Foundations of Financial Economics
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Final Exam
Total time: 1:30 hours. Total points: 20

Instructions:

Please sit in alternate seats. This is a closed-book, closed-note exam. Please get rid of everything but pen/pencil. In your answer explain all the steps in your reasoning. Keep answers short; I don't give more credit for long answers, and I can take points off if you add things that are wrong or irrelevant.

Formulas:

If x and y are random variables then

$$E\left(xy\right) = ExEy + cov\left(x, y\right)$$

$$\sigma^2\left(x\right) = Ex^2 - \left(Ex\right)^2$$

If x is normal distributed then $\exp(x)$ is lognormal, and

$$E\exp(x) = \exp(Ex + 0.5\sigma^2\left(x\right))$$

Brownian motion

$$z_{t+\Delta} - z_t \sim N(0, \Delta)$$

Differential

$$dz_t = lim_{\Delta \searrow 0}(z_{t+\Delta} - z_t)$$

$$dz_t^2 = dt,\ dz_t dt = 0,\ dt^\alpha = 0,\ \text{if } \alpha > 1$$

$$E_t(dz_t) = 0, var_t(dz_t) = E_t(dz_t^2) = dz_t^2 = dt$$

Ito's Lemma

$$\begin{aligned} df\left(x, t\right) &= f_t dt + f_x dx + \frac{1}{2} f_{xx} dx^2 \\ &= \left(f_t + f_x \mu_x + \frac{1}{2} f_{xx} \sigma_x^2\right) dt + f_x \sigma_x dz \end{aligned}$$

Questions:

1. Suppose that $x = [-2, 1]$ is the payoff vector of the only asset available in an economy with two states. The price of x, $p\left(x\right)$, is 2. Denote by $X \in \mathbb{R}^2$ the space of payoffs formed by x. The law of one price holds.

(2 pts) a. Obtain a discount factor $x^* \in X$ able to price all payoffs in this economy. Is this discount factor positive?

Answer:

$$x^* = \lambda x$$

and

$$p\left(x\right) = x x^{*\prime} = x x' \lambda$$

$$2 = 5\lambda \Longrightarrow \lambda = \frac{2}{5}$$

$$x^* = [-\frac{4}{5}, \frac{2}{5}]$$

(2 pts) b. Is it possible to obtain a positive discount factor $y^* \in \mathbb{R}^2_+$ that can price all payoffs in this economy? If so, calculate one and illustrate graphically. Are there any arbitrage opportunities in this economy?

Answer: Let ε be a vector that is orthogonal to X and such that the vector $y^* = x^* + \varepsilon \in \mathbb{R}^2_+$. The vector y^* is a discount vector. For instance $\varepsilon = [1, 2]$ is orthogonal to X, since the product $x\varepsilon' = 0$. Also, $y^* = [1 - \frac{4}{5}, 2 + \frac{2}{5}]$

$$p(x) = xy^{*\prime} = x(x^{*\prime} + \varepsilon') = xx^{*\prime}.$$

We proved in class that if there is a positive discount factor there are no arbitrage opportunities.

(2 pts) c. How would your results change for $x = [2, 1]$ and $p(x) = -2$. Illustrate.

Answer:

In this case there is an arbitrage opportunity because the asset pays a positive payoff in each state and has a negative cost. Also, there is no positive discount factor.

From

$$p(x) = xx^{*\prime} = xx'\lambda$$

get that $\lambda = -\frac{2}{5}$, $x^* = [-\frac{4}{5}, -\frac{2}{5}]$. Because x^* is in the negative quadrant there is no discount factor $y^* = x^* + \varepsilon \in \mathbb{R}^2_+$, where $\varepsilon \perp x^*$.

2. Consider the Gârleanu and Panageas model. In the model there are two types of people. The utility function of type A individuals is $\int_0^\infty e^{-\delta t} \frac{C_{A,t}^{1-\gamma_A}}{1-\gamma_A} dt$ and the utility function of individuals of type B is $\int_0^\infty e^{-\delta t} \frac{C_{B,t}^{1-\gamma_B}}{1-\gamma_B} dt$. Assume that $\gamma_A = 2\gamma_B$.

