Microeconomics

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About the course

Book: Hal Varian, Microeconomic Analysis, 3rd edition, Norton, International Edition, 1992.

Lectures: slides cover the book chapter by chapter. Last slide will contain an overview of the homework exercises for that chapter. We will discuss (all of) the exercises in class by using the whiteboard.

Assessment: one midterm (30%) and one final exam (70%).

Fenix: you can find all the material in the folder “All you need” (EN), including a syllabus with an overview of the program and further details.

Program

- Firms: Chapter 1 unto 5

  Ch 1 Technology

  Ch 2 Profit maximization

  Ch 3 Profit function

  Ch 4 Cost minimization

  Ch 5 Cost function

- Consumers: Chapter 7, 8 and 10

  Ch 7 Utility maximization

  Ch 8 Choice

  Ch 10 Consumers' surplus

Program

- Markets: Chapter 13, 14 and 16

Ch 13 Competitive markets

Ch 14 Monopoly

Ch 16 Oligopoly

We will use some concepts from Chapter 26 (Mathematics) and 27 (Optimization) throughout the course.

Microeconomics

Chapter 1

Technology

Fall 2024

Technology

The technology of a firm is to **use inputs as to produce outputs**. To study firms' choices we need ways to summarize their production possibilities.

One way is the production function: $y = f(\mathbf{x})$, where y is output and $\mathbf{x}$ is a vector of inputs. We often assume a certain functional form for $f(\cdot)$, such as Cobb-Douglas, CES, etc.

This chapter discusses more general ways to describe a firm's technology. It then introduces assumptions on the technology that are in line with the well-known production functions.

Production plan

Suppose the firm has n possible goods that can serve as inputs and/or outputs.

Lets define that $y_j^i > 0$ if the firm uses good j as an input. Hence, $y_j^i = 0$ if the firm does not use good j as an input. Similar for y_j^o with output.

Net output for good j: $y_j = y_j^o - y_j^i$.

Production plan: a list of net outputs of various goods, which is described by the vector **y** in R^n.

$$\mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} y_1^o - y_1^i \\ \vdots \\ y_n^o - y_n^i \end{pmatrix} = \begin{pmatrix} y_1^o \\ \vdots \\ y_n^o \end{pmatrix} - \begin{pmatrix} y_1^i \\ \vdots \\ y_n^i \end{pmatrix}$$

Production possibilities set

Production possibilities set: the set of technologically feasible production plans. This is denoted by Y, and is a subset of R^n:

$$Y = \{\mathbf{y} \text{ in } R^n : \mathbf{y} \text{ is technologically feasible}\}$$

What is technological feasible depends on the time period: some inputs may be fixed in the short run.

Short-run production possibilities set: let **z** be the short-run constraints on the inputs. Then $Y(\mathbf{z})$ denotes the feasible plans consistent with the constraints **z**. For example, consider that $y_n = \bar{y}_n$ in the short term:

$$Y(\bar{y}_n) = \{\mathbf{y} \text{ in } R^n : \mathbf{y} \text{ is technologically feasible and } y_n = \bar{y}_n\}$$

Input requirement set

Consider a firm that produces one output y, and this output is not used as an input and vice versa. Then $\mathbf{y}$ can be described by $(y, -\mathbf{x})$, where $\mathbf{x}$ is a vector of (positive) inputs that produces y units of output.

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} y_1^o - 0 \\ 0 - y_2^i \\ \vdots \\ 0 - y_n^i \end{pmatrix} = \begin{pmatrix} y_1 \\ -x_1 \\ \vdots \\ -x_n \end{pmatrix} = \begin{pmatrix} y \\ -\mathbf{x} \end{pmatrix}$$

With $(y, -\mathbf{x})$ we can define the input requirement set.

