# Microeconomics

## Chapter 7

## Utility maximization

Fall 2024

# Consumer behavior

In the **theory of the firm**, firm behavior was modeled based on profit-maximizing or cost-minimizing decisions, while considering the underlying technological constraints.

In the **theory of the consumer**, consumer behavior will be modeled based on utility-maximizing decisions, while considering the underlying economic constraints.

# Consumer preferences

If we want to analyze utility-maximizing behavior, we first need to discuss **consumer preferences**. And to discuss preferences we need to define **consumption bundles** and a **consumption set**.

**Consumption bundle**: a list of consumption goods, described by the vector **x** in $R_+^k$, where $k$ is the number of different goods, and element $x_i \geq 0$ reflects the specific consumption for good $i = 1, ..., k$.

**Consumption set**: the set of all possible consumption bundles **x** that a consumer can hypothetically choose. This set is denoted by X,

$$X = \{\mathbf{x} \text{ in } R_+^k : \mathbf{x} \text{ can be hypothetically chosen}\}.$$

# Consumer preferences

The consumer is assumed to have **preferences** on **x** in X. We assume that those preferences are represented by a binary relationship.

Let $\mathbf{x} \neq \mathbf{y}$ be two bundles in X, then the binary relationships are:

| | |
|---|---|
| $\mathbf{x} \succ \mathbf{y}$ | **x** is preferred to **y** |
| $\mathbf{x} \succeq \mathbf{y}$ | **x** is weakly preferred to **y** |
| $\mathbf{x} \sim \mathbf{y}$ | indifferent between **x** and **y** |
| $\mathbf{x} \preceq \mathbf{y}$ | **y** is weakly preferred to **x** |
| $\mathbf{x} \prec \mathbf{y}$ | **y** is preferred to **x** |

# Assumptions on preferences

**1. Completeness:** for all $\mathbf{x}$ and $\mathbf{y}$ in X, either $\mathbf{x} \succeq \mathbf{y}$ or $\mathbf{y} \succeq \mathbf{x}$ or both.

This ensures that the consumer can make comparisons: The consumer has the ability and information to evaluate alternatives.

**2. Transitivity:** for all $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ in X, if $\mathbf{x} \succeq \mathbf{y}$ and $\mathbf{y} \succeq \mathbf{z}$, then $\mathbf{x} \succeq \mathbf{z}$.

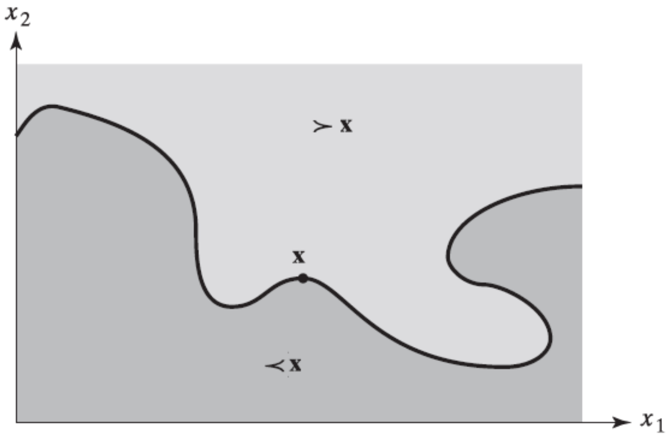
This ensures that choices are consistent: pairwise comparisons are linked together in a consistent way.

Completeness and transitivity imply that the consumer can completely rank any finite number of $\mathbf{x}$ in X, from best to worst, possibly with ties.

**3. Continuity:** for all $\mathbf{y}$ in X, the sets $\{\mathbf{x} : \mathbf{x} \succeq \mathbf{y}\}$ and $\{\mathbf{x} : \mathbf{x} \preceq \mathbf{y}\}$ are closed sets.

This ensures the absence of discontinuous consumption behavior. Continuity mostly speaks to the mathematical aspects of representing preferences by a utility function.

# Consumer preferences in a graph




The graph above reflects a hypothetical set of preferences over bundles $\mathbf{x} = (x_1, x_2)$ that satisfy the three assumptions: Completeness, transitivity, and continuity (and local non-satiation).

# Additional assumptions on preferences

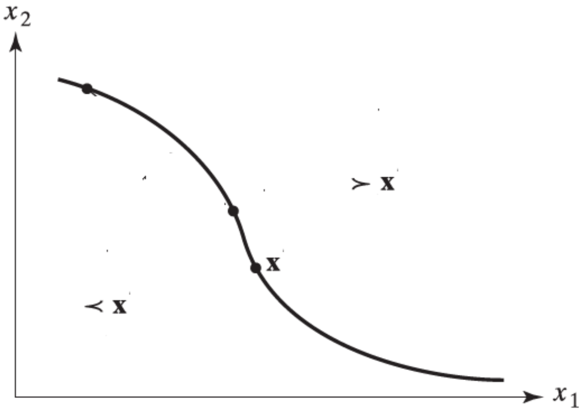
**4. Monotonicity:** if $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$ then $\mathbf{x} \succ \mathbf{y}$.

