Solution Topics for the Exam

(2 pts) 1. Consider a representative agent economy with N assets. The dividend of asset i is $D_{i,t}$ and its price is

$$P_{i,t} = E_t \left\{ \sum_{s=0}^{\infty} \frac{\beta^{1+s} u'(C_{t+1+s})}{u'(C_t)} D_{i,t+1+s} \right\}, \ \beta < 1, \ \text{for } i = 1, ..., N, \quad (1)$$

where $u(C_t)$ is the period t utility of the representative agent. Use expression (1) to get

$$P_{i,t} = E_t \left\{ \frac{\beta u'(C_{t+1})}{u'(C_t)} (D_{i,t+1} + P_{i,t+1}) \right\}, \ \text{for } i = 1, ..., N. \quad (2)$$

Answer:

$$P_{i,t} = E_t \left\{ \frac{\beta u'(C_{t+1})}{u'(C_t)} D_{i,t+1} \right\} + E_t \left\{ \sum_{s=1}^{\infty} \frac{\beta^{1+s} u'(C_{t+1+s})}{u'(C_t)} D_{i,t+1+s} \right\},$$

where

$$E_t \left\{ \sum_{s=1}^{\infty} \frac{\beta^{1+s} u'(C_{t+1+s})}{u'(C_t)} D_{i,t+1+s} \right\} = E_t \left\{ \frac{\beta u'(C_{t+1})}{u'(C_t)} \right\} \left\{ \sum_{s=0}^{\infty} \frac{\beta^{1+s} u'(C_{t+2+s})}{u'(C_{t+1})} D_{i,t+2+s} \right\}$$

$$= E_t \left\{ \frac{\beta u'(C_{t+1})}{u'(C_t)} P_{i,t+1} \right\}$$

$$\Longrightarrow P_{i,t} = E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} (D_{i,t+1} + P_{i,t+1}) \right\} \ \text{for } i = 1, ..., N, \ \text{all } t$$

(2 pts) 2. Use expression (2) to show that

$$1 = E_t \{ m_{t+1} R_{i,t+1} \}, \quad (3)$$

where m_{t+1} and $R_{i,t+1}$ denote the stochastic discount factor and the rate of return on asset i, respectively.

Answer: The rate of return on asset i is

$$R_{i,t+1} = \frac{D_{i,t+1} + P_{i,t+1}}{P_{i,t}}$$

and the discount factor is

$$m_{t+1} \equiv \beta \frac{u'(C_{t+1})}{u'(C_t)}$$

From (2) get

$$1 = E_t\left\{m_{t+1}\left(\frac{D_{i,t+1}+P_{i,t+1}}{P_{i,t}}\right)\right\}$$
$$= E_t\left(m_{t+1}R_{i,t+1}\right)$$

(2 pts) 3. Let asset N be a risk free asset. Its return, $R_{N,t+1}$, is known at time t. Show using (3) that

$$E_t\left(R_{i,t+1}\right) - R_{N,t+1} = -R_{N,t+1}cov_t\left(m_{t+1}, R_{i,t+1}\right), \text{ for } i = 1, ..., N-1 \tag{4}$$

Answer:

$$1 = E_t\left(m_{t+1}\right)E_t\left(R_{i,t+1}\right) + cov_t\left(m_{t+1}, R_{i,t+1}\right)$$

since $E_t\left(m_{t+1}\right) = \dfrac{1}{R_{N,t+1}}$

$$1 = \frac{E_t\left(R_{i,t+1}\right)}{R_{N,t+1}} + cov_t\left(m_{t+1}, R_{i,t+1}\right)$$

$$E_t\left(R_{i,t+1}\right) - R_{N,t+1} = -R_{N,t+1}cov_t\left(m_{t+1}, R_{i,t+1}\right)$$

(2 pts) 4. Use equation (4) to show that the return on the riskless asset can be lower than the expected return in a risky asset.

Answer:

$$E_t\left(R_{i,t+1}\right) > R_{N,t+1} \text{ if } cov_t\left(m_{t+1}, R_{i,t+1}\right) < 0$$

(2 pts) 5. Provide the economic intuition for the result in question (4).

Answer:

Assets whose returns covary positively with consumption make consumption more volatile, and so must promise higher expected returns to induce investors to hold them. Thus, assets that covary negatively with consumption have expected rates of return that are lower than the risk-free rate, since

$$cov_t\left(m_{t+1}, R_{i,t+1}\right) = \frac{\beta}{u'(C_t)}cov_t\left(u'(C_{t+1}), R_{i,t+1}\right) < 0$$

if and only if returns, $R_{i,t+1}$, covary negatively with the marginal utility of consumption, $u'(C_{t+1})$, or equivalently vary positively with consumption, C_{t+1}.

(2 pts) 6. Assume now that there is an additional asset, the "market portfolio", for which condition (4) holds too. This asset pays at $t+1$ dividend, C_{t+1}, but does not pay any dividend in the following periods. At t the price of this asset is $P_{m,t}$ and its return is $R_{m,t+1} = C_{t+1}/P_{m,t}$.

