Lisbon School of Economics & Management
Universidade de Lisboa

STATISTICS I

Economics / Finance/ Management

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LESSON 7

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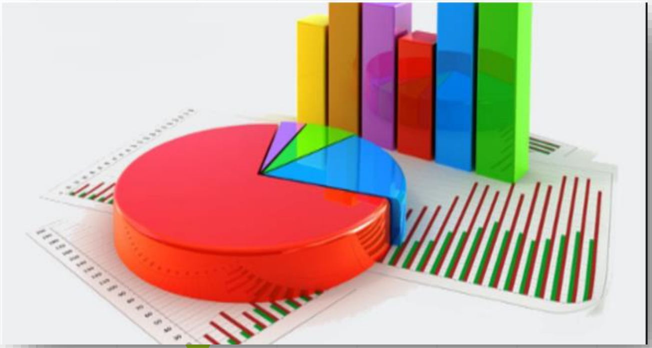

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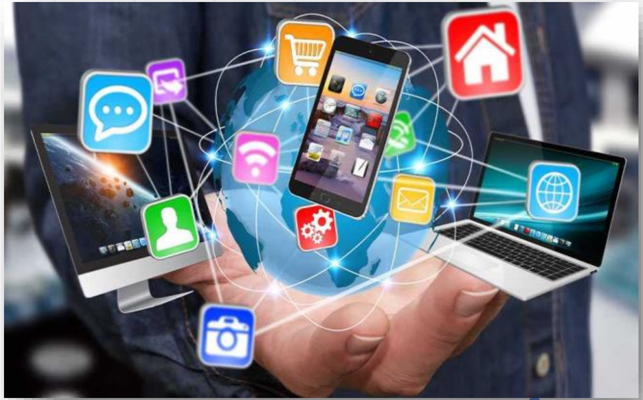

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Roadmap:

- Probability
- Random variable and two dimensional random variables:
  - Distribution
  - Joint distribution
  - Marginal distribution
  - Conditional distribution functions
- Expectations and parameters for a random variable and two dimensional random variables
- Discrete Distributions
- Continuous Distributions

Bibliography: Miller & Miller, John E. (2014) Freund’s Mathematical Statistics with applications, 8th Edition, Pearson Education, [MM]

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Two-Dimensional Random Variables: Exercises

Joint Distribution, Marginal Distribution and Conditional Distribution Functions

Chapter 4

1. Let (X, Y) be a two-dimensional random variable such that its cumulative distribution function is given by

$$F_{X,Y}(x,y) = \begin{cases} 0, & x < 0 \text{ or } y < 0 \\ \frac{x+y}{2}, & 0 \leq x < 1 \text{ and } 0 \leq y < 1 \\ \frac{1+y}{2}, & x \geq 1 \text{ and } 0 \leq y < 1 \\ \frac{x+1}{2}, & 0 \leq x < 1 \text{ and } y \geq 1 \\ 1, & x \geq 1 \text{ and } y \geq 1 \end{cases}$$

Compute the marginal cumulative distribution functions of X and Y: F_X and F_Y.

- The Marginal cumulative distribution function of X :
  $F_X(x) = P(X \leq x) = P(X \leq x, Y \leq +\infty) = \lim_{y \to +\infty} F_{X,Y}(x,y)$.
- The Marginal cumulative distribution function of Y :
  $F_Y(y) = (Y \leq y) = P(X \leq +\infty, Y \leq y) = \lim_{x \to +\infty} F_{X,Y}(x,y)$.

Exercise 1

$$F_{X,Y}(x,y) = \begin{cases} 0, & x<0 \text{ or } y<0 \\ \frac{x+y}{2}, & 0\le x<1 \text{ and } 0\le y<1 \\ \frac{1+y}{2}, & \underbrace{x\ge 1}_{} \text{ and } 0\le y<1 \\ \frac{x+1}{2}, & 0\le x<1 \text{ and } y\ge 1 \\ 1, & \underbrace{x\ge 1}_{x\to+\infty} \text{ and } y\ge 1 \end{cases} \quad y\to+\infty$$

$$F_X(x) = \lim_{y\to+\infty} F_{X,Y}(x,y) = \begin{cases} 0 & (x<0) \\ \frac{x+1}{2} & (0\leqslant x<1) \\ 1 & (x\geqslant 1) \end{cases}$$

$$F_Y(y) = \lim_{x\to+\infty} F_{X,Y}(x,y) = \begin{cases} 0 & (y<0) \\ \frac{1+y}{2} & (0\leqslant y<1) \\ 1 & (y\geqslant 1) \end{cases}$$

2. Let X and Y be two independent random variables such that

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{x}{2}, & 0 \leq x < 2 \\ 1, & x \geq 2 \end{cases}, \quad \text{and} \quad F_Y(y) = \begin{cases} 0, & y < 0 \\ \frac{y}{3}, & 0 < y < 3 \\ 1, & y \geq 3 \end{cases}.$$

a) Find the joint density function of X and Y.

b) Compute the marginal density function of X and Y

Exercise 2 a)

a)

Because $X \perp Y$ we have:

$$F_{X,Y}(x,y) = F_X(x)\,F_Y(y) = \begin{cases} 0 & (x<0 \vee y<0) \\ \frac{xy}{6} & (0\leq x<2,\ 0\leq y<3) \\ \frac{x}{2} & (0\leq x<2,\ y\geq 3) \\ \frac{y}{3} & (x\geq 2,\ 0\leq y<3) \\ 1 & (x\geq 2,\ y\geq 3) \end{cases}$$

The joint density is:

↳ Note: The solutions only show $F_{X,Y}(x,y)$

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y) = \begin{cases} \frac{1}{6} & (0<x<2,\ 0<y<3) \\ 0 & (\text{otherwise}) \end{cases}$$

Auxiliary calculations:

$$\frac{\partial^2}{\partial x \partial y}(0) = \frac{\partial^2}{\partial x \partial y}(1) = 0$$

$$\frac{\partial^2}{\partial x \partial y}\left(\frac{xy}{6}\right) = \frac{\partial}{\partial x}\left(\frac{x}{6}\right) = \frac{1}{6}$$

$$\frac{\partial^2}{\partial x \partial y}\left(\frac{x}{2}\right) = \frac{\partial^2}{\partial x \partial y}\left(\frac{y}{3}\right) = 0$$

Exercise 2 b)

b)

$$f_X(x) = \frac{d}{dx} F_X(x) = \begin{cases} \frac{1}{2} & (0 < x < 2) \\ 0 & (\text{otherwise}) \end{cases}$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} \frac{1}{3} & (0 < y < 3) \\ 0 & (\text{otherwise}) \end{cases}$$

Alternative solution:

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dy = \begin{cases} \int_0^3 \frac{1}{6}\, dy = \frac{1}{6}\left[y\right]_{y=0}^{y=3} = \frac{3}{6} = \frac{1}{2} & (0 < x < 2) \\ 0 & (\text{otherwise}) \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dx = \begin{cases} \int_0^2 \frac{1}{6}\, dy = \frac{1}{6}\left[x\right]_{x=0}^{x=2} = \frac{2}{6} = \frac{1}{3} & (0 < y < 3) \\ 0 & (\text{elsewhere}) \end{cases}$$

3. If the values of the joint probability function of X and Y are as shown in the table

X	0	1	2
Y			
0	1/12	1/6	1/24
1	1/4	1/4	1/40
2	1/8	1/20	0
3	1/120	0	0

(a) find:

i. $P(X = 1, Y = 2)$;

ii. $P(X = 0, 1 \leq Y < 3)$;

iii. $P(X + Y \leq 1)$;

iv. $P(X > Y)$.

