Foundations of Financial Economics
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Final Exam
Total time: 1:30 hours. Total points: 20
Instructions:

Please sit in alternate seats. This is a closed-book, closed-note exam. Please get rid of everything but pen/pencil. In your answer explain all the steps in your reasoning. Keep answers short; I don't give more credit for long answers, and I can take points off if you add things that are wrong or irrelevant.

Formulas:

If x and y are random variables then

$$E\left(xy\right) = ExEy + cov\left(x, y\right)$$

$$\sigma^2\left(x\right) = Ex^2 - \left(Ex\right)^2$$

If x is normal distributed then $\exp(x)$ is lognormal, and

$$E\exp(x) = \exp(Ex + 0.5\sigma^2\left(x\right))$$

Brownian motion

$$z_{t+\Delta} - z_t \sim N(0, \Delta)$$

Differential

$$dz_t = lim_{\Delta \searrow 0}(z_{t+\Delta} - z_t)$$

$$dz_t^2 = dt,\ dz_t dt = 0,\ dt^\alpha = 0,\ \text{if } \alpha > 1$$

$$E_t(dz_t) = 0, var_t(dz_t) = E_t(dz_t^2) = dz_t^2 = dt$$

Ito's Lemma

$$\begin{aligned} df\left(x, t\right) &= f_t dt + f_x dx + \frac{1}{2} f_{xx} dx^2 \\ &= \left(f_t + f_x \mu_x + \frac{1}{2} f_{xx} \sigma_x^2\right) dt + f_x \sigma_x dz \end{aligned}$$

(4 pts) 1. What is a complete market? Explain how we can complete the market with options.

Answer:

In a complete market all contingent claims are tradable. A contingent claim is an asset that pays one unit in a particular state of nature and zero in all the other states. Suppose the states of nature are indexed to the possible payoffs of a particular asset and the payoffs are discrete, specifically:

$$S_T = \left\{0, 1, 2, 3, ..., M\right\}.$$

Then, the contingent claim that pays 1 unit when $S_T = 3$ and zero when $S_T \neq 3$ is equivalent to buying 2 call options, 1 with strike price 2 and another with strike price 4, and selling 2 call options with strike price 3. The payoff of buying 1 call option with strike price 2 and selling 1 call option with strike price 3 is $(0,0,0,1,1,...)$ and the payoff of selling 1 call option with strike price 3 and buying 1 with strike price 4 is $-(0,0,0,0,1,1,...)$. The remaining M contingent claims can obtained in a similar manner.

2. Consider a model with power utility, $u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}$, and $\beta = 1$. Assume that $\ln\left(\frac{C_{t+1}}{C_t}\right) \equiv \Delta c_{t+1}$ is normal distributed. Let m_{t+1} denote the stochastic discount factor.

(2 pts) a. Show that the ratio

$$\frac{\sigma\left(m_{t+1}\right)}{E(m_{t+1})}$$

is approximately equal to $\gamma\sigma\left(\Delta c_{t+1}\right)$.

Answer:

$$m_{t+1} = \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}$$

$$\Longrightarrow m_{t+1} \text{ is lognormal distributed}$$

$$Em_{t+1} = e^{E(\log m_{t+1})+0.5\sigma^2(\log m_{t+1})}$$

$$\begin{aligned}
\sigma^2\left(m_{t+1}\right) &= E\left(m_{t+1}\right)^2 - \left(Em_{t+1}\right)^2 \\
&= e^{2E(\log m_{t+1})+2\sigma^2(\log m_{t+1})} \\
&\quad -e^{2E(\log m_{t+1})+\sigma^2(\log m_{t+1})}
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{\sigma^2\left(m_{t+1}\right)}{\left(E(m_{t+1})\right)^2} &= \frac{e^{2E(\log m_{t+1})+2\sigma^2(\log m_{t+1})} - e^{2E(\log m_{t+1})+\sigma^2(\log m_{t+1})}}{e^{2E(\log m_{t+1})+\sigma^2(\log m_{t+1})}} \\
&= e^{\sigma^2(\log m_{t+1})} - 1
\end{aligned}$$

