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MASTER IN
MATHEMATICAL FINANCE

FINAL MASTER WORK
DISSERTATION

OPTIMAL REINSURANCE IN A DIFFUSION SETTING

CRISTIANA AMARAL CORREIA

OCTOBER - 2021

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ORIENTATION:

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Abstract

The business of insurance companies is to take on the risk of policyholders (individuals or companies), receiving in return the payment of a premium. In order to protect themselves against big losses, and not be at risk of insolvency, insurance companies usually reinsure part of their portfolio by transferring part of the risk taken to another insurance company. Reinsurance works, in this way, as the insurance of the insurer itself.

The optimal reinsurance problem aims at answering two fundamental questions: (i) What type of reinsurance contract should be done; (ii) how much risk should be transferred to the reinsurance company. This master's final work, seeks to find the optimal reinsurance for 3 different optimality criterion: (i) minimizing the probability of ruin occurring in infinite time; (ii) maximizing the expected value; and (iii) minimizing the variance of the process.

To obtain the optimal reinsurance treaty, the classic risk model of Crámer and Lundberg is approximated by a diffusion process which is described by a Brownian motion process. The surplus Brownian motion process is defined by parameters that incorporate several characteristics of the underlying Crámer-Lundberg process, including the different premium calculation principles and the different types of reinsurance treaties.

In this work, the reinsurance treaties under study are the proportional quota-share treaty and the non-proportional excess of loss treaty, and the premium calculation principle considered by both, the first insurer and the reinsurer, is the expected value principle.

After building the model, the probability of ruin is analysed. The present study addresses this moment of ruin, i.e., when the surplus process hits zero or negative values, in continuous time and infinite time horizon.

The optimal reinsurance strategy is obtained numerically and a sensitivity analysis is made, using Mathematica software.

Keywords: Reinsurance optimization; surplus process; Brownian motion; Ruin probability; Quota-share and Excess of loss treaties; Expected value premium principle

Resumo

A atividade de qualquer companhia de seguros é aceitar o risco dos seus segurados (sejam pessoas físicas ou jurídicas), recebendo em troca o pagamento de um prémio. Com o intuito de se protegerem face a grandes perdas, as seguradoras podem ressegurar parte da sua carteira transferindo parte do risco para outra seguradora, diminuindo assim o risco de insolvência. O resseguro funciona, desta forma, como o seguro da própria seguradora.

O problema da otimização do resseguro visa responder a duas questões fundamentais: (i) Que tipo de contrato de resseguro deve ser feito; (ii) Quanto risco deve ser transferido para a resseguradora. O presente trabalho final de mestrado, procura encontrar o resseguro ótimo para 3 critérios de otimalidade diferentes: (i) minimizando a probabilidade de ruína ocorrer no tempo infinito; (ii) maximizando o valor esperado do processo; e (iii) minimizando a variância do processo.

Para obter o tratado de resseguro ótimo, o modelo de risco clássico de Crámer e Lundberg é aproximado por um processo de difusão, que é descrito por um processo *Brownian motion*. O *surplus Brownian motion process* é definido por parâmetros que incorporam várias características do processo Crámer-Lundberg subjacente, incluindo os diferentes princípios de cálculo de prémio e os diferentes tipos de tratados de resseguro.

Neste trabalho vamos focar a análise nos tratados de resseguro proporcional *quota-share* e não proporcional *excess of loss*. O princípio de cálculo do prémio considerado, quer para a primeira seguradora quer para a resseguradora é o princípio do valor esperado.

Após a construção do modelo, a probabilidade de ruína é analisada. O presente estudo trata desse momento de ruína quando o *surplus process* atinge valores nulos ou negativos, em tempo contínuo e num horizonte temporal infinito.

Através do *software Mathematica*, é obtida numericamente a estratégia ótima de resseguro e é realizada uma análise de sensibilidade.

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1 Introduction

The business of insurance companies is to take on the risk of policyholders (individuals or companies), receiving in return the payment of a premium. In order to protect themselves against big losses, and not be at risk of insolvency, insurance companies usually reinsure part of their portfolio by transferring part of the risk taken to another insurance company. Reinsurance works, in this way, as the insurance of the insurer itself. The reinsurer accepts the risk transferred by the insurer against the payment of a premium. Both premiums, of the insurer and reinsurer, must be higher than the expected value of the risk transmitted to it, since otherwise the companies will bankrupt with probability 1 (see Centeno [7]). There are two fundamental questions that insurance companies face when deciding reinsurance strategies: (i) what is the optimal form of reinsurance, and (ii) what is the optimal retention level, i.e., what is the optimal amount of risk that should be reinsured.

The main objective of this study is to understand, which type of reinsurance, quota-share or excess of loss treaties, minimizes the ruin probability and which is the optimal level of retention from the perspective of the insurance company in each case. Additionally, and based on the results obtained minimizing the probability of ruin, we study which of the reinsurance treaties offers the highest expected value and the smallest variance to the surplus process.

We start by considering the classic surplus process, the Crámer-Lundberg model. This model follows a compound Poisson process and can be approximated by a diffusion process. The model obtained is a Brownian motion with drift, which is a stochastic process with almost surely continuous trajectories and stationary and independent increments (see Dixit [11]).

The model we are considering incorporates the principles for calculating the premiums and different types of reinsurance treaty. The premium for a given policy is a fixed amount received by the company as compensation for the risk assumed by the insurer, Centeno [7]. This amount takes into account to the expected losses and management expenses, but it also includes a safety loading charge, which serves to compensate any random deviations of the losses in relation to their average. There are several principles for calculating the premium to be collected. In this study we assume that the insurance company applies the expected value principle for premium calculation with a safety loading $\eta > 0$, and the reinsurer also applies the expected value principle with a safety loading $\theta > 0$. As reported by Schmidli [23], the reinsurer's safety loading must be greater or equal to the insurer's safety loading, i.e., $\theta \geq \eta$. Otherwise, the insurer could reinsure the whole portfolio and make a profit without any risk.

The model also incorporates the types of reinsurance treaties under consideration. Reinsurance, as already mentioned, works as an insurance for the insurance company. Reinsurance treaties can be classified into proportional and non-proportional type treaties. The non-proportional type includes excess of loss or stop-loss reinsurance. In this study, we consider the quota-share and the excess of loss treaties.

The optimality criterion considered to obtain the optimal reinsurance treaty is the minimization of the probability of ruin. In order to do that, the moment of ruin must be analysed. Ruin occurs when the surplus process is below zero. The moment of ruin can be analyzed in finite or infinite time horizon, and discrete or continuous time. The present work studies the moment of ruin in continuous time and considers the infinite time horizon.

Hence, the aim is to find the optimal reinsurance treaty, using the criterion of minimizing the probability of ruin for each type of treaty we are considering: the quota-share and the excess of loss treaties. The latter will also be subdivided into the inferior limit M and the superior limit L contracts. Since we are working in a continuous setup, the ruin time is a first passage time for which the density and CDF can be analytically obtained. Such expression only depends on the diffusion parameters. Thus, to minimize this probability analytically, we start by defining each of the diffusion process parameters and apply it to the ruin probability expression. Minimizing the resulting expression, in some cases, it is possible to explicitly set an expression for the optimal minimum.

Assuming that the individual claims follow a Gamma distribution, and after minimizing the probability of ruin for each of the considered types of treaty, a numerical and sensitivity analysis is performed. First, a sensitivity analysis on the insurer and reinsurer loadings is carried out, with the objective of understanding which of the reinsurance treaties gives a lower ruin probability and which is the optimal level of risk to be retained by the insurance company. Then, the safety loading of the insurer and reinsurer are fixed and a sensitivity analysis is performed on the parameters of the underlying distribution with the objective of understanding the behavior of the expression of the ruin probability as a function of the underlying risk. Additionally, different values of the initial reserve are considered keeping the remaining parameters fixed, in order to understand how the value of the initial reserve influences the ruin probability.

Based on the optimal retention levels obtained by minimizing the probability of ruin for each type of reinsurance treaties under consideration, we analyse which of them offers the highest expected value, or the lowest variance for the surplus process.

This master's thesis is organized into 6 main chapters, including this introduction. Chapter 2 provides a literature review, contextualizing the present study. Chapter 3 describes how to obtain the diffusion model, starting from the classic Crámer-Lundberg risk model. There, the considered premium calculation principles and reinsurance treaties, namely quota-share and excess of loss, are formally introduced. In Chapter 4 the moment of ruin is analysed starting from the concepts of ruin and first hitting time. In this chapter, the expression for the Laplace transform of the hitting time, the probability of the density of the hitting time, the probability of ruin in infinite time, the expected value of the first hitting time and its variance are deduced. In Chapter 5 a numerical and sensitivity analysis of the optimal reinsurance treaty to several parameters of the surplus process is carried out. Finally, in Chapter 6 the main conclusions of this study are presented.

2 Literature review

Reinsurance is an important management tool for insurance companies and, according to Li and Shen [16], is an effective way to spread risk in the insurance business. As Centeno [7] stated, the main motivation that leads insurance companies to reinsure their portfolios is to protect themselves against losses that could bring negative results and, consequently, jeopardize their solvency. It is natural for an insurer to make use of the different types of reinsurance treaties that exist, and indeed there is no ideal type of treaty applicable to all cases. Each type of reinsurance only provides protection against certain factors that influence the amount of claims.

The reinsurance optimization problem is a fundamental topic that raises some questions for actuaries, and it has been widely studied, for example by Schmidli [23], Taksar and Markussen [26] and Browne [2]. Several criterion can be considered to the optimal reinsurance problem as e.g. the minimization of the ruin probability, or the maximization of the expected utility of wealth. As in the present study, authors like Shmidli [24], Zhang, Zhou, and Guo [28], X. Liang, Z. Liang and Young [17], Hipp and Vogt [13], Shmidli [23] and Taksar and Markussen [26] consider the criterion of minimizing the ruin probability. Dickson and Waters [8] also minimize the ruin probability up to some given time horizon, either in discrete or continuous time. The study of Golubin [12] minimizes the expected maximum loss. Menga, Liao and Siu [22] minimize the probability of ruin and maximize the expected utility, simultaneously. Cani and Thonhauser [5] maximize the expected discounted surplus level integrated over time. Finally, Browne [2] minimize the ruin probability and maximize the probability of achieving a given upper wealth level before hitting a given lower level.

In addition to the ruin probability, in this study the optimal treaties obtained for each type of reinsurance are compared regarding the maximization of the expected value, and the minimization of the variance, of the wealth continuous process.

The optimal reinsurance strategy can be studied both in a static or in a dynamic setting. Studies in a dynamic environment have been very popular. For example, Schmidli [23] considers dynamic proportional reinsurance strategies in a diffusion setting and also a discrete setting by means of compound Poisson process. A year later, Schmidli [24] considers the classic risk model and includes besides reinsurance, investment in a risky asset modeled by the Black-Scholes model. It considers a proportional type of reinsurance and, to determine the optimal reinsurance strategy, the Hamilton-Jacobi-Bellman (HJB) approach is used and a numerical procedure to solve the equation is applied. Mao et al. [21] extends the work of Zhang and Siu [27] and establishes Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation to determine the optimal investment and reinsurance strategy for an insurer whose wealth follows a diffusion process. Liang and Yuen [18] examine the optimal proportional reinsurance strategy in a risk model with two dependent classes of insurance business. In this study, stochastic control theory is used to derive closed-form expressions for the optimal strategy and value functions for both the compound Poisson risk model and the Brownian motion model. Hipp and Vogt [13] consider a risk process modeled by a compound Poisson process to find the optimal dynamic excess of loss reinsurance strategy minimizing the probability of ruin in infinite time and prove the existence of a smooth solution of the corresponding HJB equation. In the research of Taksar and Markussen [26], they assumed the case of proportional reinsurance and made use of the stochastic control theory to determine the optimal reinsurance policy. Cani and Thonhauser [5] look for a dynamic reinsurance strategy and, for that, they use analytical methods to identify the value function as a particular solution of the HJB equation. Liu and Yang [20] assume that: (i) the insurance company receives a premium at a constant rate, (ii) the total claims are modeled by a compound Poisson process, and (iii) the insurer can invest in the money market and in risky assets such as stocks. The optimal solution is obtained from the HJB equation. Liang et al. [17] use a stochastic Perron's method to prove that the minimum probability of ruin is the unique viscosity solution of its HJB equation with appropriate boundary conditions.