(b) find the following values of the joint cumulative distribution function of the two random variables:

i. $F(1.2, 0.9)$;

ii. $F(-3, 1.5)$;

iii. $F(2, 0)$;

iv. $F(4, 2.7)$.

Exercise 3 a)

$f_{X,Y}(x,y)$:

Y \ X	0	1	2
0	1/12	1/6	1/24
1	1/4	1/4	1/40
2	1/8	1/20	0
3	1/120	0	0

(iii) (iv) (ii) (i)

a)

(i) $P(X=1, Y=2) = f_{X,Y}(1,2) = \frac{1}{20}$

(ii) $P(X=0,\ 1 \leq Y < 3) = \sum_{(x,y):\, x=0 \,\wedge\, 1\leq y<3} f_{X,Y}(x,y) = \sum_{y=1}^{2} f_{X,Y}(0,y) =$

$= f_{X,Y}(0,1) + f_{X,Y}(0,2) = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$

(iii) $P(X+Y \leq 1) = \sum_{(x,y)\in D_{X,Y}:\, x+y\leq 1} f_{X,Y}(x,y) = f_{X,Y}(0,0) + f_{X,Y}(1,0) + f_{X,Y}(0,1) = \frac{1}{12} + \frac{1}{6} + \frac{1}{4} =$

$= \frac{2}{24} + \frac{4}{24} + \frac{1}{4} = \frac{6}{24} + \frac{1}{4} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

(iv) $P(X>Y) = \sum_{(x,y)\in D_{X,Y}:\, x>y} f_{X,Y}(x,y) = f_{X,Y}(1,0) + f_{X,Y}(2,0) + f_{X,Y}(2,1) = \frac{1}{6} + \frac{1}{24} + \frac{1}{40} = \frac{5}{24} + \frac{1}{40} =$

$= \frac{200}{960} + \frac{24}{960} = \frac{224}{960} = \cdots = \frac{7}{30}$

Exercise 3 b)

b)

(i) $F_{X,Y}(1.2,\ 0.9) = F_X(1,0) = P(X \leq 1,\ Y \leq 0) = \sum_{(x,y) \in D_{X,Y}:\ x \leq 1 \wedge y \leq 0} f_{X,Y}(x,y) = f_{X,Y}(1,0) + f_X(0,0) = \frac{1}{6} + \frac{1}{12} = \frac{3}{12} = \frac{1}{4}$

(or:) $= \sum_{(x,y) \in D_{X,Y}:\ x \leq 1.2 \wedge y \leq 0.9} f_{X,Y}(x,y) =$

(ii) $F_{X,Y}(-3,\ 1.5) = F_X(-3,1) = P(X \leq -3,\ Y \leq 1) = \sum_{(x,y):\ D_{X,Y}:\ x \leq -3 \wedge y \leq 1} f_{X,Y}(x,y) = 0$

(or:) $= \sum_{(x,y) \in D_{X,Y}:\ x \leq -3 \wedge y \leq 1.5} f_{X,Y}(x,y) =$

(iii) $F_{X,Y}(2,0) = P(X \leq 2,\ Y \leq 0) = \sum_{(x,y) \in D_{X,Y}:\ x \leq 2 \wedge y \leq 0} f_{X,Y}(x,y) = f_{X,Y}(2,0) + f_{X,Y}(1,0) + f_{X,Y}(0,0) = \frac{1}{24} + \frac{1}{6} + \frac{1}{12} = \frac{1}{24} + \frac{4}{24} + \frac{2}{24} = \frac{7}{24}$

(iv) $F_{X,Y}(4,\ 2.7) = F_{X,Y}(2,2) = P(X \leq 2,\ Y \leq 2) = 1 - P(X > 2 \vee Y > 2) = 1 - f_{X,Y}(0,3) = 1 - \frac{1}{120} = \frac{119}{120}$

(or:) $= \sum_{(x,y) \in D_{X,Y}:\ x \leq 4 \wedge y \leq 2.7} f_{X,Y}(x,y) = \sum_{(x,y) \in D_{X,Y}:\ x \leq 2 \wedge y \leq 2} f_{X,Y}(x,y) =$

X	0	1	2
Y			
0	1/12	1/6	1/24
1	1/4	1/4	1/40
2	1/8	1/20	0
3	1/120	0	0

4. If the joint probability distribution of X and Y is given by

$$f_{X,Y}(x,y) = c(x^2+y^2), \quad \text{for } x = 1,3; \quad y = -1,2.$$

a) Find the value of c;

b) Compute $P(X+Y>2)$;

c) Compute the cumulative distribution function.

$$F_{X,Y}(x,y) = P(X \le x, Y \le y)$$

Properties of the joint cumulative distribution function:

- $0 \le F_{X,Y}(x,y) \le 1$
- $F_{X,Y}(x,y)$ is non decreasing with respect to x and y :
  - $\Delta_x > 0 \Rightarrow F_{X,Y}(x+\Delta x, y) \ge F_{X,Y}(x,y)$
  - $\Delta_y > 0 \Rightarrow F_{X,Y}(x, y+\Delta y) \ge F_{X,Y}(x,y)$
- $\lim\limits_{x\to-\infty} F_{X,Y}(x,y) = 0$, $\lim\limits_{y\to-\infty} F_{X,Y}(x,y) = 0$ and $\lim\limits_{x\to+\infty, y\to+\infty} F_{X,Y}(x,y) = 1$
- $P(x_1 < X \le x_2, y_1 < Y \le y_2) = F_{X,Y}(x_2,y_2) - F_{X,Y}(x_1,y_2) - F_{X,Y}(x_2,y_1) + F_{X,Y}(x_1,y_1)$.
- $F_{X,Y}(x,y)$ is right continuous with respect to x and y: $\lim\limits_{x\to a^+} F_{X,Y}(x,y) = F_{X,Y}(a,y)$ and $\lim\limits_{y\to b^+} F_{X,Y}(x,y) = F_{X,Y}(x,b)$.