Additionally, assume that preferences are quadratic, $u(C_t) = C_t - \frac{1}{2}C_t^2$, so that

$$m_{t+1} = \beta \frac{1 - C_{t+1}}{1 - C_t}.$$

Use equation (4) to show that the following equation holds

$$E_t(R_{i,t+1}) - R_{N,t+1} = \frac{cov_t(R_{i,t+1}, R_{m,t+1})}{var_t(R_{m,t+1})} [E_t(R_{m,t+1}) - R_{N,t+1}] \quad (5)$$

Answer:

From (4)

$$E_t(R_{m,t+1}) - R_{N,t+1} = -R_{N,t+1} cov_t(m_{t+1}, R_{m,t+1})$$

solving for $R_{N,t+1}$ and substitutitng in equation (4)

$$E_t(R_{i,t+1}) - R_{N,t+1} = \frac{cov_t(m_{t+1}, R_{i,t+1})}{cov_t(m_{t+1}, R_{m,t+1})} [E_t(R_{m,t+1}) - R_{N,t+1}]$$

Replacing

$$m_{t+1} = \beta \frac{1 - C_{t+1}}{1 - C_t}$$

in the equation above

$$E_t(R_{i,t+1}) - R_{N,t+1} = \frac{cov_t(C_{t+1}, R_{i,t+1})}{cov_t(C_{t+1}, R_{m,t+1})} [E_t(R_{m,t+1}) - R_{N,t+1}]$$

Dividing the numerator and denominator by $P_{m,t}$ we get

$$E_t(R_{i,t+1}) - R_{N,t+1} = \frac{cov_t(R_{m,t+1}, R_{i,t+1})}{var_t(R_{m,t+1})} [E_t(R_{m,t+1}) - R_{N,t+1}]$$

(2 points) (7) Provide the economic interpretation of equation (5).

Answer:

This equation is the familiar CAPM relation stated in the expected return/beta language. It says that the risk premium in security i is proportional to that of the market portfolio. The proportionality is equal to the ratio of the covariance between the rate of return of the market portfolio and the rate of return of security i and the variance of the rate of return of the market portfolio.

(2 points) (8) Now assume the Campbell and Cochrane (1999) preferences

$$u(C_t, X_t) = \frac{(C_t - X_t)^{1-\gamma}}{1 - \gamma}.$$

The stochastic discount factor for these preferences is

$$m_{t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \left(\frac{S_{t+1}}{S_t} \right)^{-\gamma},$$

where $S_t = \frac{C_t - X_t}{C_t}$. The habit, X_t, is determined by the past trend of consumption. Let lower case letters denote logs, $c_t = \ln C_t$ and $s_t = \ln S_t$. Additionally, assume

$$\ln C_t - \ln C_{t-1} = \Delta c_t = g + v_t, \text{ with } v_t \sim \text{iid } N(0, \sigma_v^2)$$

and

$$\Delta s_{t+1} = -\theta \left(s_t - \overline{s} \right) + \lambda \left(s_t \right) v_{t+1}$$

where θ, $\overline{s}$, and g are parameters, all greater than zero, and $\lambda \left(s_t \right)$ a function of s_t. What is the expression for the riskless interest rate?

Answer:

$$\begin{aligned} m_{t+1} &= \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \left(\frac{S_{t+1}}{S_t} \right)^{-\gamma} \\ &= \beta \left(e^{g + v_{t+1}} \right)^{-\gamma} \left(e^{-\theta (s_t - \overline{s}) + \lambda(s_t) v_{t+1}} \right)^{-\gamma} \\ &= \beta e^{-g\gamma} e^{\theta (s_t - \overline{s})\gamma - \gamma(\lambda(s_t)+1) v_{t+1}} \end{aligned}$$

$$E_t m_{t+1} = \left\{ \beta e^{-g\gamma} e^{\theta(s_t - \overline{s})\gamma} \right\} E_t e^{-\gamma(\lambda(s_t)+1)v_{t+1}}$$

If v_{t+1} is normal then $e^{v_{t+1}}$ is lognormal. Thus, $E_t \left(e^{v_{t+1}} \right) = e^{E_t(v_{t+1}) + \frac{1}{2}\sigma_t^2(v_{t+1})}$.

$$E_t m_{t+1} = \left\{ \beta e^{-g\gamma} e^{\theta(s_t - \overline{s})\gamma} \right\} e^{\gamma^2 (\lambda(s_t)+1)^2 \frac{1}{2} \sigma_v^2}$$

The risk free interest rate is

$$R_t^f = \frac{1}{E_t m_{t+1}}$$

or

$$r_t^f = \ln R_t^f = -\ln \beta + g\gamma - \gamma\theta \left(s_t - \overline{s} \right) - \frac{1}{2} \gamma^2 \left(\lambda \left(s_t \right) + 1 \right)^2 \sigma_v^2$$

(2 pts) 9. Can these preferences explain the risk free rate puzzle ?

Answer:

In the data the risk free rate is low, around 1%, and stable. The power utility function corresponds to $\theta = \lambda \left(s_t \right) = 0$,

$$r_t^f = -\ln \beta + g\gamma - \frac{\gamma^2}{2} \sigma_v^2$$

With power utility function to satisfy the equity premium the γ must be large. This implies that a small change in g or σ_v^2 would have a large change in the risk free rate, which is something we do not see in the data. These preferences can solve the risk free rate puzzle. The large fluctuations in the risk free interest rate will not happen if the function $\lambda(s_t)$ is chosen appropriately.

(2 pts) 10. In the data the equity risk premium is countercyclical. Can these preferences explain this stylized fact? Explain.

Answer:

The equity risk premium is a positive function of the level of risk aversion of the representative agent. The relative risk aversion is

$$-C_t \frac{u_{cc}}{u_c} = \frac{\gamma C_t (C_t - X_t)^{-\gamma-1}}{(C_t - X_t)^{-\gamma}} = \frac{\gamma C_t}{C_t - X_t} = \frac{\gamma}{S_t}.$$

At the bottom of recessions $C_t \searrow X_t$ which leads to a increase in the risk aversion. On the other hand at the top of booms S_t is large and risk aversion is low.