Lecture 6: Recursive Preferences

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Overview

- Epstein and Zin (1989 JPE, 1991 Ecta) introduced a class of preferences which allow to break the link between risk aversion and intertemporal substitution.
- These preferences have proved very useful in applied work in asset pricing, portfolio choice, and macroeconomics
- There other alternatives to explain the puzzles in asset pricing like:
  - Habits (Campbell and Cochrane, 1999), Long run risks (Bansal and Yaron, 2004; Bansal, Kiku, and Yaron, 2012), Idiosyncratic risk (Constantinides and Duffie, 1996), Heterogeneous preferences (Gârleanu and Panageas, 2015), etc...

Elasticity of intertemporal substitution

- For instance

$$U_t = \sum_{s=0}^{\infty} \beta^s \frac{(c_{t+s})^{1-\gamma}}{1-\gamma}$$

The **EIS** measures the responsiveness of the growth rate of consumption to the real interest rate:

$$\frac{d \log(c_{t+1}/c_t)}{d \log R} = \frac{\frac{d(c_{t+1}/c_t)}{c_{t+1}/c_t}}{\frac{dR}{R}},$$

The Euler equation

$$1 = R\beta \frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}},$$

$$0 = \log R + \log \beta - \gamma \log\left(\frac{c_{t+1}}{c_t}\right),$$

$$\frac{d \log\left(\frac{c_{t+1}}{c_t}\right)}{d \log R} = \frac{1}{\gamma},$$

Elasticity of intertemporal substitution

$$\frac{d \log \left(\frac{c_{t+1}}{c_t} \right)}{d \log R} = \frac{1}{\gamma}$$

- Implies an inverse relationship between risk aversion and willingness to substitute consumption over time
- This is an "artificial" restriction because the 2 concepts are distinct
- If γ is high, the consumer has a low **IES** and is reluctant to shift consumption across time
- If γ is low, the consumer is more flexible and willing to adjust consumption in response to changes in interest rates

Relative risk-aversion

- The **RRA** measures how the willingness to bear risk changes with wealth
- Measures the percentage change in marginal utility due to a percentage change in consumption:

$$-\frac{\frac{d(u'(c))}{u'(c)}}{\frac{dc}{c}},$$

$$d\left(u'(c)\right) = u''(c)\,dc$$

$$-\frac{\frac{d(u'(c))}{u'(c)}}{\frac{dc}{c}} = -\frac{\frac{u''(c)dc}{u'(c)}}{\frac{dc}{c}} = -c\frac{u''(c)}{u'(c)}$$

$$-c\frac{u''(c)}{u'(c)} = \gamma c\frac{c^{-\gamma-1}}{c^{-\gamma}} = \gamma$$

Economic Implications of RRA

- Portfolio Choice:
  - Higher RRA, more investment in risk-free assets, less in stocks
  - Lower RRA, more investment in risky assets
- Insurance Demand:
  - High RRA, more likely to buy insurance
  - Low RRA, less likely to pay for insurance
- Consumption & Savings Behavior:
  - High RRA, more precautionary savings to buffer against uncertainty
  - Low RRA, more willing to consume and take risks

Value function

- The standard expected utility time-separable preferences are defined as

$$V_t = \sum_{s=0}^{\infty} \beta^s E_t u\left(c_{t+s}\right)$$

Alternatively can write it as

$$\begin{aligned} V_t &= (1-\beta) \sum_{s=0}^{\infty} \beta^s E_t u\left(c_{t+s}\right) \\ &= (1-\beta) u\left(c_t\right) + \beta E_t\left(V_{t+1}\right) \end{aligned}$$

- V_t is known as value function or lifetime utility

EZ Preferences

- EZ preferences generalize this: they are defined recursively over current (known) consumption and a certainty equivalent $H_t(V_{t+1})$ of tomorrow's utility V_{t+1}:

$$V_t = F(c_t, H_t(V_{t+1}))$$

where

$$H_t(V_{t+1}) = G^{-1}(E_t G(V_{t+1}))$$

with F and G increasing and concave, and F homogeneous of degree one

EZ Preferences

- **Observation**: F is homogeneous of degree one if

$$F(tX, tY) = tF(X, Y), \text{ for } t > 0$$

and

$$F = X \cdot F'_X + Y \cdot F'_Y$$

Also known as Euler's theorem

- Note that

$$H_t\left(V_{t+1}\right) = G^{-1}\left(E_t G\left(V_{t+1}\right)\right)$$

$$H_t(V_{t+1}) = V_{t+1}$$

if there is no uncertainty on V_{t+1}

- The more concave G is, and the more uncertain V_{t+1} is, the lower is $H_t(V_{t+1})$. Hint: Use a graph for the intuition.

