Topics for the Solution to the Exam

1. [10 values]

(a)

$C_t : \beta^t u'(C_t) - \lambda_t = 0$

$M_{i,t+1} : E_t\left[\lambda_{t+1}\left(D_{i,t+1} + S_{i,t+1}\right)\right] - \lambda_t S_{i,t} = 0$ for $i = 1, ..., N$, and all t

$Z_{j,t+1} : E_t\left[\lambda_{t+1} B_{j-1,t+1}\right] - \lambda_t B_{j,t} = 0$ for $j = 1, ..., T$ and all t

(b)

$u'(C_t)B_{j,t} = \beta E_t\left[u'(C_{t+1})B_{j-1,t+1}\right]$

$\Longrightarrow$

$u'(C_t)B_{j,t} = \beta^s E_t\left[u'(C_{t+s})B_{j-s,t+s}\right] \Rightarrow B_{j,t} = E_t\left\{\beta^s \frac{u'(C_{t+s})}{u'(C_t)} B_{j-s,t+s}\right\}$

for $s = j \Rightarrow B_{j,t} = E_t\left\{\beta^j \frac{u'(C_{t+j})}{u'(C_t)} \cdot 1\right\} \Leftrightarrow \frac{1}{R_{j,t+1}} = E_t\left\{\beta^j \frac{u'(C_{t+j})}{u'(C_t)}\right\}$

(c)

$\frac{1}{R_{2,t+1}} = E_t\left\{\beta^2 \frac{u'(C_{t+2})}{u'(C_t)}\right\} = E_t\left\{\beta \frac{u'(C_{t+1})}{u'(C_t)} \beta \frac{u'(C_{t+2})}{u'(C_{t+1})}\right\}$, since $\frac{1}{R_{1,t+2}} = E_{t+1}\left\{\beta \frac{u'(C_{t+2})}{u'(C_{t+1})}\right\}$, get $\frac{1}{R_{2,t+1}} = E_t\left\{\beta \frac{u'(C_{t+1})}{u'(C_t)} \frac{1}{R_{1,t+2}}\right\}$.

(d) $u'(C_t) = C_t^{-\gamma}$ & $\frac{c_{t+1}}{c_t} \sim N(a+\mu, \sigma^2)$ if $s_t = 0$ & $\frac{c_{t+1}}{c_t} \sim N(a+b+\mu, \sigma^2)$ if $s_t = 1$

$\frac{1}{R_{1,t+1}} = E_t\left\{\beta \frac{u'(C_{t+1})}{u'(C_t)}\right\} = \beta E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right] = \beta E_t\left[\exp\left(\log\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right)\right] = \beta E_t\left[\exp\left(-\gamma \frac{c_{t+1}}{c_t}\right)\right]$

since $-\gamma \frac{c_{t+1}}{c_t} \sim N(-\gamma\left(a+\mu\right), \gamma^2\sigma^2)$ if $s_t = 0$ & $-\gamma \frac{c_{t+1}}{c_t} \sim N(-\gamma\left(a+b+\mu\right), \sigma^2)$ if $s_t = 1$

$\frac{1}{R_{1,t+1}} = \beta e^{-\gamma(a+\mu)+\frac{\gamma^2\sigma^2}{2}}$ if $s_t = 0$ & $\frac{1}{R_{1,t+1}} = \beta e^{-\gamma(a+b+\mu)+\frac{\gamma^2\sigma^2}{2}}$ if $s_t = 1$

Let $r_{j,t} \equiv \log R_{j,t}$ & $\beta \equiv e^{-\delta}$

$r_{1,t+1} = \delta + \gamma\left(a+\mu\right) - \frac{\gamma^2\sigma^2}{2}$, if $s_t = 0$ & $r_{1,t+1} = \delta + \gamma\left(a+b+\mu\right) - \frac{\gamma^2\sigma^2}{2}$ if $s_t = 1$

(e) $\frac{1}{R_{1,t+2}} = \beta e^{-\gamma(a+\mu)+\frac{\gamma^2\sigma^2}{2}}$ if $s_{t+1} = 0$ & $\frac{1}{R_{1,t+2}} = \beta e^{-\gamma(a+b+\mu)+\frac{\gamma^2\sigma^2}{2}}$ if $s_{t+1} = 1$

$cov_t\left\{\beta \frac{u'(C_{t+1})}{u'(C_t)}, \frac{1}{R_{1,t+2}}\right\} = E_t\left[\beta \frac{u'(C_{t+1})}{u'(C_t)} \frac{1}{R_{1,t+2}}\right] - \frac{1}{R_{1,t+1}} E_t\left[\frac{1}{R_{1,t+2}}\right]$

since $\frac{1}{R_{2,t+1}} = \frac{1}{R_{1,t+1}} E_t\left\{\frac{1}{R_{1,t+2}}\right\} + cov_t\left\{\beta \frac{u'(C_{t+1})}{u'(C_t)}, \frac{1}{R_{1,t+2}}\right\}$

$\Longrightarrow \frac{1}{R_{2,t+1}} = \frac{1}{R_{1,t+1}} E_t\left[\frac{1}{R_{1,t+2}}\right] + E_t\left[\beta \frac{u'(C_{t+1})}{u'(C_t)} \frac{1}{R_{1,t+2}}\right] - \frac{1}{R_{1,t+1}} E_t\left[\frac{1}{R_{1,t+2}}\right]$

$\Longrightarrow \frac{1}{R_{2,t+1}} = E_t\left[\beta \frac{u'(C_{t+1})}{u'(C_t)} \frac{1}{R_{1,t+2}}\right]$

