Investments and Portfolio Management

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Closed-book exam. You can only use the official formulas' sheet. **Duration: 2.5h**

. . . .

Name: **Number:**

. . . .

GROUP I (35 points)

1. Explain how individual investor bias such as *overconfidence*, *anchoring* and *inertia* may affect portfolio decision making. In which way may mandatory investment policy statements (IPS) to help dealing with them? . . . . [20p]

   Answer:

   Research in psychology has documented a range of decision-making behaviours called biases.

   (i) Investors tend to overestimate their ability to identify "winning" investments. They perceive good investments as "skill" and possible bad investments as "bad luck." This is known as overconfidence and may lead to under-diversified portfolios (as investors strong believe they know who are the winners) as well as overtrading (as they also believe they can beat the market).

   (ii) Anchoring is an example of mental shortcuts humans tend to use. This may lead investors to believe past performance is an indicator of future performance, entering technical analysis trading, following trends, etc. Anchoring may also explain the tendency investors have to invest in what they know (home bias).

   (iii) On the other hand, investor may fail to take action, having second thoughts. This is related to the human desire to avoid regret. Thus, inertia can act as a barrier stopping people from saving and/or making the necessary changes in their portfolios and/or do the appropriate rebalancing.

   IPS help dealing with behavioural bias working like an auto-pilot document where portfolio managers decide asset allocations, but also rebalancing policy, etc. So, in some sense both investors and portfolio managers have their "hand tied", which is one of the aways to control for biases.

2. Choose <u>ONE</u> of the following statements and discuss whether it is true or false: .... [15p]

   I. *Any investor worried with safety is indifferent between the optimal portfolios according to Roy, Kataoka or Telser.*

   Comment: FALSE.

   Roy, Kataoka and Telser define alternative safety criteria so the optimal portfolios according to each of them may differ.

   Roy criterium is appropriate for investors who are extremely averse to returns below a limit R_L and wish to minimize the probability of occurrence of that event. One should apply these criteria when faced with safety concerned investors, evaluating which "safety" definition best fits the investor profile.

   Kataoka criterium should be used for investors that express their concerns in terms of the worst $\alpha\%$ outcomes/scenarios and choose portfolios that maximize the $\alpha\%$ quantile of the returns distribution.

   Finally, Telser criterium should be applied whenever investors like to say both R_L and $\alpha\%$ requiring one should only consider portfolios that have a probability of returns lower or equal to R_L smaller than $\alpha\%$. In Telser's case if more than one portfolio satisfies the safety constraint one should then pick the one with maximum expected return, since the investor's concern about risk was already considered.

   II. *The possibility of obtaining abnormal returns based upon technical analysis is inconsistent with all forms of market efficiency.*

   Comment: TRUE

   It goes against the hypothesis of market's efficiency in its weakest form, so also against semi-strong or strong efficiency .

   Technical analysis relies on extrapolating past data series in order to infer something about the future. In its weakest form, the hypothesis of market efficiency postulates that, at least, all past information is already included in the prices.

   Thus even in weak efficient markets technical analysts should not allow to earn abnormal returns.

GROUP II (30 points)

Any second order Taylor approximation of a generic utility function $U(W)$ around the initial wealth W_0, is quadratic in W, i.e. we always get $U(W) \approx aW^2 + bW + c$.
(i) Derive the parameters a, b and c in terms of initial wealth, utility function and its derivatives. [10p]
(ii) Show the quadratic approximation of U is *equivalent* to $V(W) = (1 + RRA_0)\,W - \frac{1}{2}ARA_0W^2$, where RRA_0 and ARA_0 stand for the U relative and absolute risk aversions evaluated at W_0. . . [10p]
(iii) Consider investments A, B and C represented by the points $A = (0\%, 5\%)$, $B = (10\%, 10\%)$ and $C = (15\%, 25\%)$ in the space $\big(\sigma, \bar{R}\big)$. to rank A, B and C according to his preferences. Analyze his risk profile and find the certainty return equivalent to his preferred investment. . . . . [10p]

Solution:

(i) The second Taylor expansion of $U(W)$ around the initial wealth W_0 is

$$
\begin{aligned}
U(W) &\approx U(W_0) + U'(W_0)(W - W_0) + \frac{1}{2}U''(W_0)(W - W_0)^2 \\
&\approx \underbrace{U(W_0) - U'(W_0)W_0 + \frac{1}{2}W_0^2U''(W_0)}_{c} + \underbrace{\big(U'(W_0) - U''(W_0)W_0\big)}_{b}\,W + \underbrace{\frac{1}{2}U''(W_0)}_{a}\,W^2
\end{aligned}
$$