Functional forms

- Most of the literature considers simple functional forms for F and G
  - Power function
$$G(x) \equiv \frac{x^{1-\alpha}}{1-\alpha}, \alpha > 0$$
  - CES
$$F(c,z) \equiv \left((1-\beta)c^{1-\gamma} + \beta z^{1-\gamma}\right)^{\frac{1}{1-\gamma}}, \gamma > 0$$
    the ES is $1/\gamma$.
- For this case get

$$V_t = F\left(c_t, H_t\left(V_{t+1}\right)\right), \text{ with } H_t\left(V_{t+1}\right) = G^{-1}\left(E_t G\left(V_{t+1}\right)\right)$$

$$V_t \equiv \left((1-\beta)c_t^{1-\gamma} + \beta\left(E_t\left(V_{t+1}^{1-\alpha}\right)\right)^{\frac{1-\gamma}{1-\alpha}}\right)^{\frac{1}{1-\gamma}}$$

Functional forms

Proposition: If c_t is deterministic we have the **standard** time-separable expected discounted utility with discount factor β, and IES $= 1/\gamma$ and risk aversion $= \gamma$. Also, when $\alpha = 0$ we have the **standard** utility function.
Proof: Given

$$V_t \equiv \left((1-\beta) c_t^{1-\gamma} + \beta \left(E_t \left(V_{t+1}^{1-\alpha} \right) \right)^{\frac{1-\gamma}{1-\alpha}} \right)^{\frac{1}{1-\gamma}}$$

If c_t is deterministic then

$$(V_t)^{1-\gamma} = (1-\beta) c_t^{1-\gamma} + \beta (V_{t+1})^{1-\gamma} = (1-\beta) \sum_{s=0}^{\infty} \beta^s c_{t+s}^{1-\gamma},$$

when $\alpha = 0$ iterate forward to get

$$(V_t)^{1-\gamma} = (1-\beta) c_t^{1-\gamma} + \beta E_t (V_{t+1})^{1-\gamma} = (1-\beta) \sum_{s=0}^{\infty} \beta^s E_t u(c_{t+s}).$$

Limits

$$G(x) \equiv \frac{x^{1-\alpha}}{1-\alpha}, \quad \alpha > 0$$

$$F(c,z) \equiv \left((1-\beta)c^{1-\gamma} + \beta z^{1-\gamma}\right)^{\frac{1}{1-\gamma}}, \quad \gamma > 0$$

Implies

$$G(x) = \log x, \quad \text{if } \alpha = 1$$

Cobb-Douglas

$$F(c,z) = c^{1-\beta} z^{\beta}, \quad \text{if } \gamma = 1$$

Limits

As

$$G(x) \equiv \frac{x^{1-\alpha}}{1-\alpha}, \quad \alpha > 0$$

$$G(x) = \log x, \quad \text{if } \alpha = 1$$

and

$$H_t(V_{t+1}) = G^{-1}(E_t G(V_{t+1}))$$

Thus:

- $\alpha > 0$

$$H_t(V_{t+1}) = \left[E_t (V_{t+1})^{1-\alpha}\right]^{\frac{1}{1-\alpha}}$$

- $\alpha = 1$

$$H_t(V_{t+1}) = \exp(E_t \log(V_{t+1}))$$

F is Cobb-Douglas if gamma=1

Define

$$F(c,z) = \left((1-\beta)c^{1-\gamma} + \beta z^{1-\gamma}\right)^{\frac{1}{1-\gamma}}$$

divide and multiply by c

$$\begin{aligned} F(c,z) &= c\left((1-\beta) + \beta x^{1-\gamma}\right)^{\frac{1}{1-\gamma}} \\ &= cf(x) \end{aligned}$$

where

$$x = z/c \text{ and } f(x) = \left(1-\beta+\beta x^{1-\gamma}\right)^{\frac{1}{1-\gamma}}$$

so

$$\begin{aligned} \frac{f'(x)}{f(x)} &= \frac{1}{1-\gamma}\frac{\left(1-\beta+\beta x^{1-\gamma}\right)^{\frac{1}{1-\gamma}-1}(1-\gamma)\beta x^{-\gamma}}{\left(1-\beta+\beta x^{1-\gamma}\right)^{\frac{1}{1-\gamma}}} \\ &= \frac{\beta x^{-\gamma}}{\left(1-\beta+\beta x^{1-\gamma}\right)} \end{aligned}$$