This implies that at least as much of every good, and strictly more of some good(s), is strictly preferred. If free disposal of unwanted goods is allowed, then this assumption seems quite harmless.

**5. Convexity:** given $\mathbf{x} \neq \mathbf{y} \neq \mathbf{z}$ in X, if $\mathbf{x} \succeq \mathbf{z}$ and $\mathbf{y} \succeq \mathbf{z}$, then $t\mathbf{x} + (1-t)\mathbf{y} \succ \mathbf{z}$ for all $0 < t < 1$.

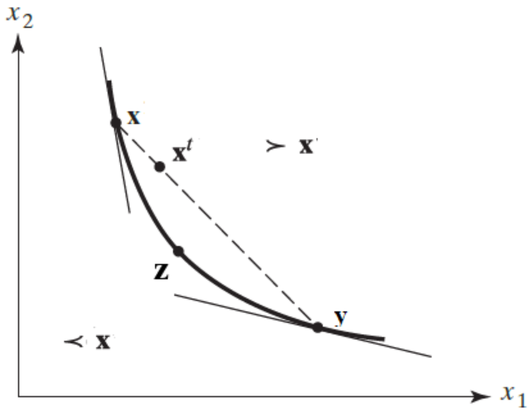
This implies that a consumers prefers a balanced consumption bundle instead of an extreme bundle: The weighted average (or mixture) of bundles $t\mathbf{x} + (1-t)\mathbf{y}$ is preferred to the extreme bundle $\mathbf{z}$.

# Consumer preferences in a graph




The graph above reflects a hypothetical set of preferences over bundles $\mathbf{x} = (x_1, x_2)$ that satisfy the four assumptions: Completeness, transitivity, continuity, and monotonicity.

# Consumer preferences in a graph




The graph above reflects a hypothetical set of preferences over bundles $\mathbf{x} = (x_1, x_2)$ that satisfy the five assumptions: Completeness, transitivity, continuity, monotonicity, and convexity.

# From preferences to utility function

Under the three assumptions completeness, transitivity and continuity, there exists a continuous **utility function** that represent those preferences.

**Existence of a utility function:** if preferences are complete, transitive, and continuous, there exists a utility function $u(\mathbf{x}) : R_+^k \to R$ that represents those preferences. That is, there exists a utility function $u(\mathbf{x})$ that satisfies $u(\mathbf{x}) \geq u(\mathbf{y}) \leftrightarrow \mathbf{x} \succeq \mathbf{y}$.

A utility function $u(\mathbf{x})$ is a convenient way to describe preferences. In particular, now we can aim to find the consumption bundle $\mathbf{x}$ that maximizes utility subject to economic constraints.

The additional two assumptions of monotonicity and convexity guarantee that the SOCs of such a constrained maximization problem are met.

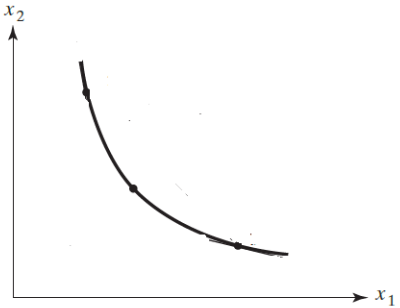
# Indifference curve

A utility function $u(\mathbf{x})$ is often represented by an indifference curve. This is simply a level set for the utility function.

**Indifference curve:** all consumption bundles that give utility level $u$,

$$I(u) = \{\mathbf{x} : u(\mathbf{x}) = u\}.$$

An indifference curve for the consumer is analogous to the isoquant for the firm.



# The ordinal character of utility functions

The only relevant feature of a utility function is its **ordinal character**: if some function $u(\mathbf{x})$ represents a consumer's preferences, then so will the function $u'(\mathbf{x}) = u(\mathbf{x}) + 5$. That is, if $u(\mathbf{x})$ satisfies $u(\mathbf{x}) \geq u(\mathbf{y}) \leftrightarrow \mathbf{x} \succeq \mathbf{y}$ then $u'(\mathbf{x})$ also satisfies $u'(\mathbf{x}) \geq u'(\mathbf{y}) \leftrightarrow \mathbf{x} \succeq \mathbf{y}$ .

No interpretation should be given to the actual numbers that are given by $u(\mathbf{x})$, only to the ordering of those numbers.

The above implies that the **utility function is invariant to positive monotonic transformations**: let $u(\mathbf{x})$ be a utility function that represents a consumer's preferences. Then $g(u(\mathbf{x}))$ also represents that consumer's preferences if $g : R \to R$ is a positive monotonic transformation: $u(\mathbf{x}) > u(\mathbf{y})$ implies $g(u(\mathbf{x})) > g(u(\mathbf{y}))$.

# Exercise

Assume completeness, transitivity, continuity, monotonicity, and convexity.

