Foundations of Financial Economics
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Final Exam
Total time: 1:30 hours. Total points: 20

Instructions:

Please sit in alternate seats. This is a closed-book, closed-note exam. Please get rid of everything but pen/pencil. In your answer explain all the steps in your reasoning. Keep answers short; I don't give more credit for long answers, and I can take points off if you add things that are wrong or irrelevant.

Formulas:

If x and y are random variables then

$$E\left(xy\right) = ExEy + cov\left(x, y\right)$$

$$\sigma^2\left(x\right) = Ex^2 - \left(Ex\right)^2$$

If x is normal distributed then $\exp(x)$ is lognormal, and

$$E\exp(x) = \exp(Ex + 0.5\sigma^2\left(x\right))$$

Brownian motion

$$z_{t+\Delta} - z_t \sim N(0, \Delta)$$

Differential

$$dz_t = lim_{\Delta\searrow 0}(z_{t+\Delta} - z_t)$$

$$dz_t^2 = dt,\ dz_t dt = 0,\ dt^\alpha = 0,\ \text{if } \alpha > 1$$

$$E_t(dz_t) = 0, var_t(dz_t) = E_t(dz_t^2) = dz_t^2 = dt$$

Ito's Lemma

$$df\left(x, t\right) = f_t dt + f_x dx + \frac{1}{2} f_{xx} dx^2 = \left(f_t + f_x \mu_x + \frac{1}{2} f_{xx} \sigma_x^2\right) dt + f_x \sigma_x dz$$

Questions:

1. Suppose that $x = [-2, 1]$ is the payoff vector of the only asset available in an economy with two states. The price of x, $p\left(x\right)$, is 2. Denote by $X \in \mathbb{R}^2$ the space of payoffs formed by x. The law of one price holds.

(2 pts) a. Obtain a discount factor $x^* \in X$ that is able to price all payoffs in this economy. Is this discount factor positive?

(2 pts) b. Is it possible to obtain a positive discount factor $y^* \in \mathbb{R}_+^2$ that can price all payoffs in this economy? If so, calculate one and illustrate graphically. Are there any arbitrage opportunities in this economy?

(2 pts) c. How would your results change for $x = [2, 1]$ and $p\left(x\right) = -2$. Illustrate.

2. Consider the Gârleanu and Panageas model. In the model there are two types of people. The utility function of type A individuals is $\int_0^\infty e^{-\delta t} \frac{C_{A,t}^{1-\gamma_A}}{1-\gamma_A} dt$

and the utility function of individuals of type B is $\int_0^\infty e^{-\delta t} \frac{C_{B,t}^{1-\gamma_B}}{1-\gamma_B} dt$. Assume that $\gamma_A = 2\gamma_B$.

(2 pts) a. Obtain the consumption level of each type of individual as an implicit function of the aggregate consumption, $C_t = C_{A,t} + C_{B,t}$. How does the consumption share of each type of individual change with the aggregate consumption? Explain.

(2 pts) b. In the context of this model the market Sharpe ratio obeys the following inequality:

$$\left| \frac{E_t(dR) - rdt}{\sigma_t(dR)} \right| \leq \gamma_m \sigma \left(\frac{dC_t}{C_t} \right),$$

where γ_m is the aggregate risk aversion. Let the inverse of the aggregate risk aversion be $\frac{1}{\gamma_m} \equiv \frac{1}{\gamma_B} \frac{C_{B,t}}{C_t} + \frac{1}{\gamma_A} \frac{C_{A,t}}{C_t}$. Is this formula in accordance with the empirical facts? Explain. If you did not solve question a) assume that in the competitive equilibrium $C_{A,t}^{-\gamma_A} = C_{B,t}^{-\gamma_B}$.

(3 pts) c. Assume that aggregate consumption follows a Brownian motion: $\frac{dC_t}{C_t} = \mu dt + \sigma dz_t$. Obtain the standard deviations of consumption growth for each type of individual as a function of the aggregate consumption growth.

3. Let $R_{j,t}$ be the risk-free real gross return between periods t and $t+j$. At the beginning of t, the return $R_{j,t}$ is known with certainty. In equilibrium

$$(R_{j,t})^{-1} = E_t \left\{ \frac{\beta^j u'(C_{t+j})}{u'(C_t)} \right\}, \text{ for } j = 1, ..., T.$$

Let $\Delta c_{t+1} \equiv \ln C_{t+1} - \ln C_t$. Assume that Δc_{t+1} is an i.i.d. Normal distribution with mean μ and variance σ^2. Let the preferences be $u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}$.

(2 pts) a. Obtain the analytical solution for the short-term (one-period) interest rate. Recall that the short-term rate is $(R_{1,t})^{-1} = E_t \left[\frac{\beta u'(C_{t+1})}{u'(C_t)} \right]$. Is the interest rate positively correlated with consumption growth?

(2 pts) b. Define the long-term (two-period) interest rate as the square root of $R_{2,t}$. Show that long-term (two-period) interest rate satisfies the equation:

$$\frac{1}{R_{2,t}} = \frac{1}{R_{1,t}} E_t \left[\frac{1}{R_{1,t+1}} \right] + cov_t \left[\frac{\beta u'(C_{t+1})}{u'(C_t)}, \frac{\beta u'(C_{t+2})}{u'(C_{t+1})} \right].$$

Is the long-term interest rate the average of the expected short-term interest rates? Does this depend on the fact that Δc_{t+1} is i.i.d. ? What if $\gamma = 0$?

(3 pts) c. Consider now that consumption growth follows the process, $\Delta c_{t+1} = \alpha s_t + \varepsilon_{t+1}$, $\alpha > 0$, where s_t is an i.i.d. process, independent from ε_{t+1}, that can take only two values, $\{0, 1\}$, with equal probability. All variables with subscript t are known at time t. The ε_{t+1} is an i.i.d. Normal distribution with mean μ and variance σ^2. Obtain the analytical solution for the long-term interest rate. Can the long-term rate be smaller than the short-term rate?