F is Cobb-Douglas if gamma=1

- And

$$\lim_{\gamma \to 1} \frac{f'(x)}{f(x)} = \lim_{\gamma \to 1} \frac{\beta x^{-\gamma}}{(1 - \beta + \beta x^{1-\gamma})}$$
$$= \lim_{\gamma \to 1} \frac{\beta}{(1 - \beta) x^{\gamma} + \beta x^{1}} = \frac{\beta}{x}$$

- Since f is continuous then

$$\lim_{\gamma \to 1} f(x) = x^{\beta}$$

or Cobb-Douglas function

$$F(c, z) = cf(x) = c(z/c)^{\beta} = c^{1-\beta} z^{\beta}$$

Risk Aversion vs IES

- In general α is the relative risk aversion coefficient for static gambles and γ is the inverse of the intertemporal elasticity of substitution for deterministic variations
- Suppose consumption is c today and consumption tomorrow is uncertain: $\{c_L, c, c, ...\}$ or $\{c_H, c, c, ...\}$, each has prob. 0.5
- Lifetime utility today

$$V_t = F\left(c, G^{-1}\left(0.5G\left(V_L\right) + 0.5G\left(V_H\right)\right)\right)$$

where

$$V_L = F\left(c_L, \overline{c}\right), V_H = F\left(c_H, \overline{c}\right)$$

- Curvature of G determines how adverse you are to the uncertainty.
    - If G is linear you only care about the expected value
    - If not, it is the certainty equivalent:

$$G\left(\widehat{V}\right) = 0.5G\left(V_L\right) + 0.5G\left(V_H\right)$$

Special Case: Deterministic consumption

- If consumption is deterministic: we have the usual standard time-separable expected discounted utility with discount factor β and IES$= 1/\gamma$, risk aversion $= \gamma$.
- Proof: Without uncertainty, then $H_t(V_{t+1}) = G^{-1}\left(E_t G\left(V_{t+1}\right)\right) = V_{t+1}$ and

$$V_t = F\left(c_t, H_t\left(V_{t+1}\right)\right)$$

With a CES functional form for F, we recover CRRA preferences:

$$V_t = \left[(1-\beta)c_t^{1-\gamma} + \beta\left(V_{t+1}\right)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}$$

$$U_t = (1-\beta)c_t^{1-\gamma} + \beta U_{t+1} = (1-\beta)\sum_{s=0}^{\infty}\beta^s c_{t+s}^{1-\gamma}$$

where

$$U_t = \left(V_t\right)^{1-\gamma}.$$

Special Case: RRA=1/IES

- if α (=RRA) $= \gamma$ (=1/IES) , then the formula

$$V_t \equiv \left((1-\beta) c_t^{1-\gamma} + \beta \left(E_t \left(V_{t+1}^{1-\alpha} \right) \right)^{\frac{1-\gamma}{1-\alpha}} \right)^{\frac{1}{1-\gamma}}$$

simplifies to

$$(V_t)^{1-\gamma} \equiv (1-\beta) c_t^{1-\gamma} + \beta \left(E_t V_{t+1}^{1-\gamma} \right)$$

- Define

$$U_t = V_t^{1-\gamma}$$

then

$$U_t = (1-\beta) c_t^{1-\gamma} + \beta E_t (U_{t+1}),$$

is the expected utility

Simple example with two lotteries

Lotteries:

- lottery A pays in each period $t = 1, 2, ...$ either c_h or c_l, with probability 0.5 and the outcome is iid across periods;
- lottery B pays starting at $t = 1$ either c_h at all future dates for sure, or c_l at all future dates for sure; there is a single draw at time $t = 1$
- With expected utility, you are indifferent between these lotteries, but with EZ lottery B is preferred iff $\alpha > \gamma$.
- In general, early resolution of uncertainty is preferred if and only if $\alpha > \gamma$ i.e. risk aversion $> \frac{1}{IES}$
- This is another way to motivate these preferences, since early resolution seems intuitively preferable.