$r_{2,t+1} = 2\left(\delta + \gamma\left(a+\mu\right) - \frac{\gamma^2\sigma^2}{2}\right)\pi\left(0 \mid 0\right) + (\gamma b + 2)\left(\delta + \gamma\left(a+\mu\right) - \frac{\gamma^2\sigma^2}{2}\right)\pi\left(1 \mid 0\right)$ if $s_t = 0$

$$r_{2,t+1} = 2\left(\delta + \gamma\left(a+b+\mu\right) - \frac{\gamma^2\sigma^2}{2}\right)\pi\left(1 \mid 1\right) + \left(\gamma b + 2\right)\left(\delta + \gamma\left(a+\mu\right) - \frac{\gamma^2\sigma^2}{2}\right)\pi\left(0 \mid 1\right)$$

if $s_t = 1$

(f) If $b = 0$ the returns on the short term and long term bonds are equal. If $b > 0$ they are different. The return on the long term bond is higher if $s_t = 0$ and lower if $s_t = 1$. Also the returns (both short and long) are procyclical, since they are higher if $s_t = 1$. The returns are higher if $s_t = 1$ and $\pi\left(s_{t+1} = 1 \mid s_t = 1\right)$ is larger.

2.

Investor i's problem is

$$\max_{\{C_{i,t},\theta_{i,k,t}\}} E_0 \sum_{t=0}^{\infty} e^{-\delta t} \frac{\left(C_{i,t}\right)^{1-\gamma} - 1}{1-\gamma}$$

$$\text{s.t. } \sum_{k=0}^{N} P_{k,t}\theta_{i,k,t} + C_{i,t} = \sum_{k=0}^{N} \theta_{i,k,t-1}\left(P_{k,t} + D_{k,t}\right)$$

$$\text{for all } t.$$

first order condition

$$e^{-\delta t}\left(C_{i,t}\right)^{-\gamma} = \lambda_t$$

$$P_{k,t}\lambda_t = E_t \lambda_{t+1}\left[P_{k,t+1} + D_{k,t+1}\right]$$

or

$$1 = E_t\left[\frac{\lambda_{t+1}}{\lambda_t}\frac{P_{k,t+1} + D_{k,t+1}}{P_{k,t}}\right]$$

$\Longrightarrow$

$$1 = E_t\left[e^{-\delta}\left(\frac{C_{i,t+1}}{C_{i,t}}\right)^{-\gamma} R_{k,t+1}\right]$$

Replacing for $\frac{C_{i,t+1}}{C_{i,t}}$

$$1 = E_t\left[e^{-\delta}\exp\ln\left(\frac{C_{i,t+1}}{C_{i,t}}\right)^{-\gamma} R_{k,t+1}\right]$$

$$1 = E_t\left[e^{-\delta}\exp\left[-\gamma\ln\left(\frac{C_{i,t+1}}{C_{i,t}}\right)\right] R_{k,t+1}\right]$$

$$1 = E_t\left[e^{-\delta}\exp\left[-\gamma\left(g_{t+1} - \frac{1}{2}\sigma^2_{\varepsilon,t+1} + \varepsilon_{i,t+1}\right)\right] R_{k,t+1}\right]$$

$$1 = E_t\exp\left[-\delta - \gamma\varepsilon_{i,t+1} + \gamma\frac{1}{2}\sigma^2_{\varepsilon,t+1} - \gamma g_{t+1} + \ln R_{k,t+1}\right]$$

$$1 = E_t\exp\left[-\delta + \left(\gamma^2 + \gamma\right)\frac{1}{2}\sigma^2_{\varepsilon,t+1} - \gamma\ln\left(\frac{C_{t+1}}{C_t}\right) + \ln R_{k,t+1}\right] \qquad (1)$$

$$1 = E_t\exp\left[\ln m_{t+1} + \ln R_{k,t+1}\right]$$

$$1 = E_t \left(m_{t+1} R_{k,t+1} \right)$$

thus: $m_{t+1} = e^{-\delta} \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} e^{\frac{\left(\gamma^2+\gamma\right)}{2} \sigma^2_{\varepsilon,t+1}}$

For instance if the cross-sectional conditional variance is

$$\sigma^2_{\varepsilon,t+1} = a \ln \left(\frac{C_{t+1}}{C_t} \right), \text{ for } a \in R$$

then (1) becomes:

$$1 = E_t \exp \left[-\delta + \left(\left(\gamma^2 + \gamma \right) \frac{a}{2} - \gamma \right) \ln \left(\frac{C_{t+1}}{C_t} \right) + \ln R_{k,t+1} \right]$$

or

$$1 = E_t \left[e^{-\delta} \left(\frac{C_{t+1}}{C_t} \right)^{-\vec{\gamma}} R_{k,t+1} \right] \tag{2}$$

where $\vec{\gamma} = \gamma - \left(\gamma^2 + \gamma \right) \frac{a}{2}$

This equation of consumption is similar to the equation of a representative-consumer economy, but with the risk aversion coefficient modified. If an econometrician were to estimate a representative agent's pricing equation without explicitly accounting for consumer heterogeneity, the econometrician would be either overestimating or underestimating the subjective discount rate and the risk aversion coefficient. For instance if the variance of the cross-sectional distribution of consumption growth increases in economic downturns ($C_t/C_{t-1} < 1$, i.e. $a < 0$) the econometrician who does not take into account the consumer heterogeneity in estimating equation (2) would be overestimating the true risk aversion coefficient. In other words, the risk aversion that must be used for pricing, $\vec{\gamma}$, is larger than γ, which might help explain puzzles, like the risk premium puzzle.