Microeconomics

Chapter 5

Cost function

Fall 2024

Cost function

The previous chapter introduced the **cost function**:

$$c(\mathbf{w}, y) = \min_{\mathbf{x}} \mathbf{wx}$$
$$\text{such that } f(\mathbf{x}) = y.$$

The **cost function** $c(\mathbf{w}, y) = \mathbf{wx}(\mathbf{w}, y)$ gives us the minimal costs for producing y units of output against input prices $\mathbf{w}$.

Where the production function describes the technological possibilities of the firm, the cost function describes its **economic possibilities**.

This chapter uses comparative statics exercises to derive insights into the cost function (and some more).

Short run versus long run

The **cost function** is,

$$c(\mathbf{w}, y) = \mathbf{w}\mathbf{x}(\mathbf{w}, y).$$

In the **short run**, some factors of production may be fixed. Let $\mathbf{x}_f$ be the fixed factors and let $\mathbf{x}_v$ be the variable factors, and break up the prices into $\mathbf{w} = (\mathbf{w}_v, \mathbf{w}_f)$. The **short-run cost function** is,

$$c(\mathbf{w}, y, \mathbf{x}_f) = \mathbf{w}_v\mathbf{x}_v(\mathbf{w}, y, \mathbf{x}_f) + \mathbf{w}_f\mathbf{x}_f.$$

where $\mathbf{x}_v(\mathbf{w}, y, \mathbf{x}_f)$ is the short-run conditional factor demand function, which now also depends upon $\mathbf{x}_f$.

Note that the short- and long run are relative concepts: They depend upon the problem that is analyzed. **The long run**, for a particular problem or firm, **is the time period over which all factors are variable**.

Various short-run cost concepts

The **short-run cost function** is,

$$c(\mathbf{w}, y, \mathbf{x}_f) = \mathbf{w}_v \mathbf{x}_v(\mathbf{w}, y, \mathbf{x}_f) + \mathbf{w}_f \mathbf{x}_f.$$

From which we can define:

$$\begin{aligned}
\text{SR total cost} = STC =& c(\mathbf{w}, y, \mathbf{x}_f), \\
\text{SR average cost} = SAC =& \frac{c(\mathbf{w}, y, \mathbf{x}_f)}{y}, \\
\text{SR average variable cost} = SAVC =& \frac{\mathbf{w}_v \mathbf{x}_v(\mathbf{w}, y, \mathbf{x}_f)}{y}, \\
\text{SR average fixed cost} = SAFC =& \frac{\mathbf{w}_f \mathbf{x}_f}{y}, \\
\text{SR marginal cost} = SMC =& \frac{\partial c(\mathbf{w}, y, \mathbf{x}_f)}{\partial y}.
\end{aligned}$$

Various long-run cost concepts

When all factors are variable, the firm can also optimize the usage of $\mathbf{x}_f$. The long-run cost function is identical to how we introduced and analyzed the cost function in Chapter 4: There were no fixed factors.

We can, however, express the long-run cost function in terms of the short-run cost function.

Let $\mathbf{x}_f(\mathbf{w}, y)$ be the optimal long-run conditional factor demand functions for the fixed factors. We can then plug $\mathbf{x}_f(\mathbf{w}, y)$ into the short-run cost function $c(\mathbf{w}, y, \mathbf{x}_f)$ to obtain the long-run cost function:

$$\begin{aligned} c(\mathbf{w}, y, \mathbf{x}_f(\mathbf{w}, y)) =& \mathbf{w}_v \mathbf{x}_v(\mathbf{w}, y, \mathbf{x}_f(\mathbf{w}, y)) + \mathbf{w}_f \mathbf{x}_f(\mathbf{w}, y) \\ =& \mathbf{w}_v \mathbf{x}_v(\mathbf{w}, y) + \mathbf{w}_f \mathbf{x}_f(\mathbf{w}, y) \\ =& \mathbf{w}\mathbf{x}(\mathbf{w}, y) \\ =& c(\mathbf{w}, y). \end{aligned}$$