Contrary to most studies carried out in continuous time, in this work we present the optimal reinsurance strategy in a static environment.

The surplus process can be modeled by the compound Poisson model or by its approximated Brownian motion. In the present study, we approximate the surplus process to the diffusion model. There are several studies that consider this approximation, among which are Zhang et al. [28], Mao et al. [21], Taksar and Markussen [26] and Li and Shen [16]. In the papers of Schmidli [23] and Meng et al. [22], the surplus process is modeled by both the classic risk model and a Brownian motion. However, there are many authors who study only the surplus process modeled through the compound Poisson process, as it is the case of Dickson and Waters [9], Schmidli [24], Dickson and Waters [8], Hipp and Vogt [13] and Golubin [12].

In this work, to simplify, the expected value premium calculation principle is considered both for the insurer and the reinsurer. For instance, Cai and Tan [4] and Hu et al. [14] also use the expected value premium principle to determine the optimal retention level of a stop-loss reinsurance. On the other hand, in the work of Liang and Yuen [18] the variance premium principle is used. In Li and Shen [16] the reinsurance premium is calculated according to the variance and standard deviation premium principles.

We consider the quota-share and the excess of loss treaties in our optimal reinsurance problem. Some authors, like Dimitrova and Kaishev [10], Hipp and Vogt [13], Golubin [12], Brachetta and Ceci [1], also use the excess of loss to study this type of problems. Others, such as Schmidli [23] and [24], Liang and Yuen [18] and Taksar and Markussen [26], consider proportional treaties. In the studies of Zhang et al. [28], and Centeno [6], the optimization problem with combinations of quota-share and excess of loss reinsurance strategies is considered.

In our numerical illustration, we show that the quota-share treaty minimizes the ruin probability when compared to the considered treaties. We also found that the quota-share treaty offers a lower variance (risk) and the superior limit L of the excess of loss, a higher expected value to the surplus process. In the work of Zhou, Dong and Xu [29], they considered the problem of minimizing the VaR and CTE of an insurer's retained risk and conclude that the quota-share after stop-loss is a better reinsurance strategy than stop-loss after quota-share. Zhang, Zhou and Guo [28] minimize the ruin probability by controlling the combinational quota-share and excess of loss reinsurance strategy and conclude that the optimal combinational reinsurance strategy must be the pure excess of loss. Golubin [12] studied the classic risk model where both insurance and reinsurance policies are chosen by the insurer in order to minimize the expected maximal loss, and show that the optimal reinsurance is excess of loss. We found that the optimal reinsurance strategy highly depends on the structure of our risk size. We illustrate such a result considering Gamma distributions with the same expected value but with different moments of higher order.

3 The surplus process of the insurer

The main purpose of this chapter is to approximate the surplus process to a diffusion model, starting from the classic risk model. The purpose is to obtain a surplus process modeled by a Brownian motion incorporating the premium calculation principles and the reinsurance treaties.

3.1 The Classic risk model

As mentioned by Schmidli [25], the Cramér-Lundberg model, also known as the classic risk model, "measures" the effect of a certain decision of the actuary on risk. The model was initially constructed by Filip Lundberg and Harald Cramér and their collective risk model was obtained as a limit of a sum of individual risk models for an increasing number of individual contracts.

In the classic risk model, the surplus process of a collective contract or a large portfolio is modeled by,

$$X_t = x + Pt - S_t, \tag{1}$$

where x is the initial surplus, $P > 0$ is the premium rate and S_t is the aggregated claims in a period of time $(0, t]$. Therefore, let $\{S_t\}_{t\geq 0}$ be the aggregate claims process, defined as follows,

$$S_t = \sum_{i=0}^{N_t} Y_i \tag{2}$$

where N_t is a Poisson process, modeling the incoming claims with rate λ, $Y_0 \equiv 0$, and, for $i = 1, 2...$, Y_i is a sequence of independent and identically distributed random variables to Y, which is strictly positive and independent of N_t, representing the size of the successive claims.

The aggregate claims process, $\{S_t\}_{t\geq 0}$, is a compound Poisson process, since the number of claims, $\{N_t\}_{t\geq 0}$, is a Poisson process. Recall that the Poisson process has independent and stationary increments and so does the compound Poisson process.

The moment-generating function of the aggregate claim process is thus given by

$$M_{S_t}(r) = M_{N_t}[\ln M_Y(r)] = \exp[\lambda t(M_Y(r) - 1)], \tag{3}$$

whose existence depends on the moment-generating function of Y_i. Calculating the derivatives of the moment-generating function at point $r = 0$, we obtain the raw moments of S_t, in particular:

$$E(S_t) = \lambda t \mu, \tag{4}$$

$$Var(S_t) = \lambda t \mu_2, \tag{5}$$

where μ and μ_2 are the first and the second raw moments of the claim size distribution Y.

3.2 The approximated diffusion model

According to Schmidli [25], it is difficult to study the characteristics of the classic risk model when the aggregate claim process is a compound Poisson process. One possibility for overcoming this difficulty is to look for approximations, namely the diffusion approximations. According to Schmidli

[25], the idea is to consider a sequence of risk models, in such way that they weakly converge to a diffusion process.

As previously stated, the classic risk model has stationary and independent increments, therefore this diffusion approximation only makes sense if the obtained process also has stationary and independent increments. So, in line with Schmidli [25], increasing the number of claims and making them smaller, the limiting process should be a Brownian motion. As reported by Dixit [11], a Brownian motion is a continuous-time stochastic process such that, given the initial value x_0 at time $t = 0$, the random variable x_t for any $t > 0$ is normally distributed with mean $x_0 + \mu t$ and variance $\sigma^2 t$, where μ represents the trend and σ the volatility. In this case, The diffusion process for the surplus process can be written as follows:

$$dX_t = \alpha dt + \sigma dW_t, \tag{6}$$

where α corresponds to the drift parameter, i.e., represents what the insurance company is effectively gaining with the premium income, and σ is the diffusion parameter, which represents the volatility of the model, W_t stands for the standard Brownian Motion.

3.3 The insurer continuous model

As stated in the previous section, the diffusion approximation for the surplus process has the parameters α and σ. The drift parameter, α, represents what the insurance company is effectively gaining with the premium income. Therefore, this parameter corresponds to the premium gained by the insurer less the risk retained by it. In what follows S represents the aggregate claims amount in one unit of time.

Inspired by Schmidli [23], the mathematical expression for the α parameter is given by:

$$\alpha = P_I(S, Z(S)) - E(Z(S)), \tag{7}$$

where $Z(S)$ is the risk retained by the insurance company, with $0 < Z(S) < S$, and $P_I(S, Z(S))$ is the premium charged by the insurer. The premium gained by the insurance company, P_I, is given by the total premium paid by the policyholder to the insurer, P_T, minus the premium paid to the reinsurance company, P_R:

$$P_I(S, Z(S)) = P_T(S) - P_R(S - Z(S)) \tag{8}$$

Finally, the σ parameter represents the volatility of the process and can be written as follows,

$$\sigma^2 = Var(Z(S)) \Leftrightarrow \sigma = \sqrt{Var(Z(S))}. \tag{9}$$

3.3.1 Premium Calculation Principles

As defined by Centeno [7], the premium for a given policy is a fixed amount received by the company as compensation for the risk assumed by the insurer. The premium corresponds in part to the claims and management expenses, but it also includes a safety loading, which works as a compensation for any random deviation of the claims in relation to their average. Now, let S be the risk. Any reasonable premium P should be composed by the net premium, $E(S)$, plus some safety loading, η, i.e., $P > E(S)$. There are several premium calculation principles. In Centeno [7] and in Schmidli [25] we can find some of the most common ones, which are presented below.

1. *Expected Value Principle:* $P = (1+\eta)E(S)$, for some safety loading $\eta > 0$
2. *Variance Principle:* $P = E(S) + \varepsilon Var(S)$, for some $\varepsilon > 0$
3. *Modified Variance Principle:* $P = E(S) + \varepsilon \frac{Var(S)}{E(S)}$, for some $\varepsilon > 0$
4. *Standard deviation Principle:* $P = E(S) + \varepsilon \sqrt{Var(S)}$, for some $\varepsilon > 0$
5. *Exponential Principle:* $P = \varepsilon^{-1} log(E[exp(\alpha S)])$, for some $\varepsilon > 0$
6. *Zero Utility Principle:* Let $u(x)$ be some strictly increasing and strictly concave function. The zero utility premium principle is the unique solution of the equation $u(x) = E[u(x) + P - S]$, where x corresponds to the initial wealth of the insurance company.
7. *Adjusted risk Principle:* Let $F(x)$ be the distribution function of the risk S and $S \geq 0$. So, $P = \int_0^\infty (1 - F(x))^\eta dx$, for some $\eta \in (0,1)$

3.3.2 Including the Reinsurance treaty

In order to protect themselves against losses that may jeopardize the continuity of their activity, insurance companies reinsure part of their business portfolio. As exposed by Centeno [7], treaties can be classified into proportional and non-proportional. Examples of proportional treaties are the quota-share and the surplus. Examples of non-proportional treaties are the excess of loss and stop loss reinsurance treaty. In this thesis the quota-share and excess of loss treaties are considered.

Consider a reinsurance treaty acting on the aggregate claim over a certain period. Let $Z(S)$ be the retained risk by the insurance company, and $H(S) = S - Z(S)$ the risk ceded to that the reinsurance entity, with $0 < Z(S) < S$.

1. Quota share treaty:

In a quota-share treaty the reinsurer is responsible for a certain fixed percentage of the underlying risk and, in exchange, it receives the correspondent share of the original premium. The reinsurance entity pays a commission, also proportional to the premiums received, to share the management and acquisition expenses. To facilitate, we assume that the value of this commission is zero.

Let the retained percentage of risk be b, with $b \in (0,1)$, and $1 - b$ be the ceded percentage to the reinsurer. For each aggregate loss S, the insurer retains $Z(S) = bS$ of the risk and transfers $H(S) = S - Z(S) = S - bS = S(1-b)$ of the risk to the reinsurer.