1. Show that indifference curves must be downwards sloping.

2. Show that two indifference curves cannot cross.

3. Show that indifference curves become less steep as we move downward and to the right along them.

# Marginal utility

Consider a setting with two goods, so that $u(\mathbf{x}) = u(x_1, x_2)$.

**Marginal utility** of good 1 or 2: how much does utility change if we change the consumption of good 1 or 2.

$$MU_i = \frac{\partial u(\mathbf{x})}{\partial x_i}, \quad \text{for } i = 1, 2.$$

# Marginal rate of substitution

**Marginal rate of substitution**: How easy (or difficult) is it for a consumer to change between the consumption of $x_1$ and $x_2$ while keeping utility constant?

Let $x_2(x_1)$ be the indifference curve at utility level $u = \bar{u}$, then:

$$MRS = \frac{\partial x_2(x_1)}{\partial x_1}.$$

$x_2(x_1)$ traces all bundles such that $u(x_1, x_2) = \bar{u}$. Hence, $x_2(x_1)$ satisfies the identity $u(x_1, x_2(x_1)) = \bar{u}$, so that the total derivative towards $x_1$ is zero:

$$\frac{\partial u(\mathbf{x})}{\partial x_1} + \frac{\partial u(\mathbf{x})}{\partial x_2}\frac{\partial x_2(x_1)}{x_1} = 0.$$

Hence, we can get an expression for the MRS without having to find $x_2(x_1)$:

$$MRS = \frac{\partial x_2(x_1)}{\partial x_1} = -\frac{\frac{\partial u(\mathbf{x})}{\partial x_1}}{\frac{\partial u(\mathbf{x})}{\partial x_2}} = -\frac{MU_1}{MU_2}.$$

The MRS for the consumer is analogous to the TRS for the firm.

# Marginal rate of substitution

We can write that:

$$MRS = -\frac{MU_1}{MU_2}.$$

Imagine that $MU_1 = 2$ and $MU_2 = 1$. Then $MRS = -\frac{MU_1}{MU_2} = -\frac{2}{1} = -2$. Note that we can also reason this intuitively from $MRS = \frac{\partial x_2(x_1)}{\partial x_1}$:

- If consumer increases $x_1$ by 1, then utility increases by 2: $MU_1 = \frac{\Delta u}{\Delta x_1} = 2 \rightarrow \Delta u = 2 \times \Delta x_1 \rightarrow$ and $\Delta x_1 = 1$, so $\Delta u = 2$.
- Consumer needs to decrease $x_2$ by 2 as to keep utility constant: $MU_2 = \frac{\Delta u}{\Delta x_2} = 1 \rightarrow \Delta u = 1 \times \Delta x_2 \rightarrow$ and if $\Delta x_2 = -2$, then $\Delta u = -2$.
- Hence, $MRS = \frac{\Delta x_2}{\Delta x_1} = \frac{-2}{1} = -2$.

# Exercise

1. Suppose $MU_1 = 9$ and $MU_2 = 3$ at utility level $u = \bar{u}$. The consumer increases $x_1$ by 2. How much does the consumer need to decrease $x_2$ as to keep utility at level $\bar{u}$?

2. Confirm that in the example above we have:

$$-\frac{MU_1}{MU_2} = \frac{\Delta x_2}{\Delta x_1}.$$

# Consumer behavior

We will generally assume that consumers aim to maximize utility. More specifically, we will assume that the consumer will want to **choose the bundle x** from the **set of affordable alternatives** to **maximize utility**.

# Budget constraint

**Budget constraint**: Let $m$ be the budget of a consumer, and let $\mathbf{p}$ be the vector of fixed prices for the $k$ goods. Then we can define the budget constraint, or the set of affordable alternatives, as:

$$B = \{\mathbf{x} : \mathbf{px} = m\}.$$

Consider a setting with two goods, then we can write $B$ as:

$$B = \{(x_1, x_2) : p_1 x_1 + p_2 x_2 = m\}.$$

Note that for any fixed $m$, we can think of $B$ as a level set, and of the budget constraint as an **isobudget** line. The constraint gives us all bundles $(x_1, x_2)$ that cost $m$:

$$B = \{(x_1, x_2) : x_2 = \left(\frac{m}{p_2}\right) - \left(\frac{p_1}{p_2}\right) x_1\}.$$

The intercept of the budget constraint $\left(\frac{m}{p_2}\right)$ gives the budget in terms of the price of $x_2$. The slope of the budget line $\left(\frac{\partial x_2(x_1)}{\partial x_1} = -\frac{p_1}{p_2}\right)$ gives the **economic rate of substitution**: when $x_1$ increases, how much does $x_2$ need to decrease as to keep spending budget $m$.

# Utility maximization

Now that we have introduced the utility function and the budget constraint, we can write the problem of utility maximization as:

$$\max_{\mathbf{x}} u(\mathbf{x}),$$
$$\text{such that } \mathbf{px} = m.$$

Since the only relevant feature about the utility function is its ordinal character, the problem of utility maximization should be interpreted as the problem of finding the most preferred bundle.