Various long run cost concepts

The **long-run cost function** is,

$$c(\mathbf{w}, y) = \mathbf{w}\mathbf{x}(\mathbf{w}, y).$$

From which we can define:

$$\begin{aligned}
\text{LR total cost} = LTC = & c(\mathbf{w}, y), \\
\text{LR average cost} = LAC = & \frac{c(\mathbf{w}, y)}{y}, \\
\text{LR average variable cost} = LAVC = & LAC, \\
\text{LR average fixed cost} = LAFC = & 0, \\
\text{LR marginal cost} = LMC = & \frac{\partial c(\mathbf{w}, y)}{\partial y}.
\end{aligned}$$

Exercise

Consider a Cobb-Douglas production function where the factor x_2 is fixed at k. Hence, the cost-minimization problem is:

$$\min_{x_1} w_1 x_1 + w_2 k \quad \text{subject to } x_1^{\alpha} k^{1-\alpha} = y.$$

1. Find the short-run conditional factor demand function $x_1(\mathbf{w}, y, x_2 = k)$.
2. Find the short-run cost function $c(\mathbf{w}, y, x_2 = k)$.
3. Find the following cost curves: SAC, SAVC, SAFC, and SMC.

Exercise

In the long run the firm is also able to choose x_2. The cost-minimization problem becomes:

$$\min_{x_1, x_2} w_1 x_1 + w_2 x_2 \quad \text{subject to } x_1^{\alpha} x_2^{1-\alpha} = y.$$

1. Find the long-run conditional factor demand functions $x_1(\mathbf{w}, y)$ and $x_2(\mathbf{w}, y)$.

2. Show that $x_1(\mathbf{w}, y)$ can also be found by plugging $x_2(\mathbf{w}, y)$ into the short-run conditional factor demand function $x_1(\mathbf{w}, y, x_2 = k)$ derived in the previous exercise. That is, show that:

$$x_1(\mathbf{w}, y) = x_1(\mathbf{w}, y, x_2(\mathbf{w}, y)).$$

Long-run and short-run cost curves

For any output y, the **long-run** cost function must be weakly lower than the **short-run** cost function since the short-run cost minimization is a constrained version of the long-run cost minimization.

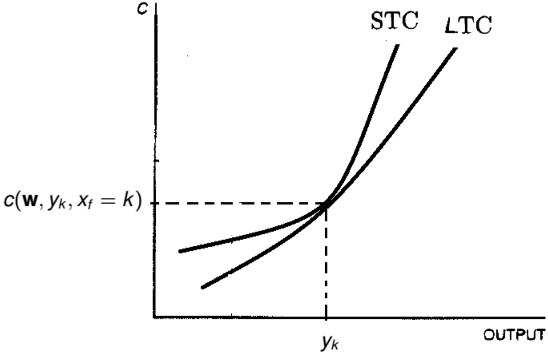
To be precise, let input x_f be fixed on the short-run at $x_f = k$, where on the long-run $x_f(\mathbf{w}, y)$ is the optimal conditional factor demand, so that:

$$\begin{aligned} STC =& c(\mathbf{w}, y, x_f = k), \\ LTC =& c(\mathbf{w}, y, x_f(\mathbf{w}, y)). \end{aligned}$$

Now let $x_f = k$ be the associated optimal conditional factor demand at $y = y_k$. Hence, $x_f(\mathbf{w}, y = y_k) = k$. Then it must be that:

$$\begin{aligned} c(\mathbf{w}, y = y_k, x_f = k) &= c(\mathbf{w}, y = y_k, x_f(\mathbf{w}, y = y_k)), \quad \text{but} \\ c(\mathbf{w}, y \neq y_k, x_f = k) &\geq c(\mathbf{w}, y \neq y_k, x_f(\mathbf{w}, y \neq y_k)). \end{aligned}$$

Long-run and short-run cost curves

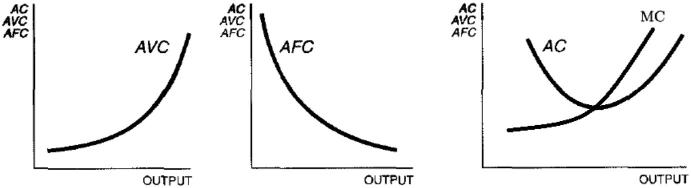


The geometry of cost curves

The **total cost** curve is increasing in y: when y increases, total costs should increase.