(2 pts) a. Obtain the consumption level of each type of individual as an implicit function of the aggregate consumption, $C_t = C_{A,t} + C_{B,t}$. How does the consumption share of each type of individual change with the aggregate consumption? Explain.

Answer:

The solution to the planner's problem gives the competitive equilibrium allocation as the first welfare theorem holds.

The planner's problem is

$$\max E_0 \int_0^\infty e^{-\delta t} \frac{C_{A,t}^{1-\gamma_A}}{1-\gamma_A} dt + \lambda E_0 \int_0^\infty e^{-\delta t} \frac{C_{B,t}^{1-\gamma_B}}{1-\gamma_B} dt$$

subject to

$$C_{A,t} + C_{B,t} = C_t$$

where λ is some positive constant.

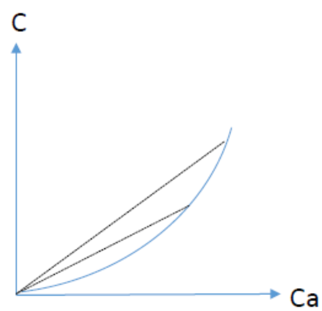
The first order condition of this problem is:

$$C_{A,t}^{-\gamma_A} = \lambda C_{B,t}^{-\gamma_B}$$

This condition together with the resource constraint determines $C_{A,t}$ and $C_{B,t}$.

$$C_{A,t} + \lambda^{\frac{1}{\gamma_B}} C_{A,t}^{\frac{\gamma_A}{\gamma_B}} = C_t \tag{1}$$
$$C_{B,t} = C_t - C_{A,t}$$

The graph of $C_{A,t} + \lambda^{\frac{1}{\gamma_B}} C_{A,t}^2$ is



The ratio $\dfrac{C_t}{C_{A,t}}$, which is given by the ray from the origin, is increasing in $C_{A,t}$. Thus, the share of type A's consumption decreases in "good times", when C_t is "large", and increases in "bad times", when C_t is "small".

(2 pts) b. In the context of this model the market Sharpe ratio obeys the following inequality:

$$\left| \frac{E_t(dR) - rdt}{\sigma_t(dR)} \right| \leq \gamma_m \sigma \left(\frac{dC_t}{C_t} \right),$$

where γ_m is the aggregate risk aversion. Let the inverse of the aggregate risk aversion be $\frac{1}{\gamma_m} \equiv \frac{1}{\gamma_B}\frac{C_{B,t}}{C_t} + \frac{1}{\gamma_A}\frac{C_{A,t}}{C_t}$. Is this formula in accordance with the empirical facts? Explain.

Remark: if you did not solve question a) assume that in the competitive equilibrium $C_{A,t}^{-\gamma_A} = C_{B,t}^{-\gamma_B}$.

Answer:

Aggregate risk aversion goes up in recessions and down in booms. Since

$$\frac{1}{\gamma_m} \equiv \frac{1}{\gamma_B}\frac{C_{B,t}}{C_t} + \frac{1}{\gamma_A}\frac{C_{A,t}}{C_t},$$

then in expansions $\frac{1}{\gamma_m}$ goes up as $\frac{C_{B,t}}{C_t}$ increases and $\frac{C_{A,t}}{C_t}$ decreases, conversely in recessions $\frac{1}{\gamma_m}$ goes down. Thus, the upper bound on the Sharpe ratio goes down in "good times" and goes up in "bad times". This is in accordance with the empirical fact that the Sharpe ratio goes down in "good times" and goes up in "bad times". Expected returns are high, prices are low, and risk premiums are high in the bottoms of recessions. Conversely, expected returns are low, prices are high, and risk premiums are low at the tops of booms.

(3 pts) c. Assume that aggregate consumption follows a Brownian motion: $\frac{dC_t}{C_t} = \mu dt + \sigma dz_t$. Obtain the standard deviations of consumption growth for each type of individual as a function of the aggregate consumption growth. Which type has a lower standard deviation of consumption growth?