# The method of Lagrange with two goods

First, write down the Lagrangian,

$$\mathcal{L} = u(\mathbf{x}) - \lambda(p_1 x_1 + p_2 x_2 - m).$$

Second, differentiate $\mathcal{L}$ wrt each endogenous variable: $x_1$, $x_2$ and $\lambda$. The FOCs for an interior solution $\mathbf{x}^*$ set these derivatives to zero,

$$\frac{\partial u(\mathbf{x})}{\partial x_1} - \lambda p_1 = 0,$$
$$\frac{\partial u(\mathbf{x})}{\partial x_2} - \lambda p_2 = 0,$$
$$p_1 x_1 + p_2 x_2 - m = 0.$$

Third, since we have 3 unknown endogenous variables ($x_1$, $x_2$ and $\lambda$) and 3 FOCs, we can solve for the endogenous variables in terms of the exogenous variables ($p_1$, $p_2$ and $m$).

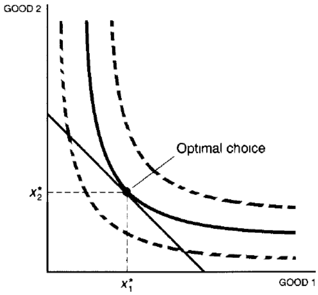
# The method of Lagrange with two goods

Dividing the first two FOCs by each other gives us the following optimality condition:

$$\frac{p_1}{p_2} = \frac{\partial u(\mathbf{x}) / \partial x_1}{\partial u(\mathbf{x}) / \partial x_2}.$$

We will combine this optimality condition with a graphical analysis of the Lagrange method to introduce the SOC and develop an economic intuition for the method more generally (again).

# The method of Lagrange graphically




The utility-maximizing consumer wants to find a **point on the budget constraint with maximal utility**: this is a point where the indifference curve is furthest to the northeast. This point $\mathbf{x}^*$ is characterized by the slopes of the two lines being equal, which is the optimality condition we have seen before:

$$\frac{p_1}{p_2} = \frac{\partial u(\mathbf{x})/\ \partial x_1}{\partial u(\mathbf{x})/\ \partial x_2}$$

# Second-order condition

The SOC for utility-maximization is that:

$$V(u) \text{ is convex} \leftrightarrow u(\mathbf{x}) \text{ is concave} \leftrightarrow \mathbf{h}^T \mathbf{D}^2 u(\mathbf{x})\mathbf{h} \leq 0.$$

With this condition we can be certain that the indifference curve is always weakly above the budget line. This is what we need for $\mathbf{x}^*$ to be utility-maximizing.

This guarantees that any change in the bundle $\mathbf{x}$ that keeps spending constant (that is, a change along the budget line) must result in weakly lower utility (that is, an indifference curve related to a weakly lower $u$).

# The method of Lagrange: economic intuition

Recall that the optimality condition is:

$$\frac{p_1}{p_2} = \frac{\partial u(\mathbf{x}) / \partial x_1}{\partial u(\mathbf{x}) / \partial x_2}.$$

The RHS is the **marginal rate of substitution** ($\frac{\partial x_2(x_1)}{\partial x_1} = -\frac{\partial u(\mathbf{x}) / \partial x_1}{\partial u(\mathbf{x}) / \partial x_2}$): when $x_1$ increases, how much does $x_2$ need to decrease as to keep utility constant.

The LHS is the **economic rate of substitution** ($\frac{\partial x_2(x_1)}{\partial x_1} = -\frac{p_1}{p_2}$): when $x_1$ increases, how much does $x_2$ need to decrease as to keep spending the same.

Hence, the optimality condition tells us that at $\mathbf{x}^*$ the economic and marginal rate of substitution need to be equal. Imagine they are not:

$$\frac{p_1}{p_2} = \frac{2}{1} \neq \frac{1}{1} = \frac{\partial u(\mathbf{x}) / \partial x_1}{\partial u(\mathbf{x}) / \partial x_2},$$

then we can consume one unit less of $x_1$ and one unit more of $x_2$, so that utility stays constant, but we still have an additional euro to spend. This cannot be $\mathbf{x}^*$.

# Marshallian demand and indirect utility

**Marshallian demand function** $\mathbf{x}(\mathbf{p}, m)$: a function that gives us the optimal choice of consumption goods as a function of prices $\mathbf{p}$ and budget $m$.

How to get this function? From the FOCs of the Langrangian we can write $\mathbf{x}$ in terms of $(\mathbf{p}, m)$.

The Marshallian demand function tells us how demand changes when prices change, while income is kept constant. This demand function can in principle be estimated with data on consumption, prices, and income.