But how about the average total, average variable, average fixed, and marginal cost curves? When discussing the geometry of these cost curves, lets consider a time horizon where at least one factor is fixed.

The geometry of cost curves



(i) SAVC: will (eventually) increase with output y due to decreasing returns to scale: Increasing y will require increasingly more inputs. Intuition: firm encounters capacity constraints because of fixed factors.

(ii) SAFC: will decrease with output y since fixed costs can be spread over more y.

(iii) SAC: is a U-shaped curve: $SAC = SAVC + SAFC$.

(iv) SMC is equal to **SAC** in the minimum of **SAC**: if $SMC < SAC$ the SAC must be declining and if $SMC > SAC$ the SAC must be increasing.

Exercise

Consider a firm that uses two inputs x_1 and x_2 to produce one output via $y = x_1^{\alpha} x_2$. However, x_2 is fixed at k. The cost-minimization problem is:

$$\min_{x_1} w_1 x_1 + w_2 k \quad \text{subject to } x_1^{\alpha} k = y.$$

1. Find the short-run conditional factor demand function $x_1(\mathbf{w}, y, k)$.
2. Find the short-run average variable cost function (*SAVC*).
3. Show the following:
    - *SAVC* is increasing in y in case of DRTS
    - *SAVC* is constant in y in case of CRTS
    - *SAVC* is decreasing in y in case of IRTS

Exercise

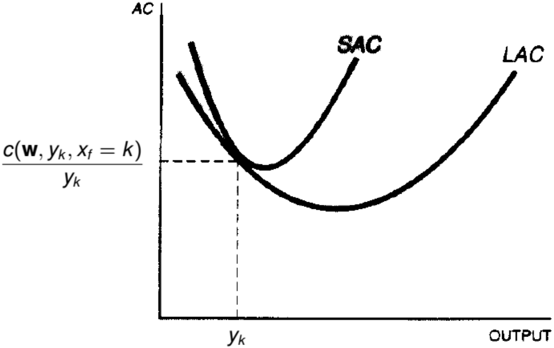
Show that SMC is equal to SAC at the point of minimum SAC. That is, let $y = y^*$ be the point of minimum SAC, and show that:

$$\frac{\partial c(y^*)}{\partial y} = \frac{c(y^*)}{y^*}.$$

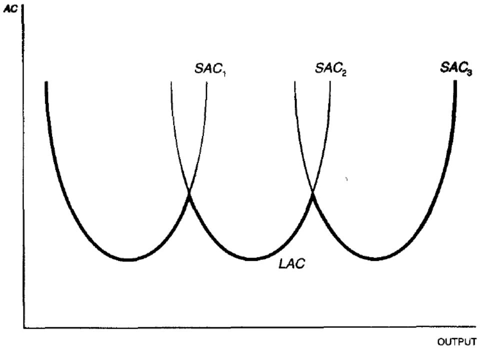
(Note that $c(y) = c(\mathbf{w}, y, \mathbf{x}_f)$ is the (short-run) cost function, but we suppressed $\mathbf{w}$ and $\mathbf{x}_f$ as they do not play a role here.)

Long-run and short-run average cost curves

We already saw that the **long-run** cost function must be weakly lower than the **short-run** cost function. This must also be true for the average cost functions.



Long-run and short-run average cost curves



Imagine that the short-run fixed input has three different levels: k_1, k_2, k_3. For instance, k_i implies i number of factories. Then we have three different *SAC*s: the SAC_i is the cost-minimizing *SAC* with the fixed input at k_i. In the long run the input is not fixed anymore, we can switch from k_1 to k_2 to k_3, so that the LAC will simply be the lower envelope of all SACs.

Conditional demand function from cost function

If you were given the conditional factor demand functions $\mathbf{x}(\mathbf{w}, y)$ finding the cost function is easy: just substitute the conditional factor demand functions into $\mathbf{wx}$:

$$\begin{aligned} c(\mathbf{w}, y) =& \mathbf{wx} \\ =& \mathbf{wx}(\mathbf{w}, y). \end{aligned}$$

It turns out that if you know the cost function, it also easy to find the conditional factor demand functions. This is what **Shephards's lemma** shows us.