Using equation (8), the premium retained by the insurance company, $P_I(S, Z(S))$ becomes,

$$P_I(S, Z(S)) = P_T(S) - P_R((1-b)S). \tag{10}$$

The distribution of retained aggregate losses is $F_{Z(S)}(s) = F_S(s/b)$ and of the transferred ones is $F_{H(S)}(s) = F_S(s/(1-b))$. The expected value of the retained aggregate losses is expressed as,

$$E(Z(S)) = \int_0^\infty Z(s) dF_S(s) = b \int_0^\infty s dF_S(s) = bE(S) \tag{11}$$

The second raw moment is expressed as,

$$E(Z(S)^2) = \int_0^\infty Z(s)^2 dF_S(s) = b^2 \int_0^\infty s^2 dF_S(s) = b^2 E(S^2) \tag{12}$$

Hence, by equations (7) and (9) the parameters α and σ are defined as follows,

$$\alpha = P_T(S) - P_R((1-b)S) - E(Z(S)) = P_T(S) - P_R((1-b)S) - bE(S) \tag{13}$$

$$\sigma^2 = Var(Z(S)) = E(Z(S)^2) - E(Z(S))^2 = b^2E(S^2) - b^2E(S^2) = b^2Var(S) = b^2\lambda E(Y^2) \tag{14}$$

Using the parameters above, the diffusion model becomes,

$$dX_t = (P_T(S) - P_R((1-b)S) - bE(S))dt + b\sqrt{\lambda E(Y^2)}dW_t, \tag{15}$$

where W_t stands for the standard Brownian motion.

2. Excess of Loss Treaty:

Now, consider a reinsurance strategy acting on the individual claims. Let $Z(Y)$ be the individual retained risk by the insurance company, and $H(Y) = Y - Z(Y)$ be the individual risk that the reinsurance entity receives, $0 < Z(Y) < Y$. In the excess of loss treaty, an amount M is retained by the insurer while the remaining part of the risk is ceded to the reinsurer of each individual claim that occurs. Losses in excess of the deductible M are the responsibility of the reinsurer, and this liability is generally limited to a certain amount L for each individual loss. When $L = +\infty$ the coverage is said to be unlimited.

Hence, for each claim Y, the insurer is ceding the amount of risk $H(Y) = Y - Z(Y)$:

$$H(Y) = \min\{L; (Y-M)_+\} = \begin{cases} 0 & \text{if } Y \leq M \\ Y - M & \text{if } M < Y \leq M + L, \\ L & \text{if } Y > M + L \end{cases} \tag{16}$$

where $(Y-M)_+ = \max(0, Y-M)$. Consequently, for each claim Y, the insurer retains the amount of risk $Z(Y)$:

$$Z(Y) = Y - H(Y) = \begin{cases} Y & \text{if } Y \leq M \\ M & \text{if } M < Y \leq M + L. \\ Y - L & \text{if } Y > M + L \end{cases} \tag{17}$$

Let $S_H = \sum_{i=0}^{N_1} H(Y_i)$ and $S_Z = \sum_{i=0}^{N_1} Z(Y_i)$ be respectively the aggregate claim amount ceded and retained. Using equation (8), the premium $P_I(Y, H(Y))$ becomes,

$$P_I(S, S_Z) = P_T(S) - P_R(S_H) \tag{18}$$

The distribution of the individual claims retained by the insurance company is

$$F_{Z(Y)}(y) = \begin{cases} F_Y(y) & \text{if } y < M \\ F_Y(y+L) & \text{if } y \geq M \end{cases} \tag{19}$$

The first moment of the individual claims is

$$\begin{aligned} E(Z(Y)) &= \int_0^M y dF_Y(y) + M \int_M^{M+L} dF_Y(y) + \int_{M+L}^{\infty} (y-L) dF_Y(y) \\ &= \int_0^M y dF_Y(y) + M(F_Y(M+L) - F_Y(M)) + \int_{M+L}^{\infty} (y-L) dF_Y(y) \\ &= \int_0^M (1-F_Y(y))dy + \int_{M+L}^{\infty} (1-F_Y(y))dy \end{aligned} \tag{20}$$

The second raw moment is

$$\begin{aligned} E(Z(Y)^2) &= \int_0^M y^2 dF_Y(y) + M^2 \int_M^{M+L} dF_Y(y) + \int_{M+L}^{\infty} (y-L)^2 dF_Y(y) \\ &= 2\int_0^M y(1-F_Y(y))dy + 2\int_{M+L}^{\infty} (y-L)(1-F_Y(y))dy \end{aligned} \tag{21}$$

By equations (7) and (9), the parameters α and σ are defined by

$$\alpha = P_T(S) - P_R(H(S)) - E(Z(S)), \tag{22}$$

$$\sigma^2 = Var(S_Z) = \lambda E(Z(Y)^2). \tag{23}$$

Using the above parameters, it is easy to obtain the diffusion model:

$$dX_t = (P_T(S) - P_R(H(S)) - E(Z(S))dt + \sqrt{\lambda E(Z(Y)^2)} dW_t, \tag{24}$$

where W_t stands for the standard Brownian motion.

4 The moment of ruin

We are now concerned with the moment of ruin. The surplus process described by the diffusion process derived in the previous chapter hits a given level c, i.e., the first moment when ruin occurs. As stated by Centeno [7], ruin occurs when the surplus process is negative in a given moment or moments in time. Hence, we will consider $c = 0$. The surplus level can be analyzed in finite or infinite time horizon.

We have seen that the diffusion model in equation (6) is an arithmetic Brownian motion. Now, let τ_c be the instant when the stochastic variable X_t reaches a fixed barrier c, assuming $X_0 > c$, i.e., τ_c is the first hitting time, given by:

$$\tau_c = \inf\{t \geq 0 : X_t \leq c\}. \tag{25}$$

In line with Burnecki, Mista and Weron [3], the ruin probability in infinite time, i.e., the probability that the capital of an insurance company ever drops below the level c, can be written as

$$\Psi(c) = P(\tau_c < \infty) = P\left(\inf_{t\geq 0} X_t \leq c\right). \tag{26}$$

In the case of finite time horizon, the process is considered in a finite time interval, i.e., ensuring that within a defined period, $0 < t < T$, the process does not go into ruin. Also by Burnecki, Mista and Weron [3], the ruin probability in finite time T is given by,

$$\Psi(c, T) = P(\tau_c \leq T) = P\left(\inf_{0\leq t\leq T} X_t \leq c\right). \tag{27}$$

Note that $\Psi(c, T) < \Psi(c)\ \forall\ T > 0$.

Laplace Transform of the hitting time

Let τ_c be the first hitting time of level $c > 0$ for a Brownian motion X_t with $X_0 > c$, drift α and variance σ^2. Then for $s > 0$, its Laplace transform is given by, (see Liao [19])

$$E[e^{-s\tau_c}] = \exp\left[-\frac{X_0 - c}{\sigma^2}(\sqrt{\alpha^2 + 2s\sigma^2} - \alpha)\right], \tag{28}$$

with $s = \lambda\alpha + \lambda^2\sigma^2/2$ and $\lambda > 0$.

Probability density of the hitting time

According to the study of Ingersoll [15] and Liao [19], the first hitting time probability density function $f(t)$ is given by

$$f(t) = \frac{X_0 - c}{\sigma\sqrt{2\pi t^3}} exp\left[-\frac{(X_0 - c - \alpha t)^2}{2\sigma^2 t}\right], \tag{29}$$

for $t > 0$. From equation (29), for $\alpha = 0$, τ has no finite moments.

The cumulative probability distribution is given by one less the probability of not hitting a given barrier c, and it is described by (see Ingersoll [15])

$$F(t) = \Phi\left[\frac{-(X_0 - c) + \alpha t}{\sigma\sqrt{t}}\right] + exp\left[\frac{2\alpha(X_0 - c)}{\sigma^2}\right]\Phi\left[\frac{-(X_0 - c) - \alpha t}{\sigma\sqrt{t}}\right], \tag{30}$$

where $\Phi(.)$ is the standard normal probability distribution.

Finiteness of the hitting time

Consider the probability that the process hits a barrier placed at positive level c, below the current value $X_0 > c$. Then, the ruin probability regarding level $c < X_0$ is given by (see Liao [19])

$$\Psi(c) = \begin{cases} exp\left[\frac{-2\alpha(X_0-c)}{\sigma^2}\right] & \text{if } \alpha > 0, \\ 1 & \text{if } \alpha \leq 0 \end{cases} \tag{31}$$

For the case of negative or zero drift, almost surely the surplus process will reach zero in a finite time. The parameter α represents what the insurance company is really gaining from the premiums, and if $\alpha = 0$ almost surely the company will bankrupt.

In the case of positive drift, the probability that X_t hits the zero level increases with the volatility and decreases with the initial level X_0 and with the drift.

Expectation of the hitting time

Consider the case when the barrier c is below the current level X_0, $c < X_0$. Then the expected first hitting time is given by (see Liao [19])

$$E(\tau_c) = \begin{cases} \frac{X_0-c}{|\alpha|} & \text{if } \alpha < 0 \\ \infty & \text{if } \alpha \geq 0 \end{cases} \tag{32}$$

We can conclude that, with probability 1, the Brownian motion process will eventually hit c, but its mean time is infinite when the drift is negative.

If the drift is negative, the variance of the first hitting time is finite. The variance depends on the volatility and, considering $c < X_0$, it is given by,

$$Var(\tau_c) = \begin{cases} \frac{\sigma^2(X_0-c)}{2\alpha^3} & \text{if } \alpha < 0. \\ \infty & \text{if } \alpha \geq 0 \end{cases} \tag{33}$$

5 Reinsurance optimization: numerical simulations

Numerical studies and sensitivity analyzes for the optimal reinsurance problem are provided in this chapter. The numerical examples and analysis were developed using the Mathematica software. The criterion considered to obtain the optimal reinsurance strategy are the minimization of the ruin probability, the maximization of the expected value and the minimization of the variance of the surplus process. Individual claims are assumed to follow a Gamma distribution.

5.1 The underlying risk: The Gamma distribution

The Gamma distribution is a family of continuous two-parameter, γ and β, probability distributions, with $\gamma > 0$, the shape parameter and $\beta > 0$, the scale parameter. It is used to model positive data values that are right skewed.

Figure 1 shows the behavior of the Gamma distribution density function when the γ parameter (shape) varies and the β parameter (scale) is fixed. The greater the value of the γ parameter, the more the distribution tends to approximate to a Gaussian. If γ is a positive integer, then we have an Erlang distribution, if $\gamma = 1$ we get the Exponential distribution.

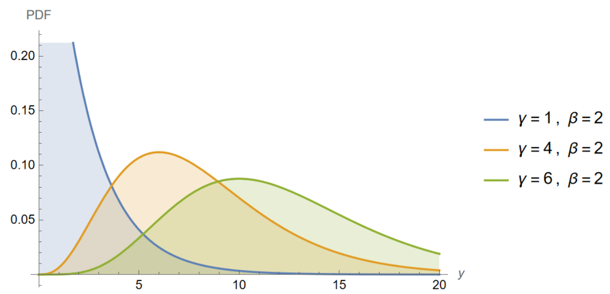

Figure 1: Gamma PDF with $\gamma = 1, 4, 6$ and $\beta = 2$

The scale parameter β indicates how much the density stretches or shrinks upwards (y axis), depending on the general magnitudes of the represented data values. In Figure 2, the shape parameter is fixed and the scale parameter varies. The larger the scale parameter the less shrank the curves are.

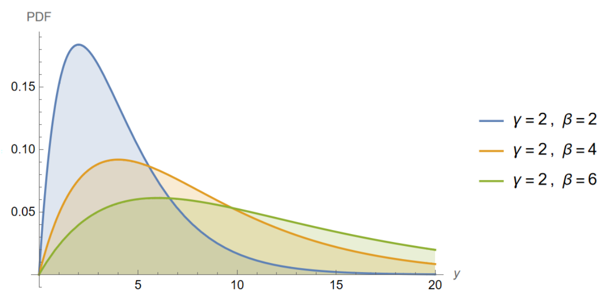

Figure 2: Gamma PDF with $\gamma = 2$ and $\beta = 2, 4, 6$

5.2 Optimal reinsurance strategy: minimizing the ruin probability

The goal now is to obtain the optimal reinsurance treaty minimizing the ruin probability in infinite time. The optimal treaty will be obtained numerically for each type of treaties considered, first for the quota-share and then for the excess of loss.