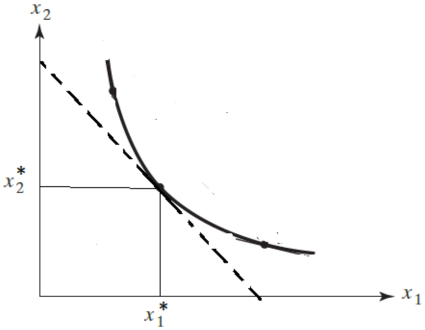
**Indirect utility function** $v(\mathbf{p}, m)$: a function that gives us the maximum utility achievable given prices $\mathbf{p}$ and budget $m$.

How to get this function? Substitute $\mathbf{x}(\mathbf{p}, m)$ into $u(\mathbf{x}) = u(\mathbf{x}(\mathbf{p}, m)) = v(\mathbf{p}, m)$

# Exercise

The graph below indicates the optimal consumption bundle. Imagine $p_1$ increases.

1. Draw the new budget constraint.
2. Show the new Marshallian demand.



# The Lagrange multiplier

Recall that the **Lagrange multiplier** $\lambda$ measures how the optimal solution to the constrained optimization problem changes when the constraint is relaxed.

When we apply this interpretation of $\lambda$ to the utility maximization problem, the optimal solution is the indirect utility function $v(\mathbf{p}, m)$ and the constraint is relaxed if we increase the budget $m$.

Hence, in this setting the Lagrange multiplier measures how utility changes, $\Delta v(\mathbf{p}, m)$, when we increase spending, $\Delta m$, so $\lambda = \frac{\Delta v(\mathbf{p}, m)}{\Delta m}$.

# The Lagrange multiplier

The proof for this interpretation of $\lambda$ follows from the envelope theorem.

Consider the Lagrangian with two goods,

$$\mathcal{L}(\mathbf{p}, m, \mathbf{x}, \lambda) = u(\mathbf{x}) - \lambda(p_1 x_1 + p_2 x_2 - m).$$

First, note that:

$$\frac{\partial \mathcal{L}(\mathbf{p}, m, \mathbf{x}, \lambda)}{\partial m} = \lambda.$$

Second, substitute the Marshallian demand functions $\mathbf{x}(\mathbf{p}, m)$ and the Lagrange multiplier $\lambda(\mathbf{p}, m)$ into the Lagrangian to obtain the Lagrangian evaluated at the optimal point: $\mathcal{L}(\mathbf{p}, m, \mathbf{x}(\mathbf{p}, m), \lambda(\mathbf{p}, m)) = \mathcal{L}(\mathbf{p}, m)$. It turns out, this is equal to:

$$\begin{aligned}
\mathcal{L}(\mathbf{p}, m) =& u(\mathbf{x}(\mathbf{p}, m)) - \lambda(\mathbf{p}, m)(p_1 x_1(\mathbf{p}, m) + p_2 x_2(\mathbf{p}, m) - m), \\
=& u(\mathbf{x}(\mathbf{p}, m)), \\
=& v(\mathbf{p}, m).
\end{aligned}$$

# The Lagrange multiplier

Third, use the logic of the envelope theorem to show that at the optimal point:

$$\frac{\partial \mathcal{L}(\mathbf{p}, m)}{\partial m} = \underbrace{\frac{\partial \mathcal{L}(\cdot)}{\partial m}}_{\text{direct effect}} + \underbrace{\frac{\partial \mathcal{L}(\cdot)}{\partial x_1}\frac{\partial x_1(\cdot)}{\partial m} + \frac{\partial \mathcal{L}(\cdot)}{\partial x_2}\frac{\partial x_2(\cdot)}{\partial m} + \frac{\partial \mathcal{L}(\cdot)}{\partial \lambda}\frac{\partial \lambda(\cdot)}{\partial m}}_{\text{indirect effect}},$$

$$= \frac{\partial \mathcal{L}(\cdot)}{\partial m},$$

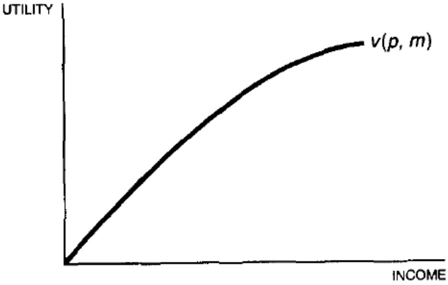
$$= \lambda(\mathbf{p}, m),$$

as the indirect effects are zero because of the FOCs of the Lagrangian.

Since $\mathcal{L}(\mathbf{p}, m) = v(\mathbf{p}, m)$, we conclude that:

$$\frac{\partial \mathcal{L}(\mathbf{p}, m)}{\partial m} = \frac{\partial v(\mathbf{p}, m)}{\partial m} = \lambda(\mathbf{p}, m).$$

# The indirect utility function




**Indirect utility function** $v(\mathbf{p}, m)$: maximum utility for each income $m$.

Since preferences satisfy monotonicity, indirect utility is increasing in $m$: If $m$ increases, $x$ increases, and so $u$ increases. We can take the inverse of the indirect utility function $u = v(\mathbf{p}, m)$, written as $m = v^{-1}(\mathbf{p}, u) = e(\mathbf{p}, u)$.

