Universidade de Lisboa
Instituto Superior de Economia e Gestão
BsC in Economics, Finance and Management

Mathematics II – 1st Semester - 2022/2023

Regular Assessment - 4th of January 2023

Duration: $(120 + \varepsilon)$ minutes, $|\varepsilon| \leq 30$

Version C

Name: ....

Student ID #: ....

Part I

- Complete the following sentences in order to obtain true propositions. The items are independent from each other.
- There is no need to justify your answers.

(a) *(6)* If $A : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear map such that

$$A(1,0,0) = (5,0,0), \quad A(0,2,0) = (0,-1,0) \quad \text{and} \quad A(0,0,3) = (0,0,3/4),$$

then the eigenvalues of A^{-1} are .... and their geometric multiplicities are ...., respectively.

(b) *(4)* The quadratic form Q associated to the matrix $\begin{pmatrix} 0 & -3 \\ -5 & 2 \end{pmatrix}$ is given by

$Q(x, y) =$ ....

(c) *(5)* The (maximal) domain of $f : D_f \to \mathbb{R}$ is the set

$$D_f = \{(x, y) \in \mathbb{R}^2 : y > (x-1)^2 \wedge (x, y) \neq (1, 5)\}.$$

One possible analytical expression for f is

$f(x, y) =$ ....

(d) *(4)* The set of accumulation points of $\left\{\left(\cos(n\pi), \left(\frac{1}{2}\right)^n\right), n \in \mathbb{N}\right\} \subset \mathbb{R}^2$ is

....

(e) *(5)* Consider $A = \{(x,y) \in \mathbb{R}^2 : x^2 + (y+2)^2 < 4 \wedge xy \neq 0\}$. The topological border (frontier) of A is analytically defined by

$\partial A =$ ....

(f) *(6)* The continuous map $f(x,y) = x^2 + y^2$ has a maximum but not a minimum when restricted to the set

$M = \{(x,y) \in \mathbb{R}^2 :$ ....$\}$

.....................'s Theorem cannot be applied because M is not ....

(g) *(4)* Geometrically, the level curve of $f(x,y,z) = x^2 + y^2 + z^2$ associated to 2 is a/an

....

(h) *(4)* If $f(x,y) = \frac{3}{(x-1)^2+y^2}$ and $g : \mathbb{R}\backslash\{3\} \to \mathbb{R}$ is the map defined by $g(x) = 2 + \frac{5}{x-3}$, then

$$\lim_{(x,y)\to(0,0)} [g \circ f(x,y)] = \text{.........}$$

(i) *(4)* If $u_n =$ (.............., ...) with $n \in \mathbb{N}$, is a non-constant sequence in $\mathbb{R}^2$ and $f(x,y) = \ln x + \cos(y)$, then $\lim\limits_{n\to+\infty} f(u_n) = 3$.

(j) *(5)* The map $f(x,y) = \sqrt{12x^2 + 8y^2}$ is positively homegeneous of degree .... In this case, the *Euler identity* says that (compute explicitly the derivatives)

....

(k) *(4)* If $f(x,y) = xy$, $x(t) = t^2$ and $y(t) = 1 + t^3$, by the *Chain rule* we get:

$\frac{df}{dt}(t) =$ ....

(l) *(5)* The gradient vector of $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $(2x\cos y, 1 - x^2 \sin y)$. If $f(x, y)$ does not have constant terms in both components, then $f(1, \pi) =$ ......

(m) *(4)* With respect to the map $f : \mathbb{R}^2 \to \mathbb{R}$, we know that $\nabla f(3, 2) = (0, 0)$ and

$$H_f(3,2) = \begin{pmatrix} \ldots\ldots & 0 \\ 0 & \ldots\ldots \end{pmatrix}.$$

Then, $f(3, 2)$ is a local maximum of f.

(n) *(3)* The point $x = 5$ is a saddle-point of the map $f(x) =$ ...., $x \in \mathbb{R}$.

(o) *(3)* The differential of order 2 of the map $f(x, y) = e^{3y}$ at the point $(0, 0)$ is given by

$$D_2 f(0,0)(h_1, h_2) = \ldots\ldots h_1^2 + \ldots\ldots. h_1 h_2 + \ldots\ldots.. h_2^2$$

(p) *(4)* The map $y(x) = \dfrac{1}{x}$, $x \in \mathbb{R}^-$, is a solution of the IVP $\begin{cases} \dot{y} = \ldots\ldots \\ y(\ldots\ldots) = -2 \end{cases}$.