Exercise 4 a)

$$f_{X,Y}(x,y) = c(x^2+y^2) \quad (x = 1,3;\ y = -1,2)$$

a) $f_{X,Y}(x,y)$ is a f.m.f. so:

$$\sum_{(x,y)\in D_{X,Y}} f_{X,Y}(x,y) = 1 \Leftrightarrow c \ \ 1^2 + (-1)^2 + c(1^2+2^2) + c(3^2+(-1)^2) + (3^2+2^2) = 1 \Leftrightarrow$$

$$\Leftrightarrow c(1+1+1+4+9+1+9+4) = 1 \Leftrightarrow$$

$$\Leftrightarrow 30c = 1 \Leftrightarrow c = \frac{1}{30}$$

Exercise 4 b)

b)

$$P(X+Y>2) = \sum_{(x,y)\in D_{X,Y}:\ x+y>2} f_{X,Y}(x,y) = f_{X,Y}(1,2) + f_{X,Y}(3,2) =$$

$$= \frac{1}{30}(1^2+2^2) + \frac{1}{30}(3^2+2^2) = \frac{1}{30}(1+4+9+4) = \frac{18}{30} = \frac{9}{15} = \frac{3}{5}$$

Exercise 4 c)

e)

$$F(x, y) = \begin{cases} 0 & (x < -1 \lor y < -1) \\ f_{X,Y}(-1,-1) = \frac{2}{30} = \frac{1}{15} & (-1 \leq x < 3;\ -1 \leq y < 2) \\ f_{X,Y}(-1,-1) + f_{X,Y}(-1,2) = \frac{2}{30} + \frac{5}{30} = \frac{7}{30} & (-1 \leq x < 3;\ y \geq 2) \\ f_{X,Y}(-1,-1) + f_{X,Y}(3,-1) = \frac{2}{30} + \frac{10}{30} = \frac{12}{30} = \frac{6}{15} = \frac{2}{5} & (x \geq 3;\ -1 \leq y < 2) \\ 1 & (x \geq 3;\ y \geq 2) \end{cases}$$

5. Given the values of the joint probability distribution of X and Y shown in the table

	$X=-1$	$X=1$
$Y=-1$	$\frac{1}{8}$	$\frac{1}{2}$
$Y=0$	0	$\frac{1}{4}$
$Y=1$	$\frac{1}{8}$	0

a) Find the marginal probability function of X.

b) Find the marginal probability function of Y.

c) Find the conditional probability function of X given $Y=-1$.

d) Compute the conditional cumulative distribution function of X given $Y=-1$.

e) Verify if X and Y are independent.

Exercise 5 a)

$f_{X,Y}(x,y)$	$X=-1$	$X=1$
$Y=-1$	$\frac{1}{8}$	$\frac{1}{2}$
$Y=0$	0	$\frac{1}{4}$
$Y=1$	$\frac{1}{8}$	0

$D_X = \{-1, 1\}$

$D_Y = \{-1, 0, 1\}$

a)

$$f_X(x) = \sum_{y \in D_Y} f_{X,Y}(x,y) = \begin{cases} \frac{1}{8} + \frac{1}{8} = \frac{2}{8} = \frac{1}{4} & (x=-1) \\ \frac{1}{2} + \frac{1}{4} = \frac{3}{4} & (x=1) \\ 0 & (\text{elsewhere}) \end{cases}$$

Exercise 5 b)

b)

$$f_Y(y) = \sum_{x \in D_X} f_{X,Y}(x,y) = \begin{cases} \frac{1}{8} + \frac{1}{2} = \frac{1}{8} + \frac{4}{8} = \frac{5}{8} & (y = -1) \\ \frac{1}{4} & (y = 0) \\ \frac{1}{8} & (y = 1) \\ 0 & (\text{elsewhere}) \end{cases}$$

Exercise 5 c)

c)

$$f_{X|Y=-1}(x) = \frac{f_{X,Y}(x,-1)}{f_Y(-1)} = \begin{cases} \frac{1}{8} / \frac{5}{8} = \frac{1}{5} & (x = -1) \\ \frac{1}{2} / \frac{5}{8} = \frac{4}{5} & (x = 1) \\ 0 & (\text{elsewhere}) \end{cases}$$

Exercise 5 d)

d)

$$\boxed{F_{X|Y=-1}(x) = \sum_{x' \in D_x :\, x' \leq x} f_{X|Y=-1}(x')}$$

$x = -1$

$x = 1$

Auxiliary calculations:

$$F_{X|Y=-1}(-1) = f_{X|Y=-1}(-1) = \frac{1}{5}$$

$$F_{X|Y=-1}(1) = f_{X|Y=-1}(-1) + f_{X|Y=-1}(1) = \frac{1}{5} + \frac{4}{5} = 1$$

Therefore:

$$F_{X|Y=-1}(x) = \begin{cases} 0 & (x < -1) \\ \frac{1}{5} & (-1 \leq x < 1) \\ 1 & (x \geq 1) \end{cases}$$

Exercise 5 e)

e)

$$f_{X,Y}(-1,0) = 0 \neq f_X(-1)\, f_Y(0) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$$

Therefore X and Y are not independent.

Also, we could have used $f_X(x) \neq f_{X|Y=-1}(x)$

6. If the values of the joint probability function of X and Y are as shown in the table

Y \ X	0	1	2
0	1/12	1/6	1/24
1	1/4	1/4	1/40
2	1/8	1/20	0
3	1/120	0	0

find

a) the marginal probability function of X;

b) the marginal probability function of Y;

c) the conditional probability function of X given $Y = 1$;

d) the conditional probability function of Y given $X = 0$.