**Expenditure function** $e(\mathbf{p}, u)$: minimum income required to achieve utility $u$.

# Expenditure minimization

You can also get the expenditure function $e(\mathbf{p}, u)$ by solving the consumers' choice problem via expenditure minimization.

**Expenditure minimization problem (EMP)**: find $\mathbf{x}$ that minimizes expenditure $\mathbf{px}$ subject to the utility constraint $u(\mathbf{x}) = u$.

$$\min_{\mathbf{x}} \mathbf{px},$$
$$\text{such that } u(\mathbf{x}) = u.$$

Note that EMP is identical to cost minimization and thus all properties and intuition carry over from Chapter 4.

So far we discussed the **utility maximization problem (UMP)**: find $\mathbf{x}$ that maximizes utility $u(\mathbf{x})$ subject to the budget constraint $\mathbf{px} = m$.

$$\max_{\mathbf{x}} u(\mathbf{x}),$$
$$\text{such that } \mathbf{px} = m.$$

# The method of Lagrange for the EMP

The upcoming slides discuss the method of Lagrange for the **EMP**. Note that its properties and intuition carry over from the cost minimization problem.

First, write down the Lagrangian,

$$\mathcal{L} = p_1 x_1 + p_2 x_2 - \lambda(u(\mathbf{x}) - u)$$

Second, differentiate $\mathcal{L}$ wrt each endogenous variable: $x_1$, $x_2$ and $\lambda$. The FOCs for an interior solution $\mathbf{x}^*$ set these derivatives to zero,

$$\begin{aligned} p_1 - \lambda \frac{\partial u(\mathbf{x})}{\partial x_1} &= 0, \\ p_2 - \lambda \frac{\partial u(\mathbf{x})}{\partial x_2} &= 0, \\ u(\mathbf{x}) - u &= 0. \end{aligned}$$

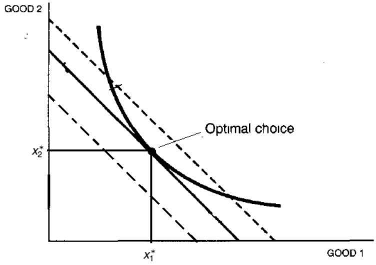
Third, since we have 3 unknown endogenous variables ($x_1$, $x_2$ and $\lambda$) and 3 FOCs, we can solve for the endogenous variables in terms of the exogenous variables ($p_1$, $p_2$ and $u$).

# The method of Lagrange for the EMP

Dividing the first two FOCs by each other gives us the following optimality condition (again):

$$\frac{p_1}{p_2} = \frac{\partial u(\mathbf{x}) / \partial x_1}{\partial u(\mathbf{x}) / \partial x_2}.$$

# The method of Lagrange for the EMP




The expenditure-minimizing consumer wants to find a **point on the indifference curve with minimal expenditures**: this is a point where the intercept of the budget line ($\frac{m}{p_2}$) is minimal. This point $\mathbf{x}^*$ is characterized by the slopes of the two lines being equal, which is the optimality condition we have seen before:

$$\frac{p_1}{p_2} = \frac{\partial u(\mathbf{x}) / \partial x_1}{\partial u(\mathbf{x}) / \partial x_2}$$

# Second-order condition for the EMP

The SOC for expenditure-minimization is that:

$$V(u) \text{ is convex} \leftrightarrow u(\mathbf{x}) \text{ is concave} \leftrightarrow \mathbf{h}^T \mathbf{D}^2 u(\mathbf{x}) \mathbf{h} \leq 0.$$

With this condition we can be certain that the indifference curve is always weakly above the budget line. This is what we need for $\mathbf{x}^*$ to be expenditure-minimizing.

This guarantees that any change in the bundle $\mathbf{x}$ that keeps utility constant (that is, a change along the indifference curve) must result in weakly higher spending (that is, a budget line related to a weakly higher $m$).

# Hicksian demand and expenditure function

**Hicksian demand function** $\mathbf{h}(\mathbf{p}, u)$: a function that gives us the optimal choice of consumption goods as a function of prices $\mathbf{p}$ and utility $u$.

How to get this function? From the FOCs of the Lagrangian for the EMP we can write $\mathbf{x}$ in terms of $(\mathbf{p}, u)$.

The Hicksian demand function tells us how demand changes when prices change, while utility is kept constant. However, if one wants to keep utility constant while prices change, one needs to change/compensate income. This is a theoretical concept and often referred to as **compensated demand**. It is also difficult to estimate since we do not have data on utility.

**Expenditure function** $e(\mathbf{p}, u)$: a function that gives us the minimum income required to achieve utility $u$ at prices $\mathbf{p}$ .

How to get this function? Substitute $\mathbf{h}(\mathbf{p}, u)$ into $\mathbf{px} = \mathbf{ph}(\mathbf{p}, u) = e(\mathbf{p}, u)$.