(ii) Two utility functions are *equivalent* if they represent the same preferences. Any linear transformation of an utility function leads to an equivalent utility function $U(W) = \alpha + \beta V(W)$, provided $\beta > 0$. In our case

$$
\begin{aligned}
U(W) &\approx c + \big(U'(W_0) - U''(W_0)W_0\big)\,W + \frac{1}{2}U''(W_0)W^2 \\
&\approx c + \frac{U'(W_0) - U''(W_0)W_0}{U'(W_0)}U'(W_0)W + \frac{1}{2}\frac{U''(W_0)}{U'(W_0)}U'(W_0)W^2 \\
&\approx c + (1 + RRA_0)\,U'(W_0)W - \frac{1}{2}ARA_0U'(W_0)W^2 \\
&\approx c + U'(W_0)\left[(1 + RRA_0)\,W - \frac{1}{2}ARA_0W^2\right] \approx c + U'(W_0)V(W) \quad \prec\approx\succ \quad V(W)\ ,
\end{aligned}
$$

where $\alpha = c$, $\beta = U'(W_0) > 0$, and we use the definitions $ARA_0 = -\frac{U''(W_0)}{U'(W_0)}$ and $RRA_0 = W_0ARA_0$.
(iii) Preferences of investors in an uncertainty setting can be evaluated by the risk-tolerance function (RTF), that is nothing more than the expected value of Utility. For any utility we have that the RTF is approximately given by $f(\sigma, \bar{R}) = \bar{R} - \frac{1}{2}RRA_0(\sigma^2 + \bar{R}^2)$. For $U(W)$, we have

$$
U(W) = a^2 + bW + c\ , \qquad U'(W) = 2aW + b\ , \qquad U''(W) = 2a
$$

$$
RRA(W) = -\frac{U''(W)}{U'(W)}W = -\frac{2aW}{2aW + b}.
$$

So, we have $f(A) = f(0\%, 5\%) = 5\% - \frac{1}{2}RRA_0(5\%)^2$, $f(B) = f(10\%, 10\%) = 10\% - \frac{1}{2}RRA_0((10\%)^2 + (10\%)^2)$, $f(C) = f(15\%, 25\%) = 25\% - \frac{1}{2}RRA_0((25\%)^2 + (15\%)^2)$ and for any concrete level level for RRA_0 we can rank the three investments.

Taking, for e.g., $a = -\frac{1}{2}$, $b = 10$ and $W_0 = 5$, we get a risk averse investor with $RRA_0 = -\frac{2aW_0}{2aW_0+b} = 1$.

$$
f(A) = 0.04875 \qquad f(B) = 0.09 \qquad f(C) = 0.2075 \qquad \Longrightarrow \qquad C \succ B \succ A\ .
$$

To find the certain equivalent return to investment C we need to solve

$$
f(certain) = f(C) \quad \Leftrightarrow \quad R_{certain} - \frac{1}{2}R_{certain}^2 = 0.2075 \quad \Leftrightarrow \quad R_{certain} = 23.515\%\ .
$$

GROUP III (135 points)

Problem 1 (75 points)

Consider as valid all standard CAPM assumptions and consider the equilibrium relationship

$$\bar{R}_i^e = 3.5\% + 12\%\beta_i \ .$$

1. Criticize the key assumptions of the standard CAPM and interpret the parameters values in the equilibrium relationship above. .... [10p]
   Answer: The most disturbing assumptions of the CAPM equilibrium model are: (i) the existence of a market portfolio with all assets that exist in positive quantity in the economy; (ii) assuming all assets in the economy are infinitely divisible and tradable; (iii) assuming all investors have homogeneous expectations about returns of risky assets. Less problematic assumptions of the standard CAPM are (iv) it assumes the same risk free return to both deposit and borrow; (v) its assumes shortselling is allowed without bounds, (vi) it assumes investors only care about expected returns and risk where risk is measured by volatility and (vii) it assumes there are no taxes or transaction costs.
   Under all these assumption the equilibrium expected return for individual assets is $\bar{R}^e = R_f + \beta_i \left[\bar{R}_m - R_f\right]$ so the 3.5% above is the risk-free return and the 12% the market risk premium.

2. Find out the *market portfolio* equilibrium expected return? .... [5p]
   Solution:
   From the equilibrium relationship we get $R_f = 3.5\%$ and $\bar{R}_m - R_f = 12\%$, thus $\bar{R}_M = 15.5\%$.