5.2.1 Minimizing ruin probability in infinite time

In order to minimize the ruin probability in infinite time, the parameters of each treaty are incorporated into the expression of the ruin probability in equation (31). Then, the optimal value of the retained risk will be find by means of the roots of the first derivative of the ruin probability. In some cases, it will be possible to explicitly obtain the roots. In other cases, only numerical results can be derived.

As mentioned, it is assumed that the insurance company applies the expected value principle for premium calculation, with safety loading $\eta > 0$, and the reinsurer also uses the expected value principle, with safety loading $\theta > 0$. The reinsurer's safety loading must be greater or equal than the insurer's safety loading, i.e., $\theta \geq \eta$ (see Schmidli [23]). Otherwise, the insurer could reinsure the whole portfolio and make a profit without any risk.

1. Quota-share treaty

In the case of the quota-share treaty, considering the expected value premium principle, the insurer's and reinsurer's premiums are, respectively, defined as follows,

$$P_T(S) = (1+\eta)\lambda E(Y), \tag{34}$$

$$P_R(H(S)) = (1+\theta)E((1-b)S) = (1+\theta)(1-b)\lambda E(Y), \tag{35}$$

From equations (34) and (35), and using the general expression for the parameters of the diffusion process in the equations (13) and (14), these become:

$$\alpha = (1+\eta)\lambda E(Y) - (1+\theta)(1-b)\lambda E(Y) - b\lambda E(Y) = \lambda E(Y)(b\theta - (\theta - \eta)), \tag{36}$$

$$\sigma^2 = Var(bS) = b^2 Var(S)) = \lambda b^2 E(Y^2) \tag{37}$$

Equations (36) and (37) will now be incorporated into the expression of the ruin probability in equation (31). The following equation provides the expression of the ruin probability as a function of the percentage of risk retained b by the insurer,

$$\Psi(b) = exp\left[\frac{-2[\lambda E(Y)(b\theta - (\theta - \eta))](X_0 - c)}{b^2 \lambda E(Y^2)}\right], \alpha > 0. \tag{38}$$

Now, letting $\Psi'(b) = 0$ and $b \in [0,1]$, we obtain:

$$b^* = 2 - \frac{2\eta}{\theta}, \tag{39}$$

representing the optimal level of retention for the quota-share treaty minimizing the probability of ruin using the expected value principle.

2. Excess of loss treaty

Consider the inferior limit M of the excess of loss treaty, when $L=\infty$. Thus, the risk retained by the insurer is $Z(Y)=min(Y,M)$ and the risk transferred to the reinsurer is $H(Y)=Y-min(Y,M)$. For this case, the insurer's and reinsurer's premiums become

$$P_T(Y)=(1+\eta)\lambda E(Y), \tag{40}$$

$$P_R(H(Y))=(1+\theta)\lambda E(H(Y))=\lambda(1+\theta)\int_M^\infty S_Y(y)dy \tag{41}$$

Using the general parameters defined before in equations (22) and (23), we obtain

$$\begin{aligned}\alpha(M)=\lambda(1+\eta)E(Y)-\lambda(1+\theta)\int_M^\infty S_Y(y)dy-\lambda\int_0^M S_Y(y)dy\\ \Leftrightarrow \alpha(M)=\lambda(\eta E(Y)-\theta\int_M^\infty S_Y(y)dy),\end{aligned} \tag{42}$$

$$\begin{aligned}\sigma^2(M)=\lambda\int_0^M y^2 f_Y(y)dy+M^2\lambda\int_M^\infty f_Y(y)dy\\ \Leftrightarrow \sigma^2(M)=2\lambda\int_0^M yS_Y(y)dy.\end{aligned} \tag{43}$$

So, applying the parameters in equations and considering $\alpha>0$, the ruin probability in infinite time becomes,

$$\Psi(M)=exp\left[\frac{-2\lambda(\eta E(Y)-\theta\int_M^\infty S_Y(y)dy)(X_o-c)}{2\lambda\int_0^M yS_Y(y)dy}\right]. \tag{44}$$

To minimize this probability we can find the roots of its derivatives, i.e., $\Psi'(M)=0$, obtaining

$$2M\alpha(M)-\theta\sigma^2(M)=0. \tag{45}$$

For this case, it was not possible to explicitly obtain an expression for the optimal retention level for the inferior limit M of the excess of loss treaty, minimizing the probability of ruin and using the expected value principle.

Consider now the superior limit L of the excess of loss treaty. Thus, the risk retained by the insurer is $Z(Y)=Y-min(Y,L)$ and the risk transferred to the reinsurer is $H(Y)=min(Y,L)$. For this case, the insurer's and reinsurer's premiums become

$$P_T(Y)=(1+\eta)\lambda E(Y), \tag{46}$$

$$P_R(H(Y))=(1+\theta)\lambda E(H(Y))=(1+\theta)\lambda\int_0^L S_Y(y)dy. \tag{47}$$

Using the general parameters defined in equations (22) and (23), we obtain:

$$\begin{aligned}\alpha(L) = (1+\eta)\lambda E(Y) - (1+\theta)\lambda \int_0^L S_Y(y)dy - \lambda \int_L^\infty S_Y(y)dy \\ \Leftrightarrow \alpha(L) = \lambda(\eta E(Y) - \theta \int_0^L S_Y(y)dy),\end{aligned} \tag{48}$$

$$\sigma^2(L) = \lambda \int_L^\infty (y-L)^2 f_Y(y)dy \Leftrightarrow \sigma^2(L) = 2\lambda \int_L^\infty (y-L)S_Y(y)dy \tag{49}$$

So, using equations (48) and (49), the probability of ruin in infinite time becomes

$$\Psi(L) = exp\left[\frac{-2\lambda(\eta E(Y) - \theta \int_0^L S_Y(y)dy)(X_o - c)}{2\lambda \int_L^\infty (y-L)S_Y(y)dy}\right]. \tag{50}$$

To minimize this probability consider again the roots of its derivative, i.e., $\Psi'(L) = 0$, obtaining

$$\theta S_Y(L)\sigma^2(L) - 2\alpha(L)E(Z(Y)) = 0 \tag{51}$$

As before, it was not possible to explicitly obtain an expression for the optimal retention level for the superior limit L of the excess of loss treaty, minimizing the probability of ruin and using the expected value principle.

Having analyzed the lower limit M and the superior limit L separately, let us now look at the excess of loss treaty more generally, including both inferior and superior limits. The insurer's and reinsurer's premiums become

$$P_T(Y) = (1+\eta)\lambda E(Y), \tag{52}$$

$$P_R(H(Y)) = (1+\theta)\lambda E(H(Y)) = (1+\theta)\lambda \int_M^{M+L} S_Y(y)dy. \tag{53}$$

Using the general parameters defined in equations (22) and (23),from equations (52) and (53) we obtain:

$$\begin{aligned}\alpha(M,L) = (1+\eta)\lambda E(Y) - (1+\theta)\lambda \int_M^{M+L} S_Y(y)dy - \lambda \int_0^M S_Y(y)dy - \lambda \int_{M+L}^\infty S_Y(y)dy \\ \Leftrightarrow \alpha(M,L) = \lambda(\eta E(Y) - \theta \int_M^{M+L} S_Y(y)dy),\end{aligned} \tag{54}$$

$$\begin{aligned}\sigma^2(M,L) = \lambda \int_0^M y^2 f_Y(y)dy + \lambda M^2 \int_M^{M+L} f_Y(y)dy + \lambda \int_{M+L}^\infty (y-L)^2 f_Y(y)dy \\ \Leftrightarrow \sigma^2(M,L) = \lambda(2\int_0^M yS_Y(y)dy + 2\int_{M+L}^\infty (y-L)S_Y(y)dy).\end{aligned} \tag{55}$$

So, using equations (54) and (55), the probability of ruin in infinite time becomes

$$\Psi(M,L) = exp\left[\frac{-2\lambda(\eta E(Y) - \theta\int_M^{M+L} S_Y(y)dy)(X_o - c)}{\lambda(2\int_0^M yS_Y(y)dy + 2\int_{M+L}^{\infty}(y-L)S_Y(y)dy)}\right]. \quad (56)$$

Once more, we minimize this probability by finding the critical points of Ψ:

$$\begin{cases} \frac{\partial\Psi}{\partial M} = 0 \\ \frac{\partial\Psi}{\partial L} = 0 \end{cases} \Leftrightarrow \begin{cases} \theta\sigma^2(M,L) - 2M\alpha(M,L) = 0 \\ (S_Y(M+L) - S_Y(M))\theta\sigma^2(M,L) - 2\alpha(M,L)(MS_Y(M+L)+ \\ \int_{M+L}^{\infty} S_Y(y)dy) = 0. \end{cases} \quad (57)$$

Again, it was not possible to explicitly obtain an expression for the optimal retention level M and L of the excess of loss treaty, minimizing the probability of ruin and using the expected value principle.

5.2.2 Sensitivity to the safety loading

We now aim at analysing the behavior of the optimal reinsurance treaties: we will use a numerical approach so the underlying risk distribution, and the initial surplus (reserve), are given. Consider that each claim amount follows a Gamma distribution and let the parameters of the distribution be $\gamma = 2$ and $\beta = 3$. Let also the initial surplus be equal to 1, i.e. $x = 1$, and the ruin barrier be $c = 0$.

Table 1 contains different values for the insurer's, η, and reinsurer's, θ, safety loading. Note that in the two first cases, the insurer's safety loading is the same, with the reinsurer's safety loading varying, and in last two, the opposite occurs.

θ	η
0.35	0.30
0.40	0.30
0.80	0.60
0.80	0.75

Table 1: Different values for θ and η

1. Quota-share treaty

Equation (39) is analytically solved, and the optimal retained risk b^* is obtained in each case of Table 1. The corresponding value of the ruin probability, $\Psi(b^*)$, is also computed. In Table 2 we can see the optimal retained risk and the respective ruin probability obtained. The graph in Figure 3 shows the behavior of ruin probability of the quota-share treaty for each case in Table 1.

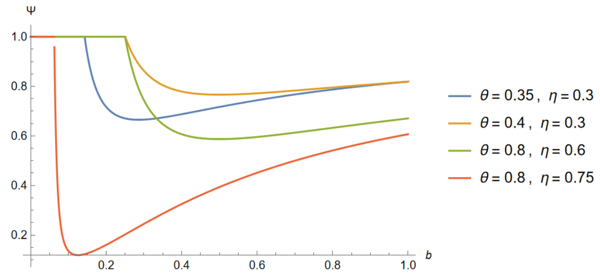

Figure 3: Ruin probability in a QS treaty

2. Excess of loss treaty

Equation (45) is numerically solved in Mathematica, and the optimal retained risk M for the inferior limit M of the excess of loss treaty is obtained in each case of Table 1. The corresponding value of the ruin probability, $\Psi(M)$, is also computed. In Table 2 we can see the optimal retained risk and the respective ruin probability obtained. The graph in Figure 4 shows the behavior of the ruin probability for the inferior limit M of the excess of loss treaty in each case in Table 1.