Resolution of uncertainty

$$V_t \equiv \left((1-\beta) c_t^{1-\gamma} + \beta \left(E_t \left(V_{t+1}^{1-\alpha} \right) \right)^{\frac{1-\gamma}{1-\alpha}} \right)^{\frac{1}{1-\gamma}}$$

- For lottery B, the utility once you know your consumption is either c_h, or c_l forever,

$$V_h = F(c_h, V_h) = \left((1-\beta) c_h^{1-\gamma} + \beta V_h^{1-\gamma} \right)^{\frac{1}{1-\gamma}}$$

  or

$$V_l = F(c_l, V_l) = \left((1-\beta) c_l^{1-\gamma} + \beta V_l^{1-\gamma} \right)^{\frac{1}{1-\gamma}}$$

- The certainty equivalent before playing the lottery is

$$G^{-1}\left(0.5G(c_h) + 0.5G(c_l)\right) = \left(0.5c_h^{1-\alpha} + 0.5c_l^{1-\alpha}\right)^{\frac{1}{1-\alpha}}$$

Resolution of uncertainty

- Given

$$W_t \equiv \left((1-\beta) c_t^{1-\gamma} + \beta \left(E_t \left(W_{t+1}^{1-\alpha} \right) \right)^{\frac{1-\gamma}{1-\alpha}} \right)^{\frac{1}{1-\gamma}}$$

- For lottery A, the values satisfy

$$W_h^{1-\gamma} = (1-\beta) c_h^{1-\gamma} + \beta \left(0.5 W_h^{1-\alpha} + 0.5 W_l^{1-\alpha} \right)^{\frac{1-\gamma}{1-\alpha}}$$

$$W_l^{1-\gamma} = (1-\beta) c_l^{1-\gamma} + \beta \left(0.5 W_h^{1-\alpha} + 0.5 W_l^{1-\alpha} \right)^{\frac{1-\gamma}{1-\alpha}}$$

Resolution of uncertainty

- We want to compare

$$G^{-1}\left(0.5G\left(c_h\right)+0.5G\left(c_l\right)\right)$$

with

$$G^{-1}\left(0.5G\left(W_h\right)+0.5G\left(W_l\right)\right)$$

- **notice** that function

$$x^{\frac{1-\gamma}{1-\alpha}}$$

is concave if $1-\gamma<1-\alpha$, i.e. $\gamma>\alpha$, and convex otherwise. As a result, if $\gamma>\alpha$

$$\begin{aligned}\left(0.5W_h^{1-\alpha}+0.5W_l^{1-\alpha}\right)^{\frac{1-\gamma}{1-\alpha}} &> 0.5\left(W_h^{1-\alpha}\right)^{\frac{1-\gamma}{1-\alpha}}+0.5\left(W_l^{1-\alpha}\right)^{\frac{1-\gamma}{1-\alpha}}\\ &= 0.5W_h^{1-\gamma}+0.5W_l^{1-\gamma}\end{aligned}$$

Resolution of uncertainty

Since

$$W_h^{1-\gamma} = (1-\beta)c_h^{1-\gamma} + \beta \left(0.5W_h^{1-\alpha} + 0.5W_l^{1-\alpha}\right)^{\frac{1-\gamma}{1-\alpha}}$$

and

$$W_l^{1-\gamma} = (1-\beta)c_l^{1-\gamma} + \beta \left(0.5W_h^{1-\alpha} + 0.5W_l^{1-\alpha}\right)^{\frac{1-\gamma}{1-\alpha}}$$

Then

$$W_h^{1-\gamma} > (1-\beta)c_h^{1-\gamma} + \beta \left(0.5W_h^{1-\gamma} + 0.5W_l^{1-\gamma}\right)$$

and

$$W_l^{1-\gamma} > (1-\beta)c_l^{1-\gamma} + \beta \left(0.5W_h^{1-\gamma} + 0.5W_l^{1-\gamma}\right)$$

Resolution of uncertainty

- Multiplying both equations by 0.5 and summing them up

$$(1-\beta)\left(0.5W_h^{1-\gamma}+0.5W_l^{1-\gamma}\right)>(1-\beta)\left(0.5c_h^{1-\gamma}+0.5c_l^{1-\gamma}\right)$$

- These results imply that if $\gamma > \alpha$ then

$$0.5W_h^{1-\gamma}+0.5W_l^{1-\gamma}>0.5c_h^{1-\gamma}+0.5c_l^{1-\gamma}$$

- In this case the certainty equivalent of lottery A is higher than the certainty equivalent of lottery B and agents prefer late to early resolution of uncertainty.

Stochastic Discount Factor

- The stochastic discount factor with these preferences turns out to be slightly different:

$$m_{t+1} = \beta \frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}} \left[\frac{V_{t+1}}{\left(E_t V_{t+1}^{1-\alpha} \right)^{\frac{1}{1-\alpha}}} \right]^{\gamma-\alpha}$$

- The first term is familiar. The second term is next period's value (lifetime utility) relative to its certainty equivalent.

Stochastic Discount Factor

This equation consists of two key components:

- **Consumption Growth Component:**

$$\frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}}$$

captures intertemporal substitution.

