities (189) and (189) with (188) we obtain:

$$|\Gamma(s_j)| < c_2(A_2R_k)^{\|x\|\langle a_j,\hat{x}\rangle}|s_j|^{b_j-1/2}e^{-\frac{\pi}{2}|\mathrm{Im}(s_j)|+|\mathrm{Re}(s_j)|} \quad \text{if } \langle a_j,\hat{x}\rangle \geq 0, \tag{191}$$

$$|\Gamma(s_j)| < c_2(A_1\|x\|)^{\|x\|\langle a_j,\hat{x}\rangle}|s_j|^{b_j-1/2}e^{-\frac{\pi}{2}|\mathrm{Im}(s_j)|+|\mathrm{Re}(s_j)|} \quad \text{if } \langle a_j,\hat{x}\rangle < 0, \tag{192}$$

$$|\Gamma(q_j)| > c_1(A_1\|x\|)^{\|x\|\langle c_j,\hat{x}\rangle}|q_j|^{d_j-1/2}e^{-\frac{\pi}{2}|\mathrm{Im}(q_j)|-|\mathrm{Re}(q_j)|} \quad \text{if } \langle c_j,\hat{x}\rangle > 0, \tag{193}$$

$$|\Gamma(q_j)| > c_1(A_1R_k)^{\|x\|\langle c_j,\hat{x}\rangle}|q_j|^{d_j-1/2}e^{-\frac{\pi}{2}|\mathrm{Im}(q_j)|-|\mathrm{Re}(q_j)|} \quad \text{if } \langle c_j,\hat{x}\rangle \leq 0. \tag{194}$$

Just like we did in step one we will fix θ and ϕ, i.e. fix $\hat{x}$, and using the above estimates and the fact that $\mathrm{Im}(s_j) = \langle a_j, y\rangle$ and $\mathrm{Im}(q_j) = \langle c_j, y\rangle$ we can bound the value of $\left|\frac{\Gamma_1\Gamma_2\Gamma_3}{\Gamma_4}t^{-z}\right|$ by:

$$\begin{aligned}
\left|\frac{\Gamma_1\Gamma_2\Gamma_3}{\Gamma_4}t^{-z}\right| &\leq \frac{\prod_{j=1}^{\mu} c_2\left(A_1\|x\|\right)^{\|x\|\langle a_j,\hat{x}\rangle}|s_j|^{b_j-1/2}e^{-\frac{\pi}{2}|\mathrm{Im}(s_j)|+|\mathrm{Re}(s_j)|}}{\prod_{j=1}^{\chi} c_1\left(A_2R_k\right)^{\|x\|\langle c_j,\hat{x}\rangle}|q_j|^{d_j-1/2}e^{-\frac{\pi}{2}|\mathrm{Im}(q_j)|-|\mathrm{Re}(q_j)|}} \\
&\times \frac{\prod_{j=\mu+1}^{m} c_2\left(A_2R_k\right)^{\|x\|\langle a_j,\hat{x}\rangle}|s_j|^{b_j-1/2}e^{-\frac{\pi}{2}|\mathrm{Im}(s_j)|+|\mathrm{Re}(s_j)|}}{\prod_{j=\chi+1}^{p} c_1\left(A_1\|x\|\right)^{\|x\|\langle c_j,\hat{x}\rangle}|q_j|^{d_j-1/2}e^{-\frac{\pi}{2}|\mathrm{Im}(q_j)|+|\mathrm{Re}(q_j)|}}|t_1|^{x_1}|t_2|^{x_2}|t_3|^{x_3}e^{|\langle y,\arg t\rangle|} \qquad (195) \\
&\leq C\|x\|^{A\|x\|}R_k^B D(x,y)E^{\|x\|}e^{F(y)} \qquad (196)
\end{aligned}$$

where A, B and C and E are constants of the form:

$$A = \left\langle \sum_{j=1}^{\mu} a_j - \sum_{j=\chi+1}^{p} c_j, \hat{x}\right\rangle \tag{197}$$

$$B = \left\langle \sum_{j=\mu+1}^{m} a_j - \sum_{j=1}^{\chi} c_j, \hat{x}\right\rangle \tag{198}$$

$$C \geq c_1^{-p}c_2^{m} \tag{199}$$

$$E = A_1^{\left\langle\sum_{j=1}^{\mu} a_j - \sum_{j=\chi+1}^{p} c_j,\hat{x}\right\rangle}A_2^{\left\langle\sum_{j=\mu+1}^{m} a_j - \sum_{j=1}^{\chi} c_j,\hat{x}\right\rangle}e^{\sum_{j=1}^{m}|\langle a_j,\hat{x}\rangle|+\sum_{j=1}^{p}|\langle c_j,\hat{x}\rangle|} \tag{200}$$

and $D(x, y)$ and $F(y)$ are the functions:

$$D(x,y) = \frac{\prod_{j=1}^{m}|\langle a_j,x\rangle + b_j + i\langle a_j,y\rangle|^{b_j-1/2}}{\prod_{j=1}^{p}|\langle c_j,x\rangle + d_j + i\langle c_j,y\rangle|^{d_j-1/2}} \tag{201}$$

$$F(y) = |\langle y,\arg t\rangle| - \frac{\pi}{2}\left(\sum_{j=1}^{m}|\langle a_j y\rangle| - \sum_{j=1}^{p}|\langle c_j,y\rangle|\right) \tag{202}$$

As we shown in (153) to (155), $F(y)$ is negative for all $y \in \sigma$, therefore since $\|y\| = R_k$, as $R_k \to \infty$, $e^{F(y)}$ goes to zero at an exponentially rate. If $\|x\|$ increases as R_k increases, we can just use the arguments from step 1 to show that $\|\xi\|$ goes to zero, hence $E^{\|x\|}$ must be bounded. Since $D(x, y)$ increases at most at a polynomial rate, our last requirement is to show that $\|x\|^{A\|x\|} R_k^B$ increases, at most, at a slower rate than $e^{F(y)}$ decreases.