# The Lagrange multiplier

Recall that the **Lagrange multiplier** $\lambda$ measures how the optimal solution to the constrained optimization problem changes when the constraint is relaxed.

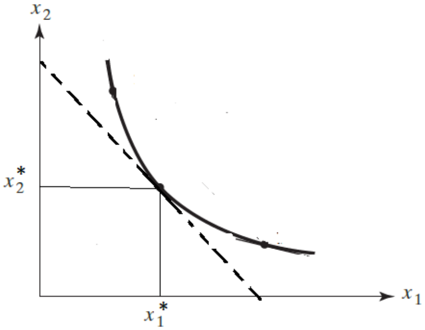
When we apply this interpretation of $\lambda$ to the expenditure minimization problem, the optimal solution is the expenditure function $e(\mathbf{p}, m)$ and the constraint is relaxed if we increase utility $u$.

Hence, in this setting the Lagrange multiplier measures how expenditures change, $\Delta e(\mathbf{p}, m)$, when we increase utility, $\Delta u$, so $\lambda = \frac{\Delta e(\mathbf{p}, m)}{\Delta u}$.

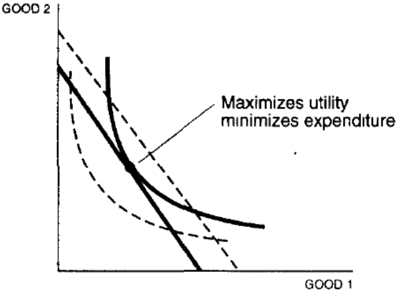
# Exercise

The graph below indicates the optimal consumption bundle. Imagine $p_1$ increases.

1. Draw the new budget constraint.
2. Show the new Hicksian demand.



# Duality: EMP=UMP

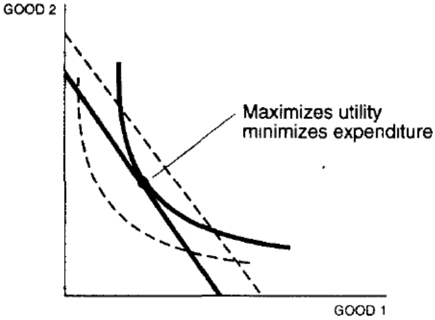



**EMP**: Given the indifference curve, slide the budget constraint to the south-west region until it just touches the indifference curve.

**UMP**: Given the budget constraint, slide the indifference curve to the north-east region until it just touches the budget constraint.

Under the assumptions we made, EMP and UMP will find the same optimal point $\mathbf{x}^*$. This is referred to as **duality**.

# Two important identities




**UMP**: Let $\mathbf{x}(\mathbf{p}, m)$ and $v(\mathbf{p}, m)$ be the solutions.

**EMP**: Let $\mathbf{h}(\mathbf{p}, u)$ and $e(\mathbf{p}, u)$ be the solutions.

**Duality** ensures that (1) $\mathbf{x}(\mathbf{p}, m) = \mathbf{h}(\mathbf{p}, u)$ if $u = v(\mathbf{p}, m)$, and
(2) $\mathbf{h}(\mathbf{p}, u) = \mathbf{x}(\mathbf{p}, m)$ if $m = e(\mathbf{p}, u)$.

# Two important identities

(1) $x_i(\mathbf{p}, m) = h_i(\mathbf{p}, v(\mathbf{p}, m))$: The Marshallian demand at income $m$ is the same as the Hicksian demand at utility $v(\mathbf{p}, m)$.

(2) $h_i(\mathbf{p}, u) = x_i(\mathbf{p}, e(\mathbf{p}, u))$: The Hicksian demand at utility $u$ is the same as the Marshallian demand at income $e(\mathbf{p}, u)$.

It is identity (2) that gives rise to the term compensated demand for the Hicksian demand: It is the Marshallian demand when income changes are arranged as to achieve some target level of utility.

## Marshallian demand from indirect utility

If you were given the Marshallian demand $\mathbf{x}(\mathbf{p}, m)$, finding the indirect utility function is easy: just substitute the Marshallian demand into $u(\mathbf{x})$,

$$u(\mathbf{x}(\mathbf{p}, m)) = v(\mathbf{p}, m).$$

It turns out that if you know the indirect utility function, it also easy to find the Marshallian demand. This is what **Roy's identity** shows us.

# Roy's identity

**Roy's identity** shows that we can find the Marshallian demand function from the indirect utility function as follows:

$$-\frac{\partial v(\mathbf{p}, m)/\ \partial p_i}{\partial v(\mathbf{p}, m)/\ \partial m} = x_i(\mathbf{p}, m).$$

In words, **Roy's identity** is that the Marshallian demand function can be found from the fraction that contains the derivative of the indirect utility function towards the price and income. This is similar to Hotelling's and Shephard's lemma, but then applied to utility maximization.