3. Consider now the volatility of the *market portfolio* to be $\sigma_m = 25\%$. Write down the Capital Market Line (CML) equation and explain what it represents. .... [7.5p]
   Solution: The CML represent the equilibrium efficient frontier, i.e. the set of all efficient portfolios in an equilibrium situation. In the CAPM equilibrium all efficient portfolios are simply combinations of the risk-less asset with the market portfolio. The EF equation is, thus, $\bar{R} = R_f + SR_m\sigma$, where SR_m stands for the Sharpe ratio of the market portfolio. From the market date we get

$$\bar{R} = R_f + \frac{\bar{R}_m - R_f}{\sigma_m}\sigma \quad \Leftrightarrow \quad \bar{R} = 3.5\% + \frac{12\%}{25\%}\sigma \quad \Leftrightarrow \quad \bar{R} = 3.5\% + 0.48\sigma$$

4. Mr. X risk tolerance function is $f(\sigma, \bar{R}) = 7\bar{R} - 7\left(\sigma^2 + \bar{R}^2\right)$.
   (a) From his risk tolerance function what can you conclude about Mr. X risk profile? ... [5p]
   Solution: Since we have

$$\frac{\partial f}{\partial \sigma} = -14\sigma < 0$$

   one can conclude Mr. X is risk averse.

(b) Show the certainty equivalent of Mr. X of an investment in the market portfolio M is $R_c \approx 7,39\%$ and compute the associated risk premium. What can you conclude about his risk profile? .... [10p]
Solution: The risk tolerance function of Mr. X gives us the expected utility of a particular investment choice in the plan $(\sigma, \bar{R})$. If he chooses to invest in market portfolio, for which we have $\bar{R}_m = 15.5\%$ and $\sigma_m = 25\%$, his expected utility is

$$f(25\%, 15.5\%) = 7 \times 0.155 - 7\left(0.25^2 + 0.155^2\right) = 0.4793$$

For the certainty equivalent we must have $\sigma_c = 0$. Using $R_c \approx 7,39\%$ we obtain the same expected utility

$$f(0, R_c) = f(0, 7.39\%) = 7 \times 0.0739\% - 7 \times 0.0739^2 = 0.4793\ .$$

Given the certainty equivalent the risk premium is $\pi_m = \bar{R}_m - R_c = 15.5\% - 7.39\% \approx 8.11\%$. A positive risk premium tell ius Mr. X is risk averse.

(c) Suppose now Mr. X considers not only the market portfolio but all efficient portfolios. What is the optimal investment for him? Explain[1] . ... [12.5p]
Solution: The optimal investment is from all efficient portfolios the one with the highest expected return. So we must solve

$$\begin{aligned} &\max\ f(\sigma, \bar{R}) = 7\bar{R} - 7\left(\sigma^2 + \bar{R}^2\right) \quad \Leftrightarrow \quad \max\left[7(0.035 + 0.48\sigma) - 7\sigma^2 - 7(0.035 + 0.48\sigma)^2\right] \\ &\ \text{s.t.}\quad \bar{R} = 3.5\% + 0.48\sigma \end{aligned}$$

From the first-order-condition we get

$$7\times0.48 - 2\times7\sigma - 2\times7\times0.48(0.035+0.48\sigma) = 0 \quad \Leftrightarrow \quad 17.2256\sigma = 3.1248 \quad \Leftrightarrow \quad \sigma^* = 0.1814$$

The optimal to Mr. X is to invest 72.56% – since $\frac{0.1814}{0.25} = 0.7256$ – and deposit the remaining 27.44%.

(d) His wife, Ms. X, would prefer to maximize the probability of returns' above 3.5%.

  (i) Identify the criteria underlying Ms X concerns? Explain. .... [5p]
  Answer: Maximising the probability of returns' above 3.5% is equivalent to minimising the probability of returns less or equal to 3.5%. To find her optimal investment we should, therefore, use Roy's safety first criteria $\min \Pr\left(R \leq R_L\right)$ with $R_L = 3.5\%$.

  (ii) What would you recommend her? Any chance her husband's portfolio would make her happy? .... [7.5p]
  Answer: Given that her $R_L = R_f = 3.5\%$ all efficient portfolios (except 100% investment in deposit) will have the same likelihood ratio of her desired outcome. In case this is her only request, one can recommend any point on the efficient frontier, including Mr. X optimal portfolio.

5. In this world there is a risky asset, A, with $\beta_A = 0.7$.

(a) Determine its equilibrium expected return. .... [5p]
Solution: The equilibrium expected return is

$$\bar{R}^e_A = 3.5\% + 12\% \times 0.7 = 11.9\%\ .$$

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[1] If you did not answer Question 3, consider $\bar{R}_p = 0.035 + 0.48\sigma_p$ as the CML for Question 4.

(b) A market model has been fit to the observed returns of A. The parameters are $\alpha_A = 2\%$, $\beta_A = 0.7$, $\sigma_{\epsilon A} = 5\%$. What is your investment recommendation on asset A? ....[7.5p]
Solution: From the parameters we obtain that the market implied expected return

$$\bar{R}_A = 2\% + 0.7 \times 15.5\% = 12.85\% \ ,$$

and, since we have $\bar{R}_A > \bar{R}^e_A$, we can conclude that according to CAPM the asset is underpriced. So, the recommendation is to buy.