First thing to note is that by construction A defined in (197) is negative and B defined in (198) is positive. Secondly by definition, for any $x + iy \in \sigma$ we have $\Delta_x = \langle \Delta, \hat{x} \rangle < 0$, therefore since $A + B = \Delta_x < 0$, we have $-B/A < 1$. Consider the function $f_R(x) = x^{Ax} R^{Bx}$, its derivative is given by $f_R'(x) = f_R(x)(A \log x + A + B \log R)$ with a zero of value $x_M = R^{-B/a}/e$. In other words, $f_R(x)$ has a maximum at x_M of $f_R(x_M) = \left(e^{-\frac{B}{A}}\right)^{R^{-B/A}}$. Therefore even if we choose $\|x\| = x_M$, the term $\|x\|^{A\|x\|} R_k^B$ will increase at a rate lower that $e^{F(y)}$ decreases. We conclude that as $k \to \infty$ the second integral of (169) vanishes.

Step 3:

We will now undertake the third integral of (169) where the singularities of the form ω are present. Let's define $P_{j,\nu} = \{x + yi \in \mathbb{C}^3 : \langle \underline{a}_j, x \rangle = -b_j - \nu,\ \langle \underline{a}_j, y \rangle = 0\}$ and $V_{j,\nu} \supset P_{j,\nu}$ as the open sets that contain the discontinuity given by the values in $P_{j,\nu}$. If we fix the angles θ_x, ϕ_x, θ_y, ϕ_y, for the aforementioned discontinuity, we can restate the previews equations as $\|x\| \langle \underline{a}_j, \hat{x} \rangle = -b_j - \nu$ and $\|y\| \langle \underline{a}_j, \hat{y} \rangle = 0$.

For the case where $P_{j,\nu} \cap O_k \neq \emptyset$, we have $\|x\| = R_k$ and the real part of the singularities will be given by the one dimensional segments $T_{k,j,\nu} = \{x \in \mathbb{R}^3 : \|x\| = R_k,\ \langle \underline{a}_j, x \rangle = -b_j - \nu\}$. Hence, we can chose the radii R_k's, such that, for each point $x = R_{k_1} \hat{x} = R_k(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$ belonging the two dimensional segment of $T_{k_1,j,\nu}$, when we increase the radius from R_{k_1} to R_{k_2}, the points with the same angles θ and ϕ but now with the higher length R_{k_2}, i.e. $R_{k_2}\hat{x}$, will not intersect $T_{k_2,j,\mu}$, for any $\mu \in \mathbb{N}$ and $k_2 > k_1$, and will intersect the remaining $T_{k_1,i,\mu}$ at most one time, for any $\mu \in \mathbb{N}$, $j \neq i$ and $k_2 > k_1$. We can construct the radii R_k's in order for this event to occur, because the set containing all the two dimensional segments $T_{k,j,\nu}$ is countable. In fact, we can define its (surjective) enumerating function as:

$$\begin{aligned} \eta\colon \quad \mathbb{N}^3 &\longrightarrow \{T_{k,j,\nu}\}_{k,j,\nu} \\ (k, j, \nu) &\longmapsto T_{k,j,\nu}, \end{aligned} \tag{203}$$

where $k \in \mathbb{N}$ represents the radii R_k, $j \in \{1, ..., m\}$ represents the Gamma function and $\nu \in \mathbb{N}$ the zero in the Gamma function.

This entails that as $k \to \infty$ the upper bound (176) deduced in step 1 will still hold for any angles θ_x and ϕ_x.

The case where $P_{j,\nu} \cap B_k \neq \emptyset$ is slightly more complicated. Let's consider $S_{j,\nu} = \{i \in \{1, 2, ..., m\} : \forall_{z \in V_{j,\nu}} \langle \underline{a}_i, z\rangle \notin \mathbb{Z}_0^-\}$, the multi-index set of the gammas that don't have a singularity under $s_i(z)$, and $Z_{j,\nu} = \{i \in \{1, 2, ..., m\} : \forall_{z \in V_{j,\nu}} \langle \underline{a}_i, z\rangle \in \mathbb{Z}_0^-\}$, the multi-index set of the gammas that are singularities have a singularity under $s_i(z)$. Note that the reason for this distinction is that in some instances we may be working with multiple singularity simultaneously in one neighborhood $V_{j,\nu} \cap B_k \neq \emptyset$.