Exercise 6 a)

X	0	1	2
Y			
0	1/12	1/6	1/24
1	1/4	1/4	1/40
2	1/8	1/20	0
3	1/120	0	0

a)

$$f_X(0) = \frac{1}{12} + \frac{1}{4} + \frac{1}{8} + \frac{1}{120} = \cdots = \frac{7}{15}$$

$$f_X(1) = \frac{1}{6} + \frac{1}{4} + \frac{1}{20} = \cdots = \frac{7}{15}$$

$$f_X(2) = \frac{1}{24} + \frac{1}{40} = \cdots = \frac{1}{15}$$

<u>Conclusion</u>:

$$f_X(x) = \begin{cases} \frac{7}{15} & (x = 0, 1) \\ \frac{1}{15} & (x = 2) \\ 0 & (\text{elsewhere}) \end{cases}$$

Exercise 6 b)

b)

$$f_Y(0) = \frac{1}{12} + \frac{1}{6} + \frac{1}{24} = \frac{7}{24}$$

$$f_Y(1) = \frac{1}{4} + \frac{1}{4} + \frac{1}{40} = \frac{21}{40}$$

$$f_Y(2) = \frac{1}{8} + \frac{1}{20} = \frac{7}{40}$$

$$f_Y(3) = \frac{1}{120}$$

<u>Conclusion</u>:

$$f_Y(y) = \begin{cases} \frac{7}{24} & (y = 0) \\ \frac{21}{40} & (y = 1) \\ \frac{7}{40} & (y = 2) \\ \frac{1}{120} & (y = 3) \\ 0 & (\text{elsewhere}) \end{cases}$$

Exercise 6 c)

c)

$$f_{X|Y=1}(x) = \frac{f_{X,Y}(x,1)}{f_Y(1)} = \begin{cases} f_{X,Y}(0,1)/f_Y(1) = \frac{1}{4} / \frac{21}{40} = \frac{10}{21} & (x=0) \\ f_{X,Y}(1,1)/f_Y(1) = \frac{1}{4} / \frac{21}{40} = \frac{10}{21} & (x=1) \\ f_{X,Y}(2,1)/f_Y(1) = \frac{1}{40} / \frac{21}{40} = \frac{1}{21} & (x=2) \\ 0 & (\text{elsewhere}) \end{cases}$$

Therefore:

$$f_{X|Y=1}(x) = \begin{cases} \frac{10}{21} & (x = 1) \\ \frac{1}{21} & (x = 2) \\ 0 & (\text{elsewhere}) \end{cases}$$

Exercise 6 d)

d)

$$f_{Y|X=0}(y) = \frac{f_{X,Y}(0,y)}{f_X(0)} = \begin{cases} f_{X,Y}(0,0)/f_X(0) = \frac{1}{12}/\frac{7}{15} = \frac{5}{28} & x = 0 \\ f_{X,Y}(0,1)/f_X(0) = \frac{1}{4}/\frac{7}{15} = \frac{15}{28} & x = 1 \\ f_{X,Y}(0,2)/f_X(0) = \frac{1}{8}/\frac{7}{15} = \frac{15}{56} & x = 2 \\ f_{X,Y}(0,3)/f_X(0) = \frac{1}{120}/\frac{7}{15} = \frac{1}{56} & x = 3 \\ 0 & (\text{elsewhere}) \end{cases}$$

Therefore:

$$f_{Y|X=0}(y) = \begin{cases} \frac{5}{28} & (y = 0) \\ \frac{15}{28} & (y = 1) \\ \frac{15}{56} & (y = 2) \\ \frac{1}{56} & (y = 3) \\ 0 & (\text{elsewhere}) \end{cases}$$

7. If the joint cumulative distribution function of X and Y is given by

$$F_{X,Y}(x,y) = \begin{cases} (1-e^{-x^2})(1-e^{-y^2}) & \text{, for } x \geq 0,\ y \geq 0 \\ 0 & \text{, elsewhere} \end{cases}$$

(a) Find the marginal cumulative distribution functions of the two random variables X and Y.

(b) Find the joint density function of the two random variables X and Y.

(c) Find $P(1 < X \leq 2, 1 < Y \leq 2)$.

(d) Verify if X and Y are independent random variables.

Exercise 7 a)

$$F_{X,Y}(x,y) = \begin{cases} (1-e^{-x^2})(1-e^{-y^2}) & \text{, for } x \geq 0,\ y \geq 0 \\ 0 & \text{, elsewhere} \end{cases}$$

a)

$$F_X(x) = \lim_{y \to +\infty} F_{X,Y}(x,y) = \begin{cases} 1-e^{-x^2} & (x \geq 0) \\ 0 & (x < 0) \end{cases}$$

Auxiliary calculation:

$$\lim_{y \to +\infty} \left((1-e^{-x^2})(1-e^{-y^2})\right) = 1-e^{-x^2}$$

$$F_Y(y) = \lim_{x \to +\infty} F_{X,Y}(x,y) = \begin{cases} 1-e^{-y^2} & (y \geq 0) \\ 0 & (y < 0) \end{cases}$$

Auxiliary calculation:

$$\lim_{x \to +\infty} \left((1-e^{-x^2})(1-e^{-y^2})\right) = 1-e^{-y^2}$$

Exercise 7 b)

b)

$$f_{X,Y}(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y} F_{X,Y}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left((1-e^{-x^2})(1-e^{-y^2})\right)\right) =$$

$$= \frac{\partial}{\partial x}\left((1-e^{-x^2})\frac{\partial}{\partial y}(1-e^{-y^2})\right) =$$

$$= \frac{\partial}{\partial x}\left((1-e^{-x^2})\, 2ye^{-y^2}\right) = 2ye^{-y^2}\frac{\partial}{\partial x}(1-e^{-x^2}) =$$

$$= 2ye^{-y^2}\, 2xe^{-x^2} = 4xye^{-y^2}e^{-x^2} \quad (x>0,\ y>0);$$

$f_{X,Y}(x,y) = 0$ otherwise

Exercise 7 c)

e)

$$P(1<X<2,\ 1<Y<2) = \int_1^2\int_1^2 f_{X,Y}(x,y)\,dx\,dy =$$

$$= \int_1^2\int_1^2 4xy\,e^{-y^2}e^{-x^2}\,dy\,dx = 4\int_1^2 x\,e^{-x^2}\int_1^2 y\,e^{-y^2}\,dy\,dx =$$

$$= 4\int_1^2 x\,e^{-x^2}\,\tfrac{1}{2}\left[-e^{-y^2}\right]_1^2 dx = -2\int_1^2 x\,e^{-x^2}\left(e^{-4}-e^{-1}\right)dx =$$

$$= -2\left(e^{-4}-e^{-1}\right)\int_1^2 x\,e^{-x^2}\,dx = -2\left(e^{-4}-e^{-1}\right)\tfrac{1}{2}\left[-e^{-x^2}\right]_1^2 =$$

$$= \left(e^{-4}-e^{-1}\right)\left(e^{-4}-e^{-1}\right) = \left(e^{-4}-e^{-1}\right)^2 \simeq 0.122$$

Exercise 7 c)