# Proof Roy's identity

Consider the Lagrangian with two goods,

$$\mathcal{L}(\mathbf{p}, m, \mathbf{x}, \lambda) = u(\mathbf{x}) - \lambda(p_1 x_1 + p_2 x_2 - m).$$

First, note that:

$$\begin{aligned}\frac{\partial \mathcal{L}(\mathbf{p}, m, \mathbf{x}, \lambda)}{\partial p_1} &= -\lambda x_1, \\ \frac{\partial \mathcal{L}(\mathbf{p}, m, \mathbf{x}, \lambda)}{\partial m} &= \lambda.\end{aligned}$$

Second, substitute the Marshallian demand functions $\mathbf{x}(\mathbf{p}, m)$ and the Lagrange multiplier $\lambda(\mathbf{p}, m)$ into the Lagrangian to obtain the Lagrangian evaluated at the optimal point: $\mathcal{L}(\mathbf{p}, m, \mathbf{x}(\mathbf{p}, m), \lambda(\mathbf{p}, m)) = \mathcal{L}(\mathbf{p}, m)$. It turns out, this is equal to:

$$\begin{aligned}\mathcal{L}(\mathbf{p}, m) &= u(\mathbf{x}(\mathbf{p}, m)) - \lambda(\mathbf{p}, m)(p_1 x_1(\mathbf{p}, m) + p_2 x_2(\mathbf{p}, m) - m), \\ &= u(\mathbf{x}(\mathbf{p}, m)), \\ &= v(\mathbf{p}, m).\end{aligned}$$

# Proof Roy's identity

Third, use the logic of the envelope theorem to show that at the optimal point:

$$\frac{\partial \mathcal{L}(\mathbf{p},m)}{\partial p_1} = \underbrace{\frac{\partial \mathcal{L}(\cdot)}{\partial p_1}}_{\text{direct effect}} + \underbrace{\frac{\partial \mathcal{L}(\cdot)}{\partial x_1}\frac{\partial x_1(\cdot)}{\partial p_1} + \frac{\partial \mathcal{L}(\cdot)}{\partial x_2}\frac{\partial x_2(\cdot)}{\partial p_1} + \frac{\partial \mathcal{L}(\cdot)}{\partial \lambda}\frac{\partial \lambda(\cdot)}{\partial p_1}}_{\text{indirect effect}}$$

$$= \frac{\partial \mathcal{L}(\cdot)}{\partial p_1} = -\lambda(\mathbf{p},m)x_1(\mathbf{p},m),$$

$$\frac{\partial \mathcal{L}(\mathbf{p},m)}{\partial m} = \underbrace{\frac{\partial \mathcal{L}(\cdot)}{\partial m}}_{\text{direct effect}} + \underbrace{\frac{\partial \mathcal{L}(\cdot)}{\partial x_1}\frac{\partial x_1(\cdot)}{\partial m} + \frac{\partial \mathcal{L}(\cdot)}{\partial x_2}\frac{\partial x_2(\cdot)}{\partial m} + \frac{\partial \mathcal{L}(\cdot)}{\partial \lambda}\frac{\partial \lambda(\cdot)}{\partial m}}_{\text{indirect effect}}$$

$$= \frac{\partial \mathcal{L}(\cdot)}{\partial m} = \lambda(\mathbf{p},m),$$

as the indirect effects are zero because of the FOCs of the Lagrangian.

Since $\mathcal{L}(\mathbf{p},m) = v(\mathbf{p},m)$, we conclude that:

$$-\frac{\partial \mathcal{L}(\mathbf{p},m)/\,\partial p_i}{\partial \mathcal{L}(\mathbf{p},m)/\,\partial m} = -\frac{\partial v(\mathbf{p},m)/\,\partial p_i}{\partial v(\mathbf{p},m)/\,\partial m} = x_i(\mathbf{p},m).$$

# Exercise

Consider the following utility maximization problem:

$$\max_{x_1, x_2} x_1^{\alpha} x_2^{1-\alpha},$$

$$\text{such that } p_1 x_1 + p_2 x_2 = m.$$

1. Find the Marshallian demand functions.
2. Find the indirect utility function.
3. Find $\lambda(\mathbf{p}, m)$. Show that the derivative of the indirect utility function towards $m$ is equal to $\lambda(\mathbf{p}, m)$. Use this to provide an economic interpretation of $\lambda(\mathbf{p}, m)$.
4. Show Roy's identity for $x_1(\mathbf{p}, m)$.

# Exercise

Consider the following expenditure minimization problem:

$$\min_{x_1, x_2} p_1 x_1 + p_2 x_2,$$

$$\text{such that } x_1^{\alpha} x_2^{1-\alpha} = u.$$

1. Find the Hicksian demand functions.

2. Find the expenditure function.

3. Plug the indirect utility function into the Hicksian demand function. What do you find? Plug the expenditure function into the Marshallian demand function. What do you find?

# Homework exercises

Exercises: 7.4(a)-(b), 8.5, and exercises on the slides