Since we are dealing with a case where $x \in K \setminus \mathbb{Z}_0^-$, where K is compact, by the inequalities (150) the estimate upper bound for the modulus of ωt^{-z} is given by:

$$\left| \frac{\prod_{j=1}^m \Gamma(s_j(z))}{\prod_{k=1}^p \Gamma(q_k(z))} t^{-z} \right| \leq \frac{\prod_{i \in S_{j,\nu}} (|\langle \underline{a}_i, x\rangle| + 1)^{\langle \underline{a}_i, x\rangle + b_i - 1/2}}{\prod_{k=1}^p \left(|\langle \underline{c}_j, x\rangle| + 1\right)^{\langle \underline{c}_j, x\rangle + b_j - 1/2}} \times \exp\left\{ |\langle y, \arg t\rangle| - \frac{\pi}{2} \left(\sum_{j=1}^m |\langle \underline{a}_j, y\rangle| - \sum_{j=1}^p |\langle \underline{c}_j, y\rangle| \right) \right\} \times \left| \prod_{i \in Z_{j,\nu}} \Gamma(\langle \underline{a}_i, z\rangle + b_i) \right| \tag{204}$$

Consider α defined in (30), for each $\underline{t}$ in the domain U defined (144) we have $\|\arg \underline{t}\| < (\pi/2)\alpha$ if we apply the Cauchy-Schwartz inequality, $|\langle y, \arg t\rangle| \leq \|\arg \underline{t}\| \|y\|$, we get:

$$|\langle y, \arg \underline{t}\rangle| \leq \frac{\pi}{2}\alpha' \|y\| \qquad \text{for some} \qquad \alpha' < \alpha \tag{205}$$

Consider the parametrization given by $x = p_1^0$, $y = v_0\lambda + v_1\theta + p_2^0$ of the plane $P_{j,\nu} \cap B_k$ where (p_1^0 and p_2^0 are points and v_1 and v_2 are vectors). The neighborhood $V_{j\nu}$ around $P_{j\nu} \cap B_k$ can be defined as the intersection of the parallel planes $\{x + iy \in \mathbb{C}^3 : x = p_1, y = v_0\lambda + v_1\theta + p_2\}$ with B_k where p_1 and p_2 are in the neighborhood of p_1^0 and p_2^0. If we take into account that $\sum_{Z_{j,\nu}} \langle \underline{a}_j, y\rangle$ vanishes on $P_{j,\nu}$, we have:

$$\left(\sum_{i \in S_{j,\nu}} |\langle \underline{a}_i, y\rangle| - \sum_{k=1}^p |\langle \underline{c}_i, y\rangle| \right) \Bigg|_{V_{j,\nu}} = \left(\sum_{i \in S_{j,\nu}} |\langle \underline{a}_i, v_0\rangle| - \sum_{k=1}^p |\langle \underline{c}_i, v_0\rangle| \right) \|\lambda\| + \left(\sum_{i \in S_{j,\nu}} |\langle \underline{a}_i, v_1\rangle| - \sum_{k=1}^p |\langle \underline{c}_i, v_1\rangle| \right) \|\theta\| + K_1 \tag{206}$$

$$= \left(\sum_{i \in S_{j,\nu}} |\langle \underline{a}_i, y \rangle| - \sum_{k=1}^{p} |\langle \underline{c}_i, y \rangle| \right) \Bigg|_{P_{j,\nu}} + K_2 \tag{207}$$

$$= \left(\sum_{i=1}^{m} |\langle \underline{a}_i, y \rangle| - \sum_{k=1}^{p} |\langle \underline{c}_i, y \rangle| \right) \Bigg|_{P_{j,\nu}} + K_3 \tag{208}$$

$$\geq \alpha \|y\| + K_4 \tag{209}$$

where the K_1, K_2, K_3, and K_4 are bounded constants. Hence, for any $x + iy \in V_{j,\nu}$ the following inequality will hold:

$$|\langle y, \arg \underline{t} \rangle| - \frac{\pi}{2} \left(\sum_{i \in S_{j,\nu}} |\langle \underline{a}_i, y \rangle| - \sum_{k=1}^{p} |\langle \underline{c}_k, y \rangle| \right) \leq \frac{\pi}{2}(\alpha' - \alpha)\|y\| + K_4 \tag{210}$$

In other words, $\left| \frac{\prod_{i \in S_{j,\nu}} \Gamma(s_i(z))}{\prod_{k=1}^{p} \Gamma(q_k(z))} t^{-z} \right|$ decreases exponentially as R_k increases.

What remains to be shown is that the forms ξ_1, $\xi_{\{2,3\}}$ and $\xi_{\{1,2,3\}}$ (the proofs for the remaining forms are analogous) decrease as R_k increases, whether or not they are in $\mathcal{U}_\delta$ or its complement. Of necessity, for the latter case, we will have to show that the forms are bounded at any point where the Gamma functions are discontinuous.