- **Risk Adjustment Component:**

$$\left[\frac{V_{t+1}}{\left(E_t V_{t+1}^{1-\alpha}\right)^{\frac{1}{1-\alpha}}}\right]^{\gamma-\alpha}$$

adjusts for risk preferences.

Stochastic Discount Factor

- Let U_t be the expected lifetime utility

$$U = E_t \sum_{s=t}^{\infty} \beta^s u\left(c\left(s^t\right)\right)$$

Discount factor general formula:

$$m_{t+1} = \frac{\beta u'\left(c\left(s^{t+1}\right)\right)}{u'\left(c\left(s^t\right)\right)} = \frac{1}{\pi\left(s^{t+1} \middle| s^t\right)} \frac{\frac{\partial U_t}{\partial c_{t+1}}}{\frac{\partial U_t}{\partial c_t}}$$

Stochastic Discount Factor

- The stochastic discount factor of

$$V_t = \left((1-\beta) c_t^{1-\gamma} + \beta \left(H_t \left(V_{t+1} \right) \right)^{1-\gamma} \right)^{\frac{1}{1-\gamma}}$$

is

$$m_{t+1} = \frac{1}{\pi(s^{t+1}|s^t)} \frac{\frac{\partial V_t}{\partial c_{t+1}}}{\frac{\partial V_t}{\partial c_t}} = \frac{1}{\pi(s^{t+1}|s^t)} \frac{\frac{\partial V_t}{\partial V_{t+1}} \frac{\partial V_{t+1}}{\partial c_{t+1}}}{\frac{\partial V_t}{\partial c_t}}$$

- One of the terms of the ratio is

$$\frac{\partial V_t}{\partial c_t} = (1-\beta) V_t^{\gamma} c_t^{-\gamma}$$

Stochastic Discount Factor

- Preferences

$$V_t = \left((1-\beta) c_t^{1-\gamma} + \beta \left(H_t \left(V_{t+1} \right) \right)^{1-\gamma} \right)^{\frac{1}{1-\gamma}}$$

where the certainty equivalent

$$H_t(V_{t+1}) = \left[E_t \left(V_{t+1} \right)^{1-\alpha} \right]^{\frac{1}{1-\alpha}}$$

$$= \left[\begin{array}{c} \pi\left(s_1\right)\left(V_{t+1}\left(s_1\right)\right)^{1-\alpha} + \pi\left(s_1\right)\left(V_{t+1}\left(s_1\right)\right)^{1-\alpha} \\ + \ldots + \pi\left(s_N\right)\left(V_{t+1}\left(s_N\right)\right)^{1-\alpha} \end{array} \right]^{\frac{1}{1-\alpha}}$$

Another term of the ratio above

$$\frac{\partial V_t}{\partial V_{t+1}(s_1)} = \frac{1}{1-\gamma}(V_t)^{\gamma}(1-\gamma)\beta(H_t(V_{t+1}))^{-\gamma}$$
$$\frac{1}{1-\alpha}\left[E_t(V_{t+1})^{1-\alpha}\right]^{\frac{1}{1-\alpha}-1}\pi(s_1)(1-\alpha)(V_{t+1}(s_1))^{-\alpha}$$

$$\frac{\partial V_t}{\partial V_{t+1}(s_i)} = (V_t)^{\gamma}\beta[H_t(V_{t+1})]^{-\gamma}$$
$$[H_t(V_{t+1})]^{\alpha}\pi(s_i)(V_{t+1}(s_i))^{-\alpha}$$

$$\frac{\partial V_t}{\partial V_{t+1}(s_i)} = (V_t)^{\gamma}\beta[H_t(V_{t+1})]^{\alpha-\gamma}\pi(s_i)(V_{t+1}(s_i))^{-\alpha}$$

- Thus

$$m_{t+1} = \frac{1}{\pi(s^{t+1}|s^t)} \frac{\frac{\partial V_t}{\partial V_{t+1}} \frac{\partial V_{t+1}}{\partial c_{t+1}}}{\frac{\partial V_t}{\partial c_t}} = \frac{\left[\beta V_t^{\gamma} (H_t(V_{t+1}))^{\alpha-\gamma} V_{t+1}^{-\alpha}\right]\left[(1-\beta) V_{t+1}^{\gamma} c_{t+1}^{-\gamma}\right]}{(1-\beta) V_t^{\gamma} c_t^{-\gamma}}$$

$$= \beta \frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}} \frac{(H_t(V_{t+1}))^{\alpha-\gamma}}{V_{t+1}^{\alpha-\gamma}} = \beta \frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}} \left(\frac{\left(E_t V_{t+1}^{1-\alpha}\right)^{\frac{1}{1-\alpha}}}{V_{t+1}} \right)^{\alpha-\gamma}$$

$$m_{t+1} = \beta \frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}} \left[\frac{V_{t+1}}{\left(E_t V_{t+1}^{1-\alpha}\right)^{\frac{1}{1-\alpha}}} \right]^{\gamma-\alpha}$$