Or :

$$P(1<X<2,\ 1<Y<2) = F_{X,Y}(2,2) - F_{X,Y}(2,1) - F_{X,Y}(1,2) + F_{X,Y}(1,1)$$

$$= (1-e^{-4})(1-e^{-4}) - (1-e^{-4})(1-e^{-1}) - (1-e^{-1})(1-e^{-4}) + (1-e^{-1})(1-e^{-1}) =$$

$$= (1-e^{-4})^2 - 2(1-e^{-1})(1-e^{-4}) + (1-e^{-1})^2 \simeq 0.122$$

Exercise 7 d)

d)

$$F_X(x) = \begin{cases} 1-e^{-x^2} & (x \geqslant 0) \\ 0 & (x < 0) \end{cases}$$

$$F_Y(y) = \begin{cases} 1-e^{-y^2} & (y \geqslant 0) \\ 0 & (y < 0) \end{cases}$$

$$F_X(x)\, F_Y(y) = \begin{cases} (1-e^{-x^2})(1-e^{-y^2}) & (x \geqslant 0,\ y \geqslant 0) \\ 0 & (\text{elsewhere}) \end{cases} = F_{X,Y}(x,y)$$

Conclusion: $X \perp Y$ because $\forall (x,y) \in \mathbb{R}^2 \Rightarrow F_{X,Y}(x,y) = F_X(x)\, F_Y(y)$

8. If the joint probability density of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{4}(2x+y) & , \text{for } 0 < x < 1,\ 0 < y < 2 \\ 0 & , \text{elsewhere} \end{cases}$$

find

a) the marginal density of X;

b) the conditional density of Y given $X = 1/4$

c) the marginal density of Y;

d) the conditional density of X given $Y = 1$;

e) $P(X > \frac{1}{4} | X < \frac{3}{4})$.

f) the joint cumulative distribution function of X and Y.

Exercise 8 a)

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{4}(2x+y) & \text{, for } 0 < x < 1,\ 0 < y < 2 \\ 0 & \text{, elsewhere} \end{cases}$$

a)

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy = \int_0^2 \frac{1}{4}(2x+y)\,dy =$$

$$= \frac{1}{4}\int_0^2 2x + y\,dy = \frac{1}{4}\left[2xy + \frac{y^2}{2}\right]_{y=0}^{y=2} =$$

$$= \frac{1}{4}(4x+2) = x + \frac{1}{2} \qquad (0 < x < 1)$$

Exercise 8 b)

b)

$$f_{Y|X=\frac{1}{4}}(y) = \frac{f_{X,Y}(\frac{1}{4}, y)}{f_X(\frac{1}{4})} = \frac{\frac{1}{4}(\frac{1}{2}+y)}{\frac{1}{4}+\frac{1}{2}} = \frac{\frac{1}{8}+\frac{1}{4}y}{\frac{3}{4}} =$$

$$= \frac{1}{6} + \frac{1}{3}y \quad (0 < y < 2)$$

Exercise 8 c)

c)

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dx = \int_0^1 \frac{1}{4}(2x+y)\, dx =$$

$$= \frac{1}{4}\int_0^1 2x + y\, dx = \frac{1}{4}\left[x^2 + xy\right]_{x=0}^{x=1} =$$

$$= \frac{1}{4}(1+y) = \frac{1}{4} + \frac{1}{4}y \qquad (0 < y < 2)$$

Exercise 8 d)

d)

$$f_{X|Y=1}(x) = \frac{f_{X,Y}(x,1)}{f_Y(1)} = \frac{\frac{1}{4}(2x+1)}{\frac{1}{4}+\frac{1}{4}} = \frac{\frac{1}{4}+\frac{1}{2}x}{\frac{1}{2}} =$$

$$= \frac{1}{2} + x \qquad (0 < x < 1)$$

Exercise 8 e)

$$P(X > \tfrac{1}{4} \mid X < \tfrac{3}{4}) = \frac{P(\frac{1}{4} < X < \frac{3}{4})}{P(X < \frac{3}{4})} = \frac{\frac{1}{2}}{\frac{21}{32}} = \frac{32}{42} = \frac{16}{21}$$

$$P(\tfrac{1}{4} < x < \tfrac{3}{4}) = \int_{\frac{1}{4}}^{\frac{3}{4}} f_x(x)\,dx = \int_{\frac{1}{4}}^{\frac{3}{4}} x + \tfrac{1}{2}\,dx =$$

$$= \left[\frac{x^2}{2} + \frac{1}{2}x\right]_{1/4}^{3/4} = \frac{1}{2}\left(\left(\frac{3}{4}\right)^2 + \frac{3}{4} - \left(\left(\frac{1}{4}\right)^2 + \frac{1}{4}\right)\right) =$$

$$= \frac{1}{2}\left(\frac{9}{16} + \frac{3}{4} - \left(\frac{1}{16} + \frac{1}{4}\right)\right) = \frac{1}{2}\left(\frac{9}{16} + \frac{12}{16} - \frac{1}{16} - \frac{4}{16}\right) =$$

$$= \frac{1}{2}(1) = \frac{1}{2}$$

$$P(X < \tfrac{3}{4}) = \int_{-\infty}^{\frac{3}{4}} f_x(x)\,dx = \int_{0}^{\frac{3}{4}} x + \tfrac{1}{2}\,dx = \left[\frac{x^2}{2} + \frac{1}{2}x\right]_0^{\frac{3}{4}} =$$

$$= \frac{1}{2}\left(\left(\frac{3}{4}\right)^2 + \frac{3}{4}\right) = \frac{1}{2}\left(\frac{9}{16} + \frac{12}{16}\right) = \frac{1}{2}\left(\frac{21}{16}\right) = \frac{21}{32}$$

Exercise 8 f)

Auxiliary calculations:

$0 < x < 1 \,,\; 0 < y < 2$

$$F_{X,Y}(x,y) = \int_0^x \int_0^y f_{X,Y}(u,v)\,dv\,du = \int_0^x \int_0^y \frac{1}{4}(2u+v)\,dv\,du =$$

$$= \frac{1}{4}\int_0^x \left[2uv + \frac{v^2}{2}\right]_{v=0}^{v=y} du = \frac{1}{4}\int_0^x 2uy + \frac{y^2}{2}\,du = \frac{1}{4}\left[2y\frac{u^2}{2} + \frac{y^2}{2}u\right]_{u=0}^{u=x} =$$

$$= \frac{1}{4}\left(yx^2 + \frac{y^2}{2}x\right) = \frac{1}{4}yx^2 + \frac{1}{8}xy^2$$

Exercise 8 f)

$\underline{0 < x < 1, \quad y \geq 2:}$

$$F_{X,Y}(x,y) = \int_0^x \underbrace{\int_{-\infty}^{+\infty} f_{X,Y}(u,y)\,dy}_{f_X(u)}\,du =$$

$$= \int_0^x f_X(u)\,du = \int_0^x u + \frac{1}{2}\,du =$$

$$= \left[\frac{u^2}{2} + \frac{1}{2}u\right]_0^x = \frac{1}{2}x^2 + \frac{1}{2}x$$