These properties will be demonstrated in the following lemmas:

Lemma 1. *There exists a sequence of radii R_k, such that $R_k \to \infty$ and for which*

$$\lim_{k \to \infty} \int_{\sigma_{\{2,3\}} \cap S_k} \xi_1 = 0 \tag{211}$$

Proof. For the form ξ_1 it is easier to recall that $\xi_1 = \rho_1 \wedge \omega$ which can be written as:

$$\xi_1 = \frac{|f_1|^2}{\|f\|^2} \wedge h dz = \frac{\frac{1}{|\Gamma_1|^2}}{\frac{1}{|\Gamma_1|^2} + \frac{1}{|\Gamma_2|^2} + \frac{1}{|\Gamma_3|^2}} \wedge \frac{\Gamma_1 \Gamma_2 \Gamma_3}{\Gamma_4} t^{-z} dz \tag{212}$$

where Γ_1, Γ_2, Γ_3 and Γ_4 are a product of a subset of Gamma functions present in the numerator and denominator of ω defined at the beginning of the proof of the theorem 7. Knowing this, the following inequality for the norm of ξ_1 will hold true for any $z \in \mathcal{U}_\delta$:

$$\|\xi_1\| \leq \frac{1}{1 + \frac{|\Gamma_1|^2}{|\Gamma_2|^2} + \frac{|\Gamma_1|^2}{|\Gamma_3|^2}} \|\omega\| \tag{213}$$

Since $\frac{1}{1+a^2/b^2+a^2/c^2} \leq 1$, for any $a, b, c \in \mathbb{R}$, then the first factor of the inequality (213) is bounded by 1 and the second factor, as we deduced previously, decreases exponentially as R_k increases.

On the other hand, the complementary set, $\mathcal{U}_\delta^c$, will include the complex planes $L_j^\nu = \{z \in \mathbb{C}^3 : s_j(z) = -\nu\}$ of D_1, D_2 and D_3. Recall from (39), we have $D_1 \cap \sigma_1 = \emptyset$, $D_2 \cap \sigma_2 = \emptyset$ and $D_3 \cap \sigma_3 = \emptyset$ which implies that $\sigma_{\{2,3\}} \cap (D_2 \cup D_3) = \emptyset$. Since we are working in $\sigma_{\{2,3\}}$ we will only consider the planes L_j^ν in D_1. We will represent Γ_1 as the product $\Gamma_1' \cdot \Gamma_1''$, where Γ_1' is the product of the functions $\Gamma(s_j(z))$ in Γ_1 without singularities on L_j^ν and Γ_1'' is the product of the functions $\Gamma(s_j(z))$ in Γ_1 with singularities on L_j^ν, where we will denote by s the number of factors in Γ_1''. Taking this notation into account, the following inequality for the norm of ξ_1 will hold true for any $z \in \mathcal{U}_\delta^c$:

$$\|\xi_1\| \leq \overbrace{\frac{1}{1 + \frac{|\Gamma_1'|^2 |\Gamma_1''|^2}{|\Gamma_2|^2} + \frac{|\Gamma_1'|^2 |\Gamma_1''|^2}{|\Gamma_3|^2}}}^{-2s} \overbrace{|\Gamma_1''|}^{s} \left\| \frac{\Gamma_1' \Gamma_2 \Gamma_3}{\Gamma_4} t^{-z} \right\| \tag{214}$$

The first factor has a zero of order $2s$, the second factor has a pole of order s, and the third factor decreases exponentially as R_k increases. Therefore the product of the first and second factors have a zero of order s. This in conjunction with the previous deductions completes the proof of lemma 1. □

Lemma 2. *There exists a sequence of radii R_k, such that $R_k \to \infty$ and for which*

$$\lim_{k\to\infty} \int_{\sigma_1 \cap S_k} \xi_{\{2,3\}} = 0 \tag{215}$$

Proof. The proof of lemma 2 will be similar to lemma 1. From equation (157) and (158) the form $\xi_{\{2,3\}}$ can be written down as:

$$\xi_{\{2,3\}} = \frac{\overline{f}_2 d\overline{f}_3 d - \overline{f}_3 d\overline{f}_2}{f_1 ||f||^4} \wedge h dz = \frac{-\frac{1}{\overline{\Gamma}_2} \frac{d\overline{\Gamma}_3}{\overline{\Gamma}_3^2} + \frac{1}{\overline{\Gamma}_3} \frac{d\overline{\Gamma}_2}{\overline{\Gamma}_2^2}}{\frac{1}{\Gamma_1} \left(1/|\Gamma_1|^2 + 1/|\Gamma_2|^2 + 1/|\Gamma_3|^2\right)^2} \wedge \frac{t^{-z}}{\Gamma_4} dz \tag{216}$$

Hence, the following inequality for the norm of $\xi_{\{2,3\}}$ will hold true for any $z \in \mathcal{U}_\delta$:

$$\left\|\xi_{\{2,3\}}\right\| = \left\| \frac{\frac{1}{|\Gamma_2|^2 |\Gamma_3|^2} \left(\frac{d\overline{\Gamma}_2}{\overline{\Gamma}_2} - \frac{d\overline{\Gamma}_3}{\overline{\Gamma}_3} \right)}{\frac{1}{|\Gamma_1|^4 |\Gamma_2|^4 |\Gamma_3|^4} \left(|\Gamma_1|^2 |\Gamma_2|^2 + |\Gamma_1|^2 |\Gamma_3|^2 + |\Gamma_2|^2 |\Gamma_3|^2 \right)^2} \right\| \left\| \frac{\Gamma_1 \Gamma_2 \Gamma_3}{\Gamma_4} t^{-z} \right\| \tag{217}$$

$$\leq \frac{|\Gamma_1|^4 |\Gamma_2|^2 |\Gamma_3|^2}{\left(|\Gamma_1|^2 |\Gamma_2|^2 + |\Gamma_1|^2 |\Gamma_3|^2 + |\Gamma_2|^2 |\Gamma_3|^2 \right)^2} \left\| \frac{d\overline{\Gamma}_2}{\overline{\Gamma}_2} - \frac{d\overline{\Gamma}_3}{\overline{\Gamma}_3} \right\| \|\omega\| \tag{218}$$

Since $\frac{a^2bc}{a^2b^2+a^2c^2+b^2c^2} \leq 1$, for $a, b, c \geq 0$, then the first factor of the inequality (218) is bounded by 1. The second factor is the difference between two digamma functions, $\psi(z) = -\gamma + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+z} \right)$, taking into account the inequality $\psi(z) \leq \log(z - 1 + e^\gamma)$, the second factor increases no faster than logarithm of R_k. The third factor as we deduced previously decreases exponentially as R_k increases.