Stochastic Discount Factor

- It is usual to write the stochastic discount factor

$$m_{t+1} = \beta \frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}} \left(\frac{V_{t+1}}{H_t \left(V_{t+1} \right)} \right)^{\gamma - \alpha}$$

as

$$m_{t+1} = \beta^{\theta} \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma\theta} R_{m,t+1}^{\theta - 1}$$

where $R_{m,t+1}$, (known as the market return), is:

$$R_{m,t+1} = \frac{c_{t+1}^{\gamma}}{\beta c_t^{\gamma}} \left(\frac{V_{t+1}}{H_t \left(V_{t+1} \right)} \right)^{1-\gamma}$$

and

$$\theta = \frac{1-\alpha}{1-\gamma}.$$

Stochastic Discount Factor

- Now take logs

$$m_{t+1} = \beta^{\theta} \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma\theta} R_{m,t+1}^{\theta-1}$$

to obtain

$$\log m_{t+1} = \theta \log \beta - \gamma\theta \log \left(\frac{c_{t+1}}{c_t} \right) + (\theta - 1) \log R_{m,t+1}$$

- In lecture 2 (where $\alpha = \gamma \iff \theta = 1$)

$$m_{t+1} = \beta \frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}}$$

$$\log m_{t+1} = \log \beta - \gamma \log \left(\frac{c_{t+1}}{c_t} \right)$$

Risk free rate

- Assume that both $\log\left(\frac{c_{t+1}}{c_t}\right)$ is normal distributed and $\log\left(R_{m,t+1}\right)$ is normal distributed and they are independent distributed.
- **Remember**: If z is normal distributed then $\exp(z)$ is lognormal. Also $E\exp(z) = \exp(Ez + 0.5\sigma^2(z))$.
- Let $\Delta c_{t+1} = \log\left(\frac{c_{t+1}}{c_t}\right)$ and $\log\left(R_{m,t+1}\right) = r_{m,t+1}$.
- Then $\exp(\Delta c_{t+1})$ and $\exp(r_{m,t+1})$ are lognormal distributed.

$$\left(R^f_{t+1}\right)^{-1} = E_t\left(m_{t+1}\right)$$

$$\left(R^f_{t+1}\right)^{-1} = E_t \exp\left(\log\left(\beta^\theta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma\theta} R^{\theta-1}_{m,t+1}\right)\right)$$

$$\left(R^f_{t+1}\right)^{-1} = e^{\log\beta^\theta} E_t e^{-\gamma\theta(\Delta c_{t+1})} E_t e^{-(1-\theta) r_{m,t+1}}$$

using the fact that for independent variables $Exy = ExEy$.

Risk free rate

$$\left(R_{t+1}^{f}\right)^{-1} = e^{\log \beta^{\theta}} e^{-\gamma\theta E_t(\Delta c_{t+1})+\frac{1}{2}(\gamma\theta)^2\sigma^2(\Delta c_{t+1})} e^{-(1-\theta)E_t r_{m,t+1}+\frac{1}{2}(1-\theta)^2\sigma^2(r_{m,t+1})}$$

- taking logarithms

$$\begin{aligned} r_{t+1}^{f} &= -\theta \log \beta + \gamma\theta E_t\left(\Delta c_{t+1}\right) - \frac{1}{2}\left(\gamma\theta\right)^2 \sigma^2\left(\Delta c_{t+1}\right) \\ &\quad + \left(1-\theta\right) E_t r_{m,t+1} - \frac{1}{2}\left(1-\theta\right)^2 \sigma^2\left(r_{m,t+1}\right) \end{aligned}$$

- if $r_{m,t+1}$ and Δc_{t+1} are jointly lognormal distributed (not independent) then there is an additional term

$$r_{t+1}^{f} = \ldots - \gamma\theta\left(1-\theta\right) cov_t\left(\Delta c_{t+1}, r_{m,t+1}\right)$$

Equity premium

- Assume that consumption growth and asset returns are jointly log-normally distributed like in lecture 2.