Input requirement set: the set of all input bundles $\mathbf{x}$ that can produce at least y units of outputs.

$$V(y) = \{\mathbf{x} \text{ in } R^n_+ : (y, -\mathbf{x}) \text{ is in } Y\}$$

Isoquant

Isoquant: all input bundles **x** that produce exactly y units of output.

$$Q(y) = \{\mathbf{x} \text{ in } R^n_+ : \mathbf{x} \text{ is in } V(y) \text{ but not in } V(y') \text{ for } y' > y\}$$

An isoquant is an example of a **level set**. There are many of them in economics. They reduce the dimension by one, which is useful for graphs.

Level set

Consider a function $y = f(\mathbf{x})$, a **level set** $L(\bar{y})$ for a fixed value $y = \bar{y}$ is described by:

$$L(\bar{y}) = \{\mathbf{x} : f(\mathbf{x}) = \bar{y}\}$$

For instance, you may be used to plot all the consumption bundles (x_1, x_2) that give exactly utility level $u = \bar{u}$, which is called an indifference curve of the utility function $u(x_1, x_2)$:

$$L(\bar{u}) = \{(x_1, x_2) : u(x_1, x_2) = \bar{u}\}$$

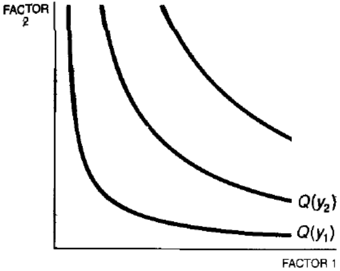
Production function

Production function: picks out the maximum output as a function of the inputs.

$$f(\mathbf{x}) = \{y \text{ in } R\text{: } y \text{ is the maximum output associated with } \mathbf{x} \text{ in } Y\}$$

Example: Cobb-Douglas technology

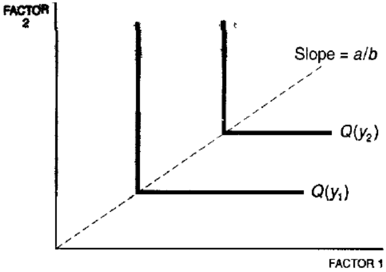
$$Y = \{(y, -x_1, -x_2) \text{ in } R^3 : x_1^a x_2^{1-a} \geq y\}$$
$$V(y) = \{(x_1, x_2) \text{ in } R_+^2 : x_1^a x_2^{1-a} \geq y\}$$
$$Q(y) = \{(x_1, x_2) \text{ in } R_+^2 : x_1^a x_2^{1-a} = y\}$$
$$f(x_1, x_2) = x_1^a x_2^{1-a}.$$



A

Example: Leontief technology

$$\begin{aligned} Y &= \{(y, -x_1, -x_2) \text{ in } R^3 : \min(ax_1, bx_2) \geq y\} \\ V(y) &= \{(x_1, x_2) \text{ in } R^2_+ : \min(ax_1, bx_2) \geq y\} \\ Q(y) &= \{(x_1, x_2) \text{ in } R^2_+ : \min(ax_1, bx_2) = y\} \\ f(x_1, x_2) &= \min(ax_1, bx_2). \end{aligned}$$



B

Exercise

Consider a Leontief production function $y = \min(ax_1, bx_2)$. Draw the short-run production possibilities set $Y(z)$ while $x_1 = z$. Hence, draw a graph that contains the following set:

$$Y(z) = \{(y, -x_2) \text{ in } R^2\text{: } \min(ax_1, bx_2) \geq y \text{ , } x_1 = z\}$$

Example of an input requirement set

Suppose that we can produce output y by using two inputs x_1 and x_2.

Technique A: one unit of x_1 and two units of x_2 produce one unit of y.
Technique B: two units of x_1 and one unit of x_2 produce one unit of y.