For the complementary set, $\mathcal{U}_\delta^c$, from (39), we have $D_1 \cap \sigma_1 = \emptyset$. Since we are working in σ_1 which may intersect D_2 and D_3, without loss of generality, we will consider the planes L_j^ν of the divisors D_2. As we did in the previews lemma, we will represent Γ_2 as the product $\Gamma_2' \cdot \Gamma_2''$, where Γ_2' is the product of the $\Gamma(s_j(z))$ in Γ_2 without singularities on L_j^ν and Γ_2'' is the product of the $\Gamma(s_j(z))$ in Γ_2 with singularities on L_j^ν, where we will denote by s the number of factors in Γ_2''. We should also take note that by definition ψ_2 will have one pole in L_j^ν. Taking this notation into account, the following inequality for the norm of $\xi_{\{2,3\}}$ will hold true for any $z \in \mathcal{U}_\delta^c$:

$$\left\|\xi_{\{2,3\}}\right\| \leq \overbrace{\frac{|\Gamma_1|^4|\Gamma_2|^2|\Gamma_3|^2}{\left(|\Gamma_1|^2|\Gamma_2|^2+|\Gamma_1|^2|\Gamma_3|^2+|\Gamma_2|^2|\Gamma_3|^2\right)^2}}^{2s-4s=-2s} \overbrace{\|\psi_2-\psi_3\|}^{1} \overbrace{|\Gamma_2''|}^{s} \left\|\frac{\Gamma_1\Gamma_2'\Gamma_3}{\Gamma_4}t^{-z}\right\| \quad (219)$$

The first factor has a zero of order $2s$, the second factor has a first order pole, the third factor has a pole of order s, and the fourth factor decreases exponentially as R_k increases. Therefore the product of the first, second and third factor has a zero of order $s-1$, and taking into account that $s \geq 1$, then $\left\|\xi_{\{2,3\}}\right\|$ is bounded and decreases exponentially. This in conjunction with the previous deductions completes the proof of lemma 2. $\square$

Lemma 3. *There exists a sequence of radii R_k, such that $R_k \to \infty$ and for which*

$$\lim_{k\to\infty} \int_{\sigma_\emptyset \cap S_k} \xi_{\{1,2,3\}} = 0 \quad (220)$$

Proof. The proof of lemma 3 is slightly more complex than the previews two. First recall, from (160), (163), (164) and (165), that $\xi_{\{1,2,3\}} = \xi^1_{\{1,2,3\}} + \xi^2_{\{1,2,3\}} + \xi^3_{\{1,2,3\}}$, where

$$\xi^1_{\{1,2,3\}} = \frac{\overline{f}_1 d\overline{f}_2 d\overline{f}_3 - \overline{f}_2 d\overline{f}_1 d\overline{f}_3 + \overline{f}_3 d\overline{f}_1 d\overline{f}_2}{||f||^6} \wedge h dz \quad (221)$$

$$= \frac{\frac{1}{\overline{\Gamma}_1}\frac{d\overline{\Gamma}_2}{\overline{\Gamma}_2^2}\frac{d\overline{\Gamma}_3}{\overline{\Gamma}_3^2} - \overline{\Gamma}_2\frac{d\overline{\Gamma}_1}{\overline{\Gamma}_1^2}\frac{d\overline{\Gamma}_3}{\overline{\Gamma}_3^2} - \overline{\Gamma}_3\frac{d\overline{\Gamma}_1}{\overline{\Gamma}_1^2}\frac{d\overline{\Gamma}_2}{\overline{\Gamma}_2^2}}{\left(1/|\Gamma_1|^2+1/|\Gamma_2|^2+1/|\Gamma_3|^2\right)^3} \wedge \frac{t^{-z}}{\Gamma_4}dz \quad (222)$$

$$\xi^2_{\{1,2,3\}} = -2\frac{\overline{f}_1\overline{f}_3 d\overline{f}_2\left(f_1 d\overline{f}_1 + f_2 d\overline{f}_2 + f_3 d\overline{f}_3\right)}{||f||^8} \wedge h dz \quad (223)$$

$$= -2\frac{\frac{1}{\overline{\Gamma}_1}\frac{1}{\overline{\Gamma}_3}\frac{d\overline{\Gamma}_2}{\overline{\Gamma}_2^2}\left(\frac{1}{\Gamma_1}\frac{d\overline{\Gamma}_1}{\overline{\Gamma}_1^2} + \frac{1}{\Gamma_2}\frac{d\overline{\Gamma_2}}{\overline{\Gamma}_2^2} + \frac{1}{\Gamma_3}\frac{d\overline{\Gamma_3}}{\overline{\Gamma}_3^2}\right)}{\left(1/|\Gamma_1|^2+1/|\Gamma_2|^2+1/|\Gamma_3|^2\right)^4} \wedge \frac{t^{-z}}{\Gamma_4}dz \quad (224)$$