$\underline{x \geq 1, \quad 0 < y < 2}$

$$F_{X,Y}(x,y) = \int_0^y \underbrace{\int_{-\infty}^{+\infty} f_{X,Y}(x,v)\,dx}_{f_Y(v)}\,dv = \int_0^y \frac{1}{4} + \frac{1}{4}v\,dv =$$

$$= \left[\frac{1}{4}v + \frac{1}{8}v^2\right]_0^y = \frac{1}{4}y + \frac{1}{8}y^2$$

Exercise 8 f)

Conclusion:

$$F_{X,Y}(x,y) = \begin{cases} 0 & (x<0 \vee y<0) \\ \frac{1}{4}yx^2 + \frac{1}{8}xy^2 & (0 \leqslant x < 1,\ 0 \leqslant y < 2) \\ \frac{1}{2}x^2 + \frac{1}{2}x & (0 \leqslant x < 1 \wedge y \geqslant 2) \\ \frac{1}{4}y + \frac{1}{8}y^2 & (x \geqslant 1,\ 0 \leqslant y < 2) \\ 1 & (x \geqslant 1 \wedge y \geqslant 2) \end{cases}$$

9. Consider the joint probability density function of X and Y given by

$$f_{X,Y}(x,y) = \begin{cases} kxy & \text{, for } 0 < x < 2,\ 0 < y < 1 \\ 0 & \text{, elsewhere} \end{cases}$$

a) Determine k so that $f_{X,Y}$ can serve as a joint probability density function.

b) Determine the joint distribution function of X and Y.

c) Determine the marginal cumulative distribution function of X and Y.

d) Determine the marginal probability density function of X and Y.

e) Verify if X and Y are independent.

f) Compute Compute $P(X < 1.25, Y \leq 0.5)$, $P\left(Y > \frac{X}{4} + \frac{1}{2}\right)$ and $P\left(Y < \frac{X}{4} + \frac{1}{2}\right)$.

g) Compute $f_{X|Y=y}$ and $f_{Y|X=x}$.

Exercise 9 a)

$$f_{X,Y}(x,y) = \begin{cases} kxy & \text{, for } 0 < x < 2,\ 0 < y < 1 \\ 0 & \text{, elsewhere} \end{cases}$$

a) For $f_{X,Y}(x,y)$ to be a p.d.f. we must have:

$$\int_0^2 \int_0^1 f_{X,Y}(x,y)\,dy\,dx = 1 \Leftrightarrow$$

$$\Leftrightarrow \int_0^2 \int_0^1 Kxy\,dy\,dx = 1 \Leftrightarrow K\int_0^2 x \int_0^1 y\,dy\,dx = 1 \Leftrightarrow$$

$$\Leftrightarrow K\int_0^2 x \frac{1}{2}\left[y^2\right]_0^1 dx = 1 \Leftrightarrow K\int_0^2 x\,dx = 2 \Leftrightarrow$$

$$\Leftrightarrow K\frac{1}{2}\left[x^2\right]_0^2 = 2 \Leftrightarrow K(4-0) = 4 \Leftrightarrow K = 1$$

Exercise 9 b)

b) $$f_{X,Y}(x,y) = \begin{cases} xy & (0 < x < 2,\ 0 < y < 1) \\ 0 & (\text{otherwise}) \end{cases}$$

$$F_{X,Y}(x,y) = \int_0^x \int_0^y f_{X,Y}(u,v)\,dv\,du = \begin{cases} 0 & (x < 0 \lor y < 0) \\ \dfrac{x^2 y^2}{4} & (0 \leq x < 2,\ 0 \leq y < 1) \\ y^2 & (x \geq 2,\ 0 \leq y < 1) \\ \dfrac{x^2}{4} & (0 \leq x < 2,\ y \geq 1) \\ 1 & (x \geq 2,\ y \geq 1) \end{cases}$$

Exercise 9 b)

Auxiliary calculation:

<u>$0 < x < 2,\ 0 < y < 1$:</u>

$$F_{X,Y}(x,y) = \int_0^x \int_0^y u\, v \; dv\, du = \int_0^x \frac{u}{2} \left[v^2 \right]_{v=0}^{v=y} du =$$

$$= \frac{1}{2} \int_0^x u\, y^2\, du = \frac{1}{4} \left[u^2 y^2 \right]_{u=0}^{u=x} = \frac{x^2 y^2}{4}$$

<u>$x \geq 2,\ 0 \leq y < 1$:</u> $F_{X,Y}(x,y) = \frac{2^2 y^2}{4} = y^2$

<u>$0 \leq x < 2,\ y \geq 1$:</u> $F_{X,Y}(x,y) = \frac{x^2 \cdot 1}{4} = \frac{x^2}{4}$

Exercise 9 c)

Auxiliary calculation: $F_X(x) = \lim_{y \to +\infty} F(x, y) = \frac{x^2}{4} \quad (0 < x < 2)$

$$F_X(x) = \begin{cases} 0 & (x < 0) \\ x^2/4 & (0 \leq x < 2) \\ 1 & (x \geq 2) \end{cases}$$

Auxiliary calculation: $F_Y(y) = \lim_{x \to +\infty} F_{X,Y}(x, y) = y^2 \quad (0 < y < 2)$

$$F_Y(y) = \begin{cases} 0 & (y < 0) \\ y^2 & (0 \leq y < 1) \\ 1 & (y \geq 1) \end{cases}$$

Exercise 9 d)

d)

$$f_X(x) = \frac{\partial}{\partial x} F_X(x) = \begin{cases} \frac{x}{2} & (0 < x < 2) \\ 0 & (\text{otherwise}) \end{cases}$$

$$f_Y(y) = \frac{\partial}{\partial y} F_Y(y) = \begin{cases} 2y & (0 < y < 1) \\ 0 & (\text{otherwise}) \end{cases}$$

Exercise 9 e)

$X \perp Y$ because:

$$f_X(x)\, f_Y(y) = \begin{cases} \frac{x}{2}\, 2y \\ 0 \end{cases} = \begin{cases} xy & (0<x<2,\ 0<y<1) \\ 0 & (\text{otherwise}) \end{cases} = f_{X,Y}(x,y)$$

Exercise 9 f)

f)

$$P(X < 1.25, Y \leq 0.5) = F_{X,Y}(1.25, 0.5) = \frac{1.25^2 \times 0.5^2}{4} = \frac{\left(\frac{5}{4}\right)^2 \times \left(\frac{1}{2}\right)^2}{4} = \frac{\frac{25}{16} \times \frac{1}{4}}{4}$$

$$= \frac{25}{64} \times \frac{1}{4} = \frac{25}{256}$$

$$P\left(Y > \frac{X}{4} + \frac{1}{2}\right) = 1 - P\left(Y \leq \frac{X}{4} + \frac{1}{2}\right) = 1 - \frac{17}{24} = \frac{7}{27}$$