Answer:

From (1)

$$d\left(C_{A,t} + \lambda^{\frac{1}{\gamma_B}} C_{A,t}^{\frac{\gamma_A}{\gamma_B}}\right) = dC_t$$

Applying Ito's Lemma

$$\left(C_{A,t} + \lambda^{\frac{1}{\gamma_B}} \frac{\gamma_A}{\gamma_B} C_{A,t}^{\frac{\gamma_A}{\gamma_B}}\right) \frac{dC_{A,t}}{C_{A,t}} + \frac{\lambda^{\frac{1}{\gamma_B}}}{2}\left(\frac{\gamma_A}{\gamma_B} - 1\right) C_{A,t}^{\frac{\gamma_A}{\gamma_B}} \frac{\gamma_A}{\gamma_B} \frac{dC_{A,t}^2}{C_{A,t}^2} = dC_t$$

Now

$$\begin{aligned}
\left(C_{A,t} + \lambda^{\frac{1}{\gamma_B}} \frac{\gamma_A}{\gamma_B} C_{A,t}^{\frac{\gamma_A}{\gamma_B}}\right) &= \frac{\gamma_A}{\gamma_B}\left(\frac{\gamma_B}{\gamma_A} C_{A,t} + \lambda^{\frac{1}{\gamma_B}} C_{A,t}^{\frac{\gamma_A}{\gamma_B}}\right) \\
&= \gamma_A\left(\frac{C_{A,t}}{\gamma_A} + \frac{C_{B,t}}{\gamma_B}\right) \\
&= \frac{\gamma_A}{\gamma_m}
\end{aligned}$$

Thus,

$$\frac{dC_{A,t}}{C_{A,t}} = \frac{\gamma_m}{\gamma_A} \frac{dC_t}{C_t} + [...]dt$$

This implies

$$\sigma\left(\frac{dC_{A,t}}{C_{A,t}}\right) = \frac{\gamma_m}{\gamma_A} \sigma\left(\frac{dC_t}{C_t}\right) \text{ and } \sigma\left(\frac{dC_{B,t}}{C_{B,t}}\right) = \frac{\gamma_m}{\gamma_B} \sigma\left(\frac{dC_t}{C_t}\right)$$

Individuals type A will have will have a lower standard deviation of consumption growth since $\frac{1}{\gamma_A} < \frac{1}{\gamma_B}$.

3. Let $R_{j,t}$ be the risk-free real gross return between periods t and $t+j$. At the beginning of t, the return $R_{j,t}$ is known with certainty. In equilibrium

$$(R_{j,t})^{-1} = E_t\left\{\frac{\beta^j u'(C_{t+j})}{u'(C_t)}\right\}, \text{ for } j = 1, ..., T.$$

Let $\Delta c_{t+1} \equiv \ln C_{t+1} - \ln C_t$. Assume that Δc_{t+1} is an i.i.d. Normal distribution with mean μ and variance σ^2. Let the preferences be $u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}$.

(2 pts) a. Obtain the analytical solution for the short-term (one-period) interest rate. Recall that the short-term rate is $(R_{1,t+1})^{-1} = E_t\left[\frac{\beta u'(C_{t+1})}{u'(C_t)}\right]$. Is the interest rate positively correlated with consumption growth?

Answer:

$$(R_{1,t})^{-1} = E_t \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\alpha} = E_t e^{\ln \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\alpha}} = \beta e^{-\alpha\mu + \frac{\alpha^2}{2}\sigma^2}$$

or

$$\ln R_{1,t} = -\ln \beta + \alpha\mu - \frac{\alpha^2}{2}\sigma^2.$$

From the formula we obtain that ceteris paribus a higher expected consumption growth is associated with a higher interest rate.

(2 pts) b. Define the long-term (two-period) interest rate as the square root of $R_{2,t+1}$. Show that long-term (two-period) interest rate satisfies the equation:

$$\frac{1}{R_{2,t}} = \frac{1}{R_{1,t}} E_t\left[\frac{1}{R_{1,t+2}}\right] + cov_t\left[\frac{\beta u'(C_{t+1})}{u'(C_t)}, \frac{\beta u'(C_{t+2})}{u'(C_{t+1})}\right].$$

Is the long-term interest rate the average of the expected short-term interest rates? Does this depend on the fact that Δc_{t+1} is i.i.d. ? What if $\gamma = 0$?