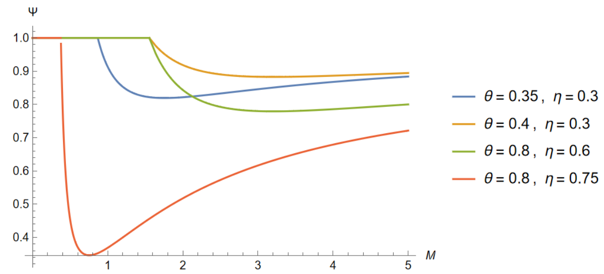

Figure 4: Ruin probability for the inferior limit M in a XL treaty

Equation (51) is numerically solved in Mathematica, and the optimal retained risk L for the inferior limit L of the excess of loss treaty is obtained in each scenario of Table 1. The corresponding value of the ruin probability, $\Psi(L)$, is also computed. In Table 2 we can see the optimal retained risk and the respective ruin probability obtained. The graph in Figure 5 shows the behavior of the ruin probability for the superior limit L of the excess of loss treaty in each case in Table 1.

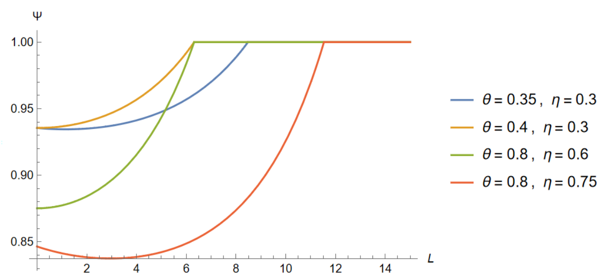

Figure 5: Ruin probability for the superior limit L in a XL treaty

Finally, consider the excess of loss reinsurance treaty in general. Equation (57) is numerically solved in Mathematica, and the optimal retained risk M and L are obtained in each case of Table 1. The corresponding value of the ruin probability, $\Psi(M, L)$, is also computed. In Table 2 we can see the results obtained.

Table 2 summarizes, the values obtained for the four cases and for each type of the reinsurance treaties under study presenting the optimal levels of risk retention and the respective probability of ruin.

Parameters	b	$\Psi(b)$	M	$\Psi(M)$	L	$\Psi(L)$	M	L	$\Psi(M,L)$
$\theta = 0.35$, $\eta = 0.30$	0.29	0.66	1.75	0.82	1.15	0.93	1.75	∞	0.82
$\theta = 0.40$, $\eta = 0.30$	0.50	0.77	3.20	0.88	0	0.94	3.20	∞	0.88
$\theta = 0.80$, $\eta = 0.60$	0.50	0.59	3.20	0.78	0	0.88	3.20	∞	0.78
$\theta = 0.80$, $\eta = 0.75$	0.13	0.12	0.75	0.35	2.96	0.84	0.75	∞	0.35

Table 2: Ruin probability for the optimal treaties for the cases in Table 1

Numerical results

Analyzing the values obtained above, we verify that for the quota-share treaty and for the inferior limit M of the excess of loss, it appears that for higher risk retention levels, from the perspective of the insurer, the probability of ruin is also higher. From an economic point of view, this means that when the insurance company is more exposed to risk, the greater is the probability of having negative results and being at risk of ruin.

In general, for the treaties considered, except for the superior limit L of the XL, and taking into account the cases in which the insurer's safety loading is fixed, it is possible to verify that with a higher reinsurer's safety loading, the higher is the risk retained by the insurance company, i.e., the lower is the ceded risk and, consequently, the greater the ruin probability. This results are in line with expectations, since when the amount to be paid for reinsurance increases, the insurer ends up assigning less risk to the reinsurer.

For the treaties considered with the exception of the superior limit L of the XL and, taking into account the cases in which the reinsurer's safety loading is fixed, it is possible to verify that the greater the safety loading of the insurer, the lower the risk retained by it and, consequently, the lower the ruin probability. This is because with a higher insurer's safety loading, the insurer is charging a higher price to the policyholder. Due to this, the policyholder chooses to transmit less risk to the insurance entity.

Take a closer look to the superior limit L of the XL, more specifically, for the cases where $\theta = 0.4$, $\eta = 0.30$ and for $\theta = 0.8$, $\eta = 0.6$. For these cases, the optimal is the insurer to retain all the risk. In section 5.2.5., we will analyze in detail the reason for this to happen.

Now, comparing the results obtained in the superior limit L of the XL with the ones in the XL in general and, looking for the cases where $\theta = 0.35$, $\eta = 0.30$, and $\theta = 0.80$, $\eta = 0.75$. We verify that exist an optimal value for both cases in the superior limit L. This tell us that the insurer cede the smallest risk to the reinsurer, and then not cede the larger risk. Also for the XL in general, we find that the value of L in the excess of loss treaty is always equal to infinity, which means, that the insurer does not cede the smallest risk to the reinsurer, and cede the higher risk.

Based on the analyzes presented in Table 2, we conclude that the quota-share treaty offers a lower ruin probability to the insurer when compared to the other treaties considered. The results obtained are justified by the lower risk retained with the quota-share treaty, which in turn reduces the probability of the insurance company obtaining negative results.

5.2.3 Sensitivity to the losses distribution

At this point, we intend to analyze the behaviour of the optimal reinsurance: we will use the numerical approach so the insurer's safety loading and reinsurer's safety loading, and the initial surplus, are given. Consider the insurer's safety loading, $\eta = 0.6$, and the reinsurer's safety loading, $\theta = 0.8$ and let the initial surplus of the process equals 1, i.e. $x = 1$, and the barrier be $c = 0$.

Table 5 contains different scenarios, with values for γ and β parameters of the Gamma distribution describing the loss of each claim and the expected value, variance, skewness and kurtosis of the underlying distribution.

Scenarios	γ	β	Expected Value	Variance	Skewness	Kurtosis
A	0.5	2	1	2	2.83	15
B	1	2	2	4	2	9
C	2	2	4	8	1.41	6
D	4	2	8	16	1	4.5
E	6	2	12	24	0.82	4
F	2	4	8	32	1.41	6
G	2	6	12	72	1.41	6
H	2	10	20	200	1.41	6

Table 3: Different scenarios for the gamma parameters γ and β

Note that scenario A, B, C, D and E have the same value for the scale parameter, β. And scenarios F, G and H take the same values for the shape parameter, γ.

1. *Quota-share treaty*

Equation (39) is analytically solved, and the optimal retained risk b^* is obtained in each scenario of Table 3. The corresponding value of the ruin probability, $\Psi(b^*)$, is also computed. In Table 4, we can see the optimal retained risk and the respective ruin probability obtained.

Figure 6 shows the behaviour of the ruin probability in a quota-share treaty taking into account the scenarios where the scale parameter is fixed. Note that for these scenarios, the optimal level of retained risk, b, and the respective ruin probability, $\Psi(b)$ is always the same. And consequently, the graph is the same for all scenarios A, B ,C , D and E. This is because we are working with the Gamma distribution, and the quotient between the expected value of Y and the variance, results in the relation of $\frac{1}{\beta}$, that is, when we only vary the shape parameter, the graph does not change. Then, in Figure 7 we also see the behaviour of the ruin probability in a quota-share treaty, now taking into account the scenarios where the shape parameter is fixed. For the scenarios considered, the optimal retained risk b is the same.

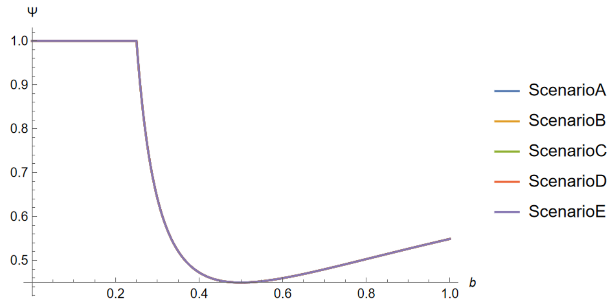

Figure 6: Ruin probability in a QS treaty when the scale parameter is fixed

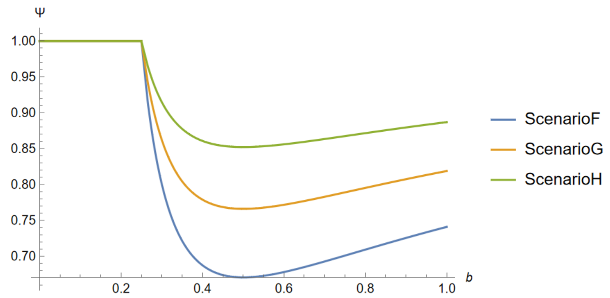

Figure 7: Ruin probability in a QS treaty when the shape parameter is fixed

2. Excess of loss treaty

Equation (45) is numerically solved in Mathematica, and the optimal retained risk M for the inferior limit M in the excess of loss treaty is obtained in each scenario of Table 3. The corresponding value of the ruin probability, $\Psi(M)$, is also computed. We can see the results obtained in Table 4.

In Figure 8 we analyse the behaviour of the ruin probability for the inferior limit M in an excess of loss treaty considering the scenarios with scale parameter fixed. After that, in Figure 9 we study the behaviour of the ruin probability for the inferior limit M in an excess of loss treaty considering the shape parameter fixed.

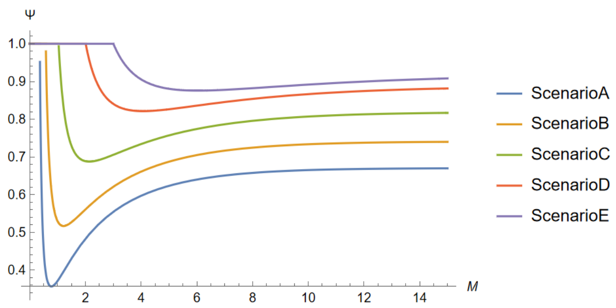

Figure 8: Ruin probability for the inferior limit M in a XL with the scale parameter fixed

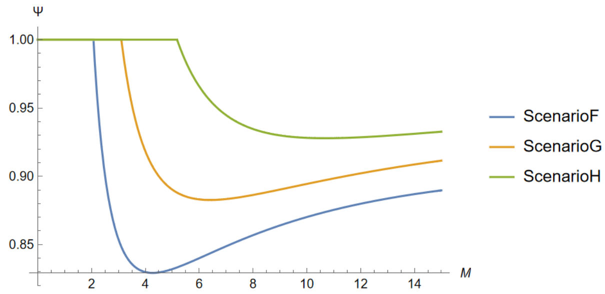

Figure 9: Ruin probability for the inferior limit M in a XL with the shape parameter fixed

Look now to the superior limit L of the excess of loss reinsurance treaty. Equation (51) is numerically solved in Mathematica, and the optimal retained risk L is obtained in each scenario of Table 3. The corresponding value of the ruin probability, $\Psi(L)$, is also computed. In Table 2 we can see the optimal retained risk and the respective ruin probability obtained. In Table 4 We can see the obtained results.

Figure 10 shows the behaviour of the ruin probability for the superior limit L in a excess of loss treaty considering the scale parameter fixed. Then, in Figure 11 we have the ruin probability also for the superior limit L in a excess of loss treaty when the shape parameter is fixed.