The input requirement set can be written as:

$$V(1) = \{(1, 2), (2, 1)\}$$

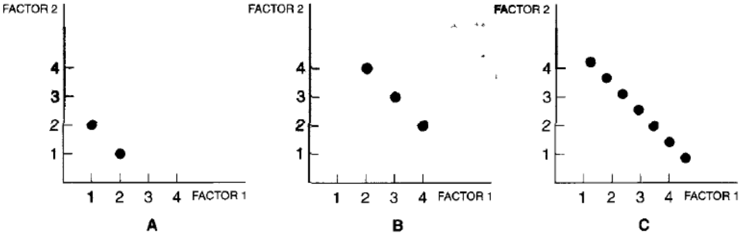
Can we write the input requirement set for $y = 2$ (and any y) units of output as follows?:

$$V(2) = \{(2, 4), (4, 2)\}$$
$$V(y) = \{(y, 2y), (2y, y)\}$$

Perhaps this is incomplete, since what about using mixtures of technique A and B? Let y_A (y_B) be the amount produced via technique A (B), then:

$$V(2) = \{(2, 4), (3, 3), (4, 2)\}$$
$$V(y) = \{(y_A + 2y_B, 2y_A + y_B) : y = y_A + y_B\}$$

Example of an input requirement set



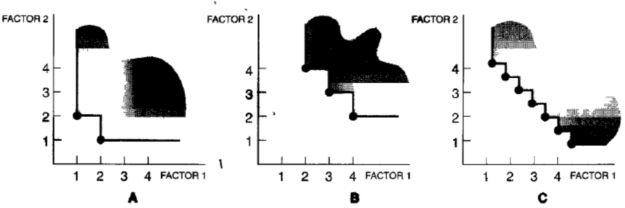
Input requirement sets. Panel A depicts $V(1)$, panel B depicts $V(2)$, and panel C depicts $V(y)$ for a larger value of y.

Monotonic input requirement set

Suppose $\mathbf{x} = (3, 2)$. We can produce one unit of output with $(1, 2)$ via technique A and have a leftover input of $(2, 0)$.

If **free disposal** of inputs is allowed, it is reasonable to assume that if we can produce y with $\mathbf{x}$, and $\mathbf{x}' \geq \mathbf{x}$, we can also produce y with $\mathbf{x}'$.

Monotonicity: if $\mathbf{x}$ is in $V(y)$, and $\mathbf{x}' \geq \mathbf{x}$, then $\mathbf{x}'$ is in $V(y)$.



Monotonicity. Here are the same three input requirement sets if we also assume monotonicity.

Convex input requirement set

Now we want to produce 100 units of output. If we replicate technique A or B 100 times (without mixture), the input requirement set is:

$$V(100) = \{(100, 200), (200, 100)\}$$

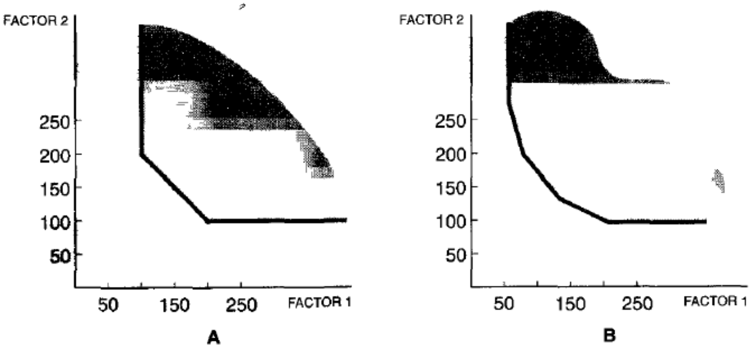
But, again, consider the possibility of a mixture of techniques. How about using technique A 25 times and B 75 times. In this case, $0.25 \times (100, 200) + 0.75 \times (200, 100) = (175, 125)$ is also in $V(100)$. More generally,

$$t \begin{pmatrix} 100 \\ 200 \end{pmatrix} + (1 - t) \begin{pmatrix} 200 \\ 100 \end{pmatrix} = \begin{pmatrix} t100 + (1 - t)200 \\ t200 + (1 - t)100 \end{pmatrix}$$

may be in $V(100)$ for $0 \leq t \leq 1$ in case a mixture of techniques is possible.

Convexity: if $\mathbf{x}$ and $\mathbf{x}'$ are in $V(y)$, and $V(y)$ is a convex set, then $t\mathbf{x} + (1 - t)\mathbf{x}'$ is in $V(y)$ for all $0 \leq t \leq 1$.