$$\xi^3_{\{1,2,3\}} = \frac{\overline{f}_1\overline{f}_2\overline{f}_3\left(f_1 d\overline{f}_1 + f_2 d\overline{f}_2 + f_3 d\overline{f}_3\right)^2}{||f||^{10}} \wedge h dz \quad (225)$$

$$= \frac{\frac{1}{\overline{\Gamma}_1}\frac{1}{\overline{\Gamma}_2}\frac{1}{\overline{\Gamma}_3}\left(\frac{1}{\Gamma_1}\frac{d\overline{\Gamma}_1}{\overline{\Gamma}_1^2} + \frac{1}{\Gamma_2}\frac{d\overline{\Gamma_2}}{\overline{\Gamma}_2^2} + \frac{1}{\Gamma_3}\frac{d\overline{\Gamma_3}}{\overline{\Gamma}_3^2}\right)^2}{\left(1/|\Gamma_1|^2+1/|\Gamma_2|^2+1/|\Gamma_3|^2\right)^5} \wedge \frac{t^{-z}}{\Gamma_4}dz \quad (226)$$

Analogously to what we did in the previous lemmas we will compute an upper bound for the norm of the three forms. Starting with $\xi^1_{\{1,2,3\}}$:

$$||\xi^1_{\{1,2,3\}}|| = \left\| \frac{\frac{1}{\overline{\Gamma}_1\overline{\Gamma}_2\overline{\Gamma}_3}\left(\frac{d\overline{\Gamma}_2}{\overline{\Gamma}_2}\frac{d\overline{\Gamma}_3}{\overline{\Gamma}_3} - \frac{d\overline{\Gamma}_1}{\overline{\Gamma}_1}\frac{d\overline{\Gamma}_3}{\overline{\Gamma}_3} + \frac{d\overline{\Gamma}_1}{\overline{\Gamma}_1}\frac{d\overline{\Gamma}_2}{\overline{\Gamma}_2}\right)}{\frac{1}{|\Gamma_1|^6|\Gamma_2|^6|\Gamma_3|^6}\left(|\Gamma_1|^2|\Gamma_2|^2 + |\Gamma_1|^2|\Gamma_3|^2 + |\Gamma_2|^2|\Gamma_3|^2\right)^3} \wedge \frac{t^{-z}}{\Gamma_4}dz \right\| \tag{227}$$

$$\leq \frac{|\Gamma_1|^4|\Gamma_2|^4|\Gamma_3|^4}{\left(|\Gamma_1|^2|\Gamma_2|^2 + |\Gamma_1|^2|\Gamma_3|^2 + |\Gamma_2|^2|\Gamma_3|^2\right)^3}\left\|\frac{d\overline{\Gamma}_2}{\overline{\Gamma}_2}\frac{d\overline{\Gamma}_3}{\overline{\Gamma}_3} - \frac{d\overline{\Gamma}_1}{\overline{\Gamma}_1}\frac{d\overline{\Gamma}_3}{\overline{\Gamma}_3} + \frac{d\overline{\Gamma}_1}{\overline{\Gamma}_1}\frac{d\overline{\Gamma}_2}{\overline{\Gamma}_2}\right\| \|\omega\| \tag{228}$$

For $\xi^2_{\{1,2,3\}}$ we get the estimation:

$$\left\|\xi^2_{\{1,2,3\}}\right\| = 2\left\| \frac{\frac{1}{\overline{\Gamma}_1}\frac{1}{\overline{\Gamma}_2}\frac{1}{\overline{\Gamma}_3}\frac{d\overline{\Gamma}_2}{\overline{\Gamma}_2}\left(\frac{1}{\overline{\Gamma}_1^2}\frac{d\overline{\Gamma}_1}{\overline{\Gamma}_1} + \frac{1}{\overline{\Gamma}_2^2}\frac{d\overline{\Gamma}_2}{\overline{\Gamma}_2} + \frac{1}{\overline{\Gamma}_3^2}\frac{d\overline{\Gamma}_3}{\overline{\Gamma}_3}\right)}{\left(1/|\Gamma_1|^2 + 1/|\Gamma_2|^2 + 1/|\Gamma_3|^2\right)^4} \wedge \frac{t^{-z}}{\Gamma_4}dz \right\| \tag{229}$$

$$\leq 2\frac{\frac{1}{|\Gamma_1|^2|\Gamma_2|^2|\Gamma_3|^2}}{\left(1/|\Gamma_1|^2 + 1/|\Gamma_2|^2 + 1/|\Gamma_3|^2\right)^4}\left\|\frac{d\overline{\Gamma}_2}{\overline{\Gamma}_2}\right\|\left\|\frac{1}{\overline{\Gamma}_1^2}\frac{d\overline{\Gamma}_1}{\overline{\Gamma}_1} + \frac{1}{\overline{\Gamma}_2^2}\frac{d\overline{\Gamma}_2}{\overline{\Gamma}_2} + \frac{1}{\overline{\Gamma}_3^2}\frac{d\overline{\Gamma}_3}{\overline{\Gamma}_3}\right\| \|\omega\| \tag{230}$$