Answer:

$$\begin{aligned}
\frac{1}{R_{2,t}} &= E_t\left\{\frac{\beta^2 u'(C_{t+2})}{u'(C_t)}\right\} = E_t\left\{\frac{\beta u'(C_{t+1})}{u'(C_t)} \frac{\beta u'(C_{t+2})}{u'(C_{t+1})}\right\} \\
&= E_t\left\{\frac{\beta u'(C_{t+1})}{u'(C_t)}\right\} E_t\left\{\frac{\beta u'(C_{t+2})}{u'(C_{t+1})}\right\} + cov_t\left[\frac{\beta u'(C_{t+1})}{u'(C_t)}, \frac{\beta u'(C_{t+2})}{u'(C_{t+1})}\right]
\end{aligned}$$

or

$$\frac{1}{R_{2,t}} = \frac{1}{R_{1,t}} E_t\left[\frac{1}{R_{1,t+1}}\right] + cov_t\left[\frac{\beta u'(C_{t+1})}{u'(C_t)}, \frac{\beta u'(C_{t+2})}{u'(C_{t+1})}\right]$$

The long-term interest rate is the average of the expected short-term interest rates only if the $cov_t\left[\frac{\beta u'(C_{t+1})}{u'(C_t)}, \frac{\beta u'(C_{t+2})}{u'(C_{t+1})}\right]$ is zero. That happens if Δc_{t+1} is i.i.d. or if the utility function is linear. If the utility function is linear then $\frac{\beta u'(C_{t+1})}{u'(C_t)}$ is a constant.

(3 pts) c. Consider now that consumption growth follows the process, $\Delta c_{t+1} = \alpha s_t + \varepsilon_{t+1}$, $\alpha > 0$, where s_t is an i.i.d. process, independent from ε_{t+1}, that can take only two values, $\{0, 1\}$, with equal probability. All variables with subscript t are known at time t. The ε_{t+1} is an i.i.d. Normal distribution with mean μ and variance σ^2. Obtain the analytical solution for the long-term interest rate. Can the long-term rate be smaller than the short-term rate?

Answer:

$$
\begin{aligned}
\frac{1}{R_{2,t}} &= \beta^2 E_t \left\{ \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \left(\frac{C_{t+2}}{C_{t+1}} \right)^{-\gamma} \right\} \\
&= \beta^2 E_t \left\{ e^{-\gamma \Delta c_{t+1}} e^{-\gamma \Delta c_{t+2}} \right\} \\
&= \beta^2 E_t \left(e^{-\gamma \alpha s_t - \gamma \varepsilon_{t+1}} \right) E_t \left(e^{-\gamma \alpha s_{t+1}} e^{-\gamma \varepsilon_{t+2}} \right) \\
&= \beta^2 E_t \left(e^{-\gamma \alpha s_t - \gamma \varepsilon_{t+1}} \right) E_t \left(e^{-\gamma \alpha s_{t+1}} E_{t+1} e^{-\gamma \varepsilon_{t+2}} \right) \\
&= \beta \left(e^{-\gamma \alpha s_t} e^{-\alpha \mu + \frac{\alpha^2}{2} \sigma^2} \right) \beta \left(e^{-\alpha \mu + \frac{\alpha^2}{2} \sigma^2} E_t e^{-\gamma \alpha s_{t+1}} \right)
\end{aligned}
$$

or

$$\frac{1}{R_{2,t}} = \frac{1}{R_{1,t}} E_t \left[\frac{1}{R_{1,t+1}} \right]$$

where

$$\frac{1}{R_{1,t}} = \beta \left(e^{-\gamma \alpha s_t} e^{-\alpha \mu + \frac{\alpha^2}{2} \sigma^2} \right)$$

and

$$E_t \left[\frac{1}{R_{1,t+1}} \right] = \beta \left(e^{-\alpha \mu + \frac{\alpha^2}{2} \sigma^2} E_t e^{-\gamma \alpha s_{t+1}} \right).$$

Thus, when $s_t = 1$ then $R_{1,t} > (R_{2,t})^{1/2}$. The opposite holds when $s_t = 0$.