Now use the approximation (first order Taylor approximation)

$$e^x \approx 1 + x$$

Replace above to get

$$\frac{\sigma^2\left(m_{t+1}\right)}{\left(E(m_{t+1})\right)^2} \approx \sigma^2\left(\log m_{t+1}\right),$$

or

$$\frac{\sigma\left(m_{t+1}\right)}{E(m_{t+1})} \approx \gamma\sigma\left(\Delta c_{t+1}\right).$$

(2 pts) b. Consider the market Sharpe ratio

$$\left|\frac{ER_m - R_f}{\sigma(R_m)}\right| \leq \frac{\sigma\left(m\right)}{E(m)}$$

where R_m is the return on the market portfolio. The postwar NYSE index excess return is around 8% per year, with standard deviation around 16%. The standard deviation of log consumption growth is about 1%. Is this a challenge to the model? Explain.

Answer:

$$0.5 = \frac{0.08}{0.16} = \frac{ER_m - R_f}{\sigma(R_m)} \leq \gamma\sigma\left(\Delta c_{t+1}\right) = 0.01\gamma$$

Need to assume that the relative risk aversion is extremely large, i.e. $\gamma > 50$.

(2 pts) c. Assume that $E\left(\Delta c_{t+1}\right) = 2$. What do the data and the model imply for the riskless interest rate? Discuss.

$$\left(R_{t+1}^f\right)^{-1} = E_t(m_{t+1})$$

$$\begin{aligned}\left(R_{t+1}^f\right)^{-1} &= E_t e^{\ln m_{t+1}} = E_t e^{\ln(C_{t+1}/C_t)^{-\gamma}} = E_t e^{-\gamma\Delta c_{t+1}} \\ &= e^{-\gamma E_t(\Delta c_{t+1}) + 0.5\gamma^2\sigma_t^2(\Delta c_{t+1})}\end{aligned}$$

or

$$-r_{t+1}^f \equiv -\ln R_{t+1}^f = -\gamma E_t\left(\Delta c_{t+1}\right) + 0.5\gamma^2\sigma_t^2\left(\Delta c_{t+1}\right)$$

As γ is very large then fluctuations in $E_t\left(\Delta c_{t+1}\right)$ or $\sigma_t^2\left(\Delta c_{t+1}\right)$ lead to large fluctuations in r_{t+1}^f which is something we do not observe in the data. In the data the risk free interest rate is pretty stable.

3. Consider the Epstein and Zin utility model

$$V_t \equiv \left((1-\beta)C_t^{1-\gamma} + \beta\left(H_t\left(V_{t+1}\right)\right)^{1-\gamma}\right)^{\frac{1}{1-\gamma}}$$

where, $H_t\left(V_{t+1}\right) = \left(E_t V_{t+1}^{1-\alpha}\right)^{\frac{1}{1-\alpha}}$.

(2 pts) a. Show that when $\alpha = 0$ we have the usual standard time-separable expected discounted utility with discount factor β and risk aversion γ.

Answer:

$$\begin{aligned}\left(V_t\right)^{1-\gamma} &= (1-\beta)C_t^{1-\gamma} + \beta E_t\left(V_{t+1}\right)^{1-\gamma} \\ &= (1-\beta)C_t^{1-\gamma} + \beta E_t\left\{(1-\beta)C_{t+1}^{1-\gamma} + \beta E_{t+1}\left(V_{t+1}\right)^{1-\gamma}\right\}\end{aligned}$$

By repeated substitution we obtain

$$\left(V_t\right)^{1-\gamma} = (1-\beta)\sum_{s=0}^{\infty}\beta^s E_t C_{t+s}^{1-\gamma}.$$

$$\frac{\left(V_t\right)^{1-\gamma}}{(1-\beta)(1-\gamma)} = \sum_{s=0}^{\infty}\beta^s E_t\frac{C_{t+s}^{1-\gamma}}{1-\gamma}.$$

Now, the sequence $\{C_0, C_1, ...\}$ that maximizes the right hand side also maximizes the V_t.