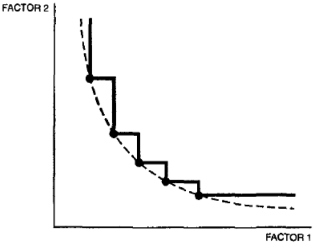
Convex input requirement set



Convex input requirement sets. If $\mathbf{x}$ and $\mathbf{x}'$ can produce y units of output, then any weighted average $t\mathbf{x} + (1-t)\mathbf{x}'$ can also produce y units of output. Panel A depicts a convex input requirement set with two underlying activities; panel B depicts a convex input requirement set with many activities.

Parametric representations of technology

We summarize the monotonic and convex input set by a “smoothed” input set. This looks like an isoquant from a Cobb-Douglas function.



Smoothing an isoquant. An input requirement set and a “smooth” approximation to it.

Do not take these functional forms literally. The engineering data describes the production plans, and we pick the functional form that best describes this data.

Concave and convex **functions** and convex **sets**

Let $\mathbf{x^1}$ and $\mathbf{x^2}$ be two points in the domain of function f. Define the following convex combinations:

$$\begin{aligned} \mathbf{x^t} &= t\mathbf{x^1} + (1-t)\mathbf{x^2} \\ y^t &= tf(\mathbf{x^1}) + (1-t)f(\mathbf{x^2}) \end{aligned}$$

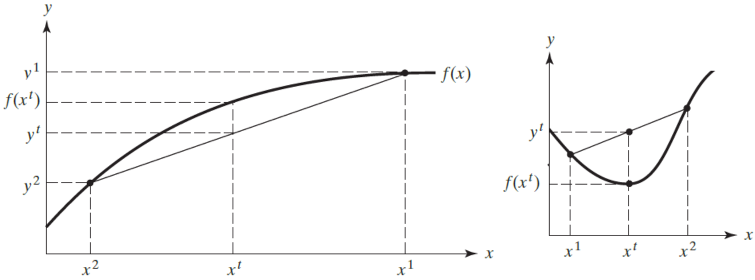
Concave function: f is a concave function if for all $\mathbf{x^1}$ and $\mathbf{x^2}$ we have that for $0 \leq t \leq 1$

$$f(\mathbf{x^t}) \geq y^t$$

Convex function: f is a convex function if for all $\mathbf{x^1}$ and $\mathbf{x^2}$ we have that for $0 \leq t \leq 1$

$$f(\mathbf{x^t}) \leq y^t$$

Concave and convex **functions** and convex **sets**



Concave (left) and convex (right)

Relationship between concave and convex functions and convex sets:

- f is concave $\leftrightarrow$ points on and below the graph form a convex set
- f is convex $\leftrightarrow$ points on and above the graph form a convex set

Concave and convex **functions** and convex **sets**

Upper contour set for function f: let $f(\bar{\mathbf{x}}) = \bar{y}$ for some $\mathbf{x} = \bar{\mathbf{x}}$, then

$$\{\mathbf{x}\colon f(\mathbf{x}) \geq \bar{y}\}$$

Lower contour set for function f: let $f(\bar{\mathbf{x}}) = \bar{y}$ for some $\mathbf{x} = \bar{\mathbf{x}}$, then

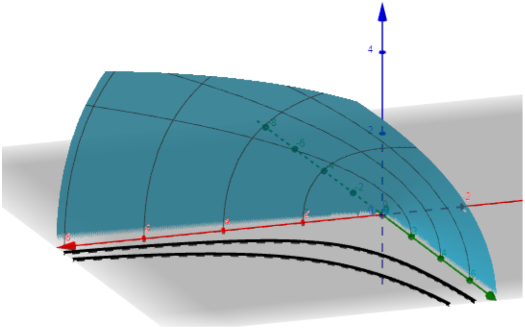
$$\{\mathbf{x}\colon f(\mathbf{x}) \leq \bar{y}\}$$

Further relationships between concave and convex functions and convex sets:

- f is concave $\leftrightarrow$ upper contour set is a convex set
- f is convex $\leftrightarrow$ lower contour set is a convex set

Let $f(\mathbf{x})$ be a production function, then we defined the upper contour set as the input requirement set. It follows that a **(quasi)concave production function** $f(\mathbf{x})$ like a cobb-douglas has a **convex input requirement set**.