Exercise 9 f)

Auxiliary calculation:

$$P\left(Y \leq \frac{X}{4}+\frac{1}{2}\right) = \int_0^2 \int_0^{\frac{x}{4}+\frac{1}{2}} f_{X,Y}(x,y)\, dy\, dx =$$

$$= \int_0^2 \int_0^{\frac{x}{4}+\frac{1}{2}} x y \, dy\, dx = \frac{1}{2} \int_0^2 x \left[y^2\right]_{y=0}^{y=\frac{x}{4}+\frac{1}{2}} dx =$$

$$= \frac{1}{2} \int_0^2 x \left(\frac{x}{4}+\frac{1}{2}\right)^2 dx = \frac{1}{2} \int_0^2 x\left(\frac{x^2}{16}+\frac{1}{4}+\frac{x}{4}\right) dx =$$

$$= \frac{1}{2}\left(\frac{1}{16}\int_0^2 x^3 dx + \frac{1}{4}\int_0^2 x\, dx + \frac{1}{4}\int_0^2 x^2 dx\right) =$$

$$= \frac{1}{2}\left(\frac{1}{16}\left[\frac{x^4}{4}\right]_0^2 + \frac{1}{4}\left[\frac{x^2}{2}\right]_0^2 + \frac{1}{4}\left[\frac{x^3}{3}\right]_0^2\right) =$$

$$= \frac{1}{2}\left(\frac{16}{64} + \frac{4}{8} + \frac{8}{12}\right) = \frac{1}{2}\left(\frac{48}{64} + \frac{8}{12}\right) =$$

$$= \frac{1}{2}\left(\frac{576+512}{768}\right) = \frac{1088}{1536} \underset{\div 8}{=} \frac{136}{192} \underset{\div 2}{=} \frac{68}{96} \underset{\div 4}{=} \frac{17}{24}$$

$$P\left(Y < \frac{X}{4}+\frac{1}{2}\right) = P\left(Y \leq \frac{X}{4}+\frac{1}{2}\right) = \frac{17}{24}$$

Exercise 9 g)

g)

$$f_{X|Y=y}(x) = \begin{cases} \dfrac{f_{X,Y}(x,y)}{f_Y(y)} = \dfrac{xy}{2y} = \dfrac{x}{2} & (0<x<2,\ \text{fixed } y) \\ 0 & (\text{otherwise}) \end{cases}$$

$$f_{Y|X=x}(y) = \begin{cases} \dfrac{f_{X,Y}(x,y)}{f_X(x)} = \dfrac{xy}{\frac{x}{2}} = 2y & (0<y<1,\ \text{fixed } x) \\ 0 & (\text{otherwise}) \end{cases}$$

10. Let X and Y be two independent random variables such that

$$f_X(x) = \begin{cases} \frac{1}{2}, & 0 < x < 2 \\ 0, & \text{elsewhere} \end{cases}, \quad \text{and} \quad f_Y(y) = \begin{cases} \frac{1}{3}, & 0 < y < 3 \\ 0, & \text{elsewhere} \end{cases}.$$

Find the joint density function of X and Y.

Exercise 10

$X \perp Y$

$$f_X(x) = \begin{cases} \frac{1}{2}, & 0 < x < 2 \\ 0, & \text{elsewhere} \end{cases}, \quad \text{and} \quad f_Y(y) = \begin{cases} \frac{1}{3}, & 0 < y < 3 \\ 0, & \text{elsewhere} \end{cases}.$$

Since $X \perp Y$ we have:

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y) = \begin{cases} \frac{1}{6} & (0 < x < 2,\ 0 < y < 3) \\ 0 & (\text{otherwise}) \end{cases}$$

11. If the joint probability density of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} 24xy & , \text{for } 0 < x < 1,\ 0 < y < 1,\ x + y < 1 \\ 0 & , \text{elsewhere} \end{cases}$$

a) Compute $P(X + Y < 1/2)$.

b) Compute $F_{X|Y=\frac{1}{2}}(x)$

Exercise 11 a)

$$f_{X,Y}(x,y) = \begin{cases} 24xy & \text{, for } 0 < x < 1,\ 0 < y < 1,\ x + y < 1 \\ 0 & \text{, elsewhere} \end{cases}$$

a) $x + y < \frac{1}{2} \Leftrightarrow x < \frac{1}{2} - y$ $\qquad x + y < \frac{1}{2} \rightarrow \begin{cases} x < \frac{1}{2} \\ y < \frac{1}{2} \end{cases}$

$$P(X+Y < \tfrac{1}{2}) = \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}-y} 24xy \, dx \, dy =$$

$$= 24 \int_0^{\frac{1}{2}} y \int_0^{\frac{1}{2}-y} x \, dx \, dy = 24 \int_0^{\frac{1}{2}} y \tfrac{1}{2} \left[x^2 \right]_{x=0}^{x=\frac{1}{2}-y} dy =$$

$$= 12 \int_0^{\frac{1}{2}} y \left(\tfrac{1}{2} - y \right)^2 dy = 12 \int_0^{\frac{1}{2}} y \left(\tfrac{1}{4} + y^2 - y \right) dy =$$

$$= 12 \int_0^{\frac{1}{2}} \tfrac{1}{4} y + y^3 - y^2 \, dy = 12 \left[\tfrac{1}{8} y^2 + \tfrac{1}{4} y^4 - \frac{y^3}{3} \right]_{y=0}^{y=\frac{1}{2}}$$

$$= 12 \left(\tfrac{1}{8} \times \tfrac{1}{4} + \tfrac{1}{4} \times \tfrac{1}{16} - \tfrac{1}{3} \times \tfrac{1}{8} \right) = 12 \left(\tfrac{1}{32} + \tfrac{1}{64} - \tfrac{1}{24} \right) =$$

$$= \frac{12}{32} + \frac{12}{64} - \frac{12}{24} = \frac{24}{64} + \frac{12}{64} - \frac{12}{24} = \frac{36}{64} - \frac{12}{24} =$$

$$= \frac{864 - 768}{1536} = \frac{96}{1536} = \frac{1}{16}$$

Exercise 11 b)

b) $x + y < 1 \iff x < 1 - y$

$= P(X \leq x \mid Y = \frac{1}{2})$

$$F_{X|Y=\frac{1}{2}}(x) = \int_0^x f_{X|Y=\frac{1}{2}}(u)\,du = \int_0^x 8u\,du$$

because given $Y = \frac{1}{2}$ we have $x + y < 1 \iff x + \frac{1}{2} < 1 \iff x < \frac{1}{2}$

$$= \frac{8}{2}\left[u^2\right]_{u=0}^{u=x} = 4x^2 \quad (0 < x < \tfrac{1}{2})$$

