Microeconomics

Math refresher

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Overview

Common Derivative Rules

Common Function	Function	Derivative
Constant	c	0
Line	x	1
	ax	a
Square	x^2	2x
Power	x^n	$n\ x^{n-1}$
Square Root	$\sqrt{x}$	$\frac{1}{2}x^{-1/2}$
Any Root	$x^{1/n}$	$\frac{1}{n}x^{(1-n)/n}$
Exponential	e^x	e^x
	a^x	$\ln(a)\ a^x$
Logarithms	ln(x)	1/x
	$log_a(x)$	1/(x ln(a))

Other Rules: Chain Rule

$$y = f(u) \quad u = f(x) \tag{1}$$

Chain Rule

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Example

- $y = (2x+4)^3$
- $y = u^3$ and $u = 2x + 4$
- $\frac{dy}{du} = 3u^2$ and $\frac{du}{dx} = 2$
- $\frac{dy}{dx} = 3u^2 \cdot 2 = 3(2x+4) \cdot 2 = 6(2x+4)^2$

Other Rules: Product Rule

$$f(x) = u(x) \cdot v(x) \tag{2}$$

Product Rule

$f'(x) = u'v + uv'$

Example

- $f(x) = \underbrace{(3x-5)}_{\text{u(x)}} \cdot \underbrace{(4x+7)}_{\text{v(x)}}$
- $u' = 3$ and $v' = 4$
- $f'(x) = 3(4x+7) + 4(3x-5) = 12x + 21 + 12x - 20 = 24x + 1$

Other Rules: Quotient Rule

$$f(x) = \frac{u(x)}{v(x)} \tag{3}$$

Quotient Rule

$f'(x) = \frac{u'v - v'u}{v^2}$

Example

- $f(x) = \frac{(3x-5)}{(4x+7)}$
- $u' = 3$ and $v' = 4$
- $f'(x) = \frac{3(4x+7)-4(3x-5)}{(4x+7)^2} = \frac{41}{(4x+7)^2}$

Overview

Exercise 1.a - Partial Derivative

$$f(x, y, z) = xy^2 + x^3y - z \tag{4}$$

- $f_x = \frac{\partial f}{\partial x} = y^2 + 3x^2y$
- $f_y = \frac{\partial f}{\partial y} = 2xy + x^3$
- $f_z = \frac{\partial f}{\partial z} = -1$

Exercise 1.b

$$f(x,y) = lnx + \sqrt{y} \tag{5}$$

- $f_x = \frac{1}{x}$
- $f_y = \frac{1}{2}y^{-\frac{1}{2}}$
  - Simplify it further: $= \frac{1}{2\sqrt{y}}$

Exercise 1.c

$$f(x,y) = xy^2(12 - 3x - 2y) \quad (6)$$

Product Rule

f'(x) = u' v + u v'

- $f_x = y^2(12 - 3x - 2y) + xy^2(-3)$
  $= y^2(12 - 3x - 2y - 3x) = 2y^2(6 - 3x - y)$

- $f_y = 2xy(12 - 3x - 2y) + xy^2(-2)$
  $= 2xy(12 - 3x - 2y - y) = 6xy(4 - x - y)$

Exercise 1.d

$$f(x, y) = \frac{3x}{x^2 + y^n} \quad (7)$$

Quotient Rule

If $f(x) = \frac{u(x)}{v(x)}$, then $f'(x) = \frac{u'v - uv'}{v^2}$

- $f_x = \frac{3(x^2+y^n) - 3x(2x)}{(x^2+y^n)^2}$
  $= \frac{3x^2+3y^n-6x^2}{(x^2+y^n)^2} = \frac{3(y^n-x^2)}{(x^2+y^n)^2}$

- $f_y = -\frac{3x \cdot ny^{n-1}}{(x^2+y^n)^2}$

Overview

Exercise 2.a - System of Equations

$$A : x - 3y + 6z = -1$$
$$B : 2x - 5y + 10z = 0$$
$$C : 3x - 8y + 17z = 1$$

Solving by elimination: Calculate $2A - B$ and $3A - C$ to eliminate x

- $A' : 2A - B = -y + 2z = -2$
- $B' : 3A - C = -y + z = -4$

Subtract B' from A', this way you eliminate y:

- You get: $z = 2$
- Plug it in, for instance into B' to get: $-y + 2 = -4$, leding to $y = 6$
- In equations A (or B/C), you can plug in the values of z and y to obtain $x = 5$

Exercise 2.b - System of Equations

$$A : x + y + z = 0$$

$$B : 12x + 2y - 3z = 5$$

$$C : 3x + 4y + z = -4$$

Solving by elimination: Calculate 3A +B and C-A to eliminate z

- $A' : 15x + 5y = 5$
- $B' : 2x + 3y = -4$

You can eliminate x by calculating $2(A'/5) - 3B'$. You obtain $y = -2$

- Then plug it back in B' (for instance): you obtain $x = 1$
- Plug the values of x and y into A and obtain $z = 1$

Overview

Short intro on function maximization

First-order condition (FOC)

- Consider the function $y = f(x)$.
- The necessary condition for a relative extremum (maximum or minimum) is that the first-order derivative be zero, i.e. $f'(x) = 0$.