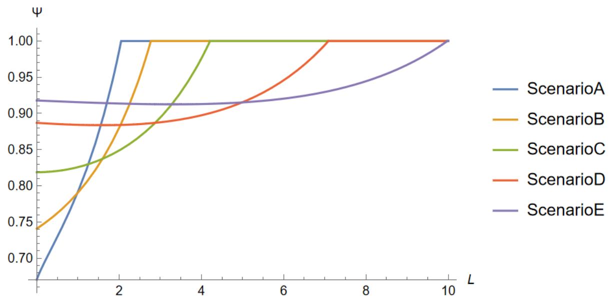

Figure 10: Ruin probability for the superior limit L in a XL with the scale parameter fixed

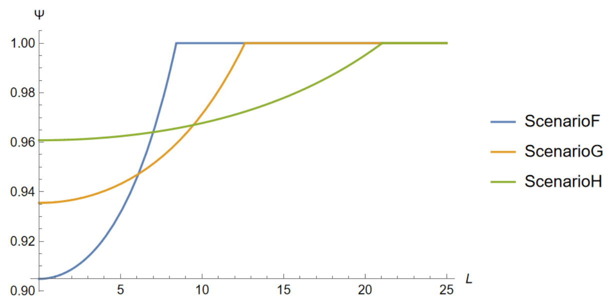

Figure 11: Ruin probability for the superior limit L in a XL with the shape parameter fixed

Finally, consider the excess of loss reinsurance treaty in general. Equation (57) is numerically solved in Mathematica, and the optimal retained risk M and L are obtained in each scenario of Table 3. The corresponding value of the ruin probability, $\Psi(M, L)$, is also computed. In Table 4 we can see the results obtained.

Table 4 summarizes the values obtained for the optimal level of retention of risk and the respective ruin probability for each optimal reinsurance treaty.

Scenario	b	$\Psi(b)$	M	$\Psi(M)$	L	$\Psi(L)$	M	L	$\Psi(M,L)$
A	0.50	0.45	0.78	0.36	0	0.67	0.78	∞	0.36
B	0.50	0.45	1.21	0.52	0	0.74	1.21	∞	0.52
C	0.50	0.45	2.14	0.69	0	0.82	2.14	∞	0.69
D	0.50	0.45	4.06	0.82	1.59	0.88	4.06	∞	0.82
E	0.50	0.45	6.03	0.88	3.33	0.91	6.03	∞	0.88
F	0.50	0.67	4.27	0.83	0	0.90	4.27	∞	0.83
G	0.50	0.77	6.41	0.88	0	0.94	6.41	∞	0.88
H	0.50	0.85	10.68	0.93	0	0.96	10.68	∞	0.93

Table 4: Ruin probability for each scenario and for the different reinsurance treaties

Numerical results

In a first analysis, the scale parameter is fixed and the shape parameter varies. Thus, for the quota-share treaty, the optimal level of risk retention, b, obtained is equal in all considered scenarios, which allows us to conclude that, for this treaty, the retention level does not depend on the distribution parameters. Additionally, we verified that the probability of ruin is also the same in all the respective scenarios. This is something explained by the moments of the Gamma distribution, since $E(Y) = \gamma\beta$ and $E(Y^2) = \gamma\beta^2$, which means that $\Psi(b) = exp\left[\frac{-2(b\theta-(\theta-\eta))(X_0-c)}{b^2\beta}\right]$, i.e., the expression of ruin probability only depends on the scale parameter. In the same analysis, but now for the inferior limit M in the XL, it appears that the retained risk by the insurer increases with increasing the shape parameter. Consequently, the probability of ruin increases, that is, as the risk shifts to the right and it is more evenly distributed along the tails of the distribution, the higher the level of risk retained by the insurer.

Now, the shape parameter is fixed and the scale parameter varies. For the case of the quota-share treaty, the optimal risk retention level remains the same with different values for the scale parameter, however, with the increase of the scale parameter, the probability of ruin also increases. Generally, for the inferior limit M in the XL, we notice that with an increasing of the scale parameter, the retained risk is higher and, consequently, the ruin probability also increases. As the scale parameter increases, so does the risk distribution in the tails, and then the retained risk is also higher.

Now, look at the superior limit L of the XL treaty. In general, we see that the optimal, for the insurer, is to retain all the risk. Possibly, because the probability of having small claims is high and, in this case, it may not make sense for the insurer to reinsure them. Further, in section 5.2.5., we will analyze this topic in more detail.

Once more, we conclude that the quota-share treaty offers a lower ruin probability compared to the other treaties considered. The results obtained are justified by the lower risk retained with the quota-share treaty, which in turn reduces the probability of the insurance company obtaining negative results.

5.2.4 Sensitivity to the initial surplus

We now aim at analysing the behaviour of the optimal reinsurance: we will use a numerical approach so the underlying risk and, the both insurer's and reinsurer's safety loading, are given. Consider that each claim amount follows a Gamma distribution with parameters $\gamma = 2$ and $\beta = 3$. Let also the insurer's safety loading, $\eta = 0.6$, and reinsurer's safety loading, $\theta = 0.8$, and the ruin barrier be, $c = 0$.

Table 5 shows the optimal level of retention for both quota-share and excess of loss (for inferior limit M, superior limit L and the treaty in general) treaties and, the respective probability of ruin, considering different values for the initial surplus of the process.

x	b	$\Psi(b)$	M	$\Psi(M)$	L	$\Psi(L)$	M	L	$\Psi(M, L)$
1	0.50	0.59	3.20	0.78	0	0.88	3.20	∞	0.78
5	0.50	0.07	3.20	0.29	0	0.51	3.20	∞	0.29
10	0.50	0.004	3.20	0.08	0	0.26	3.20	∞	0.08

Table 5: Initial surplus, x, for the different reinsurance treaties

Numerical results

As expected, the results indicate that the probability of ruin decreases when the initial reserve level is higher. With regard to the optimal level of risk retention, changing the initial reserve level of the process has no impact on any of the treaties considered, the impact being only on the value of the ruin probability.

5.2.5 Understanding the particular case of superior limit L: numerical examples

In the previous section, we found that, in most cases, it is optimal to retain all the risk ($L = 0$) or to cede all risk above a certain limit M. Thus, in this section we aim to analyse the behaviour of the optimal retention levels M and L in the XL treaty.

In Table 6, we present the parameters for three scenarios, where the underlying risk distribution and the insurer's safety loading, η, and the reinsurer's safety loading, θ, are given. Consider the insurer's safety loading, $\eta = 0.6$ and, reinsurer's safety loading, $\theta = 0.8$.

Scenario	γ	β	Expected Value	Variance	Skewness	Kurtosis
1	3	20	60	1 200	1.15	5
2	0.3	200	60	12 000	3.65	23
3	20	3	60	180	0.45	3.3

Table 6: Parameters for the scenarios 1,2 and 3

In Table 7, we present the numerical results for scenario 1, in Table 8 the numerical results for scenario 2 and, in Table 9 for scenario 3.

Treaty	Expected value	Variance	Skewness	Kurtosis	M	L	Ψ
Limit L	60	1200	1.15	5	-	7.65	0.985
XL in general	60	1200	1.15	5	30.92	652.99	0.974

Table 7: Numerical results for scenario 1

Treaty	Expected value	Variance	Skewness	Kurtosis	M	L	Ψ
Limit L	60	12 000	3.65	23	-	0	0.995
XL in general	60	12 000	3.65	23	0	0.08	0.995

Table 8: Numerical results for scenario 2

Treaty	Expected value	Variance	Skewness	Kurtosis	M	L	$\Psi(L)$
Limit L	60	180	0.45	3.3	-	24.89	0.977
XL in general	60	180	0.45	3.3	30.00	172.33	0.974

Table 9: Numerical results for scenario 3

Numerical results

In the previous sections we found that the optimal value for the superior limit L of the excess of loss treaty was equal to 0, in most cases, which means that the insurance company does not transfer any risk to the reinsurer, instead retains all the risk for itself. In this section, we intend to understand what are the measures that better explain the behaviour of the limit L.

We consider three scenarios in which the expected value of the underlying risk is the same, but the optimal L is quite different for each scenario. We note that the expected value does no consider the fatness of the tails properly. Thus, considering the same expected value of the underlying risk for scenarios 1, 2 and 3, we intend to calculate the moments of higher order, to conclude what is the impact of these measures on optimal reinsurance treaty.

Taking into account the numerical results obtained in scenario 1 and 2, we realize that for a higher kurtosis, measure for the weight of the tails of the distribution, the weight of the tails is very high, so the optimal thing, for the insurance company, would be to retain all the risk, i.e, not transfer any risk to the reinsurer. Look now to the skewness measure, which gives us information about the location of the weight of the tails, we notice that for a more positive value of this indicator, the weight of the tails is closer to zero. In other words, there is a high probability that the insurance company will have very small losses, so it makes no sense to reinsure losses that are expected to be so small.

Now consider scenario 3, with the same expected value for the underlying risk and with a identical value for the weight of the tails, when comparing the value in scenario 2. However, with a value for the asymmetry indicator smaller and less than 1. Based on the numerical results obtained, we can conclude that for an identical value for the weight of the tails, but with a less positive value for the skewness, it is possible to obtain an optimal solution for the limit L. This means that with these parameters, a insurance company has a high probability of having large claims, and therefore, it makes perfect sense to transfer part of its risk to the reinsurer.

5.3 Optimal reinsurance strategy: maximizing the expected value

After analyzing the optimal reinsurance that minimizes the ruin probability, and having obtained the ruin probability in the several scenarios, the objective now is to analyze, among the optimal solutions obtained, which one maximizes the expected value of the wealth process, i.e., the drift of the diffusion.

Table 10 represents the expected value of the surplus process defined by equation (36) for the quota-share treaty, in equation (42) for the inferior limit M of the excess of loss, by equation (48) for the superior limit L of the excess of loss and by equation (54) for the excess of loss treaty in general.

Parameters	b	$\alpha(b)$	M	$\alpha(M)$	L	$\alpha(L)$	M	L	$\alpha(M,L)$
$\theta = 0.35$, $\eta = 0.30$	0.29	0.30	1.75	0.30	1.15	1.41	1.75	∞	0.30
$\theta = 0.40$, $\eta = 0.30$	0.50	0.60	3.20	0.53	0.00	1.80	3.20	∞	0.53
$\theta = 0.80$, $\eta = 0.60$	0.50	1.20	3.20	1.07	0.00	3.60	3.20	∞	1.07
$\theta = 0.80$, $\eta = 0.75$	1.25	0.30	0.75	0.30	2.96	2.37	0.75	∞	0.30

Table 10: Expected value, α, of the surplus process

Numerical results

As noted earlier, the drift of the surplus process represents the amount effectively received by the insurer. Therefore, the greater the value of the premiums received by the insurer, the greater the expected value of the surplus process. Looking at the results obtained in Table 10, we verify that the superior limit L for the excess of loss treaty maximizes the expected value of the process. The results obtained are in line with expectations, as the insurer is not ceding risk to the reinsurer and, consequently, not paying for it. In this way, the less risk the insurer cedes, the lower the amount that need to paid to the reinsurer, and therefore, the greater the drift of the surplus process.

5.4 Optimal reinsurance strategy: minimizing the variance

Now, we repeat the previous analysis but looking at the optimal treaties that minimize the variance of the retained risk.

In Table 11, we compute the variance of the surplus process when we consider the optimal retention levels. The expression of the variance can be obtained in equations (37), (43), (49) and (55) when one fixes respectively, the quota-share, the inferior limit M of the excess of loss, the superior limit L of the excess of loss and, the excess of loss in general.