Shephard's lemma

Shephard's lemma shows that we can find the conditional factor demand functions from the cost function as follows:

$$\frac{\partial c(\mathbf{w}, y)}{\partial w_i} = x_i(\mathbf{w}, y).$$

In words, **Shephard's lemma** is that the derivative of the cost function towards the input price gives us the conditional factor demand function. This is similar to Hotelling's lemma, but then in the context of constrained optimization.

Proof Shephard's lemma

Let's consider the case with two inputs: $\mathbf{x} = (x_1, x_2)$. Consider the Lagrangian,

$$\mathcal{L}(\mathbf{w}, y, \mathbf{x}, \lambda) = w_1 x_1 + w_2 x_2 - \lambda(f(\mathbf{x}) - y).$$

First, note that:

$$\frac{\partial \mathcal{L}(\mathbf{w}, y, \mathbf{x}, \lambda)}{\partial w_1} = x_1.$$

Second, substitute the conditional factor demand functions $\mathbf{x}(\mathbf{w}, y)$ and the Lagrange multiplier $\lambda(\mathbf{w}, y)$ into the Lagrangian to obtain the Lagrangian evaluated at the optimal point: $\mathcal{L}(\mathbf{w}, y, \mathbf{x}(\mathbf{w}, y), \lambda(\mathbf{w}, y)) = \mathcal{L}(\mathbf{w}, y)$. It turns out, this is equal to:

$$\begin{aligned}
\mathcal{L}(\mathbf{w}, y) =& w_1 x_1(\mathbf{w}, y) + w_2 x_2(\mathbf{w}, y) - \lambda(\mathbf{w}, y)\Big(f(\mathbf{x}(\mathbf{w}, y)) - y\Big), \\
=& w_1 x_1(\mathbf{w}, y) + w_2 x_2(\mathbf{w}, y), \\
=& \mathbf{w}\mathbf{x}(\mathbf{w}, y), \\
=& c(\mathbf{w}, y).
\end{aligned}$$

Proof Shephard's lemma

Third, use the logic of the envelope theorem to show that at the optimal point:

$$\begin{aligned}
\frac{\partial \mathcal{L}(\mathbf{w}, y)}{\partial w_1} = & \underbrace{\frac{\partial \mathcal{L}(\cdot)}{\partial w_1}}_{\text{direct effect}} + \underbrace{\frac{\partial \mathcal{L}(\cdot)}{\partial x_1}\frac{\partial x_1(\cdot)}{\partial w_1} + \frac{\partial \mathcal{L}(\cdot)}{\partial x_2}\frac{\partial x_2(\cdot)}{\partial w_1} + \frac{\partial \mathcal{L}(\cdot)}{\partial \lambda}\frac{\partial \lambda(\cdot)}{\partial w_1}}_{\text{indirect effect}}, \\
= & \frac{\partial \mathcal{L}(\cdot)}{\partial w_1}, \\
= & x_1(\mathbf{w}, y),
\end{aligned}$$

as the indirect effects are zero because of the FOCs of the Lagrangian.

Since $\mathcal{L}(\mathbf{w}, y) = c(\mathbf{w}, y)$, we conclude that:

$$\frac{\partial \mathcal{L}(\mathbf{w}, y)}{\partial w_1} = \frac{\partial c(\mathbf{w}, y)}{\partial w_1} = x_1(\mathbf{w}, y).$$

Indeed, this is similar to the proof for the Lagrange multiplier.

Exercise

Consider the following cost-minimization problem:

$$\min_{x_1, x_2} w_1 x_1 + w_2 x_2 \quad \text{subject to } x_1^{\alpha} x_2^{1-\alpha} = y.$$

In a previous exercise you have found the conditional factor demand functions $x_1(\mathbf{w}, y)$ and $x_2(\mathbf{w}, y)$, and the cost function $c(\mathbf{w}, y)$. Now show that:

$$\frac{\partial c(\mathbf{w}, y)}{\partial w_i} = x_i(\mathbf{w}, y) \quad \text{for } i = 1, 2.$$

Homework exercises

Exercises: 5.11, and exercises on the slides