(q) *(5)* The following equality holds:

$$\int_{-\infty}^{1} \int_{0}^{1} e^{x+y} \text{ dx dy} = \ldots.$$

(r) *(3)* The absolute maximum of $f(x, y) = x$ when restricted to the set

$$M = \{(x, y) \in \mathbb{R}^2 : (x - 5)^2 + y^2 = \ldots..\}$$

occurs at $(x, y) = (7, \ldots.)$. The associated Lagrange multiplier is ....

(s) *(6)* The logistic law (associated to a given population of size p that depends on the time $t \geq 0$) states that

$$p' = ap - bp^2, \qquad a, b \in \mathbb{R}$$

where a/b may be seen as the ....of the population.

If $p(0) = 1000$, $a = 1$ and $b = 0.002$, then the solution of the previous differential equation is monotonic ....

If $a = 3$, $b = 0$ and $p(0) = 1000$, the solution of the ODE is

...., where $t \in \mathbb{R}^+$.

(t) *(6)* The graph of the solution of the IVP $\begin{cases} y'' - 4y = 0 \\ y'(0) = 0 \\ y(0) = 2 \end{cases}$ is

Part II

- Give your answers in exact form. For example, $\frac{\pi}{3}$ is an exact number while 1.047 is a decimal approximation for the same number.
- In order to receive credit, you must show all of your work. If you do not indicate the way in which you solve a problem, you may get little or no credit for it, even if your answer is correct.

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1. For $\alpha \in \mathbb{R}\backslash\{-3, 3\}$, consider the following matrix $\mathbf{A} = \begin{bmatrix} \alpha & 3 & 0 \\ 3 & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

   (a) Classify the quadratic form $Q(X) = X^T\mathbf{A}X$, $X \in \mathbb{R}^3$, as function of α.

   (b) Find the value of α for which $(1, 1, 0)$ is an eigenvector of $\mathbf{A}$ associated to 1.

2. Consider the map $f(x, y) = \begin{cases} \dfrac{x^2(x-y)}{\sqrt{x^2+y^2}} & \text{if } y > x \\ 0 & \text{if } y \leq x \end{cases}$.

   Show that:

   (a) f is continuous in $\mathbb{R}^2$.

   (b) f has a global maximum.

   (c) if $y > x$ then $x\dfrac{\partial f}{\partial x}(x, y) + y\dfrac{\partial f}{\partial y}(x, y) = 2f(x, y)$.

3. Consider the map f defined in $\mathbb{R}^2$ as

   $$f(x, y) = x^3 + y^3 + x^3y^3.$$

   (a) Compute $f(x, 0)$, $x \in \mathbb{R}$.

   (b) Identify and classify the critical points of f *(if necessary, use 3(a))*.

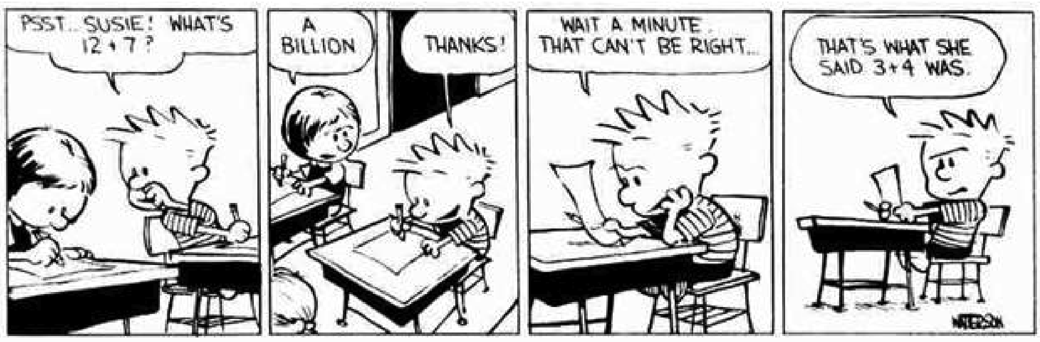
4. Let $\Omega \subset \mathbb{R}^2$ be the triangle defined by the points $(1,0), (2,0)$ and $(1,1)$. Compute

$$\int\int_{\Omega} xy \ \text{dx dy}.$$

5. Consider the following IVP (y is a function of x):

$$\begin{cases} x^2y' + xy = x^3 \\ 3y(1) = 4 \end{cases}$$

Write the solution $y(x)$ of the IVP, identifying its maximal domain.



Credits:

I	II.1(a)	II.1(b)	II.2(a)	II.2(b)	II.2(c)	II.3(a)	II.3(b)	II.4	II.5
90	10	10	15	10	10	5	15	15	20