Interpretation of the FOC

At the highest and lowest points of a curve, the tangent to the curve at such points is horizontal. The slope of the curve is zero.

Question

Why is $f'(x) = 0$ not a sufficient condition for a local maximum or minimum?
Answer: Because $f'(x) = 0$ at some inflexion points.
$\Rightarrow$ The first-order condition does not distinguish between a maximum and a minimum.

Short intro on function maximization (continues)

Second-order condition (SOC): If the first-order condition is satisfied at $x = x_0$,

- $f(x_0)$ is a local maximum if $f''(x_0) < 0$
- $f(x_0)$ is a local minimum if $f''(x_0) > 0$

Interpretation of the SOC - Maximum

As you move up a curve from the left, leading to a maximum, the curve gets increasingly flatter, i.e. the slope gets smaller and smaller. This means that $f'' < 0$. For example, if f' goes from 6 to 2, it means that $f'' < 0$.

Interpretation of the SOC - Minimum

As you move down a curve from the left, leading to a minimum, the curve gets increasingly flatter. However, since the slope is negative, a flattening of the curve implies that $f'' > 0$. For example, if f' goes from -3 to -2, it means that $f'' > 0$.

Exercise 3.a

$$f(x) = a - 2bx - x^2 \tag{8}$$

- $f'(x) = -2b - 2x = 0 \iff x = -b$
- Check the second order condition: $f''(x) = -2 < 0 \Rightarrow$ It is a maximum
- The local maximum is at $x = -b$ and $f(-b) = a + b^2$

Exercise 4.a

$$f(x) = x^2\dot{h}(g(x)) \text{ where } h(g(x)) = A - 2g(x) \text{ and } g(x) = x + y + z \tag{9}$$

Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Product Rule

$$f'(x) = u'v + uv'$$

$$\begin{aligned} f'(x) &= 2x \cdot h(g(x)) + x^2 \cdot h'(g(x)) \cdot g'(x) \\ &= 2x \cdot (A - 2g(x)) + x^2 \cdot (-2) \cdot 1 \\ &= 2x(A - 2(x + y + z)) - 2x^2 \\ &= 2x(A - 3x - 2y - 2z) \end{aligned}$$

Exercise 4.b

$$f(x) = 3x\dot{h}(g(x)) - cx \text{ where } h(g(x)) = 4\big(g(x)\big)^{-1} \text{ and } g(x) = ax + by + z \tag{10}$$

$$\begin{aligned}
f'(x) &= 3h(g(x)) + 3xh'(g(x)) \cdot g'(x) - c \\
&= 3 \cdot 4\big(g(x)\big)^{-1} + 3x(-4\big(g(x)\big)^{-2}) \cdot a - c \\
&= \frac{12}{g(x)} - \frac{12ax}{g(x)^2} - c \\
&= \frac{12\big(g(x) - ax\big)}{g(x)^2} - c \\
&= \frac{12(ax + by + z - ax)}{(ax + by + z)^2} - c \\
&= \frac{12(by + z)}{(ax + by + z)^2} - c
\end{aligned}$$

Overview

Exercise 5.a

$$\frac{4x-12}{6x-x^2-9} \tag{11}$$

$$\iff \frac{4(x-3)}{-(x^2-6x+9)} = \frac{4(x-3)}{-(x-3)^2} = \frac{4}{3-x}$$

Exercise 5.b

$$\frac{2x+6}{x^2-9} \div \frac{x^2+6x+9}{(x-3)(x+3)} \qquad (12)$$

$$\iff \frac{2(x+3)}{x^2-9} \div \frac{(x+3)^2}{x^2+3x-3x-9} = \frac{2(x+3)}{(x^2-9)} \cdot \frac{x^2-9}{(x+3)^2} = \frac{2}{x+3}$$

Exercise 5.c

$$\frac{x^2 - 13x + 42}{14 - 2x} \tag{13}$$

$$\iff \frac{(x^2 - 14x + 49) + (x - 7)}{2(7 - x)} = \frac{(x - 7)^2 + (x - 7)}{-2(x - 7)} = \frac{(x - 7)(x - 7 + 1)}{-2(x - 7)} = \frac{6 - x}{2}$$

Exercise 5.d

$$\frac{2x-4}{x^2-1} \div \frac{x^2-4}{x^2+3x+2} \tag{14}$$

$$\iff \frac{2(x-2)}{(x-1)(x+1)} \div \frac{(x-2)(x+2)}{(x+1)^2+(x+1)} = \frac{2(x-2)}{(x-1)(x+1)} \cdot \frac{(x+1)(x+1+1)}{(x-2)(x+2)}$$
$$= \frac{2}{x-1} \cdot \frac{x+2}{x+2} = \frac{2}{x-1}$$