The cobb-douglas function in 3D



The figure shows $y = x_1^{0.5} x_2^{0.5}$. y is blue axis, x_1 is red axis, and x_2 is green axis. The dark lines show the 2D isoquants with $y = 2$ and $y = 3$. The cobb-douglas function is **concave** and so it has a **convex input requirement set**.

Marginal productivity

Consider a setting with two inputs, so that $f(\mathbf{x}) = f(x_1, x_2)$.

Marginal productivity of input 1 or 2: how much does output change if we change input 1 or 2.

$$MP_i = \frac{\partial f(\mathbf{x})}{\partial x_i}, \quad \text{for } i = 1, 2.$$

The technical rate of substitution

Technical rate of substitution: how easy (or difficult) is it for a firm to change between the usage of x_1 and x_2 while keeping output y constant?

Let $x_2(x_1)$ be the isoquant at constant output $y = \bar{y}$, then:

$$TRS = \frac{\partial x_2(x_1)}{\partial x_1}.$$

The function $x_2(x_1)$ satisfies the identity $f(x_1, x_2(x_1)) = \bar{y}$, so that the **total derivative** towards x_1 is zero:

$$\frac{df(\mathbf{x})}{dx_1} = \frac{\partial f(\mathbf{x})}{\partial x_1} + \frac{\partial f(\mathbf{x})}{\partial x_2}\frac{\partial x_2(x_1)}{\partial x_1} = 0.$$

Hence, we can get an expression for the TRS without having to find $x_2(x_1)$:

$$TRS = \frac{\partial x_2(x_1)}{\partial x_1} = -\frac{\frac{\partial f(\mathbf{x})}{\partial x_1}}{\frac{\partial f(\mathbf{x})}{\partial x_2}} = -\frac{MP_1}{MP_2}.$$

Partial and total derivative

The variable x_1 enters $f(x_1, x_2(x_1))$ twice: directly and indirectly via x_2. This brings us two types of derivatives.

Partial derivative: How does $f(x_1, x_2(x_1))$ change when x_1 changes while keeping x_2 fixed.

$$\frac{\partial f(\mathbf{x})}{\partial x_1}$$

Total derivative: How does $f(x_1, x_2(x_1))$ change when x_1 changes while also allowing x_2 to change.

$$\frac{df(\mathbf{x})}{dx_1} = \frac{\partial f(\mathbf{x})}{\partial x_1} + \frac{\partial f(\mathbf{x})}{\partial x_2}\frac{\partial x_2(x_1)}{\partial x_1}$$

Two observations: (1) If $x_2(x_1)$ is constant, so that $x_2(x_1) = x_2$, then the partial derivative is equal to the total derivative. **(2)** If $x_2(x_1)$ is not constant, but you substituted for $x_2(x_1)$ in $f(\cdot)$, so that $f(x_1, x_2(x_1)) = f(x_1)$, taking the partial derivative gives you the total derivative.

Exercise

1. What is the technical rate of substitution of the following cobb-douglas production function?

$$f(\mathbf{x}) = x_1^{\alpha} x_2^{1-\alpha}$$

2. Draw the graph of an isoquant that has a technical rate of substitution of zero. What does this mean for the production function?