Problem 2 (60 points)

Consider the following returns associated with two risky investments X and Y.

Prob	X Return	Y Return
0.25	0%	20%
0.50	5%	8%
0.25	30.65%	-1.1%

1. Identify possible first, second and third order dominances. Interpret. .... [15p]
   Solution:

R	Pr X	Pr Y	Dist X	Dist Y	S Dist X	S Dist Y	SS Dist X	SS Dist Y
-1,1%	0	0.25	0	0.25	0	0.25	0	0.25
0%	0.25	0	0.25	0.25	0.25	0.50	0.25	0.75
5%	0.5	0	0.75	0.25	1	0.75	1.25	1.5
8%	0	0.5	0.75	0.75	1.75	1.5	3	3
20%	0	0.25	0.75	1	2.5	2.5	5.5	5.5
30.65%	0.25	0	1	1	3.5	3.5	9	9

   From the table above we can conclude there are no first or second order stochastic dominances, but investment X stochastically dominates investment Y in third order. The financial interpretation of this result is that X is preferred to Y by all risk averse investors who have decreasing absolute risk aversion.

2. Mr. Logaritm is an investor whose preferences are well described by $U(W) = \ln(W)$.

   (a) Evaluate Mr. Logaritm's risk profile in terms of absolute and relative risk aversion. Financially interpret all results. .... [10p]
   Solution:

$$U'(W) = \frac{1}{W} > 0, \quad \text{for} \quad W > 0 \qquad U''(W) = -\frac{1}{W^2} < 0$$
$$A(W) = \frac{1}{W} \qquad A'(W) = -\frac{1}{W^2} < 0$$
$$R(W) = 1 \qquad R'(W) = 0$$

   The above mathematical expressions tell us Mr. Logartim prefers more to less ($U'(W) > 0$), he is risk averse ($U''(W) < 0$), and that he always keeps the same percentage of his wealth in risky assets, no matter the wealth level, ($R'(W) = 0$). This implies that as his wealth increases he invests more in absolute terms in risky assets, so he has absolute decreasing risk aversion ($A'(W) < 0$).

   (b) When see as <u>alternative investments</u>, what is the best choice for Mr. Logaritm? X or Y? Explain. .... [5p]
   Solution:
   Since investment X stochastically dominates investment Y in third order, X is preferres to Y by all investors with decreasing absolute risk aversion, so also by Mr. Logaritm.

(c) Show that, for an initial investment of W_0, the certainty equivalent and risk premium of Mr. Logaritm associated with investment X are given by

$$c = W_0 e^{0.09123} \qquad \text{and} \qquad \pi = W_0 \left(1.101625 - e^{0.09123}\right) .$$

Explain all steps. .... [10p]

Solution:

The certainty equivalent is the guaranteed amount that provides the same level of expected return as investment X. The risk premium is the difference between the expected outcome of an investment and its certainty equivalent. Risk averse investor require a positive risk premia on all risky investments.

To compute the expected utility of investment X we use a change of variable $W = W_0(1+R)$ to get the utility function in terms of returns R and the initial investment W_0.

$$U(W) = \ln(W) = \ln(W_0) + \ln(1+R) = U(W_0, R) .$$

$$\begin{aligned}
\mathbb{E}\left[U(X)\right] &= 0.25 \times U(W_0, 0\%) + 0.5 \times U(W_0, 5\%) + 0.25 \times U(W_0, 30.65\%) \\
&= \ln(W_0) + 0.25 \times \ln(1) + 0.5 \times \ln(1.05) + 0.25 \ln(1.3065) \\
&= \ln(W_0) + 0.09123
\end{aligned}$$

The certainty equivalent is the fixed amount c for which we have

$$U(c) = \ln(W_0) + 0.09123 \qquad \Leftrightarrow \qquad c = W_0 e^{0.09123} .$$

The risk premium is

$$\begin{aligned}
\pi &= \mathbb{E}\left[W_X\right] - c \\
&= \mathbb{E}\left[W_0(1+R_X)\right] - W_0 e^{0.09123} \\
&= W_0 \left[1 + \mathbb{E}(R_X) - e^{0.09123}\right] \\
&= W_0 \left[1 + (0.25 \times 0 + 0.5 \times 0.05 + 0.25 \times 0.3065) - e^{0.09123}\right] \\
&= W_0 \left[1 + 0.101625 - e^{0.09123}\right]
\end{aligned}$$

3. Consider now <u>combinations of X and Y with risk-free rate for deposits $R_f = 5\%$.</u>

(a) Determine the correlation between the returns of X and Y. Based upon that information should one also consider combinations of X and Y? Why or why not? .... [5p]

(b) Derive and sketch the efficient frontier in this market. .... [7.5p]

(c) What is the optimal portfolio for an investor who wishes to maximize long-term growth? Explain. .... [7.5p]