Revision 1 - Solutions

Problem 1. Consider the following models:

$$y_t = \alpha_0 + \alpha_1 t + e_t \tag{1}$$

$$y_t = \rho y_{t-1} + v_t \tag{2}$$

where e_t and v_t are i.i.d.(0,1), and $\rho = 1$. A central banker considers the two models above to explain the Gross Domestic Product (GDP).

(a) What is the name of each model?

Model (1) is a model with a linear trend (also called trend-stationary process). Model (2) is a random walk model (because $\rho = 1$).

(b) Derive the unconditional mean and variance of y_t implied by each model. Is any of the two models covariance stationary or/and weak dependent?

For model (1):

$$E(y_t) = E(\alpha_0 + \alpha_1 t + e_t) = \alpha_0 + \alpha_1 t + E(e_t) = \alpha_0 + \alpha_1 t, \quad \text{because} \;\; E(e_t) = 0$$

The unconditional mean is not constant, it is a function of time (hence the first condition of weak stationarity or covariance stationary is not satisfied and y_t described by model (1) is not covariance-stationary). Moreover:

$$Var(y_t) = Var(\alpha_0 + \alpha_1 t + e_t) = Var(e_t) = 1 \text{ (constant)}$$

For model (2) with $\rho = 1$, doing recursive substitution gives us:

$$y_t = y_{t-1} + v_t = y_{t-2} + v_{t-1} + v_t = ... = y_0 + v_1 + \ldots + v_t = y_0 + \sum_{i=1}^{t} v_i.$$

Using the above result we have

$$E(y_t) = y_0 + \sum_{i=1}^{t} E(v_i) = y_0 + 0 = y_0 \text{ (constant)}.$$

But

$$Var(y_t) = Var(\sum_{i=1}^{t} v_i) = tVar(v_i) = t$$

using the assumption above for v_t. Because the second moment is time-varying, the second condition for covariance stationarity is violated, hence y_t is not covariance stationary.

(c) Briefly explain what is meant by weak stationarity and weak dependence.

See Lecture 10, slide 7 and 9 respectively.

(d) Predicting future GDP is of major importance in decision making regarding investment, spending and hiring (among other things). Hence we are interested in the $h-$step ahead forecast given the last observed information: $E(y_{t+h}|y_t)$. Derive $E(y_{t+h}|y_t)$ from model (1) and (2) assuming $\rho = 1$.

For model (1) we have:

$$y_{t+h} = \alpha_0 + \alpha_1(t+h) + e_{t+h}$$

Taking the expectation gives:

$E(y_{t+h}|y_t) = \alpha_0 + \alpha_1(t+h) + E(e_{t+h}|y_t) = \alpha_0 + \alpha_1(t+h)$ because e_t is independent and has mean 0.

For model (2) by doing recursive substitution we have:

$$y_{t+h} = y_{t+h-1} + v_{t+h} = \ldots = y_t + v_{t+1} + \ldots + v_{t+h}$$

Hence

$$E(y_{t+h|y_t}) = E(y_t + v_{t+1} + \ldots + v_{t+h}|y_t) = y_t$$

because v_t is identically distributed and $E(v_t) = 0$.

(e) When $|\rho| < 1$, $E(y_{t+h}|y_t) = \rho^h y_t$. What happens with the $h-$step ahead forecast as $h \to \infty$ in model (2) for $|\rho| < 1$ and $\rho = 1$?

If $|\rho| < 1$, $E(y_{t+h}|y_t) = \rho^h y_t \to 0$ as $h \to \infty$, but when $rho = 1$ the $h-$step ahead forecast is the last available observation y_t (the forecast changes when a new observation is available).

(f) y_t in model (1) has trending behaviour, while y_t in model (2) with $\rho = 1$ has highly persistent behaviour. Show that y_t described by the model:

$$y_t = \delta + y_{t-1} + u_t \tag{3}$$

is highly persistent and has a clear linear trend, where u_t is i.i.d.(0,1).

This is a random walk with drift. By backward substitution we have:

$$\begin{aligned} y_t &= \delta + y_{t-1} + u_t = \delta + \delta + y_{t-2} + u_{t-1} + u_t = \ldots = \\ &= \delta t + y_0 + u_1 + \ldots_{u\ t}, \end{aligned} \tag{4}$$

where δt is the linear trend (as in model (1) above) and $y_0 + u_1 + \ldots_{u\ t}$ is the highly persistent part as in the case of the random walk in (2).