(2 pts) b. The stochastic discount factor with the Epstein and Zin utility is

$$m_{t+1} = \beta^{\theta} \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma\theta} R_{m,t+1}^{\theta-1}, \; \theta = \frac{1-\alpha}{1-\gamma}.$$

Assume that $\log \left(\frac{C_{t+1}}{C_t} \right) \equiv \Delta c_{t+1}$ is normal distributed, $\log (R_{m,t+1}) = r_{m,t+1}$ is normal distributed and they are independently distributed. What is the expression for the riskless interest rate?

Answer

$$\left(R_{t+1}^{f} \right)^{-1} = E_t \exp \left(\log \left(\beta^{\theta} \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma\theta} R_{m,t+1}^{\theta-1} \right) \right)$$

$$\left(R_{t+1}^{f} \right)^{-1} = e^{\log \beta^{\theta}} E_t e^{-\gamma\theta(\Delta c_{t+1})} E_t e^{-(1-\theta) r_{m,t+1}}$$

$$\left(R_{t+1}^{f} \right)^{-1} = e^{\log \beta^{\theta}} e^{-\gamma\theta E_t(\Delta c_{t+1}) + \frac{1}{2}(\gamma\theta)^2 \sigma^2(\Delta c_{t+1})} e^{-(1-\theta) E_t r_{m,t+1} + \frac{1}{2}(1-\theta)^2 \sigma^2 (r_{m,t+1})}$$

$$r_{t+1}^{f} = -\theta \log \beta + \gamma\theta E_t (\Delta c_{t+1}) - \frac{1}{2} (\gamma\theta)^2 \sigma^2 (\Delta c_{t+1}) + (1-\theta) E_t r_{m,t+1} - \frac{1}{2} (1-\theta)^2 \sigma^2 (r_{m,t+1})$$

(2 pts) c. Can these preferences explain the risk free rate puzzle?

Answer:

Unlike in the power utility function, where the relative risk aversion (RRA) was γ and the intertemporal elasticity of substitution (IES) was $\frac{1}{\gamma}$, with these preferences the RRA and the IES are 2 separate parameters. RRA=α and IES=$\frac{1}{\gamma}$. In this case by appropriately choosing these two parameters can have a high equity premium and a low and stable risk free rate.

(4 pts) 4. Let there be N assets with payoffs over S states of nature given by $\mathbf{R}_n = (R_{n1}, R_{n2}, ..., R_{nS}) \in \mathcal{R}^S$ for $n = 1, ..., N$. Let asset prices $\mathbf{P} = (P_1, P_2, ..., P_N) \in \mathcal{R}^N$ be given by $P_n = \sum_{s=1}^{S} \lambda_s R_{ns}$, for some $\lambda_s > 0$, $s = 1, ..., S$ and $n = 1, ..., N$. Show that $\mathbf{P}$ is arbitrage free. What is the relationship between the λ_s and the contingent claims?

Answer:

Suppose, to get a contradiction, that $\mathbf{P}$ is not arbitrage free. Then there is a portfolio $\mathbf{y} \in \mathcal{R}^N$ with a cost $\mathbf{yP} \leq 0$ and returns $\mathbf{yR}_s \geq 0$ for all $s = 1, ..., S$, with at least one strict inequality.

$$\mathbf{yP} = \sum_{n=1}^{N} y_n P_n = \sum_{n=1}^{N} y_n \sum_{s=1}^{S} \lambda_s R_{ns} = \sum_{s=1}^{S} \lambda_s \sum_{n=1}^{N} y_n R_{ns} = \sum_{s=1}^{S} \lambda_s \mathbf{yR}_s \leq 0$$

By assumption $\mathbf{yR}_s \geq 0$, for all $s = 1, ..., S$ and strictly larger for at least one state s. Since $\lambda_s > 0$, for all $s = 1, ..., S$, then $\sum_{s=1}^{S} \lambda_s \mathbf{yR}_s > 0$, which is a contradiction.

The λ_s are the prices of the contingent claims, or the stochastic discount factor divided by the probability of the state.