$$\leq 2\frac{\frac{1}{|\Gamma_1|^2|\Gamma_2|^2|\Gamma_3|^2}}{\left(1/|\Gamma_1|^2 + 1/|\Gamma_2|^2 + 1/|\Gamma_3|^2\right)^4}\left(\frac{1}{|\Gamma_1|^2} + \frac{1}{|\Gamma_2|^2} + \frac{1}{|\Gamma_3|^2}\right)$$
$$\times\left(\sup_{i\in\{1,2,3\}}\left\|\frac{d\overline{\Gamma}_i}{\overline{\Gamma}_i}\right\|^2\right)\|\omega\| \tag{231}$$

$$\leq 2\frac{|\Gamma_1|^4|\Gamma_2|^4|\Gamma_3|^4}{\left(|\Gamma_1|^2|\Gamma_2|^2 + |\Gamma_1|^2|\Gamma_3|^2 + |\Gamma_2|^2|\Gamma_3|^2\right)^3}\left(\sup_{i\in\{1,2,3\}}\left\|\frac{d\overline{\Gamma}_i}{\overline{\Gamma}_i}\right\|^2\right)\|\omega\| \tag{232}$$

and finally for $\xi^3_{\{1,2,3\}}$ if we apply the Cauchy-Schwartz inequality we get the upper bound:

$$\left\|\xi^3_{\{1,2,3\}}\right\| = \left\| \frac{\frac{1}{\overline{\Gamma}_1}\frac{1}{\overline{\Gamma}_2}\frac{1}{\overline{\Gamma}_3}\left(\frac{1}{\Gamma_1}\frac{d\overline{\Gamma}_1}{\overline{\Gamma}_1^2} + \frac{1}{\Gamma_2}\frac{d\overline{\Gamma}_2}{\overline{\Gamma}_2^2} + \frac{1}{\Gamma_3}\frac{d\overline{\Gamma}_3}{\overline{\Gamma}_3^2}\right)^2}{(1/|\Gamma_1|^2 + 1/|\Gamma_2|^2 + 1/|\Gamma_3|^2)^5} \wedge \frac{t^{-z}}{\Gamma_4}dz \right\| \tag{233}$$

$$\leq \frac{\frac{1}{|\Gamma_1|^2|\Gamma_2|^2|\Gamma_3|^2}}{(1/|\Gamma_1|^2 + 1/|\Gamma_2|^2 + 1/|\Gamma_3|^2)^5}$$
$$\times\left\|\left(\frac{1}{|\Gamma_1|^2}\frac{d\overline{\Gamma}_1}{\overline{\Gamma}_1} + \frac{1}{|\Gamma_2|^2}\frac{d\overline{\Gamma}_2}{\overline{\Gamma}_2} + \frac{1}{|\Gamma_3|^2}\frac{d\overline{\Gamma}_3}{\overline{\Gamma}_3}\right)^2\right\|\|\omega\| \tag{234}$$

$$\leq \frac{\frac{1}{|\Gamma_1|^2|\Gamma_2|^2|\Gamma_3|^2}}{(1/|\Gamma_1|^2 + 1/|\Gamma_2|^2 + 1/|\Gamma_3|^2)^5}\left(\frac{1}{|\Gamma_1|^2} + \frac{1}{|\Gamma_2|^2} + \frac{1}{|\Gamma_3|^2}\right)^2$$
$$\times\left\|\left(\frac{d\overline{\Gamma}_1}{\overline{\Gamma}_1}\right)^2 + \left(\frac{d\overline{\Gamma}_2}{\overline{\Gamma}_2}\right)^2 + \left(\frac{d\overline{\Gamma}_3}{\overline{\Gamma}_3}\right)^2\right\|\|\omega\| \tag{235}$$

$$\leq \frac{|\Gamma_1|^4|\Gamma_2|^4|\Gamma_3|^4}{\left(|\Gamma_1|^2|\Gamma_2|^2+|\Gamma_1|^2|\Gamma_3|^2+|\Gamma_2|^2|\Gamma_3|^2\right)^3} \times \left\| \left(\frac{d\overline{\Gamma}_1}{\overline{\Gamma}_1}\right)^2 + \left(\frac{d\overline{\Gamma}_2}{\overline{\Gamma}_2}\right)^2 + \left(\frac{d\overline{\Gamma}_3}{\overline{\Gamma}_3}\right)^2 \right\| \|\omega\| \tag{236}$$

For either of the three upper bounds (228), (232) or (236), since $\frac{a^2b^2c^2}{(ab+ac+bc)^3} \leq 1$, for any $a, b, c \geq 0$, the first factor is bounded by one, the second factor increases no faster than the square logarithm of R_k, and the third factor decreases exponentially as R_k increases.