$$\left[\begin{array}{c}\Delta c_{t+1}\\ r^i_{t+1}\end{array}\right]\sim N\left(\left[\begin{array}{c}\overline{\Delta c}_{t+1}\\ \overline{r}^i_{t+1}\end{array}\right],\left[\begin{array}{cc}var\left(\Delta c_{t+1}\right) & cov\left(\Delta c_{t+1}, r^i_{t+1}\right)\\ cov\left(\Delta c_{t+1}, r^i_{t+1}\right) & var\left(r^i_{t+1}\right)\end{array}\right]\right)$$

- We established (in lecture 2) that,

$$\overline{r}^i_{t+1} - r^f_{t+1} = -\frac{1}{2}var(r^i_{t+1}) + \gamma cov_t(\Delta c_{t+1}, r^i_{t+1})$$

Equity premium

- If $\log\left(R^i_{t+1}\right)$ and $\log m_{t+1}$ are normal distributed
- The Euler equations are

$$1 = \exp\left(E_t \log m_{t+1} + r^f_{t+1} + \frac{1}{2} var(\log m_{t+1})\right)$$

$$1 = \exp\left(E_t \log m_{t+1} + \overline{r}^i_{t+1} + \frac{1}{2} var(\log m_{t+1} + r^i_{t+1})\right)$$

Equity premium

- Take logs and equate these equations:

$$E_t \log m_{t+1} + r^f_{t+1} + \frac{1}{2} var(\log m_{t+1}) = E_t \log m_{t+1} + \overline{r}^i_{t+1} + \frac{1}{2} var(\log m_{t+1} + r^i_{t+1})$$

$$r^f_{t+1} + \frac{1}{2} var(\log m_{t+1}) = \overline{r}^i_{t+1} + \frac{1}{2} var(\log m_{t+1} + r^i_{t+1})$$

$$r^f_{t+1} + \frac{1}{2} var(\log m_{t+1}) = \overline{r}^i_{t+1} + \frac{1}{2} \left[\begin{array}{c} var(\log m_{t+1}) + var(r^i_{t+1}) \\ +2cov(\log m_{t+1}, r^i_{t+1}) \end{array} \right]$$

$$\overline{r}^i_{t+1} - r^f_{t+1} = -\frac{1}{2} var(r^i_{t+1}) - cov(\log m_{t+1}, r^i_{t+1})$$

$$\overline{r}^i_{t+1} + \frac{1}{2} var(r^i_{t+1}) - r^f_{t+1} = -cov(\log m_{t+1}, \log\left(R^i_{t+1}\right))$$

Equity premium

$$\log E_t R^i_{t+1} - \log R^f_{t+1} = -cov(\log m_{t+1}, \log\left(R^i_{t+1}\right))$$

Since

$$\log m_{t+1} = \theta \log \beta - \gamma\theta \log\left(\frac{c_{t+1}}{c_t}\right) + (\theta - 1)\log R_{m,t+1}$$

Then:

$$\log\left(\frac{E_t R^i_{t+1}}{R^f_{t+1}}\right) = \gamma\theta cov(\Delta c_{t+1}, r^i_{t+1}) + (1-\theta)\, cov(r_{m,t+1}, r^i_{t+1})$$

Conclusion: Epstein-Zin is a linear combination of the CAPM and the CCAPM model

Market return

For the market return, $r_{m,t+1}$, we have

$$\log\left(\frac{E_t R_{m,t+1}}{R^f_{t+1}}\right) = \gamma\theta cov(\Delta c_{t+1}, r_{m,t+1}) + (1-\theta)\,\sigma^2(r_{m,t+1})$$

or

$$\begin{aligned} E_t r_{m,t+1} + \frac{1}{2}\sigma^2\left(r_{m,t+1}\right) - r^f_{t+1} \quad &= \quad \gamma\theta cov(\Delta c_{t+1}, r_{m,t+1}) \\ &\quad + (1-\theta)\,\sigma^2(r_{m,t+1}) \end{aligned}$$

Now we compute $(1-\theta)\,E_t r_{m,t+1}$ to use later (i.e. we multiply by $(1-\theta)$)

$$\begin{aligned} (1-\theta)\,E_t r_{m,t+1} \quad &= \quad (1-\theta)\,r^f_{t+1} - \frac{(1-\theta)}{2}\sigma^2\left(r_{m,t+1}\right) + \\ &\quad (1-\theta)\,\gamma\theta cov_t(\Delta c_{t+1}, r_{m,t+1}) + (1-\theta)^2\,\sigma^2_t(r_{m,t+1}) \end{aligned}$$

