Homework

These problems were taken out from Cochrane's book, at the end of chapter 2. These problems are more complex than the ones of the exam. However, trying to solve them is a very useful way to learn and prepare for the exam.

Problem 1.

The representative consumer maximizes a CRRA utility function.

$$E_t \sum \beta^j c_{t+j}^{1-\gamma}.$$

Consumption is given by an endowment stream.

(a) Show that with log utility, the price/consumption ratio of the consumption stream is constant, no matter what the distribution of consumption growth.

(b) Suppose there is news at time t that future consumption will be higher. For $\gamma < 1$, $\gamma = 1$, and $\gamma > 1$, evaluate the effect of this news on the price. Make sense of your results. (Note: there is a real-world interpretation here. It's often regarded as a puzzle that the market declines on good economic news. This is attributed to an expectation by the market that the Fed will respond to such news by raising interest rates. Note that $\gamma > 0$ in this problem gives a completely real and frictionless interpretation to this phenomenon!)

Problem 2.

The linear quadratic permanent income model is a very useful general equilibrium model that we can solve in closed form. It specifies a production technology rather than fixed endowments, and it easily allows aggregation of disparate consumers.

The consumer maximizes

$$E \sum_{t=0}^{\infty} \beta^t \left(-\frac{1}{2}\right) (c_t - c^*)^2$$

subject to a linear technology

$$k_{t+1} = (1+r)k_t + i_t$$

$$i_t = e_t - c_t$$

e_t is an exogenous endowment or labor income stream. Assume $\beta = 1/(1+r)$; the discount rate equals the interest rate or marginal productivity of capital.

(a) Show that optimal consumption follows

$$c_t = rk_t + r\beta \sum_{j=0}^{\infty} \beta^j E_t e_{t+j}$$

$$c_t = c_{t-1} + (E_t - E_{t-1}) r\beta \sum_{j=0}^{\infty} \beta^j E_t e_{t+j}$$

i.e., consumption equals permanent income, precisely defined, and consumption follows a random walk whose innovations are equal to innovations in permanent income.

(b) Assume that the endowment e_t follows an AR(1) $e_t = \rho e_{t-1} + \varepsilon_t$. Calculate and interpret the result for $\rho = 1$ and $\rho = 0$. (The result looks like a "consumption function" relating consumption to capital and current income, except that the slope of that function depends on the persistence of income shocks. Transitory shocks will have little effect on consumption, and permanent shocks a larger effect.).

(c) Calculate the one period interest rate (it should come out to r of course) and the price of a claim to the consumption stream. e and k are the only state variables, so the price should be a function of e and k. Interpret the time-variation in the price of the consumption stream. (This consumer gets more risk averse as consumption rises to c^*. c^* is the bliss point, so at the bliss point there is no average return that can compensate the consumer for greater risk.)

Problem 3. This is not only a historically important model, it introduces a very important method. Evaluating infinite sums as in the last problem is a huge pain. In most models, conditioning information is a function of only a few state variables, x_t. Everything you could want to know about the current state of the economy, and the conditional distribution of everything you could want to know in the future is contained in the state variables. Hence, prices (at least properly scaled) have to be a function of the state variables. Instead of solving for p in terms of a huge infinite sum, you can solve the functional equation $p(x_t) = E_t[m_{t+1}(x_t, x_{t+1})(p(x_{t+1}) + d_{t+1})]$, (an equation that specifies a function in implicit form).

Consider again CRRA utility,

$$E_t \sum \beta^j c_{t+j}^{1-\gamma}$$

Consumption growth follows a two-state Markov process. The states are $\Delta c_t = c_t/c_{t-1} = h, l$, and a 2×2 matrix Π governs the set of transition probabilities, i.e. $pr(\Delta c_{t+1} = h|\Delta c_t = l) = \pi_{l \to h}$. (This is the Mehra-Prescott 1986 model)

(a) Find the riskfree rate (price of a certain real payoff of one) in this economy. This price is generated by $p_t^b = E_t(m_{t,t+1}1)$. You are looking for two values, the price in the l state and the price in the h state.

another problem: Find the value of a perpetuity, i.e. a bond that pays a coupon of one per period forever.

(b) To model a stock, we can think of an asset that pays consumption as its dividend. The value of the whole Portuguese economy (think of it as a big corporation) is the value of a claim to the consumption it can provide us. Now, the stock price

$$p_t = E_t \sum_{j=1}^{\infty} \beta^j \frac{u'(c_{t+j})}{u'(c_t)} c_{t+j} = E_t \sum_{j=1}^{\infty} \beta^j \left(\frac{c_{t+j}}{c_t} \right)^{-\gamma} c_{t+j}$$

depends on the level of consumption, not just the growth rate at each date. However, we can apply the same trick as for the perpetuity to the ratio of price to consumption, i.e. the price/dividend ratio of the stock,

$$\begin{aligned} p_t &= E_t[m_{t+1}(p_{t+1} + c_{t+1})] \\ \frac{p_t}{c_t} &= E_t[m_{t+1}(\frac{p_{t+1}}{c_{t+1}} + 1)\frac{c_{t+1}}{c_t}] \\ \frac{p_t}{c_t} &= E_t[\beta(\frac{p_{t+1}}{c_{t+1}} + 1) \left(\frac{c_{t+1}}{c_t} \right)^{1-\gamma}] \end{aligned}$$

$\frac{p}{c}$ can take on only two values, $\frac{p}{c}(h)$ and $\frac{p}{c}(l)$. Proceed as with the perpetuity to find those two values.

(c) Pick $\beta = 0.99$ and try $\gamma = 0.5$, 5 (Try more if you feel like it). Calibrate the consumption process to have a 1% mean and 1% standard deviation, and consumption growth uncorrelated over time. Calculate prices and returns in each state.

(d) Now introduce serial correlation in consumption growth with $\gamma = 5$. (You can do this by adding weight to the diagonal entries of the transition matrix Π.) What effect does this have on the model?