The elasticity of substitution

Elasticity of substitution (σ): the percentage change in the input ratio divided by the percentage change in the TRS, with output being held fixed.

$$\sigma = \frac{\Delta(\frac{x_2}{x_1})}{\frac{x_2}{x_1}} \Big/ \frac{\Delta TRS}{TRS}$$

Rewrite, take the limit of this expression as Δ goes to zero, and use the logarithmic derivative to find an easy expression for σ:

$$\sigma = \frac{TRS}{\frac{x_2}{x_1}} \frac{\Delta(\frac{x_2}{x_1})}{\Delta TRS} = \frac{TRS}{\frac{x_2}{x_1}} \frac{\partial(\frac{x_2}{x_1})}{\partial TRS} = \frac{\partial ln(\frac{x_2}{x_1})}{\partial ln(TRS)}$$

In general **elasticities** are the percentage change in y due to the percentage change in x, and can be calculated by the logarithmic derivative.

Exercise

What is the elasticity of substitution of the following cobb-douglas production function?

$$f(\mathbf{x}) = x_1^{\alpha} x_2^{1-\alpha}$$

Returns to scale

It sounds reasonable that if we scale the inputs by some amount t, we also will produce t times as much output. After all, we can replicate what we did initially t times.

Constant returns to scale: $f(t\mathbf{x}) = tf(\mathbf{x})$ for all $t \geq 0$.

Does scaling the inputs allows for more efficient means of production?

Increasing returns to scale: $f(t\mathbf{x}) > tf(\mathbf{x})$ for all $t > 1$.

Does scaling the inputs allows for less efficient means of production?

Decreasing returns to scale: $f(t\mathbf{x}) < tf(\mathbf{x})$ for all $t > 1$.

Homogeneous and homothetic technologies

Homogenous functions: a function $f(\mathbf{x})$ is homogenous of degree k if $f(t\mathbf{x}) = t^k f(\mathbf{x})$.

A function f has constant returns to scale $\leftrightarrow$ a function f is homogeneous of degree 1.

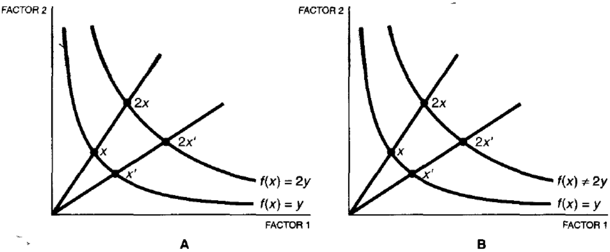
A function $g : R \to R$ is a **positive monotonic transformation** when $h(\mathbf{x}') > h(\mathbf{x})$ implies $g(h(\mathbf{x}')) > g(h(\mathbf{x}))$.

Note that $h(\mathbf{x}') = y'$ and $h(\mathbf{x}) = y$ are numbers in R, so it is simply a transformation that preserves the original order of y.

Homothetic functions: a monotonic transformation of a function that is homogeneous of degree 1.

A function $f(\mathbf{x})$ is homothetic $\leftrightarrow$ $f(\mathbf{x}) = g(h(\mathbf{x}))$, where h is homogeneous of degree 1 and g is a monotonic function.

Homogeneous and homothetic technologies



Homogeneous and homothetic functions. Panel *A* depicts a function that is homogeneous of degree 1. If $\mathbf{x}$ and $\mathbf{x}'$ can both produce y units of output, then $2\mathbf{x}$ and $2\mathbf{x}'$ can both produce $2y$ units of output. Panel *B* depicts a homothetic function. If $\mathbf{x}$ and $\mathbf{x}'$ produce the *same* level of output, y, then $2\mathbf{x}$ and $2\mathbf{x}'$ can produce the *same* level of output, but not necessarily $2y$.

For homogenous and homothetic functions the isoquants are "blown up" versions of a single isoquant. Moreover, for either of these functions the TRS is independent of the scale of production.

Exercise

1. Proof the following statement: if $f(\mathbf{x})$ is homogeneous of degree $k \geq 1$, then $\frac{\partial f(\mathbf{x})}{\partial x_i}$ is of homogeneous degree $k - 1$.

2. Use your answer to the previous question to show that the TRS of a production function with constant returns to scale is independent of the scale of production.

Homework exercises

Exercises: 1.2, 1.3, 1,8, 1.10 (show only that Y must be convex) of the book, and exercises on the slides.

All homework exercises are relevant for the midterm and exam.