Parameters	b	$\sigma^2(b)$	M	$\sigma^2(M)$	L	$\sigma^2(L)$	M	L	$\sigma^2(M,L)$
$\theta = 0.35$, $\eta = 0.30$	0.29	1.47	1.75	2.88	1.15	41.49	1.75	∞	2.88
$\theta = 0.40$, $\eta = 0.30$	0.50	4.50	3.20	8.56	0.00	54.00	3.20	∞	8.56
$\theta = 0.80$, $\eta = 0.60$	0.50	4.50	3.20	8.56	0.00	54.00	3.20	∞	8.56
$\theta = 0.80$, $\eta = 0.75$	1.25	0.28	0.75	0.56	2.96	26.77	0.75	∞	0.56

Table 11: Variance, σ^2, of the surplus process

Numerical results

The volatility parameter of the surplus process coincide with the standard variation of the aggregated claim amount. Therefore, the greater the value of the variance of the process, more exposed is the insurance company to risk. Looking at the results obtained in Table 11, we verify that, generally, the quota-share treaty minimizes the variance of the retained risk. In this case, the insurer cedes more risk to the reinsurer and, therefore, the variance of the surplus process is smaller.

Conversely, the superior limit L of the XL treaty has a higher value for the variance. This happens because the level of retained risk is higher and then the volatility of the process is also higher.

6 Conclusions

The main objectives proposed at the beginning of this master's final work were to determine the optimal level of retention and to understand which forms of reinsurance optimize the surplus process from the perspective of the insurance company. For the realization of these fundamental points, three criteria were presented throughout this work: (i) the minimization of the ruin probability, (ii) the maximization of the expected value of the surplus process, and (iii) the minimization of the variance of the surplus process.

Through a numerical and sensitivity analysis and using the Mathematica software, the optimal reinsurance strategy was studied. The main numerical results obtained, in general, were that the quota-share is the treaty that minimizes the ruin probability when compared to the excess of loss treaty in general and its inferior and superior limits, M and L respectively.

In line with what would be expected, we verified that the ruin probability decreases with the increase of the initial surplus (reserve) level.

Interestingly, for the initially defined parameters, the superior limit L of the XL treaty, contrary to the other treaties, had an optimal value equal to zero, which means that the optimal strategy would always be to retain all the risk by the insurance company. This behaviour was carefully analysed. We have concluded that L is no longer zero when distributions combine a large kurtosis and an almost negative skewness.

We noticed that for a larger kurtosis, the weight of the tails is very large, and with a more positive skewness this weight is closer to zero, so the probability of having very large losses is small. For this case, we conclude that the optimal, for the insurer, is to retain risk. Therefore, assuming identical kurtosis and smaller values for the skewness, we verified that the probability of having large claims increases, and for this case, it is optimal to reinsure part of the risk.

Based on the optimal values of risk obtained with the criterion of minimization of ruin probability for the different treaties considered, it appears that the superior limit L of XL treaty presents an expected value of wealth greater than the others. We also conclude that the quota-share treaty offers a lower variance (risk) for the surplus process.

In this way, it is concluded that the objectives initially defined were fulfilled.

References

[1] M. Brachetta and C. Ceci. Optimal excess-of-loss reinsurance for stochastic factor risk models. *Risks*, 2019.

[2] S. Browne. Optimal investment policies for a firm with a random risk process: exponential utility and minimizing the probability of ruin. *Mathematics of operations research*, 20(4), 1995.

[3] K. Burnecki, P. Mista, and A. Weron. Ruin probabilities in finite and infinite time. *Statistical Tools for Finance and Insurance. Springer,Berlin, Heidelberg*, pages 341–379, 2005.

[4] J. Cai and K. S. Tan. Optimal retention for a stop-loss reinsurance under the var and cte risk measures. *Astin Bulletin*, 37:93–112, 2007.

[5] A. Cani and S. Thonhauser. An optimal reinsurance problem in the cramér–lundberg model. *Springer*, 85:179–205, 2016.

[6] L. Centeno. On combining quota-share and excess of loss. *Austin Bulletin*, 15(1):49–63, 1985.

[7] M. L. Centeno. *Teoria do Risco na Actividade Seguradora.* Celta Editora, 2003.

[8] D. C. Dickson and H. R. Waters. Optimal dynamic reinsurance. *Astin Bulletin*, 36:415–432, 2006.

[9] D. C.M. Dickson and H. R. Waters. Reinsurance and ruin. *Insurance: Mathematics and Economics*, 19:61–80, 1996.

[10] D. S. Dimitrova and V. K. Kaishev. Optimal joint survival reinsurance: An efficient frontier approach. *Insurance: Mathematics and Economics*, 47:27–35, 2010.

[11] A. Dixit. *The art of smooth pasting.* Harwood Academic Publichers, 1993.

[12] A. Y. Golubin. Optimal insurance and reinsurance policies in the risk process. *Astin Bulletin*, 38:383–397, 2008.

[13] C. Hipp and M. Vogt. Optimal dynamic xl reinsurance. *Astin Bulletin: The Journal of the IAA*, 33:193–207, 2003.

[14] X. Hu, H. Yang, and L. Zhang. Optimal retention for a stop-loss reinsurance with incomplete information. *Insurance: Mathematics and Economics*, 65:15–21, 2015.

[15] J. E. Ingersoll. *Theory of Financial Decision Making.* Rowman and Littlefield Publishers, 1987.

[16] D. Li and C. Shen. Optimal reinsurance strategy for an insurer and a reinsurer with generalized variance premium principle. *Mathematical Problems in Engineering*, 3:1–14, 2020.

[17] X. Liang, Z. Liang, and V. R. Young. Optimal reinsurance under the mean-variance premium principle to minimize the probability of ruin. *Insurance: Mathematics and Economics*, 92:128–146, 2020.

[18] Z. Liang and K. C. Yuen. Optimal dynamic reinsurance with dependent risks: variance premium principle. *Scandinavian Actuarial Journal*, pages 18–36, 2016.

[19] M. Liao. *Applied stochastic processes*, volume 1. CRC Press, 2013.

[20] C. S. Liu and H. Yang. Optimal investment for an insurer to minimize its probability of ruin. *North American Actuarial Journal*, 8:11–31, 2004.

[21] H. Mao, J. M. Carson, K. M. Ostaszewski, Y. Luo, and Y. Wang. Determining the insurer's optimal investment and reinsurance strategy based on stochastic differential game. *Journal of Insurance Issues*, 39(2):187–202, 2016.

[22] H. Menga, P. Liao, and T. K. Siu. Continuous-time optimal reinsurance strategy with nontrivial curved structures. *Applied Mathematics and Computation*, 363, 2019.

[23] H. Schmidli. Optimal proportional reinsurance policies in a dynamic setting. *Scandinavian Actuarial Journal*, 1:55–68, 2001.

[24] H. Schmidli. On minimizing the ruin probability by investment and reinsurance. *The Annals of Applied Probability*, 12(3):890–907, 2002.

[25] H. Schmidli. *Stochastic Control in Insurance*. Springer, 2008.

[26] M. I. Taksar and C. Markussen. Optimal dynamic reinsurance policies for large insurance portfolios. *Finance Stochastic*, pages 97–121, 2003.

[27] X. Zhang and T. K. Siu. Optimal investment and reinsurance of an insurer with model uncertainty. *Insurance: Mathematics and Economics*, 45:81–88, 2009.

[28] X. Zhang, M. Zhou, and J. Guo. Optimal combinational quota-share and excess-of-loss reinsurance policies in a dynamic setting. *Applied stochastic models in business and industry*, 23:63–71, 2007.

[29] M. Zhou, H. Dong, and J. Xu. Optimal combinational of quota-share and stop-loss reinsurance contracts under var and cte with a constrained reinsurance premium. *Journal of Systems Science and Complexity*, 24:156–166, 2011.

Nomenclature

BM - Brownian motion
CVaR - Conditional Value at Risk
HJB equation - Hamilton-Jacobi-Bellman equation
HJBI equation – Hamilton-Jacobi-Bellman-Isaacs equation
SDE - Stochastic Differential equation
QS - Quota-share
VaR - Value at risk
XL - Excess of loss

Appendix

Appendix I - Optimization of ruin probability

1. Quota-Share treaty

$$\begin{cases} Z(S) = bS \\ H(S) = (1-b)S \end{cases} \tag{58}$$

$$P_T(S) = (1+\eta)\lambda E(Y) \tag{59}$$

$$P_R(H(S)) = (1+\theta)E(H(S)) = (1+\theta)\lambda E(Y)(1-b) \tag{60}$$

$$E(H(S)) = E((1-b)S) = (1-b)\lambda E(Y) \tag{61}$$

$$E(Z(S)) = E(bS) = b\lambda E(Y) \tag{62}$$

$$\begin{aligned} \alpha = (1+\eta)\lambda E(Y) - (1+\theta)\lambda(1-b)E(Y) - b\lambda E(Y) \\ = \lambda E(Y)(1+\eta-(1-b+\theta-b\theta)-b) \\ = \lambda E(Y)(\eta-\theta+b\theta) \end{aligned} \tag{63}$$

$$\begin{aligned} \sigma^2 = Var(Z(S)) = \\ = Var(bS) = b^2 Var(S) = b^2\lambda E(Y^2) \end{aligned} \tag{64}$$

$$\begin{aligned} \Psi'(b) = \frac{-\theta b^2 Var(S) + 2(\eta-\theta+b\theta)bVar(S)}{(b^2Var(S))^2} \\ = \frac{-\theta b^2 Var(S) + 2\eta bVar(S) - 2b\theta Var(S)}{b^4 Var(S)^2} \\ = \frac{-\theta b + 2\eta - 2\theta}{b^3 Var(S)} \end{aligned} \tag{65}$$

$$\begin{aligned} \Psi'(b) = 0 \\ \Leftrightarrow -\theta b + 2\eta - 2\theta = 0 \\ \Leftrightarrow b = \frac{2\eta - 2\theta}{\theta} \end{aligned} \tag{66}$$

2. Inferior limit M for excess of loss treaty

$$Z(Y) = \begin{cases} Y & \text{if } Y \leq M \\ M & \text{if } Y > M \end{cases} \tag{67}$$

$$H(Y) = \begin{cases} 0 & \text{if } Y \leq M \\ Y - M & \text{if } Y > M \end{cases} \tag{68}$$

$$P_T(Y) = (1+\eta)\lambda E(Y) \tag{69}$$

$$P_R(H(Y)) = (1+\theta)\lambda E(H(Y)) = (1+\theta)\lambda \int_M^\infty S_Y(y)dy \tag{70}$$

$$\begin{aligned} E(H(Y)) = \lambda \int_M^\infty (y-M) f_Y(y)dy \\ = \lambda \left[-(y-M)S_Y(y)\right]_M^\infty + \lambda \int_M^\infty S_Y(y)dy \\ = \lambda \int_M^\infty S_Y(y)dy \end{aligned} \tag{71}$$

$$\begin{aligned} E(Z(Y)) = \lambda \int_0^M y f_Y(y)dy + \lambda M \int_M^\infty f_Y(y)dy \\ = \lambda \left[-yS_Y(y)\right]_0^M + \lambda \int_0^M S_Y(y)dy - \lambda \left[MS_Y(y)\right]_M^\infty \\ = -\lambda M S_Y(M) + \lambda \int_0^M S_Y(y)dy + M\lambda S_Y(M) \\ = \lambda \int_0^M S_Y(y)dy \end{aligned} \tag{72}$$