Conclusion: $F_{X|Y=\frac{1}{2}}(x) = \begin{cases} 0 & (x < 0) \\ 4x^2 & (0 \leq x < \frac{1}{2}) \\ 1 & (x \geq \frac{1}{2}) \end{cases}$

Auxiliary calculations:

$$f_{X|Y=\frac{1}{2}}(x) = \frac{f_{X,Y}(x,\frac{1}{2})}{f_Y(\frac{1}{2})} = \frac{12x}{\frac{3}{2}} = \frac{24}{3}x = 8x \quad (0 < x < \tfrac{1}{2})$$

because $x + y < 1 \iff x < 1 - y$

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx = \int_0^{1-y} 24xy\,dx = 24\left[\frac{x^2}{2}y\right]_{x=0}^{x=1-y} =$$

$$= 12y(1-y)^2 \quad (0 < y < 1)$$

$$f_Y(\tfrac{1}{2}) = 6\left(1 - \tfrac{1}{2}\right)^2 = \frac{6}{4} = \frac{3}{2}$$

12. If the joint probability density of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} 24y\,(1-x-y) & , \text{for } x>0, y>0, x+y<1 \\ 0 & , \text{elsewhere} \end{cases}$$

a) Find the marginal density of X;

b) Find the marginal density of Y;

d) Determine whether the two random variables are independent.

Exercise 12 a)

$$x + y < 1 \iff \begin{cases} x < 1-y \\ y < 1-x \end{cases}$$

a)

$$f_X(x) = \int_0^{1-x} f_{X,Y}(x,y)\,dy = \int_0^{1-x} 24y(1-x-y)\,dy =$$

$$= 24\int_0^{1-x} y - xy - y^2\,dy = 24\left[\frac{y^2}{2} - x\frac{y^2}{2} - \frac{y^3}{3}\right]_{y=0}^{y=1-x}$$

$$= 24\left(\frac{(1-x)^2}{2} - x\frac{(1-x)^2}{2} - \frac{(1-x)^3}{3}\right) =$$

$$= 12(1-x)^2 - 12x(1-x)^2 - 8(1-x)^3 =$$

$$= (12 - 12x - 8(1-x))(1-x)^2 =$$

$$= (12 - 12x - 8 + 8x)(1-x)^2 =$$

$$= (4 - 4x)(1-x)^2 = 4(1-x)^3 \quad (0 < x < 1)$$

<u>Note</u>: $4(1-x)^3 = 4(1-x)^2(1-x) = -4(x-1)^2(-1)(1-x) =$
$= -4(x-1)^2(x-1) = -4(x-1)^3$ → Result in the answer sheet

b)

Exercise 12 b)

$$f_Y(y) = \int_0^{1-y} f_{X,Y}(x,y)\,dx = \int_0^{1-y} 24y(1-x-y)\,dx =$$

$$= 24y\int_0^{1-y} 1-x-y\,dx = 24y\left[x-\frac{x^2}{2}-xy\right]_{x=0}^{x=1-y}$$

$$= 24y\left(1-y-\frac{(1-y)^2}{2}-(1-y)y\right) =$$

$$= 24y\left(1-y-\frac{1+y^2-2y}{2}-y+y^2\right) =$$

$$= 24y\left(1-y-\frac{1}{2}-\frac{y^2}{2}+\cancel{y}-\cancel{y}+y^2\right) =$$

$$= 24y\left(\frac{1}{2}+\frac{y^2}{2}-y\right) = 12y+12y^3-24y^2 =$$

$$= 12(y-2y^2+y^3) = 12y(1-2y+y^2) =$$

$$= 12y(1-y)^2 \qquad (0<y<1)$$

Exercise 12 c)

$$f_X(x)\,f_Y(y) = 4(1-x)^3\,4(1-x)^3 = 8(1-x)^3 \neq f_{X,Y}(x,y) = 24y(1-x-y)$$

Conclusion: X and Y are not indelendent.

13. If X is the amount of money (in dollars) that a salesperson spends on gasoline during a day and Y is the corresponding amount of money (in dollars) for which he or she is reimbursed, the joint density of these two random variables is given by

$$f(x,y) = \begin{cases} \frac{1}{25}\left(\frac{20-x}{x}\right) & \text{, for } 10 < x < 20, \frac{x}{2} < y < x \\ 0 & \text{, elsewhere} \end{cases}$$

find

(a) the marginal density of X;

(b) the conditional density of Y given $X = 12$;

(c) the probability that the salesperson will be reimbursed at least \$8 when spending \$12.

Exercise 13 a)

a)

$$f_X(x) = \int_{\frac{x}{2}}^{x} f_{X,Y}(x,y)\,dy = \int_{\frac{x}{2}}^{x} \frac{1}{25}\left(\frac{20-x}{x}\right) dy =$$

$$= \frac{1}{25}\,\frac{20-x}{x}\,\left[\,y\,\right]_{y=\frac{x}{2}}^{y=x} = \frac{1}{25}\,\frac{20-x}{x}\left(x-\frac{x}{2}\right) =$$

$$= \frac{1}{25}\,\frac{20-x}{x}\,\frac{x}{2} = \frac{20x-x^2}{50x} = \frac{20-x}{50} \qquad (10<x<20)$$

Exercise 13 b)

b)

$$f_{Y|X=12}(y) = \frac{f_{X,Y}(12,y)}{f_X(12)} = \frac{\frac{1}{25}\left(\frac{20-12}{12}\right)}{\frac{20-12}{50}} = \frac{\frac{1}{25}\left(\frac{8}{12}\right)}{\frac{8}{50}} =$$

$$= \frac{50}{8}\,\frac{1}{25}\,\frac{8}{12} = \frac{2}{12} = \frac{1}{6} \qquad (6 < y < 12)$$

$$\left(x = 12 \Rightarrow \frac{x}{2} < Y < x \Leftrightarrow 6 < Y < 12\right)$$

c)

Exercise 13 c)

$$F_{Y|X=12}(8) = \int_{-\infty}^{8} f_{Y|X=12}(y)\,dy = \int_{6}^{8} \frac{1}{6}\,dy = \frac{1}{6}\left[\,y\,\right]_6^8 =$$

$$= \frac{8-6}{6} = \frac{2}{6} = \frac{1}{3}$$

$$P(Y \geqslant 8 \mid X = 12) = 1 - F_{Y|X=12}(8) = 1 - \frac{1}{3} = \frac{2}{3}$$

Thanks!

Questions?