Risk free rate

From a previous "slide" we got the riskless rate

- $$r^f_{t+1} = -\theta \log \beta + \gamma\theta E_t\left(\Delta c_{t+1}\right) + \left(1-\theta\right) E_t r_{m,t+1} - \frac{1}{2}\left(\gamma\theta\right)^2 \sigma^2 \left(\Delta c\right.$$
$$-\frac{1}{2}\left(1-\theta\right)^2 \sigma^2\left(r_{m,t+1}\right) - \gamma\theta\left(1-\theta\right) cov_t(\Delta c_{t+1}, r_{m,t+1})$$

- replacing $\left(1-\theta\right) E_t r_{m,t+1}$

$$\begin{aligned} r^f_{t+1} = & -\theta \log \beta + \gamma\theta E_t\left(\Delta c_{t+1}\right) + \\ & \left(1-\theta\right) r^f_{t+1} - \frac{\left(1-\theta\right)}{2}\sigma^2\left(r_{m,t+1}\right) + \\ & \left(1-\theta\right)\gamma\theta cov_t(\Delta c_{t+1}, r_{m,t+1}) + \left(1-\theta\right)^2 \sigma^2_t(r_{m,t+1}) \\ & -\frac{1}{2}\left(\gamma\theta\right)^2 \sigma^2\left(\Delta c_{t+1}\right) \\ & -\frac{1}{2}\left(1-\theta\right)^2 \sigma^2\left(r_{m,t+1}\right) - \gamma\theta\left(1-\theta\right) cov_t(\Delta c_{t+1}, r_{m,t+1}) \end{aligned}$$

Risk free rate

Rearranging

$$\begin{aligned}\theta r^f_{t+1} &= -\theta \log \beta + \gamma\theta E_t\left(\Delta c_{t+1}\right) - \frac{(1-\theta)}{2}\sigma^2\left(r_{m,t+1}\right) + \\ &\quad -\frac{1}{2}\left(\gamma\theta\right)^2\sigma^2\left(\Delta c_{t+1}\right) + \frac{1}{2}\left(1-\theta\right)^2\sigma^2\left(r_{m,t+1}\right)\end{aligned}$$

$$\begin{aligned}r^f_{t+1} &= -\log \beta + \gamma E_t\left(\Delta c_{t+1}\right) - \frac{(1-\theta)}{2\theta}\sigma^2\left(r_{m,t+1}\right) + \\ &\quad -\frac{1}{2}\gamma^2\theta\sigma^2\left(\Delta c_{t+1}\right) + \frac{1}{2\theta}\left(1-\theta\right)^2\sigma^2\left(r_{m,t+1}\right)\end{aligned}$$

$$r^f_{t+1} = -\log \beta + \gamma E_t\left(\Delta c_{t+1}\right) - \frac{1}{2}\gamma^2\theta\sigma^2\left(\Delta c_{t+1}\right) - \frac{(1-\theta)}{2}\sigma^2\left(r_{m,t+1}\right)$$

Again if $\gamma = \alpha$ then $\theta = 1$ we have the standard risk-free rate equation. If $\alpha > \gamma$ then $\theta < 1$ and the volatility from the market return reduces the real interest rate. Helps in the risk free puzzle.

EZ preference and riskless rate

US Historical data

$$\begin{aligned} E_t\left(\Delta c_{t+1}\right) &= 0.02, \\ \sigma^2\left(\Delta c_{t+1}\right) &= (0.036)^2 = 0.0013 \\ \sigma^2\left(r_{m,t+1}\right) &= (0.167)^2 = 0.0279 \end{aligned}$$

with $\beta = 0.98$, $\alpha = 2$ and $\gamma = 0.5$ (which are reasonable) get 1% riskless interest rate

EZ preference and equity premium

Equity premium

$$\log\left(\frac{E_t R_{m,t+1}}{R^f_{t+1}}\right) = \gamma\theta cov_t(\Delta c_{t+1}, r_{m,t+1}) + (1-\theta)\,\sigma^2_t(r_{m,t+1})$$

US Historical data

$$\begin{aligned} E_t(\Delta c_{t+1}) &= 0.02, \sigma_r = 0.167 \\ \sigma^2(\Delta c_{t+1}) &= (0.036)^2 = 0.0013 \\ \sigma^2(r_{m,t+1}) &= (0.167)^2 = 0.0279 \\ corr(\Delta c_{t+1}, r) &= 0.4 \end{aligned}$$

with $\beta = 0.98$, $\alpha = 2$ and $\gamma = 0.5$ (which are reasonable) get 7.4% equity premium