Since we are working in $\sigma_{\{1,2,3\}}$ which may intersect D_1, D_2 and D_3, without loss of generality for $\xi^1_{\{1,2,3\}}$ and $\xi^3_{\{1,2,3\}}$, we will consider the planes L^ν_j of the divisors D_2, for all $\xi^2_{\{1,2,3\}}$ we will have to address at least one of the other divisors set. As we did in the previews lemma, we will represent Γ_i as the product $\Gamma'_i \cdot \Gamma''_i$, where Γ'_i is the product of the $\Gamma(s_j(z))$ in Γ_i without singularities on L^ν_j and Γ''_i is the product of the $\Gamma(s_j(z))$ in Γ_i with singularities on L^ν_j, where we will denote by s the number of factors in Γ''_2. We should also take note that by definition ψ_2 will have one pole in L^ν_j. Taking this notation into account, the following inequality for the norm of $\xi_{\{1,2,3\}}$ will hold true for any $z \in \mathcal{U}^c_\delta$:

$$||\xi^1_{\{1,2,3\}}|| \leq \overbrace{\frac{|\Gamma_1|^4|\Gamma_2|^4|\Gamma_3|^4}{\left(|\Gamma_1|^2|\Gamma_2|^2+|\Gamma_1|^2|\Gamma_3|^2+|\Gamma_2|^2|\Gamma_3|^2\right)^3}}^{4s-6s=-2s} \times \overbrace{\left\| \frac{d\overline{\Gamma}_2}{\overline{\Gamma}_2}\frac{d\overline{\Gamma}_3}{\overline{\Gamma}_3} - \frac{d\overline{\Gamma}_1}{\overline{\Gamma}_1}\frac{d\overline{\Gamma}_3}{\overline{\Gamma}_3} + \frac{d\overline{\Gamma}_1}{\overline{\Gamma}_1}\frac{d\overline{\Gamma}_2}{\overline{\Gamma}_2} \right\|}^{1} \overbrace{|\Gamma''_2|}^{s} \left\| \frac{\Gamma_1\Gamma'_2\Gamma_3}{\Gamma_4} t^{-z} \right\| \tag{237}$$

The first factor has a zero of order $2s$, the second factor has a first order pole, the third factor has a pole of order s, and the fourth factor decreases exponentially as R_k increases. Therefore the product of the first, second and third factor has a zero of order $s-1$, and taking into account that $s \geq 1$, then $\left\|\xi_{\{2,3\}}\right\|$ is bounded and decreases exponentially.

For $\xi^2_{\{1,2,3\}}$ notice that

$$\left\|\xi^2_{\{1,2,3\}}\right\| \leq 2\, \overbrace{\frac{|\Gamma_1|^6|\Gamma_2|^6|\Gamma_3|^6}{\left(|\Gamma_1|^2|\Gamma_2|^2+|\Gamma_1|^2|\Gamma_3|^2+|\Gamma_2|^2|\Gamma_3|^2\right)^4}}^{6s-8s=-2s} \overbrace{\left\|\frac{d\overline{\Gamma}_2}{\overline{\Gamma}_2}\right\|}^{1\vee 0} \times \overbrace{\left\| \frac{1}{\Gamma_1^2}\frac{d\overline{\Gamma}_1}{\overline{\Gamma}_1} + \frac{1}{\Gamma_2^2}\frac{d\overline{\Gamma}_2}{\overline{\Gamma}_2} + \frac{1}{\Gamma_3^2}\frac{d\overline{\Gamma}_3}{\overline{\Gamma}_3} \right\|}^{1-2s\vee 1} \overbrace{|\Gamma''_2|}^{s\vee 0} \left\| \frac{\Gamma_1\Gamma'_2\Gamma_3}{\Gamma_4} t^{-z} \right\| \tag{238}$$

If we are working with the singularities D_1 or D_3 the first factor has a zero of order $2s$, the second factor has a first order pole, the third factor has a pole of order $1-2s$, the fourth factor as a pole of order s and the fifth factor decreases exponentially as R_k increases. Therefore the product of the first, second, third and fourth factor has a zero of order $2s-2$, and taking into account that $s \geq 1$, then $\left\|\xi^2_{\{1,2,3\}}\right\|$ is bounded and decreases exponentially. For the singularities in D_2 the first factor has a zero of order $2s$, the second and fourth factors do not have neither a pole nor a zero, the third factor has a pole of order 1 and the fifth factor decreases exponentially as R_k increases. Therefore the product of the first, second, third and fourth factor has a zero of order $2s-1$, and taking into account that $s \geq 1$, then $\left\|\xi^2_{\{1,2,3\}}\right\|$ is bounded and decreases exponentially.

Finally for $\xi^3_{\{1,2,3\}}$:

$$\left\|\xi^3_{\{1,2,3\}}\right\| \leq \overbrace{\frac{|\Gamma_1|^8|\Gamma_2|^8|\Gamma_3|^8}{(1/|\Gamma_1|^2+1/|\Gamma_2|^2+1/|\Gamma_3|^2)^5}}^{8s-10s=-2s}$$
$$\times \overbrace{\left\|\left(\frac{1}{|\Gamma_1|^2}\frac{d\overline{\Gamma}_1}{\overline{\Gamma}_1}+\frac{1}{|\Gamma_2|^2}\frac{d\overline{\Gamma}_2}{\overline{\Gamma}_2}+\frac{1}{|\Gamma_3|^2}\frac{d\overline{\Gamma}_3}{\overline{\Gamma}_3}\right)^2\right\|}^{0} \overbrace{|\Gamma_2''|}^{s} \left\|\frac{\Gamma_1\Gamma_2'\Gamma_3}{\Gamma_4}t^{-z}\right\| \tag{239}$$

The first factor has a zero of order $2s$, the second factor has a neither a pole nor a zero, the third factor has a pole of order s, and the fourth factor decreases exponentially as R_k increases. Therefore the product of the first, second and third factor has a zero of order s, and taking into account that $s \geq 1$, then $\left\|\xi^3_{\{1,2,3\}}\right\|$ is zero. This in conjunction with the previous deductions completes the proof of lemma 3. □

□