$$\begin{aligned} \alpha(M) = (1+\eta)\lambda E(Y) - (1+\theta)\lambda \int_M^\infty S_Y(y)dy - \lambda \int_0^M S_Y(y)dy \\ = (1+\eta)\lambda E(Y) - \lambda \int_M^\infty S_Y(y)dy - \lambda\theta \int_M^\infty S_Y(y)dy - \lambda \int_0^M S_Y(y)dy \\ = (1+\eta)\lambda E(Y) - \lambda\theta \int_M^\infty S_Y(y)dy - \lambda E(Y) \\ = \lambda E(Y) + \lambda\eta E(Y) - \lambda\theta \int_M^\infty S_Y(y)dy - \lambda E(Y) = \lambda(\eta E(Y) - \theta \int_M^\infty S_Y(y)dy) \end{aligned} \tag{73}$$

$$
\begin{aligned}
\sigma^2(M) &= \lambda E(Z(Y)^2) \\
&= \lambda \int_0^M y^2 f_Y(y)dy + \lambda M^2 \int_M^\infty f_Y(y)dy \\
&= -\lambda M^2 S_Y(M) + 2\lambda \int_0^M y S_Y(y)dy + \lambda M^2 S_Y(M) \\
&= 2\lambda \int_0^M y S_Y(y)dy
\end{aligned} \tag{74}
$$

$$
\Psi(M) = exp\left[\frac{-2\lambda(\eta E(Y) - \theta\int_M^\infty S_Y(y)dy)(X_o - c)}{2\lambda\int_0^M yS_Y(y)dy}\right] \tag{75}
$$

$$
\begin{aligned}
\Psi'(M) &= 0 \\
\Leftrightarrow -2\theta S_Y(M)\int_0^M yS_Y(y)dy + 2MS_Y(M)(\eta E(Y) - \theta\int_M^\infty S_Y(y)dy) &= 0 \\
\Leftrightarrow -2\theta\int_0^M yS_Y(y)dy + 2M(\eta E(Y) - \theta\int_M^\infty S_Y(y)dy) &= 0 \\
\Leftrightarrow 2M\alpha(M) - \theta\sigma^2(M) &= 0
\end{aligned} \tag{76}
$$

3. Superior limit L for excess of loss treaty

$$
Z(Y) = \begin{cases} 0 & \text{if } Y \leq L \\ Y - L & \text{if } Y > L \end{cases} \tag{77}
$$

$$
H(Y) = \begin{cases} Y & \text{if } Y \leq L \\ L & \text{if } Y > L \end{cases} \tag{78}
$$

$$
P_T(Y) = (1+\eta)\lambda E(Y) \tag{79}
$$

$$
P_R(H(Y)) = (1+\theta)\lambda E(H(Y)) = (1+\theta)\lambda\int_0^L S_Y(y)dy \tag{80}
$$

$$
\begin{aligned}
E(H(Y)) &= \lambda\int_0^L yf_Y(y)dy + \lambda\int_L^\infty Lf_Y(y)dy \\
&= \lambda\left[-yS_Y(y)\right]_0^L + \lambda LS_Y(L) + \lambda\int_0^L S_Y(y)dy \\
&= -\lambda LS_Y(L) + \lambda LS_Y(L) + \lambda\int_0^L S_Y(y)dy \\
&= \lambda\int_0^L S_Y(y)dy
\end{aligned} \tag{81}
$$

$$
\begin{aligned}
E(Z(Y)) &= \lambda \int_L^{\infty} (y-L) f_Y(y) dy \\
&= \lambda \left[-(y-L)S_Y(y)\right]_L^{\infty} + \lambda \int_L^{\infty} S_Y(y)dy \\
&= \lambda \int_L^{\infty} S_Y(y)dy
\end{aligned}
\tag{82}
$$

$$
\begin{aligned}
\alpha(L) &= (1+\eta)\lambda E(Y) - (1+\theta)\lambda \int_0^L S_Y(y)dy - \lambda \int_L^{\infty} S_Y(y)dy \\
&= (1+\eta)\lambda E(Y) - \lambda \int_0^L S_Y(y)dy - \lambda\theta \int_0^L S_Y(y)dy - \lambda \int_L^{\infty} S_Y(y)dy \\
&= (1+\eta)\lambda E(Y) - \lambda\theta \int_0^L S_Y(y)dy - \lambda E(Y) \\
&= \lambda(\eta E(Y) - \theta \int_0^L S_Y(y)dy)
\end{aligned}
\tag{83}
$$

$$
\begin{aligned}
\sigma^2(L) &= \lambda \int_L^{\infty} (y-L)^2 f_Y(y) dy \\
&= \lambda \left[-(y-L)^2 S_Y(y)\right]_L^{\infty} + \lambda \int_L^{\infty} 2(y-L)S_Y(y)dy \\
&= 2\lambda \int_L^{\infty} (y-L)S_Y(y)dy
\end{aligned}
\tag{84}
$$

$$
\Psi(L) = exp\left[\frac{-2\lambda(\eta E(Y) - \theta \int_0^L S_Y(y)dy)(X_o - c)}{2\lambda \int_L^{\infty}(y-L)S_Y(y)dy}\right]
\tag{85}
$$

$$
\begin{aligned}
&\Psi'(L) = 0 \\
\Leftrightarrow\ & \theta S_Y(L) \int_L^{\infty} (y-L)S_Y(y)dy - \alpha(L) \int_L^{\infty} S_Y(y)dy = 0 \\
\Leftrightarrow\ & 2\theta S_Y(L) \int_L^{\infty} (y-L)S_Y(y)dy - 2\alpha(L) \int_L^{\infty} S_Y(y)dy = 0 \\
\Leftrightarrow\ & \theta S_Y(L)\sigma^2(L) - 2\alpha(L) \int_L^{\infty} S_Y(y)dy = 0 \\
\Leftrightarrow\ & \theta S_Y(L)\sigma^2(L) - 2\alpha(L)E(Z(Y)) = 0
\end{aligned}
\tag{86}
$$

4. Excess of loss treaty

$$
Z(Y) = \begin{cases} Y & \text{if } Y \le M \\ M & \text{if } M < Y \le M+L \\ Y-L & \text{if } Y > M+L \end{cases}
\tag{87}
$$

$$H(Y) = \begin{cases} 0 & \text{if } Y \le M \\ Y - M & \text{if } M < Y \le M + L \\ L & \text{if } Y > M + L \end{cases} \tag{88}$$

$$P_T(Y) = (1+\eta)\lambda E(Y) \tag{89}$$

$$P_R(H(Y)) = (1+\theta)\lambda E(H(Y)) = (1+\theta)\lambda \int_M^{M+L} S_Y(y)dy \tag{90}$$

$$\begin{aligned} E(H(Y)) = \lambda \int_M^{M+L} (y-M) f_Y(y)dy + \lambda \int_{M+L}^{\infty} L f_Y(y)dy \\ = \lambda \left[-(y-M)S_Y(y)\right]_M^{M+L} + \lambda \int_M^{M+L} S_Y(y)dy + L\lambda S_Y(M+L) \\ = -L\lambda S_Y(M+L) + L\lambda S_Y(M+L) + \lambda \int_M^{M+L} S_Y(y)dy \\ = \lambda \int_M^{M+L} S_Y(y)dy \end{aligned} \tag{91}$$

$$\begin{aligned} E(Z(Y)) = \lambda \int_0^M y f_Y(y)dy + \lambda \int_M^{M+L} M f_Y(y)dy + \lambda \int_{M+L}^{\infty} (y-L) f_Y(y)dy \\ = \lambda \left[-yS_Y(y)\right]_0^M + \lambda \int_0^M S_Y(y)dy - \lambda \left[-MS_Y(y)\right]_M^{M+L} - \lambda \left[(y-L)S_Y(y)\right]_{M+L}^{\infty} \\ = \lambda (\int_0^M S_Y(y)dy + \int_{M+L}^{\infty} S_Y(y)dy) \end{aligned} \tag{92}$$

$$\begin{aligned} \alpha(M,L) = (1+\eta)\lambda E(Y) - (1+\theta)\lambda \int_M^{M+L} S_Y(y)dy - \lambda \int_0^M S_Y(y)dy - \lambda \int_{M+L}^{\infty} S_Y(y)dy \\ = (1+\eta)\lambda E(Y) - \lambda \int_M^{M+L} S_Y(y)dy - \lambda\theta \int_M^{M+L} S_Y(y)dy - \lambda \int_0^M S_Y(y)dy - \lambda \int_{M+L}^{\infty} S_Y(y)dy \\ = (1+\eta)\lambda E(Y) - \lambda\theta \int_M^{M+L} S_Y(y)dy - \lambda E(Y) \\ = \lambda(\eta E(Y) - \theta \int_M^{M+L} S_Y(y)dy) \end{aligned} \tag{93}$$

$$
\begin{aligned}
\sigma^2(M,L) &= \lambda\int_0^M y^2 f_Y(y)dy + \lambda M^2\int_M^{M+L} f_Y(y)dy + \lambda\int_{M+L}^{\infty}(y-L)^2 f_Y(y)dy \\
&= \lambda\left[-y^2 S_Y(y)\right]_0^M + 2\lambda\int_0^M yS_Y(y)dy - \lambda\left[M^2 S_Y(y)\right]_M^{M+L} \\
&\quad -\lambda\left[-(y-L)^2 S_Y(y)\right]_{M+L}^{\infty} + 2\lambda\int_{M+L}^{\infty}(y-L)S_Y(y)dy \\
&= \lambda(2\int_0^M yS_Y(y)dy + 2\int_{M+L}^{\infty}(y-L)S_Y(y)dy)
\end{aligned} \tag{94}
$$

$$
\Psi(M,L) = exp\left[\frac{-\lambda(\eta E(Y) - \theta\int_M^{M+L} S_Y(y)dy)(X_o - c)}{\lambda(\int_0^M yS_Y(y)dy + \int_{M+L}^{\infty}(y-L)S_Y(y)dy)}\right] \tag{95}
$$

$$
\begin{aligned}
&\frac{\partial\Psi}{\partial M} = 0 \\
&\Leftrightarrow 2\theta(S_Y(M+L) - S_Y(M))(\int_0^M yS_Y(y)dy + \int_{M+L}^{\infty}(y-L)S_Y(y)dy) \\
&\quad -2M(\eta E(Y) - \theta\int_M^{M+L} S_Y(y)dy)(S_Y(M+L) - S_Y(M)) = 0 \\
&\Leftrightarrow \theta\sigma^2(M,L) - 2M\alpha(M,L) = 0
\end{aligned} \tag{96}
$$

$$
\begin{aligned}
&\frac{\partial\Psi}{\partial L} = 0 \\
&\Leftrightarrow 2\theta(S_Y(M+L) - S_Y(M))(\int_0^M yS_Y(y)dy + \int_{M+L}^{\infty}(y-L)S_Y(y)dy) \\
&\quad -2(\eta E(Y) - \theta\int_M^{M+L} S_Y(y)dy)(MS_Y(M+L) + \int_{M+L}^{\infty} S_Y(y)dy) = 0 \\
&\Leftrightarrow (S_Y(M+L) - S_Y(M))\theta\sigma^2(M,L) - 2\alpha(M,L)(MS_Y(M+L) + \int_{M+L}^{\infty} S_Y(y)dy) = 0
\end{aligned} \tag{97}
$$

$$
\begin{cases} \frac{\partial\Psi}{\partial M} = 0 \\ \frac{\partial\Psi}{\partial L} = 0 \end{cases} \Leftrightarrow \begin{cases} \theta\sigma^2(M,L) - 2M\alpha(M,L) = 0 \\ (S_Y(M+L) - S_Y(M))\theta\sigma^2(M,L) - \\ -2\alpha(M,L)(MS_Y(M+L) + \int_{M+L}^{\infty} S_Y(y)dy) = 0 \end{cases} \tag{98}
$$